Kahler geometry and diffeomorphism groups: notes for a series of lectures given in CIMAT, Guanajuato, Mexico, August 2004

Lecture 1.

The theme of these lectures will be the study of *Kahler metrics of constant scalar curvature*. We begin with some background. In any co-ordinate system, the Gauss curvature of a surface in 3-space is given by a complicated formula involving the coefficients of the first fundamental form (induced Riemannian metric). A basic theorem asserts that we can always find "isothermal " coordinates, so the induced metric is

$$e^f(du^2 + dv^2),$$

and in these coordinates the Gauss curvature has the simple representation

$$K = -\frac{e^{-f}}{2}(f_{uu} + f_{vv}).$$

The existence of isothermal coordinates means that any surface in 3-space can be regarded as a *Riemann surface*; with an atlas of charts which differ by conformal, or equivalently holomorphic, maps. In this way, Riemann surfaces arise as *differential geometric objects*—i.e 2-dimensional oriented Riemannian manifolds modulo conformal equivalence. On the other hand, Riemann surfaces also arise as *algebro-geometric objects*, for example as complex plane curves given by equations $p(z_1, z_2) = 0$, where p is a polynomial in two variables. Actually we will consider *projective curves* in complex projective space $\mathbf{CP}^n$. (Recall that this is the quotient of $\mathbf{C}^{n+1} \setminus \{0\}$ by the action of scalar multiplication, and may be regarded as $\mathbf{C}^n$ compactified by the addition of points at infinity.)

Moving closer to our main theme: the starting point is a fact known from the end of the 19th. century:

Any compact Riemann surface has a metric of constant Gauss curvature (and this is essentially unique).

We want to extend these ideas to higher dimensions, in the framework of Kahler geometry. To explain what this is, recall some linear algebra. On a real, even dimensional, vector space we can consider three kinds of algebraic structure

- A complex structure;
- A Euclidean metric;
- A symplectic (i.e. skew-symmetric) form

Any two of these three structures, which are suitably compatible, define the third one. A Kahler manifold is an even dimensional manifold with each of these algebraic structures on its tangent spaces, compatible in a natural way. There are at least three different routes one can take to motivate the definition.

- In Riemannian geometry, a Kahler manifold is a Riemannian manifold whose holonomy is contained in the unitary group.
- Kahler metrics naturally occur on complex projective varieties. For example there is a standard "Fubini-Study" metric g_{FS} on $\mathbf{CP}^n$ and if $X \subset \mathbf{CP}^n$ is a complex submanifold the restriction of g_{FS} to X is Kahler.
- Starting with a symplectic manifold, a Kahler structure appears as a "complex quantisation"in that we can associate to it a complex vector space (Hilbert space) $\mathcal{H}_X$ of holomorphic functions. More precisely, we should talk about holomorphic sections of a line bundle $L \to X$ and write $\mathcal{H}_{X,L}$.

The *scalar curvature* S of a Kahler manifold is given, in local complex coordinates z_a by

$$S = i \sum_{a,b} g^{ab} \frac{\partial^2 f}{\partial z_a \partial \overline{z}_b},$$

where g_{ab} is the (Hermitian) matrix representing the metric, g^{ab} is its inverse, and e^f is the volume form (i.e. the determinant of the matrix $(g_{ab}.\ )$.

We have now set up the background for our problem:

If X is a complex projective manifold, does it have a Kahler metric of constant scalar curvature? If so, is the metric unique?

(Note that a special case of this discussion is the problem of existence of *Kahler-Einstein metrics*, where the Ricci tensor is constant. Much more is known about this, through renowned work of Calabi, Aubin, Yau, Tian,.... .)

More precisely, in our problem, we want to fix the "Kahler class" $[\omega] \in H^2(X)$. In a fixed class, the general Kahler metric is represented by a Kahler potential ϕ,

$$\omega_\phi = \omega_0 + i \sum \frac{\partial^2 \phi}{\partial z_a \partial \overline{z}_b} dz_a d\overline{z}_b.$$

Thus the scalar curvature of ω_ϕ depends on the first four derivatives of ϕ and our problem is asking about the solution of a (highly nonlinear) 4th order PDE.

While the full solution of the problem above seems a long way off, we will outline in these lectures:

- A general conceptual scheme in which to fit the problem.
- A conjecture about the correct answer.

- The uniqueness of solutions
- A partial existence theory in the case of "toric varieties"
- An algorithm for constructing solutions numerically when these exist.

To end this first lecture we will discuss the 4th and 5th items above briefly.

4th item. Consider a convex polytope $P \subset \mathbf{R}^n$. Suppose given a positive measure $d\sigma$ on the boundary of P and let $d\mu$ be standard Lebesgue measure on $\mathbf{R}^n$. Let u be a convex function on P so the Hessian u_{ij} is a positive definite matrix at each point of P. Put

$$\mathcal{F}(u) = -\int_P \log\det(u_{ij})\ d\mu + \int_{\partial P} u d\sigma - A \int_P u d\mu,$$

where

$$A = \frac{Mass(\partial P, d\sigma)}{Mass(P, d\mu)}.$$

Thus we have defined a functional $\mathcal{F}$ on the space of convex functions on P. *Question: when does this functional achieve a minimum ?* We will explain later that this question is equivalent to our problem in the case of toric varieties.

