Discussion of the Kähler-Einstein problem

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1 Introduction

Let T be an irreducible variety which parametrises a family of Fano manifolds. For simplicity we suppose that each of these has a discrete automorphism group. Write $T_{KE} \subset T$ for the subset corresponding to manifolds which admit Kahler-Einstein metrics. It is known that T_{KE} is open in the classical topology and all points in T_{KE} are "K-stable". The purpose of these notes is to make progress towards the following three goals

- **Goal 1**. T_{KE} is Zariski open.
- **Goal 2** If T_{KE} is nonempty then all K-stable points are in T_{KE}.
- **Goal 3** Without assuming *a priori* that T_{KE} is non-empty, all K-stable points are in T_{KE}.

These statements are (at least roughly) of increasing strength. The last one (in which the parameter space T is irrelevant) is the "Yau conjecture" for Fano manifolds. The first could be viewed as precise form of the statement that having a Kahler-Einstein metric is "an algebro-geometric condition". The author hopes that these notes essentially achieve Goal 1, modulo results in forthcoming joint work with X-X Chen. The scheme of proof applies to Goal 2 but we run into algebro-geometric difficulties which we do not overcome here. In Section 5 we explain how the ideas can be applied to Goal 3, making the rather large assumption that there is an appropriate differential geometric theory of manifolds with cone singularities. However the algebraic geometry is somewhat simpler in this setting.

Note that it is not actually clear from the definition that the set of K-stable points (or $\overline{K}$stable points) is Zariski open (it could *a priori* be the complement of a countable union of algebraic sets). So in some respects Goal 1 goes beyond Goals 2 and 3. See the related discussion in Section 6.

2 Sequences of Kähler-Einstein metrics

In this section we consider the following set-up. We have a sequence of polarised complex algebraic n-manifolds X_i converging to a smooth limit X_∞. We use the same symbol L for the positive line bundle over each X_i and X_∞. To be precise we might say that X_i are defined by a convergent sequence of integrable almost-complex structures on a fixed C^∞ manifold. From another point of view, if we have an appropriate moduli space $\mathcal{M}$ of complex structures then we can regard our data as a convergent sequence in $\mathcal{M}$. For simplicity, we assume that the holomorphic automorphism group of X_∞ is trivial.

Now suppose that each X_i has a constant scalar curvature Kahler (cscK) metric ω_i in the class $c_1(L)$. The problem we address is to show that if X_∞ satisfies an appropriate stability condition then it will also have a cscK metric. In reality—for real applications—our results will be limited to the case when the X_i are Fano, L is the anticanonical bundle and the metrics in question are

Kahler-Einstein. But the central argument we use applies in the more general cscK setting. So we will state our results somewhat more generally, adding differential geometric hypotheses which are known to hold in the Kahler-Einstein case.

We consider the following stability condition for X_∞. First (X_∞, L) should be K-stable. Second there should be an $\epsilon_0 > 0$ such that for all points x in X_∞ and all rational $\epsilon < \epsilon_0$ the blow-up of X_∞ at x should be K-stable with the polarisation $L - \epsilon E$. We call this condition $\overline{K}$-stability. It is known from the work of Arezzo-Pacard and Stoppa that if X_∞ has a cscK metric then it is $\overline{K}$-stable. Most likely $\overline{K}$-stability is equivalent to K-stability, but that this does not seem to be obvious.

2.1 Differential Geometric input

Let M, g be any compact Riemannian manifold and let r be a positive real number, to be thought of a scale parameter. Let $Z(r) \subset M$ be the r-neighbourhood of the set of points in M where $|Riem| > r^{-2}$ and write $\Omega(r)$ for the complement $M \setminus Z(r)$. Thus on the r-ball centred at each point of $\Omega(r)$ the curvature is bounded by r^{-2}. If we rescale the metric by a factor r^{-1} then we get a unit ball (in the rescaled metric) over which the curvature is bounded by 1.

Hypothesis V

There is a constant C such that for each of the Riemannian manifolds (X_i, ω_i) and all $r > 0$ we have

$$\mathrm{Vol} Z(r) \leq C r^4.$$

This is the essential hypothesis for our arguments. X-X Chen and the author have proved that this hypothesis holds in the Kahler-Einstein case, for complex dimension n=3 [3]. The constant C could be made to depend explicitly on standard topological invariants of X_i and certain universal constants.

Supplements to Hypothesis V

1. There is a *connected* open set $\Omega'(r) \subset \Omega(r)$ with $\mathrm{Vol}(X_i \backslash \Omega'(r) \leq C r^{6n/2n-1}$.
2. For each point in $\Omega(r)$ the r-ball is embedded by the exponential map.

The first of these is also proved in [3] when $n = 3$. The second is a "noncollapsing" hypothesis which is known to hold in the Kahler-Einstein case, after unimportant adjustment of constants.

The effect of these hypotheses is that for any (small) scale r there is a connected set with small complement such that the geometry on scale r is completely controlled at each point of this set.

Additional hypotheses

1. We suppose that the *Ricci curvature* of (X_i, ω_i) satisfies a fixed bound $|\mathrm{Ric}| \leq C$.

2. Let G by the Green's function on X_i, normalised so that $\min_{x,y} G(x,y) = 0$. (Our sign conventions are that $\Delta = \partial^2$ and the Green's function satisfies $\Delta_x G(x,y) = -\delta_y$, so that G is very positive near the diagonal.) We suppose there is a fixed bound

$$\int_{X_i} G(x,y)dy \leq C,$$

for all $x \in X_i$ and all i.

3. There is an L^∞ bound, for every holomorphic section s of L^k over X_i

$$\|s\|_{L^\infty} \leq Ck^{n/2}\|s\|_{L^2}$$

These are all known to hold in the Kahler-Einstein case (the first being trivial, of course.) See the discussion in Appendix 1. We next state another fact that will be important in our arguments. Fix r_0 so that the volume of $\Omega(r_0) \subset X_i$ is at least half the volume of X_i.

Proposition 1 *For $r \geq r_0$ and any integer k there is a constant $C(r,k)$ such that for any holomorphic section of $L^k \to X_i$ we have*

$$\|s\|_{L^2}^2 \leq C(r,k) \int_{\Omega(r)} |s|^2.$$

We sketch a proof in the Kahler-Einstein case. By "additional hypothesis 3" above we can fix $r_1 = r_1(k) < r$ so that

$$\|s\|_{L^2}^2 \leq C'(r_1,k) \int_{\Omega'(r(k))} |s|^2.$$

for a suitable $C'(r_1,k)$. Now suppose the statement is false. Then without loss of generality there are holomorphic sections s_i of $L^k \to X_i$ with

$$\int_{\Omega'(r_1)} |s_i|^2 = 1$$

but

$$\int_{\Omega(r)} |s_i|^2 \to 0,$$

as $i \to \infty$. Now take the Gromov-Haussdorf limit of a subsequence of the X_i, which we may as well suppose is the whole sequence. The sequence of domains $\Omega'(r_1) \subset X_i$ can be supposed to converge to a domain Ω' in the smooth part of this limit. The holomorphic sections are s_i are bounded (by additional hypothesis 3) so we can suppose they converge over a slightly larger set than Ω'. Then the limit gives a non-trivial holomorphic section over the connected set Ω' vanishing on a proper open subset, which is a contradiction.

This argument is not completely satisfactory; first because it does not yield an explicit estimate and second because it relies on the metric convergence theory of Gromov *et al.* It is possible to write out a longer but more direct proof, just using the Hypotheses above, which does not have these drawbacks.

Now choose an arbitrary Kahler metric ω_∞^* on X_∞ and a convergent sequence of "reference metrics" ω_i^* on X_i. So we can write $\omega_i = \omega_i^* + \sqrt{-1}\partial\overline{\partial}\phi_i$ for a sequence of Kahler potentials ϕ_i. We write $\mathrm{Osc}(\phi_i)$ for the difference between the maximum and minimum values. Now what we would like to show is

Goal 2' Assume X_i satisfy the hypotheses above. If X_∞ is $\overline{K}$-stable then there is a fixed bound on $\mathrm{Osc}(\phi_i)$, for all i.

We will not quite achieve this goal here. If we did then in the Kahler-Einstein case the estimates of Yau give bounds on all higher derivatives and we would deduce that X_∞ has a KE metric, achieving "Goal 2" (with the small modification that we use $\overline{K}$-stability).

2.2 The main argument

2.2.1 The Chow weight and stability

Now we review the central concept in our approach. Let V be a finite-dimensional Hermitian vector space which we may as well take to be the standard $\mathbf{C}^{N+1}$ and $Z \subset \mathbf{P}(V) = \mathbf{P}^N$ a projective manifold. Suppose A is a Hermitian endomorphism of V and let H be the corresponding Hamiltonian function on $\mathbf{CP}^N$ given by

$$H(\underline{z}) = \frac{1}{|\underline{z}|^2}\sum A_{\alpha\beta} z_\alpha \overline{z}_\beta.$$

Then we define the *Chow weight* of Z with respect to A to be

$$\mathrm{Ch}(Z, A) = -\frac{1}{\mathrm{Vol}(Z)}\int_Z H d\mu_{FS} + \frac{1}{N+1}\mathrm{Tr}(A).$$

Here $d\mu_{FS}$ is the volume form on Z induced by the Fubini-Study metric. Notice that the factors are chosen so that the Chow weight vanishes if A is a multiple of the identity. We write $|A|$ for the operator norm of A, i.e. the modulus of the largest eigenvalue. Thus $|H| \leq |A|$ on $\mathbf{P}^N$.

In the case when Z is preserved by the 1-parameter subgroup generated by A the Chow weight is an algebro-geometric notion, independent of the choice of Hermitian metric. Now we recall the definition of K-stability. Given a polarised manifold (X, L) we consider a *test configuration* which is a $\mathbf{C}^*$-equivariant family $\pi : \mathcal{X} \to \mathbf{C}$ with $\pi^{-1}(1) = X$ and an equivariant, ample, $\mathbf{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$ which restricts to L on the fibre $\pi^{-1}(1)$. Then for sufficiently large k we can represent this family by a 1-parameter subgroup acting on projective schemes, with the fibre $\pi^{-1}(1)$ embedded by the linear system $H^0(X, L^k)$. Thus the central fibre $Z = \pi^{-1}(0)$ is embedded in $\mathbf{P}^N$, where $N = N_k$, and preserved by a 1-parameter subgroup with generator $A = A_k$. To fix signs, in moving towards the central fibre we flow along the *decreasing* gradient flow of the function H.

The Chow weights $\mathrm{Ch}(Z, A_k)$ have a finite limit as $k \to \infty$ which is, by defintion, the *Futaki invariant* of the test configuration. The pair (X, L) is K-stable if this Futaki invariant is positive for all non-trivial test configurations.

Remarks

- The central fibre will in general be a scheme. The definition of the Chow weight extends immediately to algebraic cycles and we use the cycle corresponding to the scheme to define the Chow weight.
- By saying that $\mathcal{L}$ is a $\mathbf{Q}$-line bundle we mean that only some power $\mathcal{L}^m$ is defined as a genuine line bundle. Thus the index k will only run through multiples of m in the discussion above.

Three important properties of the Chow weight are as follows.

1. The Chow weight is continuous as a function on the Chow variety parametrising algebraic cycles.
2. For fixed A the Chow weight $\mathrm{Ch}(\ , A)$ is constant on each connected component of the fixed point set of the 1-parameter subgroup $\exp(tA)$ acting on the Chow variety.
3. If $Z_t = \exp(tA)Z$ then the Chow weight $\mathrm{Ch}(Z_t, A)$ is a decreasing function of t.

