Differential Geometric Methods in Low-dimensional Topology

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1 Introduction

This is a survey of various applications of analytical and geometric techniques to problems in manifold topology. The author has been involved in only some of these developments, but it seems more illuminating not to confine the discussion to these.

We begin by recalling the notion of a *manifold.* Suppose we are provided with a large collection of small paper discs. Then we can construct a wide variety of complicated objects by pasting these discs together in various fashions. Mathematically, the paper discs generalise to disjoint copies U_α of the unit ball B^n in some fixed Euclidean space $\mathbf{R}^n$, where α ranges over some index set. The pasting data generalises to a collection of homeomorphisms

$$\phi_{\alpha\beta} : U'_{\alpha\beta} \to U''_{\alpha\beta},$$

where $U'_{\alpha\beta} \subset U_\alpha$ and $U''_{\alpha\beta} \subset U_\beta$ are open subsets. Then we form a space M by identifying, in an abstract way, each point x in each U'_α with its image $\phi_{\alpha\beta}(x)$ in U''_β, and such a space is called an n-dimensional manifold. The essential point is that a manifold is locally modelled on Euclidean space, so we can transfer many familiar constructions from multi-dimensional geometry and calculus to this wider setting. It is important to emphasise that this notion of a manifold does not just derive from mathematicians fancy, but grows naturally out of many diverse applications, often in Mathematical Physics. Most obviously, one formulates General Relativity in terms of a four-dimensional space-time manifold.

The basic problem of geometric topology is to *classify* manifolds. More precisely, for our discussion, we want to consider manifolds constructed using differentiable maps (which allow us to do calculus): these lead to the definition of a "smooth" manifold, and the natural equivalence relation is that of "diffeomorphism". So, for each dimension n we are interested in classifying smooth n-dimensional manifolds up to diffeomorphism. For example, in two dimensions, any ellipsoid in $\mathbf{R}^3$ is diffeomorphic to the sphere, but a hyperboloid is not.

This classification problem has two complementary parts. In one direction, one seeks *invariants* of manifolds: the oldest example being the Euler characteristic which is an integer $\chi(M)$ one can assign to any (compact) manifold M, such that $\chi(M) = \chi(M')$ if M and M' are diffeomorphic. In the complementary direction, one seeks to construct diffeomorphisms $f : M \to M'$ showing that a pair of manifolds M, M' are equivalent, under suitable hypotheses.

Over the 100 years since Poincaré introduced the notion of a manifold, and hence this classification problem, many different strands have been developed. In this article we focus on constructions using differential geometry and analysis. The interesting feature here is that these methods call in techniques and ideas from other subjects, which do not ostensibly enter into the classification problem as we have formulated it. This means that we consider manifolds with some additional auxiliary structure such as a Riemannian metric, though this structure may disappear from the statement of the final result. A striking this, which probably has deep origins, is that these techniques are usually most relevant in "low-dimensional" topology, specifically when we consider n-dimensional manifolds with $n \leq 4$. In "high dimensions" ($n \geq 5$) a very rich theory was developed, particularly in the period 1950-1970. In brief, the subject of *algebraic topology* gives a systematic understanding of possible invariants and a fundamental result of Smale, the "h-cobordism theorem" yields a very powerful and general abstract technique for constructing diffeomorphisms between manifolds with the same invariants.

2 Two dimensions

The classification of two-dimensional manifolds is comparatively straightforward and has been known in some form since the mid 19th. century. Nevertheless, it is interesting to see how geometric and analytical techniques can be brought to bear on this, as a model for developments in higher dimensions.

Consider first the issue of invariants. Suppose we have a closed surface $S \subset \mathbf{R}^3$. We can consider the flow of an imaginary fluid on the surface, or in mathematical terms a vector field v (the velocity field of the fluid) defined on S and everywhere tangent to S. In this way, we are lead to study a pair of partial differential equations

$$\mathrm{div}(v) = 0 \quad , \quad \mathrm{curl}(v) = 0$$

for a tangent vector field v, corresponding to incompressible, irrotational flows. These are linear equations so the solutions form a vector space of dimension $d(S)$ (which could *a priori* be infinite). It turns out that $d(S)$ is finite and is unchanged if we continuously deform the surface in 3-space. Moreover we can extend the ideas further to an abstract two-dimensional manifold M equipped with a Riemannian metric. This metric is just the data required to define lengths and angles between tangent vectors at the same point and in turn the notions of divergence and curl. There is an enormous space of possible Riemannian metrics. In local co-ordinates u_1, u_2 (i.e. a local identification of the surface with a ball in $\mathbf{R}^2$), a metric is given by any functions $g_{ij}(u_1, u_2)$ for $i, j = 1, 2$, subject only to the constraint that for each fixed u_1, u_2 the matrix with entries g_{ij} is symmetric and positive definite. The upshot is that , changing notation, we now have an integer $d(M, g)$ where g denotes any choice of Riemannian metric out of this enormous space of possibilities. Now the crucial thing is that one can show that $d(M, g)$ does not change if we deform the metric in a continuous fashion. So we conclude that this dimension is actually an invariant $d(M)$ of the manifold M.