5th. item

Recall the "quantisation" vector space $\mathcal{H}_{X,L}$ i.e. $H^0(X;L)$. A kahler potential ϕ can be thought of more geometrically as a Hermitian metric on the fibres of L. This defines a Hermitian metric on $\mathcal{H}_{X,L}$ by the standard L^2 norm:

$$\|s\|^2 = \int_X |s|^2_\phi \omega_\phi^n.$$

On the other hand (provided L is "very ample") we get a projective embedding $X \to \mathbf{CP}^n = \mathbf{P}(\mathcal{H}^*_{X,L})$ defined by the sections of L and a metric on $\mathcal{H}_{X,L}$ defines a Fubini-Study metric on $\mathbf{CP}^n$ and hence on X.

Let $\mathcal{M}_{X,L}$ denote the set of Hermitian metrics on $\mathcal{H}_{X,L}$. Putting these constructions together we get a map

$$\Psi : \mathcal{M}_{X,L} \to \mathcal{M}_{X,L}.$$

We an replace L by L^k throughout, where k is large. (This corresponds to the "classical limit" $h \to 0$.) Thus we get Ψ_k on $\mathcal{M}_{L^k,X}$. We let $\Psi_k^{(n)}$ denote the n-fold composite of Ψ_k.

Proposition Suppose X has a constant scalar curvature metric ω. Then for large enough k and any initial H the composites $\Psi_k^{(n)}(H)$ converge as $n \to \infty$ to some limit, which corresponds to a metric ω_k. Now

$$\omega = \lim_{k\to\infty} \frac{\omega_k}{k}.$$

Lecture 2

In this lecture we outline how the constant scalar curvature problem can be fitted into a general formal package, involving "moment maps" for group actions.

We begin with a simple example. Let V be the set of n-by-n complex matrices and consider the action of $GL(n, \mathbf{C})$ on V by conjugation. We ask: when does a matrix $X \in V$ minimise the norm in its orbit? A little calculation shows that if this occurs then $[X, X^*] = 0$. A simple but fundamental observation is *a closed orbit contains a point of minimal norm.* This ties in with well-known facts from elementary matrix theory. Clearly we can extend the ideas to any representation of $GL(n, \mathbf{C})$.

We will now explain how to fit the above into a more general and abstract theory. Let (M, Ω) be a symplectic manifold. The basic Hamiltonian construction starts with a function H on M and constructs the vector field $v = v_H$ such that

$$i_v(\Omega) = dH.$$

Then v generates a flow $f_t : M \to M$ of Ω-preserving maps, i.e. an action of the group $\mathbf{R}$ on (M, Ω).

Now suppose a Lie group G acts on (M, Ω), so for ξ in $Lie(G)$ we have a vector field v_ξ on M. A moment map for the action is a map

$$\mu : M \to Lie(G)^*,$$

with the property that for any ξ

$$d(\xi, \mu) = i_{v_\xi}(\Omega).$$

We also suppose that μ is G-equivariant.

(An example of this set-up occurs if $M = T^*\mathbf{R}^3$ the symplectic manifold corresponding to the motion of a particle in 3-space. Then $G = SO(3)$ acts and the moment map gives the usual notion of angular momentum.)

Given any $c \in Lie(G)^*$ let $\Gamma \subset G$ be the stabiliser of c. A simple fact is that the space

$$\mu^{-1}(c)/\Gamma,$$

has a natural symplectic structure, the "reduced system", taking account of the symmetry. We will consider the case when $c = 0$ so $\Gamma = G$. We then have the Marsden-Weinstein "symplectic quotient"

$$M//G = \mu^{-1}(0)/G.$$

Now suppose that M is Kahler and G preserves the complex structure. The Lie algebra homomorphism from $Lie(G)$ to $Vect(M)$ extends to the complexification. If G is compact (say) there is a complexified group G^c and the Lie algebra action exponentiates to define an action of G^c, preserving the complex structure but *not* the symplectic form.

The basic principle in this situation, very roughly stated, is that the symplectic quotient $M//G$ and complex quotient M/G^c are "approximately equivalent". That is to say, *"most" G^c-orbits contain a point where $\mu = 0$ and this point is unique up to the action of G.*

To connect this with the example at the beginning, we consider the $U(n)$ action on the vector space V.

Exercise. The moment map for this action is

$$\mu(X) = i[X, X^*] \in u(n) \equiv u(n)^*.$$

Sometimes we are more interested in the action on $P(V)$, the moment map is then given by

$$\mu(X) = \frac{i[X, X^*]}{|X|^2}.$$

Thus we see that*the points of minimal norm appear as the zeros of the moment map.* To understand this more generally we develop a little more theory. Suppose that $L \to M$ is a complex line bundle with unitary connection having curvature $-i\Omega$. One interpretation of a moment map is as a lift of the G action on M to L. We define a vector field $\tilde{v}_\xi$ on the total space of L by

$$\tilde{v}_\xi = \hat{v}_\xi + (\mu, \xi)V,$$

where V is the canonical vector field which generates the $U(1)$ action on the fibres of L. Now if M is Kahler we get a G^c action on L and likewise L^*. Given a point $x \in M$ we choose a point $\hat{x}$ in L^* lying over x and we define a function F on G^c by

$$F(g) = \log |g(\hat{x})|^2.$$

This is preserved by the action of G so we get an induced function

$$F : G^c/G \to \mathbf{R}.$$

In the case when $M = P(V)$ the line bundle L^* is the tautological bundle so the complement of the zero section in L^* is identified with $V \setminus \{0\}$. Then the function essentially gets back to the (log of) the norms of vectors, as we began.