Concerning the second item: one point of view is that $\mathrm{Ch}(\ , A)$ is a Hamiltonian function for the action of $\exp(itA)$ with respect to a symplectic structure on the Chow variety–in fact this is induced from a symplectic form on an ambient projective space. Then we have the familiar fact that the Hamiltonian is constant on each component of the fixed set. From another point of view the Chow weight is a cohomological invariant of a space with a torus action and an equivariant line bundle and then the deformation invariance is clear from this.

Concerning the third item: we can think more generally of a function H on a compact Riemannian manifold M and a submanifold $N \subset M$. Let $f_t : M \to M$ be the flow generated by the gradient of H, let $N_t = f_t(N)$ and

$$I(t) = \int_{N_t} H d\mu,$$

where $d\mu$ is the induced Riemannian volume form. Write $v^\perp$ for the component of $gradH$ orthogonal to N. Then we have

$$I'(0) = -\int_N |v^\perp|^2 + h(v^\perp) d\mu,$$

where h denotes the mean curvature of N. So if N is a minimal submanifold we have $I'(0) \leq 0$. In the case at hand, when $M = \mathbf{CP}^N$ and $N = Z$, the desired assertion now follows from the fact that complex submanifolds are minimal.

For fixed Z the Chow weight can be regarded as the derivative of a function $\mathcal{F}_Z$ on the space of Hermitian metrics on $\mathbf{C}^{N+1}$ and the third property is the assertion that $\mathcal{F}_Z$ is convex.

2.2.2 Bound on the Chow weight

The input for this is a slightly refinement of the theory of Luo-Tian-Zelditch. Let $L \to Z$ be a positive line bundle over a compact complex manifold with a Hermitian metric on the fibres whose curvature yields a Kahler form ω with volume form $d\mu$. For integers $k > 0$ we define the density of states function by

$$\rho_k = \sum |s_\alpha|^2,$$

where s_α is an orthonormal basis of $H^0(L^k)$ and we use the volume form $k^n d\mu$. The basic fact is that ρ_k has an asymptotic expansion

$$\rho_k \sim 1 + a_1 k^{-1} + a_2 k^{-2} + \dots, \qquad (*)$$

as $k \to \infty$. The a_i are functions on Z, determined by the curvature tensor of the metric, and in particular a_1 is half the scalar curvature.

The refinement we need is that this is statement is local in Z.

Theorem 1 *Suppose that the unit ball centred at some point $z_0 \in Z$ is embedded and that the modulus of the Riemann curvature is bounded by* 1 *on this ball. Suppose also that the metric has constant scalar curvature and satisfies a bound on the Ricci tensor $|Ric| \leq C$ throughout Z. Then the asymptotic expansion * holds in the half-sized ball centred at z_0 with constants depending only on C and the dimension n.*

Thus, for example, we can find a k_0 depending only on C, valid for all such Z of dimension n, such that $|\rho_k - 1| \leq .01$ at z_0, once $k \geq k_0$.

We will discuss the proof in Appendix 2 below. The constant scalar curvature assumption is not really fundamental here: it is only used to give control of all derivatives of the metric in the ball.

Now consider the sequence of manifolds X_i in our problem. We use the standard L^2 norm on $H^0(X_i, L^k)$, so X_i becomes a variety in a projectivisation of a Hermitian vector space which we identify with the fixed $\mathbf{C}^{N+1}$.

Theorem 2 *Assuming the hypotheses above there is a constant C such that*

$$|\mathrm{Ch}(X_i, A)| \leq C k^{-2} (\log k) |A|$$

for all A, i.

The proof of Theorem 2 is given in Section 3 below.

2.3 Limit in the Hilbert scheme

With these preliminaries stated we can begin the main argument. We change point of view slightly and let V be a fixed hermitian vector space of dimension $N+1 = \dim H^0(X_i, L^k)$; which we often identify with $\mathbf{C}^{N+1}$. For each i we choose a hermitian isomorphism of V with $H^0(X_i, L^k)$ so we get a sequence of manifolds which we still denote by X_i in the fixed projective space, with its fixed Fubini-Study metric. On the other hand we can fix some arbitrary embedding of X_∞ as a projective variety $X^*_\infty \subset \mathbf{P}(V^*)$ and chose a sequence X^*_i say converging to X^*_∞. Thus X_i and X^*_i are isomorphic projective varieties and there is a $g_i \in PGL(V^*)$ such that $g_i(X^*_i) = X_i$. Our hypothesis on the automorphism group implies that the g_i are uniquely determined. We may suppose that our reference metrics ω^*_i are those induced from the Fubini-Study metric on X^*_∞.

Now we regard the X_i as points in the appropriate Hilbert Scheme. By compactness of the Hilbert scheme they have a convergent subsequence, which we may as well suppose is the whole sequence. The limit is some scheme W. There are two possibilities: either the g_i converge in $PGL(V^*)$, in which case W is a variety isomorphic to X_∞, or the g_i do not converge, in which case W is not isomorphic to X_∞.

Proposition 2 *There is a k_0 such if $k \geq k_0$ and the sequence g_i converges then the sequence* $\mathrm{Osc}(\phi_i)$ *is bounded.*

This is the foundation of our argument. Supposing that the ϕ_i are *not* bounded we obtain for each large enough k a limit W in a *different orbit* in the Hilbert scheme and our overall goal is to use these to show that X_∞ cannot be K-stable.

To prove Proposition 1 we apply a standard analytical argument. Let ω, ω' be two cohomologous Kahler metrics on a compact manifold Z so $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\phi$. Let $\phi_{\max}, \phi_{\min}$ be the maximum and minimum values of ϕ and set $f = \phi - \phi_{\max}$. Since $\omega + \sqrt{-1}\partial\overline{\partial} f > 0$ we have$\Delta f \geq -n$ (with sign convention $\Delta = \partial^2$). If z is a point where ϕ attains its maximum we have

$$0 = f(z) = \mathrm{Av}_\omega(f) - \int_Z G(z,w)\Delta f dw,$$

so

$$\int_Z f d\mu \geq -C,$$

where C depends only on n, the volume of Z and the Green's function bound for ω. Here $d\mu$ is the volume form of ω. In terms of ϕ,

$$\int_Z \phi_{\max} - \phi \; d\mu \leq C'$$

Now interchange the roles of ω, ω'. This gives

$$\int_Z \phi - \phi_{\min}\ d\mu' \leq C'$$

where $d\mu'$ is the volume form of ω' and C' depends on the Green's function bound for ω'. Suppose there is a open set set $\Sigma \subset Z$ on which $d\mu' > \epsilon d\mu$, for some $\epsilon > 0$. Then

$$\int_\Sigma \phi - \phi_{\min} d\mu \leq C'/\epsilon,$$

so

$$\int_\Sigma \phi_{max} - \phi_{\min} d\mu \leq C + C'/\epsilon,$$

and $\phi_{\max} - \phi_{\min} \leq (C + C'/\epsilon)\mathrm{Vol}(\Sigma)^{-1}$.

Now we apply this in our situation to the reference metrics and cscK metrics on X_i. The Green's function bounds are controlled by hypothesis. So if we can show that there is a set $\Sigma \subset X_i$ of a fixed volume (that is, independent of i) with respect to the cscK metric and such that the measure of the reference metric is bounded below on Σ by a fixed multiple of that of the cscK metric then we obtain a uniform bound on the Kahler potentials. Now we apply our Hypothesis (Hypothesis V) in a rather weak form, to say that there is some r_0 such that the volume of $Z(r_0)$ is less that half (say) the volume of X_i. In particular there is some ball in X_i of radius r_0 on which the curvature is bounded by r_0^{-2}. Scaling by the fixed amount r_0 we can apply Theorem 1 to find some fixed k_0 such that if $k \geq k_0$ then ρ_k is very close to 1 (in C^2-norm) over the half-sized ball. This implies that the Fubini-Study metric induced by the projective embedding is close to the cscK metric and in particular the volume form of the Fubini-Study metric is bounded below by some fixed multiple of that of the cscK metric, on this half-sized ball. We take this half-sized ball as our set Σ

Now suppose that the g_i are convergent. This means that the map $g_i : X_i^* \to X_i$ distorts the Fubini-Study volume by a bounded amount. Then it follows that the reference volume form on Σ is bounded below by a fixed multiple of the cScK volume form and we obtain the uniform bound on the $\mathrm{Var}(\phi_i)$.

Now, given k, fix $r_0(k) = bk^{-1/2}$ and $\Omega_i' = \Omega'(r_0) \subset X_i$. The number b is fixed so that the Fubini-Study metric induced by the projective embedding $f_i : X_i \to \mathbf{P}^N$ is uniformly equivalent to the cscK metric on Ω_i'. We can suppose k chosen so large that the volume of Ω_i in the Fubini-Study metric is at least $3/4$ of the volume of X.

Proposition 3 *There is an irreducible component B of W with* $\mathrm{Vol}(B) > \frac{1}{2}\mathrm{Vol}(W)$ *and so that $f_i(\Omega_i')$ converges to an open subset $\Omega_\infty' \subset B$, and B is smooth and reduced at points of Ω_∞'.*

Notice that B is trivially unique by the volume condition, so we know that W contains one "big" irreducible component.

Proposition 4 *The component B does not lie in any hyperplane in $\mathbf{P}^N$.*

Suppose that B lies in a hyperplane. For each i, this hyperplane corresponds to a holomorphic section s of L^k over X_i, with L^2 norm 1. We have

$$\frac{|s|^2}{\rho} \leq \delta_i,$$

on $\Omega(r)$ where $\delta_i \to 0$ as $i \to \infty$. But ρ is bounded above on $\Omega(r)$ so we get $\max_{\Omega(r)} |s|^2 \leq C\delta_i$. For large i, this contradicts the third supplementary hypothesis, since the L^2 norm of s is 1.

Now take an integer $m > 0$ and set $k' = mk$. Thus we have another sequence of embeddings

$$f_i' : X_i \to \mathbf{P}^{N'}$$

say. Here $\mathbf{P}^{N'}$ again has a fixed Fubini-Study metric and the embeddings are determined, up to the action of the unitary group, by the L^2 norm on sections of $L^{k'}$ over X_i. We get another limit $W' \subset \mathbf{P}^{N'}$. Let $\Lambda \to W$ be the restriction of the hyperplane bundle of $\mathbf{P}^N$ to W and likewise for $\Lambda' \to W'$.

Proposition 5 *Suppose that W is reduced and irreducible. Then for all sufficiently large m the schemes W, W' are isomorphic. More precisely the pair (W', Λ') is isomorphic to (W, Λ^m).*

We can choose m so large that the natural map $s^m(H^0(W, L^k)) \to H^0(W, L^{km})$ is surjective. Hence, for each large i

$$p : s^m(H^0(X_i, L^k)) \to H^0(X_i, L^{mk})$$

is surjective. Let K be the kernel of p. The L^2 norm on $H^0(X_i, L^k)$ induces a stsndard norm on the symmetric power, let J be the orthogonal complement of K. Using the obvious isomorphism from J to $H^0(X_i, L^{mk})$ we get an induced norm, $\| \|_0$ say, on $H^0(L^{mk})$. To fix ideas suppose for example that $m = 2$. Let (s_α) be an orthonormal basis of $H^0(X_i, L^k)$. A section σ of L^{2k} can be written, in general, in a variety of different ways as $\sigma = \sum c_{\alpha,\beta} s_\alpha s_\beta$ and we have

$$\|\sigma\|_0^2 = \min \sum |c_{\alpha\beta}|^2,$$

where the minimum is taken over all such representations of σ.