All the ideas above are now very well understood. The dimension $d(M)$ is just $2 - \chi(M)$ where χ is the Euler characteristic, which can be defined in many other ways. The ideas extend to higher dimensions in the form of "de Rham cohomology" and "Hodge theory", and the more general setting involves the machinery of "differential forms" rather than vector fields. At a more sophisticated level, one encounters the Dirac equation for fields of spinors on a manifold, and the Atiyah-Singer index theorem. One gets many invariants of manifolds, of any dimension, in this way, by studying the solutions of linear partial differential equations, but broadly speaking these can all be obtained in other ways, using the tools of algebraic topology.

Next we turn to the complementary question of constructing diffeomorphisms between 2-dimensional manifolds. Suppose for example that we want to show that any manifold M with $\chi(M) = 2$ is diffeomorphic to the standard sphere. One geometric approach to this goes via proving the existence of a particular Riemannian metric on the manifold. In classical differential geometry one defines, at each point of a surface in $\mathbf{R}^3$, the Gauss curvature of the surface: a natural generalisation of the notion of the curvature of a curve in the plane. The content of Gauss' famous "Theorem Egregium" is essentially that the Gauss curvature can be defined for any Riemannian metric on a general 2-dimensional manifold M. So we can search for metrics with constant Gauss curvature and in particular, in the case at hand, with Gauss curvature 1. Then it is quite an easy exercise in differential geometry to show that if we have such a metric g on our manifold there is a *unique* diffeomorphism $f : M \to S^2$ (up to rotations of the sphere) which takes g to the standard metric: in particular M is indeed diffeomorphic to the sphere.

So much for the overall strategy of this approach: we are left with the crucial problem of how to prove the existence of a Riemannian metric of Gauss curvature 1 on an abstract manifold M^2, using only the hypothesis that $\chi(M) = 2$. This can be viewed as solving a complicated nonlinear partial differential equation for the unknowns g_{ij}. The easiest way to proceed is to bring in another kind of structure, that of a Riemann surface, but we will not go into details. Suffice it to say that the hypothesis $\chi(M) = 2$ enters through the assertion that there are no non zero abstract "fluid flows"of the kind considered above, and the Fredholm Alternative from Functional Analysis.

3 Three dimensions

Exciting recent developments make it natural to include some brief discussion of 3-dimensional manifolds in our account, although this is not an area the author has contributed to personally.

First, the question of invariants. Over the past twenty years new 3-manifold invariants of various kinds have been discovered, having fundamental connections with geometry. On the one hand there are invariants such as the Casson invariant and Floer homology groups which are the 3-dimensional counterparts of the ideas in 4-dimensions discussed below. On the other hand there are the "Jones-Witten invariants" which, in Witten's point of view, arise from certain Quantum Field Theories.

Second, the question of constructing diffeomorphisms between 3-manifolds. The famous problem here is the "Poincaré conjecture" which is that any simply connected compact 3-manifold is diffeomorphic to the 3-dimensional sphere. This is the natural analogue of the question about 2-dimensional manifolds discussed above, with the "simply connected" hypothesis in place of the condition on the Euler characteristic. There has been striking progress on this problem recently, through work of G. Perelman [5], which makes it seem very likely that this famous problem has now been resolved, and the strategy of proof follows that in our two-dimensional model. A Riemannian metric in higher dimensions has, in place of the simple Gauss curvature, a complicated curvature tensor. From this one forms a slightly simpler object: the Ricci tensor R_{ij}. This is what enters into Einstein's formulation of General Relativity, and one can write down an analogue of Einstein's equation in the context of Riemannian geometry: $R_{ij} = \lambda g_{ij}$, where λ is a constant. In three dimensions it turns out that the Ricci tensor contains the same information as the full curvature tensor, and using this it is easy to show that a simply-connected 3-manifold which admits a solution of the Einstein equation is diffeomorphic to the 3-sphere. So the problem is how to construct such Riemannian metrics.