If G is a compact Lie group then G^c/G is a "symmetric space". The geodesics in G^c/G correspond to 1-parameter subgroups of G^c.

Exercise The function F is convex along geodesics in G^c/G. Minima of F correspond to zeros of μ.

[In fact, along a geodesic corresponding to a 1-parameter subgroup $\exp(i\xi t)$, the second derivative $F''(t)$ is equal to $\|v_\xi\|^2$, evaluated at the appropriate point.]

There is a good deal more we could say about this theory: relations with the notion of "stability" in algebraic geometry (closed orbits) etc., etc. But the above covers the main ideas we need, and we press on.

A well known infinite-dimensional example of this set-up occurs in *gauge theory* (Atiyah-Bott/ Hitchin-Kobayashi/Narasimhan-Seshadri/Donaldson-Uhlenbeck-Yau/.....). We want to develop the constant scalar curvature problem in an analogous way.

Let (X, ω) be a symplectic manifold and $\mathcal{J}$ be the set of compatible almost-complex structures on X. Thus $\mathcal{J}$ is the space of sections of a bundle Σ over M with fibre $Sp(2n, \mathbf{R})/U(n)$, the "Siegel generalised upper half space". A crucial fact for us is that this has a $Sp(2n, \mathbf{R}$-invariant Kahler structure. (For example, if $n = 1$ the manifold $SL(2, \mathbf{R})/U(1)$ is just the upper half plane and we take the standard Poincaré metric.) A consequence of this is that $\mathcal{J}$ is formally an (infinite-dimensional) Kahler manifold.

Suppose for simplicity that $H^1(X) = 0$ and let $\mathcal{G}_0$ be the group of *symplectomorphisms* of (X, ω). This group acts in an obvious way on $\mathcal{J}$ preserving the Kahler structure. The Lie algebra of $\mathcal{G}_0$ can be identified, by the Hamiltonian construction, with the functions of integral zero on X.

Restrict attention to the subset $\mathcal{J}_{int}$ consisting of integrable almost complex structures.

Proposition(Fujiki, Quillen) The scalar curvature is a moment map for the action of $\mathcal{G}_0$ on $\mathcal{J}_{int}$.

Now turn to the "$\mathcal{G}_0^c$-orbits". The problem here is that there is no group $\mathcal{G}_0^c$. However the orbits, if there were such a group are characterised as integral submanifolds of a certain distribution in $\mathcal{J}_{int}$ and these integral submanifolds *do* exist. Suppose J is a point in $\mathcal{J}_{int}$, so X can be regarded as a Kahler manifold with this fixed complex structure. Let $\mathcal{K}_0$ be the set of Kahler potentials

$$\mathcal{K}_0 = \{\phi : \omega_\phi = \omega + i\bar{\partial}\partial\phi > 0, \int \phi = 0\}.$$

Now let $\tilde{\mathcal{K}}_0$ be the set of pairs (ϕ, f) where $\phi \in \mathcal{K}_0$ and $f : X \to X$ with

$$f^*(\omega_\phi) = \omega.$$

There is a natural map

$$\nu : \tilde{\mathcal{K}}_0 \to \mathcal{J}_{int},$$

defined by $\nu(\phi, f) = f^*(J)$. The image of ν is the desired integral submanifold through J. Thus the space

$$\mathcal{G}_0^c/\mathcal{G}_0$$

is identified with $\mathcal{K}_0$ and our problem fits into the "standard pattern".

Lecture 3

In this lecture we will outline three applications of the ideas discussed the day before.

I. Reductive automorphism groups.

In our general setting, where groups G, G^c act on a Kahler manifold M, suppose $x \in M$ is a point with $\mu(x) = 0$. Let H be the stabiliser of x in G^c.

Proposition
The Lie group H is "reductive";i.e. the complexification of a compact group.

The analogue in the infinite dimensional picture is an old theorem of Matsushima.
Theorem
If X is a compact Kahler manifold of constant scalar curvature and $H^1(X) = 0$ then $Aut(X)$ is reductive.

(Here $Aut(X)$ denotes the group of holomorphic automorphisms of X.)

For example, if X is the "blow-up" of the projective plane at one point then $Aut(X)$ is the group of projective transformations which fix this point, which is not reductive; so this X does not have a metric of constant scalar curvature. This is a convenient place for us to recall a renowned result of Tian. Let X be the plane blown up at r points. Then the set of Kahler classes is a cone in $H^2(X) = \mathbf{R}^{r+1}$. For $r \leq 8$ there is a distinguished point in the cone given by $c_1(X)$. Tian proved that there is a Kahler-Einstein (and a fortiori constant scalar curvature) metric on X in this special class if and only if $Aut(X)$ is reductive.

The proof of the Proposition is elementary. In fact we show that H is the complexification of $H \cap G$. This is the same as saying that if $\xi_1, \xi_2 \in Lie(G)$ and $v_{\xi_1} + Jv_{\xi_2}$ vanishes at the given point x then v_{ξ_1}, v_{ξ_2} each vanish at x. This will follow if we can show that for any ξ_1, ξ_2

$$\Omega(v_{\xi_1}, v_{\xi_2}) = 0,$$

at x. This is easiest to see if we think of an equivariant moment map as a Lie algebra homomorphism

$$\mu^* : Lie(G) \to C^\infty(M),$$

lifting $\xi \mapsto v_\xi$. Then

$$\Omega(v_{\xi_1}, v_{\xi_2}) = \{\mu^*(\xi_1), \mu^*(\xi_2)\},$$

where $\{\ ,\ \}$ is the Poisson bracket. By the homomorphism property this is $\mu^*([\xi_1, \xi_2])$ which vanishes at x by hypothesis.