Now we also have an L^2 norm on $H^0(X_i, L^{km})$. We claim that it suffices to prove that these two norms are uniformly equivalent, in the sequence, i.e.

$$C^{-1} \|\sigma\|_0 \leq \|\sigma\| \leq C \|\sigma\|_0,$$

for a fixed C independent of i. To see this consider the standard Hermitian vector space $\mathbf{C}^{N'+1}$. The embeddings $f_i' : X_i \to \mathbf{P}^{N'}$ are given by choosing an isometry between the standard metric and the L^2 norm on $H^0(X_i, L^{mk})$. On the other hand we can compose the embeddings f_i with the Veronese embedding and these are given by choosing an isomorphism between the standard metric and $\| \ \|_0$. So if the norms are uniformly equivalent the embeddings f_i' differ from the composites $V \circ f_i$ by a bounded sequence in $PGL(N'+1)$ and it follows that the limits W, W' are equivalent.

Now in one direction it is clear from the third additional hypothesis that $\|\sigma\|_{L^2} \leq C\|\sigma\|_0$. The difficulty comes in the other direction. Suppose that we have a sequence σ_i with $\|\sigma_i\|_{L^2} \to 0$ and $\|\sigma_i\|_0 = 1$. We can suppose that σ_i converge to an non-zero element σ orthogonal to the kernel of $s^m(H^0(W, L^k)) \to H^0(W, L^{mk})$. Then σ is a polynomial on $\mathbf{P}^N$ which does not vanish on W but does vanish on an open set in B. If W is reduced and irreducible this is impossible.

2.3.1 General theory for group actions

Suppose that a reductive group G with maximal compact K, acts on a projective variety $S \subset \mathbf{P}(V)$ via a linear action on V. In our application S will be a Hilbert scheme or, more precisely, the corresponding underlying variety. Suppose that we have sequences x_i, y_i in S and $g_i \in G$ such that $x_i = g_i y_i$ and that $x_i \to x, y_i \to y$ as $i \to \infty$. What can we say about x and y?. A familiar case is when the y_i are all equal so we have $x_i = g_i(y)$. This is the case which is relevant to the proof of the "Hilbert criterion" in Geometric Invariant Theory, and one shows that there is some 1-parameter subgroup $\lambda : \mathbf{C} \to G$ such that $\lambda_t(y) \to x$ as $t \to \infty$. In fact we can choose this to be an algebraic 1-parameter subgroup, factoring through $\mathbf{C}^*$. We might say that there is an orbit of λ "going from" y to x. But more complicated situations can arise.

Example

Suppose S is a toric surface so $G = \mathbf{C}^* \times \mathbf{C}^*$ has an open dense orbit $S \setminus D$. Any two points $x, y \in S$ are limits of sequences x_i, y_i in the open orbit so with $x_i = g_i(y_i)$ as above. Fix a base point $s_0 \in S \setminus D$ and identify the Lie algebra of K with $\mathbf{R}^2$. The "fan" of S is a decomposition of $\mathbf{R}^2$ into wedge-shaped regions W_α, each corresponding to a fixed point $\sigma_\alpha \in D$ of the action. If v is a vector in the interior of a W_α then

$$\lim_{t \to \infty} \exp(itv)(s_0) = \sigma_\alpha.$$

Now let v' be another vector in the interior of some W_β. It might be that there is a vector u such that u is in the interior of W_α and $-u$ is in the interior of W_β. In that case $\sigma_\alpha, \sigma_\beta$ can be connected by the orbit of the 1-parameter subgroup generated by iu, in the sense that

$$\lim_{t \to \infty} \exp(itu)s_0 = \sigma_\alpha, \lim_{t \to -\infty} \exp(itu)s_0 = \sigma_\beta.$$

But if $W_\alpha \cap (-W_\beta)$ is empty we cannot find a single such 1-parameter subgroup. What we can do is to take $u = \frac{1}{2}(v - v')$ and consider orbits of the 1-parameter subgroup λ_t generated by u. Then $\sigma_\alpha, \sigma_\beta$ can be connected by a "chain" of orbits of this fixed subgroup, in the sense that there are $\alpha_0, \alpha_1, \dots, \alpha_p$ with $\alpha_0 = \alpha, \alpha_p = \beta$ and points $s_1, \dots, s_{p-1}$ such that

$$\lim_{t \to -\infty} \lambda_t(s_i) = \alpha_{i-1} \quad \lim_{t \to \infty} \lambda_t s_i) = \alpha_i.$$

As before, we can arrange that the 1-parameter subgroup is algebraic.

With these examples in mind we seek a general result. Given an algebraic 1-parameter subgroup λ in G we write $G_\lambda \subset G$ for its centraliser and $S_\lambda \subset S$ for the set of points fixed by λ. Thus G_λ acts on S_λ. If $\mathcal{O} \subset S$ is a non-trivial λ-orbit then the closure $\overline{\mathcal{O}}$ is a rational curve containing a point $\mathcal{O}^-$ in one component of S_λ and a point $\mathcal{O}^+$ in another component. We say that the orbit goes from $\mathcal{O}^-$ to $\mathcal{O}^+$. More generally we want to consider "positive semi-orbits" by which we mean a subset of S of the form

$$\{\lambda_t z : t \geq 0\}.$$

The closure contains an extra point $\mathcal{O}^+$ in S_λ (the limit as $t \to \infty$) and we say that the semi-orbit goes from z to $\mathcal{O}^+$. Likewise for "negative semi-orbits" which go from a point of S_λ to a point not in S_λ.

Now suppose we have two points w, w' in the same connected component of S_λ, an orbit or positive semi-orbit going from some point z to w and an orbit or negative semi-orbit going from w' to some other point z'. Then we say that the orbits are matching. Suppose we have $\mathcal{O}_1, \dots \mathcal{O}_p$ where $\mathcal{O}_1$ is either an orbit or a positive semi-orbit, $\mathcal{O}_p$ is either an orbit of negative semi-orbit and $\mathcal{O}_j$ are orbits for $1 < j < p$. If consecutive pairs $\mathcal{O}_j, \mathcal{O}_{j+1}$ are matching we say that we have a chain of orbits. This chain has an initial point $z_1 \in \overline{\mathcal{O}_1}$ and a terminal point z_p in $\overline{\mathcal{O}_p}$. Finally we say that the chain goes from $x \in S$ to $y \in S$ if x is either equal to z_1 or they are both in the same component of S_λ and likewise for y and z_p.

With all this terminology in place we can state

Proposition 6 *Suppose, as above, that G acts on S and $x_i \to x, y_i \to y, x_i = g_i(y_i)$. Then we can find an algebraic 1-parameter subgroup λ in G, elements $k, k' \in K$ and a chain of orbits going from kx to $k'y$. Moreover we can choose λ to be the complexification of a 1-parameter subgroup of K.*

One general setting for this is the theory of gradient flows. We fix a metric on $\mathbf{CP}^M$ preserved by a maximal compact $K \subset G$. There is a family of functions F_ξ for $\xi \in \mathbf{k}$ so that the action of 1-parameter subgroup generated by $i\xi$ is the gradient flow of F_ξ. We can do this either on $\mathbf{CP}^M$ or on S, and the possible singularities of S are not really relevant. Saying that two points a, b are in the same G-orbit is the same as saying that for some $k \in K$ the points a, bk are joined by a gradient flow line of some F_ξ, where we may suppose that ξ has unit length. Now the standard theory (as in finite dimensional Floer theory) for families of gradient flow lines of a single function is that any sequence of such contains a subsequence converging to a "broken trajectory". In the case of a Morse function, with isolated critical points, the different links of the limit have the same endpoints but in general they have endpoints in the same connected component of the critical set. The same ideas applies to a compact family of functions and prove the proposition.

For another point of view on this, consider the closure of a generic G-orbit in S. This is a subvariety of S and defines a point in the appropriate Chow variety. Varying the generic orbit we get a subset $\mathcal{M}$ of the Chow variety whose closure is a subvariety $\overline{\mathcal{M}}$. Points in $\overline{\mathcal{M}} \setminus \mathcal{M}$ correspond to degenerations of the generic orbit. Such a degeneration will be reducible, given by a union of orbit closures. This is what is happening in the case above, when the limits x, y are joined by a chain, moving from one component to another.

For example, consider the theory of $SL(2, \mathbf{C})$ bundles over a curve, represented in terms of a G action on some S. A generic bundle will have line sub-bundles of degree at most a certain number $d+1$, but we may have a special bundle E with a line sub-bundle L of degree d. Thus E is an extension

$$0 \to L \to E \to L^{-1} \to 0.$$

There may well be also a non-trivial extension

$$0 \to L^{-1} \to F \to L \to 0.$$

The closures of the orbits corresponding to E, F meet along points corresponding to $L \oplus L^{-1}$. In the simplest case the generic orbit splits into the union of these two orbits. But it might happen that E has another non-generic sub-bundle leading to another component. Or it might happen that F has a non-generic sub-bundle, and so on.

2.3.2 Partial proof in the reduced irreducible case

We apply the ideas of the previous subsection to the points X^*_∞, W in the Chow variety. We get a 1-parameter subgroup with Hermitian generator A and a chain of orbits "going from" X^*_∞ to W. The Chow weight is constant on each connected component of the fixed point set. If $\mathrm{Ch}_1, \dots \mathrm{Ch}_p$ are these Chow weights we get, by the monotonicity property

$$\mathrm{Ch}_1 \leq \ldots \leq \mathrm{Ch}_p \leq Ch(W, A) \leq Ck^{-2} \log k|A|.$$

Since (for large k) X is Chow stable, we know that $\mathrm{Ch}_1 > 0$. By itself this does not give any contradiction: what we need to do is to increase k. So we consider $k' = mk$ as above, for large m.

Suppose we know that W is reduced and irreducible, so $W' = W$. We now encounter a major difficulty, which we are not able to deal with fully. When $p = 1$ things work well. There are two sub-cases. In sub-case (1), W has an $\mathbf{C}^*$ action and is the central fibre of a test configuration for X. We can use this *same* test configuration with the higher power k' and the generator A' has $|A'| = m|A|$. We deduce that the Futaki invariant of W is 0, contradicting the K-stability of X. In subcase (2), W need not have a $\mathbf{C}^*$ action but there are schemes Y, Y' with $\mathbf{C}^*$-actions, in the same component of the fixed point set, such that Y is the central fibre of a test configuration for X and Y' is the central fibre of a test configuration for W. Again we can use these same test configurations for the higher power mk. The inequality above tells us that the Futaki invariant of Y is less than or equal to zero and we get our contradiction.

The difficulty is that when $p > 1$ we cannot read off, or at least not so easily, a chain for mk from that for k. Consider a $\mathbf{C}^*$-orbit going from some scheme Z_0 to another Z_∞. We can regard this as $\mathbf{C}^*$-equivariant family $\pi : \mathcal{Z} \to \mathbf{CP}^1$. Conversely suppose we have such a family. Given a power k we can form the vector bundle $E_k \to \mathbf{CP}^1$ by the direct image of sections of L^k. The condition that we can represent this family by a $\mathbf{C}^*$ orbit in the Hilbert scheme is that the bundle E_k is projectively trivial. But this does not imply that $E_{k'}$ is projectively trivial for all $k' = mk$. This gives an obstruction to passing a chain from the power k to mk. If we do not encounter this obstruction then the same argument as above, for $p = 1$, works.

