Math 273 Lecture 12

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Let $p > q > 0$ and suppose that $-\frac{p}{q} = [r_0, r_1, \dots, r_k]$, where this notation represents the continued fraction

$$r_0 - \cfrac{1}{r_1 - \cfrac{1}{\ddots \frac{1}{r_k}}}$$

and $r_i \leq -2$ for all i. Last time we proved that there is an injective map

$$\pi_0 \mathrm{Tight}(S^1 \times D^2, -\frac{p}{q}) \to \pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$$

and that the latter set was finite; our goal this time is to relate this to the set of tight contact structures on a lens space and determine the size of each of these sets.

Definition 1. The lens space $L(p, q)$ is the 3-manifold constructed by gluing solid tori V_0 and V_1 along their boundaries via the orientation-reversing map

$$\begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix} \in -1 \cdot \mathrm{SL}_2(\mathbb{Z})$$

where $pq' - qp' = 1$. (Any two choices of (p', q') differ by a Dehn twist along V_1 and hence produce the same manifold.) It sends a meridian $(1, 0)^{\mathsf{T}}$ of ∂V_0 to the curve $(-q, p)^{\mathsf{T}} \subset \partial V_1$, so it is easy to see that $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$.

Proposition 2. *Let* $-\frac{p'}{q'} = [r_0, r_1, \dots, r_k + 1]$. *Then there is an injective map*

$$\pi_0 \mathrm{Tight}(L(p, q)) \hookrightarrow \pi_0 \mathrm{Tight}(S^1 \times D^2, -\frac{p'}{q'}).$$

Proof. Let ξ be a tight contact structure on $L(p, q)$ and let γ be a Legendrian curve which is topologically isotopic to the core of V_0 and has negative twisting number n. Shrinking V_0 if necessary, we realize it as a standard neighborhood of γ with two dividing curves on its boundary of slope $\frac{1}{n}$. Since $\partial V_0 = \partial V_1$, this means that ∂V_1 has two dividing curves in the homology class

$$\begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix} \begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} -qn + q' \\ pn - p' \end{pmatrix},$$

i.e. of slope $-\frac{p|n|+p'}{q|n|+q'}$ with respect to V_1, which is strictly between $-\frac{p}{q}$ and $-\frac{p'}{q'}$, and we have

$$-\frac{p}{q} < -\frac{p|n|+p'}{q|n|+q'} < -\frac{p'}{q'} \leq -1$$

where the last inequality follows from the fact that $1 < \frac{p}{q} < \infty$ and $\frac{p}{q}$ is connected to $\frac{p'}{q'}$ by an edge of the Farey tessellation.

As before, we can find a convex torus in V_1 parallel to ∂V_1 with two dividing curves of slope -1, and this torus bounds a solid torus $N \subset V_1$ whose contact structure is unique up to isotopy. On the complement $\overline{V_1 \backslash N} \cong T^2 \times I$, however, ξ has boundary slopes $-\frac{p|n|+p'}{q|n|+q'}$ and -1, and since $-\frac{p'}{q'}$ lies in between them we can find another convex torus $T \subset N$ parallel to ∂V_1 of slope $-\frac{p'}{q'}$.

If we now use the torus T to redefine the splitting $L(p,q) = V_0 \cup V_1$, it follows that

$$\xi|_{V_1} \in \text{Tight}(S^1 \times D^2, -\frac{p'}{q'}).$$

The boundary slope on V_0 is given by

$$\begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix}^{-1} \begin{pmatrix} q' \\ -p' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so V_0 is a solid torus with boundary slope ∞, meaning ∂V_0 has two longitudinal dividing curves. But this means that $\xi|_{V_0}$ is unique up to isotopy rel boundary, so the contact structure $\xi|_{V_1}$ determines ξ uniquely. □

It follows now that

$$|\pi_0 \text{Tight}(L(p,q))| \leq \left|\pi_0 \text{Tight}^{\min}(T^2 \times I, -\frac{p'}{q'}, -1)\right|,$$

so we will now work to improve our upper bound on the latter quantity by showing that certain basic slices "commute" just as different stabilizations of a Legendrian knot do. Recall that given any $m > 1$ and a tight contact structure $\xi \in \text{Tight}^{\min}(T^2 \times I, -m, -1)$, we know how to decompose

$$T^2 \times I \cong (T^2 \times [1,2]) \cup (T^2 \times [2,3]) \cup \ldots \cup (T^2 \times [m-1,m]),$$

where each $T^2 \times [i, i+1]$ is a basic slice with boundary slopes $s_0 = -i$ and $s_1 = -(i+1)$. Since there are two basic slices for each pair of boundary slopes, this gave us an upper bound of 2^{m-1} possibilities for ξ.