The original proof of Matsushima's Theorem is (probably) identical to this, when one translates to the infinite dimensional setting. However the general picture makes the manipulations much more transparent.

II. Uniqueness (i)

Recall that in the general picture we have, associated to each G^c orbit, a function F on G^c/G, convex along geodesics. In the infinite dimensional picture we can identify $\mathcal{G}_0^c/\mathcal{G}_0$ with the space $\mathcal{K}_0$ of Kahler potentials (modulo constants). In the infinite dimensional case the analogue of F is the "Mabuchi functional" $\mathcal{M}$ which is characterised (up to a constant) by its variation

$$\delta\mathcal{M} = \int_X (S - \hat{S})\delta\phi \,\frac{\omega_\phi^n}{n!}.$$

Here $\omega_\phi = \omega_0 + i\overline{\partial}\partial\phi$ is the Kahler metric determined by ϕ and S is the scalar curvature of this metric. We denote the average of S, which is a topological invariant, by $\hat{S}$. This formula actually defines $\mathcal{M}$ on the space $\mathcal{K}$ of Kahler potentials and it will be more convenient to work there in what follows.

We would like to prove uniqueness of constant scalar curvature metrics by exploiting the fact that $\mathcal{M}$ is convex along geodesics in $\mathcal{K}$. But what are the geodesics in $\mathcal{K}$? These were first studied by Semmes. Let A denote the cylinder $A = S^1 \times [0, 1]$, regarded as a Riemann surface with boundary. A geodesic segment in $\mathcal{K}$ corresponds to a function Φ on $X \times A$, which is invariant under rotations in the S^1 variable and which satisfies the equation

$$\left(\pi^*(\omega_0) + i\overline{\partial}\partial\Phi\right)^{n+1} = 0,$$

where π is the projection to X. This is the *homogeneous complex Monge-Ampère equation.* The problem of joining to points in $\mathcal{K}$ by a geodesic segment is equivalent to solving this equation with given boundary values on $\partial A \times X$. This is a very difficult problem in PDE and analysis. It is perhaps not true that there is always a smooth solution. However Chen showed that there is a $C^{1,1}$ solution and used this to deduce uniqueness under some extra hypotheses. In more recent work, Chen and Tian have studied the singularities that may occur and removed the extra hypotheses.

Uniqueness (ii)

There is an alternative "finite dimensional" approach, which makes contact with the construction algorithm described in the first lecture. Suppose for simplicity that $H^1(X) = 0$ and $Aut(X)$ is trivial. We first consider the symplectic manifold (X, ω) with a line bundle $L \to X$ having curvature $-i\omega$. It is convenient to work now with the group $\mathcal{G}$ of connection and metric preserving bundle diffeomorphisms from L to L. These cover symplectomorphisms of X so we have an extension

$$S^1 \to \mathcal{G} \to \mathcal{G}_0.$$

We fix a positive integer k. Define $\mathcal{J}$ as before and let $\mathcal{Z}$ denote the set of pairs

$$(J, (s_0, \dots s_N))$$

where $J \in \mathcal{J}$ and $s_0, \dots, s_N$ are sections of L^k. We let $\mathcal{Z}_{int}$ be the subset of data where $J \in \mathcal{J}_{int}$ and $s_0, \dots, s_N$ is a basis of holomorphic sections of L^k regarded as a holomorphic line bundle over X, with the complex structure J. There is a natural symplectic form on the space of sections $\Gamma(L^k)$. Pulling back under the projection to $\Gamma(L^k)^{N+1}$ we get a symplectic form on $\mathcal{Z}_{int}$.

Now we have commuting actions of two groups $\mathcal{G}$ and $SU(N+1)$ on $\mathcal{Z}_{int}$, preserving the symplectic form. The moment maps are

- $$\mu_{SU}(J, \underline{s}) = p(i\langle s_\alpha, s_\beta \rangle),$$

  where p is the projection of a matrix to its trace-free part.

- $$\mu_{\mathcal{G}}(J, \underline{s}) = (\Delta + k)\left(\sum |s_\alpha|^2\right).$$

Taking G to be any of the three groups $SU(N+1), \mathcal{G}, SU(N+1) \times \mathcal{G}$, we can ask our standard question: does a G^c-orbit contain a zero of the G-moment map? Here we need to give an interpretation to the $\mathcal{G}^c$-orbits. These correspond to equivalence classes of data under the usual notion of isomorphism of holomorphic line bundles with holomorphic sections. The quotient $\mathcal{G}^c/\mathcal{G}$ is identified with the set of Hermitian metrics on a fixed holomorphic line bundle, and moving in a $\mathcal{G}^c$ orbit is the same as varying this metric, with fixed holomorphic data.

- For $G = SU(N+1)$ we get a rather trivial question, just a matter of choosing an orthonormal basis.

- For $G = \mathcal{G}$ we also get an easy problem. The question is equivalent to the following. We are given a holomorphic line bundle L and a basis of holomorphic sections $\underline{s}$ and we want to choose a Hermitian metric on L whose corresponding curvature form is positive and such that $\sum |s_\alpha|^2 = 1$. This condition uniquely determines the metric and the corresponding Kahler metric is just the metric induced from the Fubini-Study metric by the embedding $X \to \mathbf{CP}^N$.