We can think about this difficulty in terms of the "splitting of orbits" discussion of the previous subsection. In our original family T of Fano manifolds there is a Zariski open set T_0 where the orbit in the Hilbert scheme, for the given value k_0, is generic (for example, has maximal degree) among all those in the family. If the limit X lies in T_0 we are in good shape, for in that case W is the closure of the orbit corresponding to X and we are in subcase (1) above. More subtly, there is a Zariski-open T_1 containing T_0 consisting of orbits characterised in the following way. A point $Y \in T$ gives an orbit O in the Hilbert scheme. There may be various ways to augment this to a union of orbit closures $\overline{O} \cup \overline{0}_i \cup \ldots \overline{O}_q$ which is a degeneration of the generic orbit. We say Y is in T_1 if there is one such way in which all the $\overline{O}_i$ meet $\overline{0}$. Then we can argue that if X is in T_1 and one takes a suitable sequence X_i then only the good case occurs, either with sub-case (1) or (2).

This line of argument can be pushed a bit further and may, in fact, cover many cases where the pattern of orbit degeneration is suitably generic. But it seems impossible to verify this in a practical case. However if we restrict attention to our "Goal 1" then we can avoid this difficulty of chains.

To sum up, our approach to Goal 2 meets two difficulties

- **Difficulty 1** The limiting scheme W may not be reduced and irreducible in which case taking a larger value $k' = mk$ we may a get a different limit W'.
- **Difficulty 2** If X is joined to W by a long chain then we have a problem in passing this to to the higher power k'.

In Section 4 we attack the first difficulty, understanding how W' and W are related.

3 Proof of Theorem 2

We write $X = X_i$ and ω for the *cscK* metric in the fixed Kahler class. We work with the line bundles $L^k \to X$ and write

$$\sum_\alpha |s_\alpha|^2 = \rho$$

where s_α is any orthornomal basis of holomorphic sections of L^k. Let the scalar curvature of X be $2c$ and write

$$\rho = 1 + ck^{-1} + \eta.$$

Let V_0 be the volume of X with respect to the metric ω so

$$\dim H^0(X, L^k) = V_0 k^n P(k),$$

say, where $P(k) = 1 + ck^{-1} + O(k^{-2})$. The metric ω_{FS} induced from the Fubini-Study metric is

$$\omega_{FS} = k\omega + \sqrt{-1}\partial\overline{\partial}\log\rho$$

Then one finds from the definition that

$$V_0 \mathrm{Ch}(X, A) = \int_X H \frac{\rho}{P}\omega^n - \int_X H(\omega + i\sqrt{-1}\partial\overline{\partial}\log\rho)^n.$$

Given b write $r_0 = bk^{-1/2}$ and let $\Omega = \Omega(r_0), Z = Z(r_0)$. Writing I for the first integral and J for the second, we have

$$V_0|\mathrm{Ch}| = |I_\Omega + I_Z - J_\Omega - J_Z|,$$

in an obvious notation. Thus

$$V_0|\mathrm{Ch}| \leq |I_\Omega - J_\Omega| + |I_Z| + |J_Z|.$$

To estimate these integrals we use

Lemma 1 *Suppose F is a function on Ω such that $|F| \leq k^{-2}r^{-4}$ in $\Omega(r) \subset \Omega$ for all $r \geq r_0$. Then*

$$\int_\Omega |F| d\mu \leq Ck^{-2}\log k,$$

where C depends on b and the constant appearing in Hypothesis V.

To see this, let ν be the distribution function of $|F|$, i.e.

$$\nu(t) = \mathrm{Vol}\{x \in \Omega : |F(x)| \geq t\}$$

We have $|F| \leq b^{-4}$ in Ω so

$$I = \int_0^{b^{-4}} \nu(t) dt.$$

If $|F(x) \geq t$ then x is not in $\Omega(k^{-1/2}t^{-1/4})$. Thus $\nu(t)$ is no more than the volume of $Z(k^{-1/2}t^{-1/4})$ which is at most $Ck^{-2}t^{-1}$ by Hypothesis V. Thus

$$\nu(t) \leq Ck^{-2}t^{-1}.$$

On the other hand, certainly $\nu(t) \leq \mathrm{Vol}(X)$ so

$$\nu(t) \leq C \min(k^{-2} t^{-1}, 1),$$

which gives

$$\int_0^{b^{-2}} \nu(t) \leq C k^{-2} \log k.$$

Now take $r > r_0 = b k^{-1/2}$ and consider a point in Ω_r. Rescaling Theorem 1 we see that $|\eta| \leq C k^{-2} r^{-4}$ for a universal constant C. Likewise the second derivative of η is bounded by $C k^{-2} r^{-6} \leq C k^{-2} r^{-4}$. Write

$$(\omega + k^{-1} \sqrt{-1} \partial \overline{\partial} \log \rho^n = (1 + F_1) \omega^n,$$

and $\frac{\rho}{P} = 1 + F_2$. So $|F_1|$ and $|F_2|$ are both bounded by $C k^{-2} r^{-4}$. We apply the Lemma three times. First consider $|I_\Omega - J_\Omega|$. This is bounded by

$$\int_\Omega |H| \left((1 + F_1) - (1 + F_2) \right) d\mu \leq |A| \int_\Omega |F_1| + |F_2| d\mu.$$

So the Lemma immediately gives $|I - J| \leq C k^{-2} \log k |A|$. The other two terms are a little less obvious. We have

$$|I_Z| \leq |A| k^{-n} \mathrm{Vol}(Z, d\mu_{FS}),$$

and hence

$$|I_Z| \leq |A| (V_0 - k^{-n} \mathrm{Vol}(\Omega, d\mu_{FS}).$$

Now

$$k^{-n} \mathrm{Vol}(\Omega, d\mu_{FS}) = \int_\Omega (1 + F_1) d\mu = \mathrm{Vol}(\Omega, d\mu) + O(k^{-2} \log k),$$

by the Lemma, applied to F_1. Now the Hypothesis V tells us that the volume of the complement of Ω is $O(k^{-2})$ so

$$k^{-n} \mathrm{Vol}(\Omega, d\mu_{FS} - V_0 = O(k^{-2}) \log k,$$

and we deduce that $|I_Z| \leq C |A| k^{-2} \log k$. Similarly we write

$$|J_Z| \leq |A| P(k)^{-1} \int_Z \rho d\mu.$$

Now

$$\int_Z \frac{\rho}{P} d\mu = \int_X \frac{\rho}{P} d\mu - \int_\Omega \frac{\rho}{P} d\mu = V_0 - \int_\Omega \frac{\rho}{P} d\mu = V_0 - \mathrm{Vol}(\Omega, d\mu) - \int_\Omega F_2 d\mu,$$

and we argue as before.

4 Algebro-geometric discussion

4.1 The basic construction

Suppose we have the following set-up

- a flat family of schemes $\pi : \mathcal{W} \to \Delta$ over the disc Δ with smooth fibres $W_t = \pi^{-1}(t)$ for $t \neq 0$ and with central fibre W_0 containing a component B which is reduced at generic points.
- an embedding of the family in a projective space $\mathbf{P}^N$ such that B is not contained (as a set) in any proper linear subspace of $\mathbf{P}^N$.

Now fix $m > 0$ so large that the restriction map $H^0(\mathbf{P}^N, \mathcal{O}(m)) \to H^0(W_0, \mathcal{O}(m))$ is surjective and all the higher cohomology groups $H^i(W_0, \mathcal{O}(m))$ vanish. Suppose that s is a non-trivial section of $\mathcal{O}(m)$ over W_0 and let $\overline{s}$ be an extension to a section over a neighbourhood of W_0 in $\mathcal{W}$. Then $\overline{s}$ has a certain order of vanishing $m_{\overline{s}}$ on B (which will be 0 if s does not vanish identically on B).

Lemma 2 *For a given s there is an $m(s)$ such that $m_{\overline{s}} \leq m(s)$ for all extensions $\overline{s}$.*

This should follows from the fact that no extension can vanish identically on the fibres W_t.

Now of course we can define $m(s)$ to be the least such bound. This order-of vanishing function defines a flag in $H^0(W_0, \mathcal{O}(m))$ and we choose a basis adapted to the flag. For each of these basis elements we choose an extension over a neighbourhood of W_0 in $\mathcal{W}$. This defines another projective embedding of the family in $\mathbf{P}(U^*)$ for a fixed subspace U of $s^m H^0(W, \mathcal{O}(1))$. The function $m(s)$ defines a 1-parameter subgroup λ in $GL(U^*)$, diagonal in our basis. So we have projective maps $\lambda_t : \mathbf{P}(U^*) \to \mathbf{P}(U^*)$ for $t \neq 0$. Now we consider the family of varieties $\lambda_t(W_t) \subset \mathbf{P}(U^*)$ for $t \neq 0$. By general principles we can fill in the "missing fibre" over 0 to get a new flat family of schemes $\pi' : \mathcal{W}' \to \Delta$. Up to isomorphism this is independent of the choices made. There is a birational map from B to the central fibre W'_0 whose image is contained in a component B' of W'_0. The crucial feature of the construction is that B' does not lie in any proper linear subspace in $\mathbf{P}(U^*)$.

We have been vague above about the correspondence between the weighted flag and the 1-parameter subgroup. This is best defined by an example. Suppose that there is just a 1-dimensional subspace $\mathbf{C}p$ in U of polynomials which vanish on B and that p vanishes to order 1. Choose a basis $p = p_0, p_1, \ldots p_{N'}$ of U and take the dual basis to identify $\mathbf{P}(U^*)$ with $\mathbf{P}^{N'}$. Then in this case λ_t acts as multiplication by $\mathrm{diag}(t^{-1}, 1, \ldots 1)$. That is to say, when t is small λ_t moves points *away from* the hyperplane $p = 0$ containing B.

For a simple explicit example consider a family of cubics W_t in $\mathbf{P}^2$ which degenerate when $t = 0$ to the union of a line and conic, in general position. Take B to be the conic and $m = 2$. Write A for the line. Then in this case U is the whole space of quadratic polynomials and we have $W_t \subset \mathbf{P}^5$. As above, there is just a one dimensional subspace $\mathbf{C}p$ vanishing on B and our 1-parameter subgroup has fixed point set a $\mathbf{P}^4 \subset \mathbf{P}^5$, containing B, and an isolated point P. The central fibre W_0 of the original family, viewed as a subvariety of $\mathbf{P}^5$, is the union $B \subset \mathbf{P}^4$ and a conic, A, in a $\mathbf{P}^2 \subset \mathbf{P}^5$. The $\mathbf{P}^2$ and $\mathbf{P}^4$ meet in a line, spanning the two intersection points $B \cap A$. Near to one of these intersection points the family of curves W_t is modelled on the family of plane curves $\{xy = t\}$ where $x = 0$ corresponds to points in $\mathbf{P}^4$. The action of the 1-parameter subgroup fixes x and maps y to $t^{-1}y$ so in the model we get the t-independent family $\{xy = 1\}$, after applying the one-parameter subgroup. What happens is that the in the new central fibre W_0' the component A disappears and B' is a rational curve with an ordinary double point at P.

We want to relate this algebro-geometric construction to the discussion of Chapter 2. Recall that we had there, for a given k, a sequence X_i of projective varieties converging to a scheme W. Then the issue is to understand what happens if we do the same with a larger value $k' = mk$ and get another scheme W'. Suppose it happens that our sequence X_i can be taken to be fibres $W_{t(i)}$ in family $\mathcal{W}$ as above, for some sequence $t(i) \to 0$. Then for large m, *the scheme W' obtained from the differential geometric discussion agrees with W_0', obtained from the algebro-geometric construction above.* (See below for the proof.)