Perelman's work follows a strategy developed over many years by R. Hamilton. One introduces an extra "time" variable t and considers a 1-parameter family of Riemannian metrics on a 3-manifold satisfying the evolution equation

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij}.$$

Starting with an arbitrary initial metric at $t = 0$ one seeks to show that, after suitable rescaling, the metrics generated by this evolution equation converge to a solution of Einstein's equation. There are immense difficulties in carrying this through, but it appears that the crucial problems have been overcome by Perelman. This approach is not limited to the Poincaré conjecture. In the 1970's W.Thurston formulated a "Geometrisation conjecture" which asserts that any 3-manifold can be decomposed in a standard and well-controlled way into pieces each of which admits an Einstein metric or one of a small family of other special structures. This is a much more wide-ranging conjecture which in a sense gives a complete classification of 3-manifolds and it is this which is the natural target for Perelman and Hamilton's work.

4 Four dimensions

4.1 Invariants

We now turn to four dimensional manifolds, the topic to which the author has contributed. Standard algebraic topology provides certain tools. We restrict attention to compact, simply-connected 4-manifolds with a fixed orientation. Then the algebro-topological data associated to such a manifold M is the free Abelian group $H_2(M)$ and the intersection form Q, which is a symmetric bilinear form on $H_2(M)$. The natural "grand problem" in the field is to classify, for each algebraic isomorphism class of the data (H_2, Q) the possible diffeomorphism classes of manifolds. Roughly speaking, almost nothing was known about this question until the early 1980's but now we know a substantial amount through the emergence of the *instanton* and *Seiberg-Witten* invariants. However the problem itself still seems way out of reach, as we will discuss further below.

The general strategy by which these invariants are defined follows the same pattern as in the two-dimensional model discussed above. The crucial ingredient is the existence of certain geometrical objects and partial differential equations governing them, which play the role of the vector field and the irrotational, divergence-free conditions there. As in that model, the objects arise naturally from considerations in Mathematical Physics, although now that of fields and elementary particles rather than fluids.

A great achievement of 19th. century Mathematical Physics was the formulation of electro-magnetic theory in terms of a pair of vector fields $\mathbf{E}, \mathbf{B}$ governed by Maxwell's equations. Further insight in the 20th. century lead to the ideas that, first, the equations could be formulated in a four-dimensional setting, with space and time on an equal footing, involving a single field tensor F. Second that this field has an essentially geometric origin. The geometry involves the introduction of a complex line bundle L over space-time, and the wave-functions of Quantum Theory are viewed as sections of L. Thus the value $\psi(x)$ of a wave function at a point is not naturally a complex number but lies in a one-dimensional complex vector space L_x, and there is no completely canonical way to identify L_x with $\mathbf{C}$. The basic geometrical structure is a *connection* on this line bundle and the field tensor F is the curvature of this connection. Mathematically, these ideas are underpinned by the general theory of bundles and connections which had been developed by differential geometers and which grow naturally out of classical differential geometry and notions such as the Gauss curvature. In Physics, these ideas lead to natural generalisations in "Gauge Theory" where one simply replaces the one dimensional vector spaces L_x by vector spaces of some fixed higher dimension. (The extension can also be formulated in the language of symmetry groups such as $SU(2)$, $SU(3)$.)

These general notions of bundles, connections and curvature can be formulated over manifolds of any dimension but there is a crucial special feature of

4-dimensions. The field tensor is a skew-symmetric tensor and in a four dimensional space, with a positive definite metric, these skew symmetric tensors decompose naturally into "self-dual" and "anti-self-dual" parts. So we can write $F = F_+ + F_-$ and we consider the special condition that $F_+ = 0$. If we go back to make a "space-time decomposition" and express F in terms of a pair of vector $\mathbf{E}, \mathbf{B}$, this condition is just $\mathbf{E} = \mathbf{B}$, but the crucial thing is that the condition is actually a natural one in four-dimensions, independent of the decomposition.

Putting these ideas together, we see that if we consider a Riemannian metric on our 4-manifold M, a bundle E over M and a connection A on E, we can write down a natural condition

$$F_+(A) = 0,$$

for the curvature tensor $F(A)$. This is a partial differential equation for the connection A. If the dimension of the fibres of E is 1, as in electromagnetic theory, the equation is linear and we essentially recover part of the familiar Hodge Theory. But for higher dimensional fibres we get more subtle, nonlinear equations; the "Yang-Mills instanton equations". The basic strategy is to extract invariants of the manifold M from a study of the solutions (instantons) to these equations.