- For $G = SU(N+1) \times \mathcal{G}$ we get an interesting problem which can be thought of in two different ways.

1. First solve the equation $\mu_{SU} = 0$. The remaining condition $\mu_{\mathcal{G}} = 0$ gives an equation which "converges" to the constant scalar curvature equation as $k \to \infty$.

2. First solve the equation $\mu_{\mathcal{G}} = 0$. The remaining condition $\mu_{SU} = 0$ gives a *finite-dimensional problem* of the standard form.

We explain a little more about these two points of view.

1. Suppose given a line bundle L over a complex manifold X with a hermitian metric on L whose curvature is $-i\omega$ where ω is a Kahler form. Define functions ρ_k on X by

$$\rho_k = \sum_0^N |s_{alpha}|^2,$$

where s_α is any orthonormal basis of $H^0(X; L^k)$. Then ρ_k is an invariant of the Kahler structure.

Proposition (Tian, Zelditch, Lu) As $k \to \infty$ there is an asymptotic expansion

$$\rho_k \sim k^n + \frac{S}{2\pi} k^{n-1} + \dots, \quad (*)$$

where S is the scalar curvature of ω.

2. The finite dimensional problem has been considered before in different contexts by Luo and Zhang. It is equivalent to the following, given $X \subset \mathbf{CP}^N$ find a projective transformation g such that '$g(X)$ has centre of mass at zero, where we take the standard embedding

$$\mathbf{CP}^N \to su(N+1).$$

Theorem Suppose there is a constant scalar curvature Kahler metric ω on X. Then for large k there is an ω_k with $\rho_k(\omega_k) = \mathit{constant}$ and $\omega_k \to \omega$ as $k \to \infty$.

The uniqueness is a consequence of this and the uniqueness of the solution to the finite-dimensional problem. The proof the theorem combines arguments with the asymptotic expansion (*) and analysis of the family of finite dimensional problems as $k \to \infty$.

III. Conjectural answer

Return again to our finite-dimensional set-up, where G acts on M and on $L \to M$. There is a numerical criterion for an orbit $G^c.x$ to contain a zero of the moment map. Let $\lambda : \mathbf{C}^* \to G^c$ be a 1-parameter subgroup and let

$$y = lim_{t \to \infty} \lambda(t) x.$$

Then λ maps $\mathbf{C}^*$ to the stabiliser of y in G^c. But this stabiliser acts on the fibre of L over y, so we get a 1-dimensional representation of $\mathbf{C}^*$ which has an integer weight W_λ. The condition for finding a zero of μ is that $W_\lambda > 0$ for all λ.

The direct translation of the definition of the weight to the infinite dimensional case leads to the notion of the *Futaki invariant.* Suppose Y is a compact Kahler manifold and $Aut(Y)$ is reductive. Fix a maximal compact subgroup Γ in $Aut(Y)$ and consider any Γ invariant Kahler metric. Then we define

$$Fut : Lie(\Gamma) \to \mathbf{R}$$

by the integral

$$\int_Y (S - \hat{S}) H,$$

where S is the scalar curvature and H is the Hamiltonian function corresponding to an element of $Lie(\Gamma)$. This complexifies to a map $Lie(Aut(G)) \to \mathbf{C}$ which is a *holomorphic invariant* independent of the choice of metric.

The answer which we expect to be correct involves generalising this Futaki invariant to singular spaces. Suppose Y is any variety (or even scheme) with a $\mathbf{C}^*$ action. For $k > 0$ we have a $\mathbf{C}^*$ action on the vector space $H^0(Y; L^k)$. Thus we have a pair of integers, d_k the dimension of the vector space and w_k the total weight of the action. By general theory, the ratio $w_k / k d_k$ has an expansion for large k

$$\frac{w_k}{k d_k} = F_0 + F_1 k^{-1} + F_2 k^{-2} + \dots .$$

We define the *generalised Futaki invariant* to be the coefficient F_1. Standard results show that this agrees with the other definition when Y is smooth.

Now, given X, define a *test configuration* to be a variety $\mathcal{X}$ with a $\mathbf{C}^*$ action and an equivariant map $\pi : \mathcal{X} \to \mathbf{C}$, where $\mathbf{C}^*$ acts on $\mathbf{C}$ in the standard way, such that $\pi^{-1}(t)$ is isomorphic to X for $t \neq 0$. Then there is a $\mathbf{C}^*$ action on the central fibre $\pi^{-1}(0)$ and this has a Futaki invariant W, as above.

We say X is *K-stable* if $W \geq 0$ for all test configurations, with equality if and only if $\mathcal{X} = X \times \mathbf{C}$.

Conjecture *X admits a constant scaalr curvature metric if and only if it is K-stable.*

This extends a conjecture of Tian in the Kahler-Einstein case.

Lecture 4

In this lecture we get a tighter grip on some of the ideas we have discussed, in the particular case of *toric varieties*. This brings us to some "elementary" questions in convex geometry, which we describe first.

Suppose given the following data.

- A convex polytope $P \subset \mathbf{R}^n$.
- A bounded function A on P.
- A measure σ on the boundary of P, equal to a multiple of the Lebesgue measure on each face. (Note that σ this is completely determined by specifying the mass of each face.)