If W_0 is reduced and irreducible then the algebro-construction above is vacuous–any section vanishing on B vanishes on the whole of W_0. This is essentially a restatement of the argument at the end of Chapter 1. Suppose that W_0 is irreducible but not reduced. Thus, when m is large, there are polynomials p such that p^α vanishes on W_0 but p does not. When we run the construction we get a new situation in which p does not vanish on W_0': i.e. there is a smaller set of nilpotents in the structure sheaf of W_0'. It seems likely that one can show that for large enough m we get a W_0' which is reduced and irreducible. Trusting in this idea we will largely ignore nilpotents in what follows.

The construction above is related to the following. Suppose for simplicity that the total space $\mathcal{W}$ is smooth and W_0 has two components A, B meeting in a normal crossing along $D = A \cap B$. We regard these as divisors in $\mathcal{W}$ so defining line bundles $L_A, L_B \to \mathcal{W}$. Then $L_A \otimes L_B$ is trivial since it corresponds to the divisor of the function t on $\mathcal{W}$. Now consider the line bundle $\mathcal{L}' = \mathcal{O}(m) \otimes L_A^w \to \mathcal{W}$ for $w > 0$. Restricted to B this is $\mathcal{O}(m)[wD]$ i.e. the sections are sections of $\mathcal{O}(m)$ with poles of order m along D. Restricted to A it is $\mathcal{O}(m) \otimes [-wD]$, i.e. sections are forced to have zeros along D. Thus as we increase w we "move" sections from A to B. For fixed m, w we can use the line bundle $\mathcal{L}'$ to define a family of *rational maps* $W_t \to \mathbf{P}(U^*)$ and hence a new family $\mathcal{W}'' \to \Delta$. This agrees with $\mathcal{W}'$ in some cases, but the "weighted flag' construction seems to be more general.

Write $D \subset B$ for the set where B meets the other components of W_0, $N \subset B$ for the support of the nilpotent elements and $\Omega = B \setminus (N \cup D)$. Then we have *the rational map from B to B' is an isomorphism on from Ω to its image.*

4.2 The case of an action

Now we restrict attention to the case of a family $\mathcal{W}$ which is compatible with a $\mathbf{C}^*$-action. Thus we can take the base to be $\mathbf{C}$ and we suppose that π is a $\mathbf{C}^*$ equivariant map, where the action on W is induced by a linear action on $\mathbf{P}^N$ with generator $A \in \mathrm{End}(\mathbf{C}^{N+1})$. We let $\overline{\lambda}, \underline{\lambda}$ be the maximal and minimal eigenvalues of A and $Z_{\max}, Z_{\min} \subset \mathbf{P}^{N+1}$ be the spans of the corresponding eigenvectors. Our signs are chosen in the following way: for the orbit of a *generic* point in $\mathbf{C} \times \mathbf{P}^N$, as the $\mathbf{C}$-component moves towards 0 the $\mathbf{P}^N$ component moves towards $Z_{\min}$. The only points which flow towards $Z_{\max}$ are those which lie in $Z_{\max}$, which are of course fixed by the action.

In this situation we have an action on $H^0(\mathbf{P}^N, \mathcal{O}(m)$ with generator $A_{(m)}$ say. We can choose the subspace U to be invariant. Then it follows from the definition that the weighted flag in U is $\mathbf{C}^*$-invariant which means that we can choose the 1-parameter subgroup λ to commute with action $\exp(A_{(m)}t)$.

Proposition 7 *The family $\mathcal{W}'$ produced by this construction is compatible with the $\mathbf{C}^*$ action induced by the product subgroup $\lambda(t)\exp(A_{(m)}t)$.*

We want to pay special attention to the points in $Z_{\max}$.

Lemma 3 *$Z_{\max}$ and $Z_{\min}$ both intersect B.*

This follows immediately from the fact that B is invariant under the action and does not lie in any proper linear subspace.

Now say that a point in $B \cap Z_{\max}$ is an *interior maximum* if does not lie in A. Such points may not exist but after performing our construction they do.

Proposition 8 *For sufficiently large m there is an interior maximum in B'.*

Write $U^* = U_1^* \oplus U_2^*$ where λ acts trivially on U_1^* and with strictly negative weight on U_2. Changing the action by λ only decreases the weights. Let p be a point in $Z_{\max} \cap B$. Since $p \in B$ we have $p \in \mathbf{P}(U_1)^*$. Thus p remains a maximum when considered as a point of W'. On the other hand if $W' = B' \cup A'$ then it is clear from the construction that any point in the interior of A' (i.e. not in the closure of B') lies in $\mathbf{P}(U_2^*)$. Thus p is in the interior of B'.

Given this Proposition we can, and will below, suppose without loss of generality that there is an interior maximum in B.

4.3 More formal treatment

4.3.1 Sequences in a fixed orbit

We will now switch attention to a slightly different problem.

Goal 4

Suppose X_∞ is a Fano manifold which is $\overline{K}$-stable. If there is a sequence of Kahler metrics g_i on X_∞ with $\|\mathrm{Ric}(g_i) - g_i\|_{C^2} \to 0$ then X_∞ has a Kahler-Einstein metric.

Now we have a sequence of metrics on a *fixed* manifold X_∞. It is not hard to see that all the same estimates hold for this sequence as for the actual Kahler-Einstein metrics considered before. In fact we do not really care about the detailed hypotheses on the metrics, because we mean the discussion for illustration rather than actual applications. In Section 5 we will consider a variant of this set-up which could have such applications.

Now we get a sequence X_i of projective varieties all isomorphic to a fixed X_∞^* so $X_i = g_i(X_\infty^*)$ and we are supposing that the g_i diverge. The limit W is the central fibre of a test-configuration for X_∞^* with a $\mathbf{C}^*$-action generated by A. Then we now show more carefully that our algebro-geometric construction to produce a new W' agrees with the differential geometry.

We know that W has a $\mathbf{C}^*$-action. Let us assume that this is the whole of $\mathrm{Aut}(W)$, i.e. that this is the stabiliser of the point $[W]$ in the Hilbert scheme. By Luna's Theorem, there is a $\mathbf{C}^*$-invariant slice T for the action of the projective general linear group at W. This means that given our sequence $[X_i] \to [W]$ there are $h_i \to 1$ such that $[h_i(X_i)] \in T$ and the $[h_i(X_i)]$ are in a fixed $\mathbf{C}^*$-orbit in T.

Recall that what we have to do is to compare the standard norm on the sections of $\Lambda^m \to W$ with that defined by the L^2 over X_i, using the cScK metric. Fix r such that $\Omega_r \subset X_i$ contains at least half the volume. By Additional Hypothesis 3 and Proposition 1 , the L^2 norm of sections over all of X_i is equivalent, with fixed constants, to that over Ω_r. In turn this latter is equivalent to that defined using the Fubini-Study metric. This is distorted a bounded amount by the maps h_i so we get equivalent norms if we replace our sequence X_i by $h_i(X_i)$. Thus we may as well suppose that X_i lies in the slice T. Then we are in the situation imagined above, with a family deformation of W over a disc and the X_i are the fibres corresponding to a sequence of parameters $t_i \to 0$. We follow through the procedure of Section 4.1, so we have a basis s_a of the sections adapted to the flag defined by the order of vanishing on $B \subset W$. The $\Omega_r \subset X_i$ converge to a domain in the smooth part of B. If s_a vanishes with multiplicity m_a on B then in local co-ordinates around a typical point of $\Omega_r \subset X_i$

$$s_a = t_i^{m_a}(f_{a,0} + t_i f_{a,1} + \dots)$$

with $f_{a,0}$ not identically zero and $m_1 \geq m_2 \geq \dots$. Now we claim that the $f_{a,0}$ are linearly independent. For suppose we have a relation

$$\sum \lambda_a f_{a,0} = 0.$$

Let a_0 be the smallest index with $\lambda_0 \neq 0$, so we can suppose the relation is

$$-f_{a_0,0} = \lambda_{a_0+1} f_{a_0+1,0} + \dots.$$

Then

$$s_{a_0} + \lambda_{a_0+1} t^{m_{a_0} - m_{a_0+1}} - \dots$$

gives an extension of s_{a_0} over W_0 which vanishes to strictly higher order along B contradicting our choices.

Now if consider the sections

$$\tilde{s}_a = t^{-m_a} s_a.$$

The L^2 inner product of $\tilde{s}_a, \tilde{s}_b$ over Ω_r is approximately

$$M_{a,b} = \int_{\Omega_r} f_{a,0}\overline{f_{b,0}}$$

The linear independence of the $f_{a,0}$ implies that the matrix M is bounded above and below $c^{-1} \leq M \leq c$. But this precisely says that the embedding of X_{t_i} defined by the L^2 norm differs from $\lambda_{t_i} X_{t_i}$ by a bounded sequence. So we see that W', defined algebro-geometrically is equivalent to the limit of the embeddings defined by the L^2 norm.

4.3.2 Extension of Stoppa's argument

This subsection is based on the work of J. Stoppa [7]. Suppose as above that we have a test configuration $\mathcal{W}$ with central fibre W which contains a large component B and there is an interior maximum p in B. If we embed the test configuration in projective space we can regard p as a fixed point of the action and hence it defines a section of $\mathcal{W} \to \mathbf{C}$. Blowing up $\mathcal{W}$ along this section we get a new scheme $\hat{\mathcal{W}} \to \mathbf{C}$. Write $\widehat{W}$ for the central fibre of $\hat{\mathcal{W}}$. Then $\hat{W}$ contains the blow-up $\tilde{W}$ of W at p but in general the two are not equal. Instead $\widehat{W} = \tilde{W} \cup P$ where the component P is glued to $\tilde{W}$ along the exceptional divisor $D \subset \tilde{W}$.

Example Consider the hypersurface M in $\mathbf{C}^4$ defined by the equation $x_4 = x_1^2 + x_2^2 + x_3^2$. Let $\mathbf{C}^*$ act on $\mathbf{C}^4$ by

$$(x_1, x_2, x_3, x_4) \mapsto (t^{-a}x_1, t^{-a}x_2, t^{-a}x_3, t^{-b}x_4).$$

If $a, b > 0$ then the origin is a repulsive fixed point of the kind we are considering. We get a family of hypersurfaces M_t with equations

$$x_4 = t^{2a-b}(x_1^2 + x_2^2 + x_3^2).$$

If $2a - b > 0$ the limiting equation is $x_4 = 0$ which corresponds to a smooth central fibre. If $2a - b < 0$ the limiting equation is $x_1^2 + x_2^2 + x_3^2 = 0$ so the central fibre is singular; the product of $\mathbf{C}$ with a quadric cone. Now for $t \neq 0$ the blow-up of M_t at the origin can be regarded as a subset of $\mathbf{C}^4 \times \mathbf{P}^3$. So we get a subset V of $\mathbf{C}^4 \times \mathbf{P}^3 \times (\mathbf{C} \setminus \{0\})$ and we need to take the closure of this in $\mathbf{C}^4 \times \mathbf{P}^3 \times \mathbf{C}$. Consider a vector (r_1, r_2, r_3, r_4) with $r_1^2 + r_2^2 + r_3^2 \neq 0$. The line in $\mathbf{C}^4$ generated by this vector meets M_t at

$$t^{b-2a}\frac{r_4}{r_1^2 + r_2^2 + r_3^2} t^{b-2a}(r_1, r_2, r_3, r_4).$$

We can make this point as close to the origin as we like by taking t small. It follows that the closure of V contains the whole $\mathbf{P}^3$ factor at $(0,0) \in \mathbf{C}^4 \times \mathbf{C}$. Thus in this case $P = \mathbf{P}^3$.