Lemma 3. *$|\pi_0 \text{Tight}^{\min}(T^2 \times I, -m, -1)| = m$.*

Proof. We will apply the change of basis

$$\begin{pmatrix} 0 & -1 \\ 1 & m+1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

to change each slope $-\frac{k}{1}$ to $-\frac{1}{m+1-k}$, so that we can describe these contact structures in terms of Legendrian knots. We construct each contact structure the following way: start with a standard neighborhood N of a $tb = -1$ Legendrian unknot $U \subset (S^3, \xi_{\mathrm{st}})$, take a positive stabilization U_+ or a negative stabilization U_- inside N, identify a standard neighborhood N' of this stabilization and remove it from N. The result is a tight contact structure on $T^2 \times I$ with boundary slopes $-\frac{1}{1}$ and $-\frac{1}{2}$, and since the destabilization corresponds to pushing $U_\pm$ across a bypass it is easy to see that this contact structure is a basic slice. The basic slices corresponding to U_+ and U_- are not isotopic rel boundary, however, or else the knots would be Legendrian isotopic even though their rotation numbers are ± 1.

We can perform this process $m-1$ times to factor a tight $T^2 \times I$ with boundary slopes -1 and $-\frac{1}{m}$ into $m-1$ basic slices, where if we have a knot K at some step then we remove a standard neighborhood of $K_\pm$ from a standard neighborhood of K to get the next basic slice, and the choice of basic slice is determined by the sign of the stabilization. The resulting contact structure on $T^2 \times I$ is the solid torus N minus a standard neighborhood of an unknot U_m isotopic to the core of N, with $tb(U_m) = -m$ because we have obtained U_m by stabilizing U a total of $m-1$ times. But we know that there are exactly m Legendrian unknots with $tb = -m$, namely one for each possible rotation number $r = -m+1, -m+3, \ldots, m-3, m-1$. We conclude that there are exactly m contact structures on $T^2 \times I$ with the specified boundary slopes. $\square$

Proposition 4. *$|\pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)| \leq (-r_0 - 1) \ldots (-r_{k-1} - 1)(-r_k)$, where $-\frac{p}{q} = [r_0, \ldots, r_k]$.*

Proof. By the same argument as in the above lemma, but with a more complicated change of basis, there are exactly $-r_k$ minimally twisting contact structures on $T^2 \times I$ with boundary slopes $-\frac{p}{q}$ and

$$-\frac{p'}{q'} = [r_0, \ldots, r_{k-1}, -1] = [r_0, \ldots, r_{k-1} + 1].$$

Given an element of $\pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$, we identify and remove one of these contact structures, leaving an element of $\pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p'}{q'}, -1)$, and the proposition follows by induction on k. $\square$

In particular, using the inequality $|\pi_0 \mathrm{Tight}(L(p,q))| \leq |\pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p'}{q'}, -1)|$ where $-\frac{p'}{q'} = [r_0, r_1, \ldots, r_k + 1]$, we have now shown that

$$|\pi_0 \mathrm{Tight}(L(p,q))| \leq (-r_0 - 1) \ldots (-r_{k-1} - 1)(-r_k - 1).$$

If we can show equality, then it will follow that the associated bounds for $S^1 \times D^2$ and minimally twisting $T^2 \times I$ are tight as well, and in particular that

$$|\pi_0 \mathrm{Tight}(S^1 \times D^2, -\frac{p}{q})| = (-r_0 - 1) \ldots (-r_{k-1} - 1)(-r_k).$$

We claim there are in fact $\prod(-r_i - 1)$ tight contact structures on $L(p,q)$, and in fact that all of them can be obtained by *Legendrian surgery.*

Definition 5. Let $K \subset S^3$ be a knot, and identify a meridian μ and a longitude λ on $\partial N(K)$ where λ lies in a Seifert surface for K. We perform *Dehn surgery* on K with slope $\frac{a}{b}$ by constructing a 3-manifold

$$Y = (S^1 \times D^2) \cup_f \overline{S^3 \backslash N(K)}$$

where the gluing map $f : S^1 \times \partial D^2 \to \partial N(K)$ sends $\{*\} \times \partial D^2$ to the curve $a\mu + b\lambda$.

Example 6. The lens space $L(p,q)$ can be constructed by $-\frac{p}{q}$-surgery on an unknot in S^3.

For nullhomologous knots a Seifert surface provides a canonical framing λ_Σ, but arbitrary Legendrian knots come with another preferred framing λ_{tb}: the *Thurston-Bennequin* or *contact* framing specified by the oriented normal vectors to K inside ξ. For a Legendrian knot in S^3, this framing can be expressed as $\lambda_{tb} = tb(K) \cdot \mu + \lambda_\Sigma$.

Definition 7. Let $K \subset (Y,\xi)$ be Legendrian. A *contact $\frac{a}{b}$-surgery* (Y', ξ') on K is constructed by performing a topological $\frac{a}{b}$-surgery on K with respect to the contact framing and extending the contact structure on $\overline{Y \backslash N(K)}$ across $S^1 \times D^2$ by a tight contact structure on $S^1 \times D^2$.