Suppose that

$$\int_P A \, d\mu = Mass(\partial P, \sigma) \tag{1}$$

and that

$$\text{Centre of Mass}(P, A \, d\mu) = \text{Centre of Mass}(\partial P, \sigma). \tag{2}$$

Define a functional $L = L_{A,\sigma}$ by

$$L(f) = \int_{\partial P} f \, d\sigma - \int_P A \, f \, d\mu.$$

Notice that Equations (1) and (2) are equivalent to the condition that L vanishes on affine-linear functions (i.e. functions of the form $\lambda(x) = \lambda_i x_i + C$).

We will consider the restriction of the functional L to *convex functions* on P. Fix a base point $p_0 \in P$ and say that a function f is *normalised* if $f \geq 0$ and $f(p_0) = 0$. We say a function f is a *piecewise linear* (PL) convex function if it can be written as

$$f = Max(\lambda_1, \lambda_2, \ldots, \lambda_R),$$

where λ_i are affine-linear. Finally, we say that a function is a *rational piecewise linear* (abbreviated to QPL) convex function if the λ_i can be chosen to have all co-efficients rational.

Now we can state three conditions which may or may not be satisfied by our data.

Condition (*) There is some $\epsilon > 0$ such that

$$L(f) \geq \epsilon \int_{\partial P} f \, d\sigma,$$

for all normalised convex functions f on $\overline{P}$.

Condition(**) $L(f) > 0$ for every PL convex function which is not affine-linear

Condition (Q)** $L(f) > 0$ for every QPL convex function which is not affine-linear.

Clearly

$$(*) \Rightarrow (**) \Rightarrow (**Q).$$

Theorem 1 Suppose $n = 2$, that A is a constant and that the data P, σ is rational. Then $(**Q) \Rightarrow (*)$.

The lecturers proof of Theorem 1 consists of a sequence of intricate, but elementary, arguments.

Question What if $n > 2$?

Now return to Kahler geometry. One can show that if a constant scalar curvature metric on an algebraic variety exists it is an absolute minimum of the Mabuchi functional $\mathcal{M}$. Thus an interesting question to ask, for any X is whether the Mabuchi functional is bounded below: we expect this to be intimately related to the existence of constant scalar curvature metrics. Now recall that an algebraic variety X of complex dimension n is called a toric variety if it admits an action of the group $(\mathbf{C}^*)^n$ with an dense open orbit. In this lecture we will explain that Theorem 1 above implies

Theorem 2 Suppose X is a (smooth) toric variety of complex dimension 2. If X is K-stable then the Mabuchi functional is bounded below.

The extension of this to higher dimensions is more-or-less equivalent to answering the "Question" above.

We now recall some background on toric varieties and the correspondence between these and polytopes in $\mathbf{R}^n$. This can be viewed either algebro-geometrically or differential geometrically.

Algebraic Geometry We consider a polytope P with vertices in $\mathbf{Z}^n$. We let $c\overline{P}$ be the cone on $\overline{P}$ in $\mathbf{R}^{n+1}$ and let R be the commutative ring associated to the semi-group $c\overline{P} \cap \mathbf{Z}^{n+1}$. Then R is a finitely-generated graded ring and so corresponds to a projective variety X. There is a positive line bundle L over X such that $H^0(X;L^k)$ has a basis indexed by the points of $k\overline{P} \cap \mathbf{Z}^n$.

Differential Geometry Starting with an integer polytope P as above, we take the $2n$-manifold $P \times T^n$, where T^n is the n-torus. There is a way to compactify this which gives a compact smooth manifold X if P satisfies certain "Delzant" conditions. The symplectic form

$$\omega = \sum dx^i d\eta_i$$

(where η_i are standard co-ordinates on the circles) and the T^n action both extend to X. The integral structure induces a canonical measure σ on ∂P and canonical linear functions d_α defining the faces of P. Let

$$u_0 = \sum_\alpha d_\alpha \log d_\alpha,$$

(where the index α runs over the faces). Now (Guillemin, Abreu) let u be any convex function on $\overline{P}$ such that $u - u_0$ is smooth (plus some other condition we will not state). Then u defines a Kahler metric on X with metric tensor

$$g = \sum u_{ij} dx^i dx^j + u^{ij} d\eta_i d\eta_j.$$

Here $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$ and u^{ij} are the entries of the inverse matrix to (u_{ij}).

Let A be the constant given by the ratio of the masses of P and ∂P, so (1) is satisfied. One can show that the Mabuchi functional is given by

$$\mathcal{M}(u) = -\int_P \log\det(u_{ij}) + L(u)$$

where $L = L_{A,\sigma}$ with data A, σ specified above.

Proposition 1 If X is K-stable then the data (P, σ, A) satisfies (2) and (**Q).

Proposition 2 If the data satisfies (*) then $\mathcal{M}$ is bounded below.

Clearly Propositions 1,2 and Theorem 1 imply Theorem 2. Note that one can prove also that a minimising sequence for $\mathcal{M}$ has a *convergent subsequence* in a certain (rather weak) sense.

Outline proof of Proposition 1.