However *let us suppose for the moment that we have a case where* $\widehat{W} = \tilde{W}$, for example when B is smooth at p.

We have a choice of polarisation of our test configuration $\hat{\mathcal{W}}$. We write this as $L - \epsilon E$. That is, L is the original line bundle on W lifted to $\hat{W}$ and $\epsilon = \gamma^{-1}$ for an integer γ. What this really means is that we use sections of $L^{r\gamma} \otimes [-rE]$. If $\gamma \geq \gamma_0$ and r is sufficiently large then these sections give a projective embedding of the test configuration. Eventually we will have to be very careful about the ranges of the two variables r, γ but we leave that for the moment.

We will always choose γ so large that the blow-up of the original manifold X_∞ with exceptional divisor of size γ^{-1} is K-stable. Our basic strategy is this. Let W' be obtained from W by the procedure discussed above. Here we take $m = r\gamma$ and without loss of generality are supposing that $k_0 = 1$. Thus W' is embedded $\mathbf{P}(V^*)$ where V is a subspace of $H^0(W', \Lambda')$ say (which could well be the whole space but we do not know this). The point p lies in the interior of $B \subset W$ so gives a corresponding point p' in W' and a neighbourhood of p' in W' is isomorphic to a neighbourhood of p in B. Over this neighbourhood the hyperplane bundle Λ' is identified with $L^{r\gamma}$. Write $\mathcal{S}_r$ for the skyscraper sheaf at p' consisting of r-jets of functions. Let us assume for the moment that

1. The evaluation map from V to $\mathcal{S}_r \otimes L^{r\gamma}$ is surjective.
2. The kernel of this evaluation map yields a projective embedding of the blow-up $\tilde{W}'$ of W' at p'.

Then it follows that we can construct a test-configuration, embedded in projective space, for the blow up of X_∞ and with central fibre $\tilde{W}'$.

What we want to do now is to compare the Chow weights of $\tilde{W}'$ and W', with a formula involving the parameters r, γ. This is the same general idea as in Stoppa's work but the difference is that the scheme W' *varies.* It is given by performing our basic construction with polynomials of degree $m = r\gamma$. It comes with an embedding in a projective space of "known" dimension $\dim = \dim(r, \gamma)$ say (given by the Hilbert series of the original manifold X). So dim is a polynomial of degree n in $r\gamma$ with positive leading term C, say, and all other co-efficients "known". We have a circle action on this space and we write $\mathrm{Tr} = \mathrm{Tr}(r, \gamma)$ for the trace of the generator. This function Tr is "uknown" in that it depends on the details of way we modified the original circle action on W. Let p be our interior maximum point in W which we can also regard as a point in W' and let w be the weight of the action on the fibre of $L \to W$ at p. As noted above, when we modify the circle action we only decrease the eigenvalues so we have an upper bound

$$\mathrm{Tr} \leq w r\gamma \dim .$$

Write V for the integral of the volume of W' in its given projective embedding. So V is known, in fact $V = Cr^n\gamma^n$. To simplify notation let us suppose that $V = r^n\gamma^n$. Write I for the integral of the Hamiltonian over W'. In the language above, I is "unknown". By definition the Chow weight is

$$\mathrm{Ch}(W') = \frac{\mathrm{Tr}}{\dim} - \frac{I}{V}.$$

Now we compare W' with the blow-up $\tilde{W}'$. By the assumptions above the difference in the trace of the actions is given by the action on $\mathcal{S}_r$, tensored with the fibre of $L^{r\gamma}$. This depends only on a neighbourhood of $p \in W'$, which is the same as a neighbourhood of p in W. Flatness implies the dimension is the same as that for a smooth point so we have fixed universal constants (which we can write down) such that

$$d(r) = \dim \mathcal{S}_r = d_0 r^n + d_1 r^{n-1} + \dots$$

The crucial fact is that $d_1 > 0$. Write

$$d(r) = d_0 r^n + d_1 r^{n-1} + \eta(r),$$

with $\eta(r) = O(r^{n-2})$, We have a similar polynomial $f(r)$ for the trace of the induced action on $\mathcal{S}_r$, determined once and for all by the action on W at p. Write

$$f(r) = f_0 r^{n+1} + \epsilon(r),$$

where $\epsilon(r) = O(r^n)$.

The blow-up $\tilde{W}'$ is embedded in a space of dimension $\dim - d(r)$ and the trace of the action on this space is

$$\mathrm{Tr} - f(r) - wr\gamma d(r).$$

The volume of $\tilde{W}'$ is $V - d_0 r^n$ and the integral of the Hamiltonian is

$$I - (f_0 + w\gamma d_0) r^{n+1}.$$

So the Chow weight $\mathrm{Ch}(\tilde{W}')$ is

$$\frac{\mathrm{Tr} - (f_0 r^{n+1} + \epsilon(r) + wr\gamma(d_0 r^n + d_1 r^{n-1} + \eta(r)))}{\dim - (d_0 r^n + d_1 r^{n-1} + \eta(r))} - \frac{I - r^{n+1}(f_0 + w\gamma d_0)}{V - d_0 r^n}. \quad (**)$$

Lemma 4 *We have*

$$\mathrm{Ch}(\tilde{W}') = \frac{1}{1 - d_0\gamma^{-n}} \mathrm{Ch}(W') - wd_1\gamma^{1-n} - \frac{\mathrm{Tr}}{\dim}\left(d_1 r^{-1}\gamma^{-n} + \alpha(r,\gamma)\right) + \beta(r,\gamma),$$

where

$$|\alpha(r,\gamma)| \le c(r^{-2}\gamma^{-n} + r^{-1}\gamma^{-n-1})$$

and $|\beta(r,\gamma)| \le c(r^{-1}\gamma^{1-n} + \gamma^{-n})$, *with a constant* c *independent of* r, γ.

This is a completely elementary, but slightly complicated, estimate. The constant c depends only on the d_i, f_i the weight w and the Hilbert polynomial of the original manifold X.

Now we give the proof of Lemma 4. Write the expression in (**) as (1)-(2).

The first step is to see that the terms $\epsilon(r), \eta(r)$ appearing in the numerator of (1) can be absorbed in to the term β in the statement of the Lemma.

The second step is to see that the term $\eta(r)$ in the denominator of (1) can be absorbed into the term α.

After these two steps we can consider the simpler expression

$$\frac{\mathrm{Tr} - (f_0 r^{n+1} + wr\gamma(d_0 r^n + d_1 r^{n-1})}{\dim - (d_0 r^n + d_1 r^{n-1})} - \frac{I - r^{n+1}(f_0 + w\gamma d_0)}{V - d_0 r^n}.$$

The third step is to see that the terms involving f_0 in the numerators of (1) and (2) cancel, modulo terms which we are allowed to absorb in β. The fourth step is to see similarly that the terms involving d_0 in the numerators of (1),(2) cancel modulo terms in β.

At this point we can consider the simpler expression

$$\frac{\mathrm{Tr} - wr\gamma d_1 r^{n-1}}{\dim - (d_0 r^n + d_1 r^{n-1})} - \frac{I}{V - d_0 r^n}.$$

The fifth step is to see that the term involving d_1 in the numerator of (1) leads to the term $wd_1\gamma^{1-n}$ in the formula staed in the Lemma, up to terms which can be absorbed in β. (Recall here that we are assuming $V = r^n\gamma^n$ and we know that $\dim = r^n\gamma^n + O(r^{n-1}\gamma^{n-1})$.)

Thus, setting aside that term, we have to consider

$$Q = \frac{\mathrm{Tr}}{\dim - (d_0 r^n + d_1 r^{n-1})} - \frac{I}{V - d_0 r^n},$$

and we need to show that

$$Q = \frac{1}{1 - d_0\gamma^{-n}}\left(\frac{\mathrm{Tr}}{\dim} - \frac{I}{V}\right) - \frac{\mathrm{Tr}}{\dim}(d_1 r^{-1}\gamma^{-n} + \alpha) + \beta,$$

using notation in an obvious way. This is an exercise for the reader.

Now set $h = w - \frac{\mathrm{Tr}}{\dim} r^{-1}\gamma^{-1}$, so $h > 0$. We claim that

$$\mathrm{Ch}(\tilde{W}') \leq (1 - d_0\gamma^{-n})^{-1}\mathrm{Ch}(W') - \frac{d_1 h}{3}\gamma^{1-n} + O(\gamma^{-n}) + O(\gamma^{1-n} r^{-1}).$$

To see this consider two cases. In the first case suppose that $\frac{\mathrm{Tr}}{\dim} \geq -10|w|r\gamma$, say. Thus in view of the upper bound we have

$$\left|\frac{\mathrm{Tr}}{\dim}\right| \leq 10|w|r\gamma.$$

This means that

$$\frac{\mathrm{Tr}}{\dim}(\alpha) = O(\gamma^{-n}) + O(r^{-1}\gamma^{1-n}).$$

We can absorb the term α in β and we get the inequality with the better term $d_1 h\gamma^{1-n}$. In the case when $\frac{\mathrm{Tr}}{\dim} \leq -10|w|r\gamma$ we have $h \geq 9|w|$ and $h \leq -(11/10)r^{-1}\gamma^{-1}\mathrm{Tr}/\dim$. Now we absorb the term α in a different way

$$(d_1 r^{-1}\gamma^{-n} + \alpha) \geq \frac{d_1}{2} r^{-1}\gamma^{-n},$$

once r, γ are large enough. Then we get

$$\mathrm{Ch}(\tilde{W}') \leq (1 - d_0\gamma^{-n})^{-1}\mathrm{Ch}(W') - \Phi\gamma^{1-n} + O(\gamma^{-n}) + O(r^{-1}\gamma^{1-n}),$$

where

$$\Phi = -\frac{h}{9}d_1 + \frac{10}{22}hd_1 \geq \frac{34}{99}hd_1,$$

as required. (Of course, the constants chosen here are rather arbitrary.)

Let A' be the generator of the action for W' and let A'_0 be its trace-free part. The Chow weights of W' with respect to A' and A'_0 are the same. The following result is proved in the next subsection.

Proposition 9

$$w - \frac{1}{r\gamma}\frac{I}{\mathrm{Vol}} \geq c\frac{\|A'_0\|}{r\gamma}$$

for a fixed $c > 0$.

It follows that

$$h \geq c\frac{\|A'_0\|}{r\gamma} - \frac{1}{r\gamma}|\mathrm{Ch}(W')|.$$

Since by our basic estimate $\mathrm{Ch}(W') \leq C(r\gamma)^{-2}\log(r\gamma)\|A'_0\|$ we get

$$h \geq \frac{c}{2}\frac{\|A'_0\|}{r\gamma},$$

say, once $r\gamma$ is large.