A variety of mathematical techniques are involved in extracting discrete invariants from the instantons. On the one hand there are fundamental analytical results of K. Uhlenbeck which give information about compactness of the space of solutions. A prerequisite here is the fact that the instanton equations are elliptic equations and roughly speaking Uhlenbeck's work allows the extension of standard ideas for linear elliptic equations to this nonlinear setting. On the other hand, there is a general and more abstract body of ideas which allow one to extend techniques of differential topology to certain infinite dimensional "Fredholm" problems. In particular, under suitable technical hypotheses, one gets discrete invariants from the solution spaces to the equations in much the same way as one can define the degree of a map $f : S^n \to S^n$ by "counting" (with signs) the points in a generic preimage $f^{-1}(y)$ (*i.e.* by counting the solutions of the equation $f(x) = y$). Of course the crucial thing is that these discrete invariants are unchanged by continuous deformations of the data. This translates in our problem to independence of the choice of Riemannian metric on M.

The upshot of all this technical work was that one obtained, under suitable hypotheses, new invariants of M which took the form of a collection of polynomials functions on the homology group $H_2(M)$[2]. The fact that we get a collection of polynomials comes from the fact that we have a choice of bundles E to consider. In the late 1980's these instanton invariants were used to give much new information about the "grand problem" above: for example by showing that certain large families of 4-manifolds with the same intersection form where all distinct up to diffeomorphism.

In 1994, Seiberg and Witten introduced some different equations in four dimensions, guided by considerations from Quantum Field Theory [7]. These share many of the features of the instanton equations, in that they are formulated in terms of a connection on a bundle over the 4-manifold, but now the bundle has fibre dimension 1, just as in electromagnetism. The new subtlety is that one considers a spinor field ψ in addition to the connection on the bundle. This extra field can be thought of a something like the wave function of quantum mechanics but its' spinorial nature is crucial. The Seiberg-Witten equations take the shape

$$F_+(A) = \psi^*\psi \quad , D_A\psi = 0,$$

where D_A is the linear Dirac operator coupled to the connection and $\psi^*\psi$ denotes a certain quadratic form mapping spinors to self-dual 2-forms. Invariants of the underlying 4-manifold X can be extracted from the solutions to the Seiberg-Witten equations in a similar manner to the instanton case, but the newer theory has some decisive technical advantages. The invariants that result take the shape of certain distinguished classes ("basic classes") $\kappa_i \in H_2(X)$ with associated integers n_i. Witten made a wide ranging conjecture, backed up by almost overwhelming evidence from examples, as to precisely how these Seiberg-Witten invariants determine the polynomials given by the instanton theory. With these insights, the extent of the information which can be obtained from these methods has become much clearer, and the whole theory seems to have attained a reasonably mature form. (There is scope for exploiting the older instanton theory, and its relation with the Seiberg-Witten theory, particularly in applications of these ideas to 3-dimensions, as in recent work of Kronheimer and Mrowka [4], which establishes a result very like the Poincaré conjecture for a slightly different class of 3-manifolds.)

4.2 Constructing diffeomorphisms

As we have emphasised in this article the problem of classifying manifolds has two complementary parts. In four dimensions we have now a good supply of invariants but what is almost entirely lacking is any way of constructing diffeomorphisms between manifolds, under suitable hypotheses on the invariants. We can write down many families of 4-manifolds with the same instanton and Seiberg-Witten invariants and we have *no idea* whether they are diffeomorphic or not. Something completely new is almost certainly needed to make substantial further headway with the "grand problem", but whether this will come in 1 year, 10 years or 100 years is anybody's guess. The only progress so far in this direction seems to lie in the special case of *symplectic* 4-manifolds. A symplectic structure on a 4-manifold is a closed 2-form ω which is everywhere nondegerate, a notable supply of examples being furnished by complex algebraic surfaces with Kahler metrics. Until the 1980's there were as many inacessible questions about symplectic 4-manifolds as for the general case. But now we know a great deal more, principally through fundamental advances of Gromov and Taubes. These involve a network of ideas closely related to those above. In one direction, Gromov's fundamental paper [3] introduced the use of pseudoholomorphic curves as a tool to study symplectic manifolds. This development has had extremely wide-ranging consequences and uses some of the general ideas exploited in the instanton and Seiberg-Witten theories. One result of Gromov is particularly relevant to the classification problem because he shows that a symplectic 4-manifold satisfying suitable hypotheses, notably the existence of a certain kind of embedded 2-sphere, must be equivalent to the standard complex projective plane. The proof goes by moving the 2-sphere in a family of pseudoholomorphic curves sweeping out the manifold, and is quite parallel to arguments from the classification of algebraic surfaces. In the other direction, Taubes [6] discovered a fundamental connection between Gromov's pseudoholomorphic curves and the Seiberg-Witten equations, and was able to use this to establish the existence of the required embedded sphere. Some of the author's recent work [1] has been motivated in part by the desire to extend this technique to more general 4-manifolds, but so far without very conclusive results.

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