First K-stability implies that the Futaki invariant of X is zero, but one can show that this Futaki invariant corresponds to the difference of the centre of masses of $(P, Ad\mu)$ and $(\partial P, \sigma)$

The real work involves condition (**Q). Suppose that this fails, so that there is a QPL function f with $L(f) \leq 0$. We think of the graph of $C - f$, for suitable C, as defining a "roof" with P as the floorplan. Thus we construct a "house" polytope $Q_f \subset \mathbf{R}^{n+1}$. After rescaling we can suppose that this has integer vertices. By the general theory, this corresponds to some $(n+1)$-dimensional toric variety $\mathcal{X}_f$ (possibly singular). One shows that this fits into a "test configuration" as defined in the previous lecture. Let W_f be the Futaki invariant of the central fibre. The crucial thing then is to show that

$$sgn W_f = sgn L(f).$$

The proof of this uses the familiar relation between counting lattice points and volumes. For example if ν_k denotes the number of points in $\mathbf{Z}^n \cap \overline{P}$ then we have

$$\nu_k = k^n \; Vol(P) + k^{n-1} Vol(\partial P, \sigma) + O(k^{n-2}).$$

Outline proof of Proposition 2

For any function A we can consider the functional

$$\mathcal{F}_A(u) = -\int_P \log\det(u_{ij}) + L_{A,\sigma}(u),$$

(on a suitable set of functions u). One shows that this is a *convex* functional so a critical point is an absolute minimum. The Euler-Lagrange equation is

$$(u^{ij})_{ij} = -A. \tag{3}$$

In fact $-(u^{ij})_{ij}$ is the scalar curvature of the metric g. Now take an arbitary function defining a Kahler structure, for example the function u_0. Define A_0 by the fourth order differential expression in u_0, given by the left hand side of (3) (replacing u by u_0). Then, by construction, u_0 minimises $\mathcal{F}_{A_0,\sigma}$ and in particular this is bounded below, by C say. Thus for any u

$$-\int_P \log\det(u_{ij}) + L_{A_0,\sigma}(u) \geq C.$$

Replacing u by ru, for positive constant r, we get

$$-\int_P \log\det(u_{ij}) + rL_{A_0,\sigma}(u) \geq C_r, \tag{4}$$

where C_r depends on r. Now for normalised u

$$L_{A,\sigma}(u) \geq \epsilon \int_{\partial P} u d\sigma,$$

by hypothesis. On the other hand it is easy to see that, for normalised u,

$$|L_{A,\sigma}(u) - L_{A_0,\sigma}(u)| \leq K \int_{\partial P} u d\sigma,$$

for suitable K. Combining these we get

$$|L_{A,\sigma}(u) - L_{A_0,\sigma}(u)| \leq K/\epsilon L_{A,\sigma}(u),$$

for some K', hence

$$L_{A_0,\sigma}(u) \leq (1 + K/\epsilon) L_{A,\sigma}(u).$$

Now taking $r = r_0 = (1 + K/\epsilon)^{-1}$ in (4) we obtain

$$\mathcal{M}(u) \geq C_{r_0}.$$

This has been proved for normalised u , but $\mathcal{M}$ is unchanged by the addition of an affine-linear function, so the inequality holds in general.

Lecture 5

In this lecture we discuss the existence problem on toric varieties from the point of view of P.D.E. theory. We suppose we have data P, A, σ, as considered at the beginning of Lecture 4, and we set

$$L(f) = \int_{\partial P} f \; d\sigma - \int_P A \; f \; d\mu.$$

We suppose that L vanishes on affine-linear functions and that

$$L(f) \geq \lambda^{-1} \int_{\partial P} f \; d\sigma, \qquad (!)$$

for normalised convex functions f. Under this hypothesis, we would like to prove the existence of a solution to the PDE ("Abreu's equation")

$$(u^{ij})_{ij} = -A \qquad (*)$$

in P, for a convex function u which behaves like $\sum d_\alpha \log d_\alpha$, where d_α are the affine-linear functions defining the faces of P determined by the measure σ.

While we are not able to achieve the goal stated above, we will describe some progress in that direction. Recall that a very general philosophy of PDE theory is to deduce existence theorems from *a priori* estimates. Schematically, if $\mathcal{F}(f) = 0$ is some PDE we would like to solve then one attempts to show that *if* f is a solution then it obeys some appropriate inequality

$$\|f\|_{\text{suitable}} \leq C$$

where C depends on the data, not on f. If we can do this then there are various ways one can attack the existence problem. For example by embedding the equation in a family

$$\mathcal{F}_t(f_t) = 0,$$

for $t \in [0,1]$, with $\mathcal{F}_1 = \mathcal{F}$. We suppose that when $t = 0$ there is a known solution f_0 and consider the set $S \subset [0,1]$ for which a solution exists. We aim to prove that S is open and closed, hence the whole of $[0,1]$: in particular there is then a solution to our original problem. The proof that S is open usually proceeds via the implicit function theorem and reduces to understanding the linearisation of the problem. For elliptic problems this is frequently not difficult. The crux of the problem then is the proof that S is closed, which follows from suitable *a priori* estimates (extended from the original $\mathcal{F}$ to the family $\mathcal{F}_t$).

The challenge in proving these estimates for our problem can be seen as twofold.

- In *Riemannian geometry* there are many results about metrics whose Ricci tensor is controlled, but the scalar curvature is a much weaker object and, in general, does not give any control of the metric. The question is how to obtain this control from the scalar curvature under the Kahler hypothesis.
- In *PDE theory* there is an enormous literature on nonlinear second order elliptic equations. One of the basic techniques is the use of the maximum principle, but this does not operate i the same way for fourth order equations, as we have here.