On the other hand the eigenvalue-decreasing property of our algebro-geometric construction implies that

$$\|A'_0\| \geq cr\gamma.$$

So we deduce that h is bounded below. Since $\mathrm{Ch}(W') > 0$ we can find a γ_0 such that for all $\gamma \geq \gamma_0$ and $r \geq r(\gamma)$ we have

$$\mathrm{Ch}(\tilde{W}') \leq 2\mathrm{Ch}(W') - \kappa h,$$

say, for some $\kappa > 0$ depending only on γ. Now choose r so large that the blow-up of X is Chow stable, then $\mathrm{Ch}(\tilde{W}') > 0$. We get

$$2\mathrm{Ch}(W', A'_0) = 2\mathrm{Ch}(W', A) \geq \kappa c\frac{\|A'_0\|}{r\gamma}.$$

Taking r large, with fixed γ, this contradicts our basic estimate

$$\mathrm{Ch}(W', A_0') \leq C(r\gamma)^{-2} \log r\gamma \|A_0'\|.$$

What remains to be done is to establish the geometric hypotheses (1), (2) above, and to extend the argument to the case where $\hat{W} \neq \tilde{W}$. We begin with first of these, so we continue to suppose that $\hat{W} = \tilde{W}$.

First notice that the computations above have in fact nothing to do with the existence of projective embeddings so we have a fixed γ_0 such that if $\gamma > \gamma_0$ we get a good bound on the Chow weight. Now we also choose γ large enough that on the blow-up $\tilde{W}$ the line bundle $\lambda = L^\gamma \otimes \mathcal{L}_E^*$ is ample. This means that we get a projective embedding of $\tilde{W}$ using λ^r for large enough r. Further we can assume that $H^1(\lambda^r \otimes \mathcal{L}_E) = 0$. This means that evaluation of sections of λ^r on E is surjective which translates back on W to the fact that sections of $L^{r\gamma}$ generate $\mathcal{S}_r$. We can also suppose, taking r large, that all these sections of $L^{r\gamma}$ are polynomials in the co-ordinate functions of $W \subset \mathbf{P}^N$. Now γ is fixed and we are considering any $r > r(\gamma)$.

The point is that we now want to go from the fixed manifold W to W'. Since p lies in the interior of B we can suppose that near p the two varieties W, W' are arbitrarily close. The condition that the polynomials generate the skyscraper sheaf is open so since it holds for W it also holds for W'.

It remains to show that we get a projective embedding of the blow-up of W'. We just give the argument for the case when $r = 1$, so that we are trying to embed the blow-up by sections vanishing at a point. The simplification is that since the relevant point p lies in both W and W' this is the same in either case.

We consider the projective embedding of W gives by the Veronese map. We write this as

$$W \subset \mathbf{P}(E \oplus F \oplus G),$$

where E^* is the set of polynomials which vanish on B (but not on W) and $E^* \oplus F^*$ is the set which vanish at p. So we denote a point in the projective space by $[e, f, g]$. The action of the 1-parameter subgroup defined by the flag in E^* is of the form $(e, f, g) \mapsto \phi_t(e, f, g) = (\epsilon_t^{-1}(e), f, g)$ where ϵ_t is a contraction for $t < 1$. Recall the set-up is that if we have varieties $X_t \to W$ as $t \to 0$ and $\phi_t X_t \to W'$. We want to see first that we get a well-defined map from $\tilde{W}'$ to projective space. This is the statement that there is no point $(0, 0, g)$ in W' apart from p. If there were then we have a sequence $(e_i, f_i, g_i) \in \phi_{t_i} X_{t_i}$ with $t_i, e_i, f_i \to 0$ as $i \to \infty$, and $g_i \to g$. Thus $\epsilon_{t_i} e_i, f_i, g_i \in X_{t_i}$. This latter sequence has a limit in W but the limit is $(0, 0, g)$, since ϵ_{t_i} is contractive. Thus the limit is p. We know that the birational map from B to W' is regular near p. Denote this regular map by f. It follows from the definitions that $[0, 0, g] = f(p)$. But we know that $f(p) = p$ since all the X_i contain p. This completes the argument.

Now we want to see that this projection map is an embedding of W'. We will just show that this is true as a map of sets. So we want to show that we cannot have distinct points $[z] = [e, f, g], [z'] = [e', f', g']$ in W' with $[e', f'] = [e, f]$. As before we have sequences $z_i = (e_i, f_i, g_i) \to z, z_i' = (e_i', f_i', g_i') \to z'$ representing points in $\phi_{t_i}(X_i)$. Thus $[\epsilon_{t_i} e_i, f_i, g_i]$ converges to a point of W and likewise for the primed sequence.

Now suppose $f = g = 0$. Then $f' = 0$ and $[e] = [e']$ but $g' \neq 0$. The limit of the sequence $[\epsilon_i e_i', f_i', g_i']$ is $[0, 0, g']$. Since this is a point of W it must be p. As before it follows from the definitions that $[e', 0, g'] = f(p)$ and so $e' = 0$ which is a contradiction. Thus one of f, g is non-zero. Likewise one of f', g' is non-zero. These facts mean that

$$[\epsilon_i e_i, f_i, g_i] \to [0, f, g] \quad , [\epsilon_i e_i, f_i', g_i'] \to [0, f', g'].$$

From this we see that f and f' are both non-zero. But now $[f] = [f']$ so these points $[0, f, g], [0, f', g']$ project to the same point in the embedding of $\tilde{W}$, again giving a contradiction.

Now we turn to the general case when $\hat{W} = \tilde{W} \cup_D P$. The structure of $\hat{W}$ is determined by the local geometry (of the test configuration) around p, so we get the same local picture when we form $\hat{W}'$. The estimates for the change in the Chow weight, taking due account of the contribution from P, are in the end identical: this goes just as in Stoppa's work. To see the projective embedding, choose r so large that that $\mathcal{L}_D^{-r}$ is very ample on P and restriction

$$H^0(P, \mathcal{L}_D^{-r}) \to H^0(D, \mathcal{L}_D^{-r})$$

is surjective. Let K be the kernel of this restriction map. Then we can amalgamate our embedding $\tilde{W} \to \mathbf{P}(E \oplus F)$, discussed above, with the fixed projective embedding of P to get an embedding of $\tilde{W} \cup_D P$ in $\mathbf{P}(K \oplus E \oplus F)$.

This completes our outline of an approach to Goal 4.

4.3.3 Bounds on the Hamiltonian

Theorem 3 *Suppose $Z \subset \mathbf{CP}^N$ is an n-dimensional variety preserved by a $\mathbf{C}^*$ action on $\mathbf{CP}^N$. Let H be the Hamiltonian and $\overline{\lambda}, \underline{\lambda}$ be the maximum and minimum values of H on Z. Then*

$$\overline{\lambda} - \underline{\lambda} \leq \frac{1}{n+1}(\overline{\lambda} - \frac{1}{\mathrm{Vol}(Z)} \int_X H).$$

We give a proof when Z is smooth, hoping that it can be extended to the general case. For convenience write $n = q + 1$. Let V be the function supported on the interval $[\underline{\lambda}, \overline{\lambda}]$ given by the push forward by H of the volume form on X. We claim that $V^{1/q}$ is a concave function on this interval. Away from the critical values of H we can identify $V(t)$ with the volume of the symplectic quotient $H^{-1}(t)/S^1$. This is just given by varying the induced symplectic form, so we can write locally in t

$$V(t) = \int_{X//S^1} (\Omega + t\theta)^q.$$

We want to show that the second derivative of $V^{1/m}$ is negative at a regular value t which, without loss of generality, can taken to be $t=0$. Then the second derivative is

$$(q-1)V^{(1/q)-2}\left(\int \theta^2\omega^{q-2}\int \omega^q - \left(\int \theta\omega^{q-1}\right)\right)^2,$$

and this is negative by the Hodge index Theorem (since ω is a Kahler form and θ has type (1,1)). Now we just have to check the discontinuities in derivative of the piecewise polynomial function V. These correspond to codimension 2 components of the fixed set of the action and it is easy to see that the jump in the derivative is always negative.

Now the proof is finished by the elementary

Lemma 5 *Let f be a positive concave function on* $[0,1]$. *Then*

$$\int_0^1 tf(t)^m dt \geq \frac{1}{q+2}\int_0^1 f(t)^m dt.$$

The extremal case is when f is a multiple of $t-1$.

Now return to the situation considered in the previous subsection. We write $m=r\gamma$. We have to estimate $w-\frac{I}{mV}$ where w is the highest weight for the action on W, V is the volume (or, better, degree) of W' and I is the integral over W' of the Hamiltonian H' say. We apply the preceding result to the "big" component $B'\subset W'$. Then $\overline{\lambda}=mw$. It is clear that $\overline{\lambda}-\underline{\lambda}\geq \|A_0'\|$ where A_0' is the trace-free part of the generator of the $\mathbf{C}^*$ action on W' (since the maximum and minimum values are attained in B'). Write R' for the remaining components of W' so $V=\mathrm{Vol}(B')+\mathrm{Vol}(R')$. Write $\mathrm{Vol}(B')=\theta\mathrm{Vol}(W')$. We know that actually θ is very close to 1 (when m is large) but all we need is that $\theta\geq 1/2$ say. Now

$$\frac{I}{V}=\int_{W'}H'=\theta\frac{1}{\mathrm{Vol}B'}\int_{B'}H'+(1-\theta)\frac{1}{\mathrm{Vol}R'}\int_{R'}H'\leq\theta\frac{1}{\mathrm{Vol}B'}\int_{B'}H'+(1-\theta)mw,$$

since mw is the maximum value of H'. Thus

$$mw-\frac{I}{V}\geq\theta(mw-\frac{1}{\mathrm{Vol}B'}\int_{B'}H').$$

The preceding result states that

$$mw-\frac{1}{\mathrm{Vol}B'}\int_{B'}H'\geq(n+1)\|A_0'\|,$$

so we get

$$m\delta\geq\theta(n+1)\|A_0'\|\geq\frac{n+1}{2}\|A_0'\|,$$

as stated in Proposition 9.

4.4 Goal 1

Now we go back to our original situation, with a sequence of manifolds $X_i \to X_\infty$ and cscK metrics on X_i satisfying all our differential geometric hypotheses. We are interested in achieving Goal 1, so we are allowed to suppose that X_∞ is a "generic point" in its moduli space (in a sense we can make more precise later). As we explained in (2.3.2) this means that the limit W in the Hilbert scheme (using some power k) is the central fibre of a test configuration for X_∞–we do not encounter "chains". So, given m, we have on the one hand a differential geometric procedure (using the L^2 norms on sections of L^{mk}) yielding a scheme W'_{DG} say, and on the other an algebro-geometric construction yielding a new test configuration with central fibreW'_{AG}, say. If we knew that $W'_{DG} = W'_{AG}$ then just the same arguments as before would go through. The complication is that it is *not* always true that $W'_{DG} = W'_{AG}$. What we want to argue now is that this *is* true if X_∞ is generic, which is the case at hand.