Of course, we need to check that such a surgery is well-defined. For $\frac{a}{b} = 0$ this is impossible, because we would need a tight contact structure on $S^1 \times D^2$ with boundary slope 0, whereas any Legendrian curve on $S^1 \times \partial D^2$ parallel to the dividing set would bound an overtwisted disk. On the other hand, if $\frac{a}{b} = \frac{1}{n}$ then this is uniquely defined: we can choose f to send $\{*\} \times \partial D^2$ to $\mu + n\lambda_{tb}$ and $S^1 \times \{*\}$ to $\mu + (n-1)\lambda_{tb}$, and then the curve on $S^1 \times D^2$ sent to λ_{tb} is $(\{*\} \times \partial D^2) - (S^1 \times \{*\})$. This means that the contact structure on $S^1 \times D^2$ should have two dividing curves of slope -1, and there is exactly one such structure, so we conclude that contact $\frac{1}{n}$-surgery is well-defined for all n. (For general $\frac{p}{q} \neq 0$ we will have to finish the classification of tight contact structures on solid tori.)

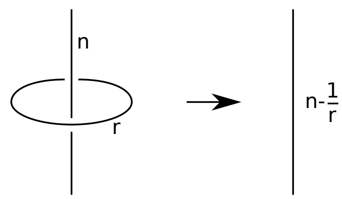
Definition 8. A *Legendrian surgery* on a Legendrian knot $K \subset Y$ is a contact (-1)-surgery along K.

Legendrian surgery is particularly interesting because of its relation to symplectic geometry. We will prove the following theorem next time.

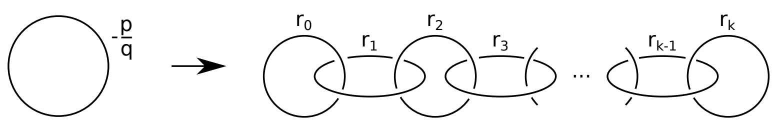
Theorem 9. *Let (X,ω) be a Stein filling of $(Y, \xi = \ker(\alpha))$, i.e. $Y = \partial X$ and $\omega|_Y = d\alpha$. If (Y', ξ') is the result of Legendrian surgery on $K \subset Y$, then there is a 4-dimensional 2-handle H attached to X along K with framing $tb(K) - 1$ such that $X \cup H$ is a Stein filling of Y'.*

Corollary 10. *If Y is Stein fillable, then so is the result of any Legendrian surgery on Y.*

Now we will use Legendrian surgery to describe some Stein fillable contact structures on $L(p,q)$. Note first that performing the following "slam dunk" move on a topological surgery diagram does not change the underlying manifold for any $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$.



Thus if $-\frac{p}{q} = [r_0, \ldots, r_k]$ then we can describe $L(p,q)$ by the following surgery diagram:



For each component K_i of this link with coefficient $r_i \leq -2$, we can realize K_i as a Legendrian knot with $tb(K_i) = r_i + 1 \leq -1$ and $r(K_i) \in \{r_i+2, r_i+3, \ldots, r_i-3, -r_i-2\}$: there are $-r_i - 1$ possible values of $r(K_i)$. The Legendrian surgery on K_i is then topologically an r_i-surgery, so Legendrian surgery on every K_i yields a Stein fillable (hence tight) contact structure on $L(p,q)$.

Let $(X,J) = B^4 \cup H_0 \cup \ldots \cup H_k$ be the Stein domain with boundary $L(p,q)$ constructed by this procedure. If $h_i \in H_2(X, \partial X)$ is a cocore of H_k, then $\langle c_1(X,J), h_i \rangle = r(K_i)$ for each i [1]. But a theorem of Lisca and Matić [2] says that if two different Stein structures on X give isotopic contact structures $TY \cap J(TY)$ on $Y = \partial X$, then their first Chern classes are identical, so every choice of $(r(K_0), \ldots, r(K_k))$ yields a distinct isotopy class of contact structure. There are $\prod(-r_i - 1)$ such choices, hence at least that many tight (indeed, Stein fillable) contact structures on $L(p,q)$. Since this matches the upper bound we have already established, we conclude:

Theorem 11. *Let $p > q > 0$ and write $-\frac{p}{q} = [r_0, \ldots, r_k]$. Then the lens space $L(p,q)$ has exactly*

$$(-r_0 - 1)(-r_1 - 1) \ldots (-r_k - 1)$$

tight contact structures up to isotopy, and each of them is Stein fillable.

Corollary 12. *If $p \geq q > 0$, then the set $\pi_0 \mathrm{Tight}(S^1 \times D^2, -\frac{p}{q})$ has cardinality*

$$(-r_0 - 1) \ldots (-r_{k-1} - 1)(-r_k).$$

References

[1] Robert E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998), no. 2, 619–693.

[2] P. Lisca and G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. 129 (1997), no. 3, 509–525.