Following all these preliminaries, the result we prove is an "interior estimate" in the case when $n = 2$ and A is constant. We show that if u is a normalised solution there are estimates, for all p,

$$|\nabla^p u| \leq C_p(\lambda, d^{-1}, P, \sigma),$$

$$(u_{ij}) \geq K(\lambda, d^{-1}, P, \sigma)^{-1},$$

where C_p and K are continuous functions of their arguments. Here d is the distance to the boundary of P and λ is the constant in (!) above.

Our problem is very much related to the theory of *Monge-Ampere equations*

$$\det(u_{ij}) = F(u, \nabla u, x),$$

which has been developed very extensively. For example if we define a linear elliptic operator by

$$P(f) = \left(u^{ij} f_i\right)_j,$$

then the equation (*) is equivalent to

$$P(L) = A,$$

where $L = \log\det(u_{ij})$. By applying deep results of Caffarelli and Caffarelli and Gutierrez (dealing with the Monge-Ampere equation and the linear operator P) one can see that our result follows if one can show three things:

1. The first derivative estimate $|\nabla u| \leq C_1(\lambda, d^{-1}, P, \sigma)$;
2. a 2-sided estimate on $L = \log\det(u_{ij})$

   $$L_- \leq L \leq L_+,$$

   where $L_\pm = L_\pm(\lambda, d^{-1}, P, \sigma)$;
3. a "modulus of convexity" estimate.

The modulus of convexity estimate one would like can be expressed as follows. For $x, y \in P$ set

$$H_x(y) = u(y) - \lambda_x(y),$$

where λ_x is the affine-linear function such that $u - \lambda_x$ is normalised at x. For $K \subset K^+$ put

$$H_K(K^+) = min_{x \in K, y \in \partial K^+} H_x(y).$$

Then one seeks an exhaustion of P by compact subsets $K_0 \subset K_1 \subset K_2 \dots$ such that

$$H_{K_m}(K_{m+1}) \geq \epsilon_m > 0.$$

(1) *The first derivative bound* This is where the hypothesis (!) enters. The boundary conditions imply the following weak form of the equation

$$\int_P \chi_{ij} u^{ij} = L(\chi)$$

for any test function χ. We take $\chi = u$, then $u^{ij} u_{ij} = n$ so we have $L(u) = nVol(P)$ and hence, for normalised u,

$$\int_{\partial P} u d\sigma \leq \lambda n Vol(P).$$

From here an elementary argument shows that

$$|\nabla u| \leq \frac{Const.}{d^{n+1}}.$$

(2) *Bounds on* $L = \log \det(u_{ij})$.

Define a vector field v on P by

$$v^j = -u^{ij}_i.$$

This can also be written as $v^j = u^{jk} L_k$. Then equation (*) asserts that the divergence of v is A. Thus if

$$w = v - \frac{A}{n} x$$

the vector field w has divergence 0. Now put

$$h = u - u_j x^j,$$

and $\tilde{L} = L + \frac{A}{n} H$. Then a short calculation shows that $u^{jk} \tilde{L}_k = w^j$. When $n = 2$ a divergence-free vector field is represented by a Hamiltonian so there is a function H with $w^j = \epsilon^{jk} H_k$ where ϵ is the alternating tensor.

The pair of functions $\tilde{L}, H$ are analogous to conjugate harmonic functions in two dimensions. They satisfy a pair of linear elliptic equations

$$P(\tilde{L}) = 0 \quad , \quad Q(H) = 0,$$

where P is the operator introduced before and Q is the similar operator

$$Q(f) = \sum u^{ij} f_{ij}.$$

Moreover the boundary conditions fix the normal component of w which translates into fixing the tangential derivative of H. Thus the boundary value of H is fixed, up to an arbitrary additive constant.

Now we can apply a general result for solutions of arbitrary elliptic equations in two dimensions, with suitable boundary values. We apply this to the equation $Q(H) = 0$. The crucial point is that we get an estimate independent of the coefficients of the operator Q. From this we deduce that $|\nabla H| \leq C$ for some C depending on P, σ, and hence $|w| \leq C$. Now consider the derivative of u as a

$$\nabla u : P \to \mathbf{R}^2.$$

It is easy to see that this is a diffeomorphism. If ξ_i are the co-ordinates on the image space we have

$$\frac{\partial \xi_i}{\partial x^j} = u_{ij},$$

hence

$$w^i = \frac{\partial \tilde{L}}{\partial \xi_i}.$$

Finally we deduce

$$|\tilde{L}(x) - \tilde{L}(x')| \leq C |(\nabla u)(x) - (\nabla u)(x')|.$$

From this it is straightforward to obtain upper and lower bounds on L.

(3) *Modulus of convexity estimate*

One approach to this goes via a sharper lower bound on L. We know that $L \to \infty$ on the boundary of P. One chooses a function ψ such that $L - \psi \to \infty$ on the boundary and such that the matrix

$$(\psi_{ij} - \psi_i \psi_j) > 0$$

throughout P. Applying the maximum principle, with a suitable choice of ψ, one can show that

$$L \geq C_1 (log(d^{-1}) + C_2.$$

Combining this with the estimate in the previous paragraph one gets

$$|\nabla u| \geq Const.d^{-\alpha},$$

for some $\alpha > 0$, and this gives a modulus of convexity estimate.

Alternatively one can get this estimate more directly from the lower bound on L using an old result, special to dimension 2, of E. Heinz. The proof of Heinz exploits the fact that there are isothermal co-ordinates (as mentioned in the first lecture) for the Riemannian metric u_{ij} on P.