As before, we assume that $\mathrm{Aut}W = \mathbf{C}^*$. We have a $\mathbf{C}^*$-invariant slice T for the projective general linear group action on the Hilbert scheme at $[W]$. We choose a representative $\tau_\infty \in T$ for the orbit of X_∞ and a sequence $\tau_i \to \tau_\infty$ representing the orbits of X_i. We know that 0 is in the closure of the $\mathbf{C}^*$ orbit of τ_∞, say $\lim_{s\to 0} s(\tau_\infty) = 0$. Now the "generic point" condition implies that for all τ near to τ_∞ the limit of $s(\tau)$ as $s \to 0$ exists. That is to say, the weights of the action are all non-negative. For if not the $\mathbf{C}^*$-orbit of τ would break up as $\tau \to \tau_\infty$ which would contradict our assumption about the orbits in the Hilbert scheme. To fix ideas, consider a case when the weights of the action are strictly positive so $s(\tau) \to 0$ as $s \to 0$ for all τ. That is, W is the central fibre of a test configuration for every scheme represented by points $\tau \in T$, and in particular for the X_i. For each τ in a neighbourhood of τ_∞ we can define a weighted flag by the order-of-vanishing construction. If these weighted flags very continuously with τ, in an obvious sense, then the same argument as before shows that $W'_{AG} = W'_{DG}$. The problem is that these flags may not vary continuously, because the order of vanishing can jump upwards in a limit. Now the key point is that the weighted flags *do* vary continuously on a Zariski open set in T so if X_∞ is generic we do not encounter this problem. The same line of argument applies when there are 0 weights in the $\mathbf{C}^*$ action on T. In this case we have a subset $\Sigma \subset T$, properly containing the origin, such that for $\sigma \in \Sigma$ the corresponding scheme, W_σ say, has a $\mathbf{C}^*$-action. The W_σ are a deformation of W among schemes with $\mathbf{C}^*$-action. If the component B of W deforms in this family then we are in essentially the same situation as before. But it could be that the W_σ do not contain a component like B. However there is a closed algebraic subset Σ' in Σ in which B does deform so there is a closed subset T' in T of points τ such that $\lim_{\to 0} s(\tau)$ lies in Σ'. Since τ_∞ lies in T' this must be a generic property, so all points near τ_∞ lie in T' and we get around the difficulty.

It seems likely that if one took the analysis further, in the case when X_∞ is not assumed to be generic, one would get an algebro-geometric description of a chain of orbits from X_∞ to W'_{DG} and that the relevant "generic" condition would then be seen to be that involving splitting of orbits in the Hilbert scheme of subschemes of $H^0(X_\infty, \mathcal{O}(mk))$.

This completes our outline of a possible route to achieve Goal 1. Note taht we may seem to only show that the Kahler-Einstein set contains a Zarsiki open set. But we can apply the same "generic condition" to subvarities and use the following principle. Suppose U is a subset of an algebraic variety A. Suppose that for any subvariety $P \subset A$ we know that if $U \cap P$ is non-empty then $U \cap P$ contains a subset $P \setminus Q$, for a subvariety $Q \subset P$. Then U is Zariski open.

5 Metrics with cone singularities

Let $0 < \beta \leq 1$. The *standard cone* with cone angle $2\pi\beta$ is the singular Riemannian metric on $\mathbf{R}^2$ written in polar co-ordinates as

$$ds^2 + \beta^2 s^2 d\theta^2.$$

If we write $s = r^\beta$ and $z = re^{i\theta}$ then the metric is

$$\beta^2 r^{2(\beta-1)}(dr^2 + r^2 d\theta^2) = \beta^2 |z|^{2(\beta-1)} \left(\frac{i}{2} dz d\overline{z}\right).$$

We write $\mathbf{C}_\beta$ for $\mathbf{C}$ endowed with this singular Kähler metric. Let D be a smooth divisor in a complex n-manifold X. We can consider Kähler metrics on $X \setminus D$ which are locally modelled on $\mathbf{C}_\beta \times \mathbf{C}^{n-1}$ around points of D. For the present we omit a precise definition of "locally modelled". The main case of interest will be when X is a Fano manifold and D is a divisor in $|-K_X|$.

Now change tack and consider a projective variety $X \subset \mathbf{CP}^N$ and a smooth divisor $D \subset X$. For $\lambda \in [0,1]$ and self-adjoint $A \in \mathrm{End}\mathbf{C}^{N+1}$ define

$$\mathrm{Ch}(X, D, \lambda, A) = \lambda \int_X H d\mu + (1-\lambda) \int_D H d\mu - C\mathrm{Tr}(A),$$

where

$$C = \frac{1}{N+1}(\lambda \mathrm{Vol}(X) + (1-\lambda)\mathrm{Vol}(D)).$$

This value of C chosen so that $\mathrm{Ch}(X, D, \lambda, A)$ vanishes when A is a multiple of the identity. This Chow weight of the pair (X, D) is simply a linear combination of the Chow weights for the individual varieties X and D. It is the derivative of the function

$$\mathcal{F}_{X,D,\lambda} = \lambda I(X) + (1-\lambda) I(D) - C \log \det h,$$

of a Hermitian metric h on $\mathbf{C}^{N+1}$. Clearly $\mathcal{F}_{X,D,\lambda}$ is convex along geodesics in the space of metrics h. This immediately gives us a notion of Chow stability of the pair (X, D). The usual proof shows that stability is equivalent to the existence of a "balanced embedding" of the pair, by which we mean one with $\mathrm{Ch}(X, D, \lambda, A) = 0$ for all A. It is also clear that if (X, D) is semistable for some parameter λ_0 and stable for another parameter $\lambda_1 > \lambda_0$ then (X, D) is stable for all $\lambda \in (\lambda_0, \lambda_1]$. The point here is that the "test configurations" in question do not depend on λ, only the Chow weight which determines whether they are destabilising.

To build a bridge between the two discussions above consider first the singular Kähler manifold $\mathbf{C}_\beta$. We can think of this as the complex manifold $\mathbf{C}$ with the trivial holomorphic line endowed with the Hermitian metric $\exp(-|z|^{2\beta})$. Thus we have an L^2 norm on holomorphic functions $f(z)$

$$\|f\|^2 = \beta^2 \int_{\mathbf{C}} |f(z)|^2 \exp(-|z|^{2\beta}) |z|^{2(\beta-1)} dz d\overline{z}.$$

It is clear that the powers $e_p = z^p$ for $p = 0, 1, \dots$ span a dense subspace of the corresponding Hilbert space and that they are orthogonal. We have $\|e_p\|^2 = I_p$ where

$$I_p = \beta^2 \int_0^\infty r^{2p+2\beta-1} \exp(-r^{2\beta}) dr,$$

and the density of states function is

$$\rho = \sum \frac{r^{2p}}{I_p} \exp(-r^{2\beta}).$$

(This discussion ignores some factors of 2π etc., as we have done throughout.) Now set $r^{2\beta} = t$ (so $t = s^2$) and write $c = \beta^{-1}$. Changing variables we find that

$$I_p = \beta \int_0^\infty t^{cp} e^{-t} dt = (cp)!,$$

where we use the notation $x! = \Gamma(x+1)$ for the extended factorial function. We define a function $f(t)$ by

$$f(t) = \sum_{p=0}^\infty \frac{1}{(cp)!} t^{cp} e^{-t} dt,$$

so $\rho = f(t)/\beta$.

Lemma 6

- *$f(t) \to \beta$ as $t \to \infty$,*
- $$\int_0^T f(t) - \beta dt \to (1-\beta)/2$$
  as $T \to \infty$.

To see this take the Laplace transform of $f(t)$

$$F(\xi) = \int_0^\infty e^{-\xi t} f(t) dt.$$

Interchanging the integral and sum this is

$$F(\xi) = \sum_{p \geq 0} (\xi + 1)^{-(1+cp)},$$

and summing the geometric series we get

$$F(\xi) = \frac{1}{(\xi + 1) - (\xi + 1)^{1-c}}.$$

Computing the Laurent series about $\xi = 0$ one finds

$$F(\xi) = \frac{1}{c\xi} + \frac{1-\beta}{2} + O(\xi),$$

and the statements follow from standard Fourier inversion theory.

Write Ω_0 for the singular metric, with fixed parameter β. We can define a new Kahler metric Ω_1 on $\mathbf{C}$ by using the embedding in the infinite dimensional projective space endowed with the L^2 norm. This is smooth metric, with Kahler potential

$$\sum \frac{|z|^{2p}}{I_p}.$$

It follows from the first item in the Lemma, together with corresponding statements about derivatives, that Ω_1 is asymptotic to the cone metric Ω_0. The second item in the Lemma asserts that

$$\int_{\mathbf{C}_\beta} \rho - 1 \Omega_0 = \frac{1-\beta}{2}.$$

From now on we write $\alpha = 1 - \beta$. Notice that by Gauss-Bonnet the integral of the scalar curvature of Ω_1 with respect to the area form Ω_1 is $2\pi\alpha$.

Now suppose we have a metric ω_0 on (X, D) with a cone singularity. We get an L^2 metric on $H^0(X; L^k)$ and an induced Fubini Study metric ω_{FS}, which is a smooth metric on X. We set $\omega_1 = k^{-1}\omega_{FS}$, so ω_1 is in the same cohoology class as ω_0. We expect that when k is large ω_1 is close to ω_0 away from D and is given near D by rounding off the singularity over a neighbourhood N of points of distance $O(k^{-1/2})$ from D, modelled transverse to D on a rescaled version of Ω_1. Let F be a smooth function on X and write $H = kF$. We consider the integral

$$\chi = k^{n+1} \int_X F \rho d\mu_0,$$

where $d\mu_0$ is the volume form of ω_0. In the case when, for a given k, H is a function derived from a self-adjoint matrix A, as in the definition of the Chow weight, this is the same as the trace of A but here we are turning things round a bit and considering a fixed function F and varying k.

Away form the divisor D we can write, as usual,

$$\rho = 1 + \frac{S}{2}k^{-1} + O(k^2).$$

On the neighbourhood N the function ρ is approximated by a rescaled version of the model discussed above. Hence

$$\int_X F\rho d\mu_0 = \int_X F d\mu_0 + k^{-1}\int_X F\frac{S}{2}d\mu_0 + \frac{\alpha}{2}k^{-1}\int_D F d\mu_0,$$

where we are also writing $d\mu_0$ for the measure on D induced by ω_0. Away from D the volume form $d\mu_1$ of ω_1 differs from $d\mu_0$ by $O(k^{-2})$. We assume the same is true for the induced volume form on D itself. Thus we get

$$\chi = \int_X H d\mu_{FS} + \frac{\alpha}{2}\int_D H d\mu_{FS} + k^n \int_X \frac{S}{2} d\mu_0 + O(k^{n-2}). \quad (*)$$

Now choose λ so that

$$\frac{\alpha}{2} = \frac{1-\lambda}{\lambda}. \quad (**)$$

Consider the expression

$$Ch_\lambda = \lambda\int_X H d\mu_{FS} + (1-\lambda)\int_D H d\mu_{FS} - C\chi,$$

where C is chosen as before so that Ch_λ vanishes when H is constant. In the case when H is derived from a matrix A this coincides with the Chow weight. Using (*), first for the given function H and second for a constant function and rearranging we find that

$$\lambda^{-1}Ch_\lambda = k^n \int_X (S-\hat{S})F d\mu_0 + O(k^{n-1}).$$

where $\hat{S}$ is the average value of the scalar curvature. In particular the leading term, for large k, vanishes for all F precisely when the scalar curvature is constant.

What this discussion suggests is that the differential geometric theory of metrics with cone singularities of angle β should be related to the algebro-geometric theory of stability for pairs (X, D) where the parameter λ is given by (**). Thus we are concerned with the range $2/3 < \lambda \leq 1$. Now we restrict to the Kahler-Einstein case of a Fano manifold X and a smooth divisor $D \in |-K_X|$. The strategy we have in mind runs as follows.

1. Show that for a sufficiently small β_0 there is a Kahler-Einstein metric with cone angle β_0. We expect that as $\beta \to 0$ these metrics converge to the complete Ricci flat metric on $X \setminus D$ due to Tian and Yau. One approach to this step is to try to perturb the Tian-Yau metric.

2. Show that if there is a KE metric with cone angle β then for large k the pair (X, D) is Chow semistable, for the corresponding parameter λ.

3. If X is K-stable then we deduce from the above two steps that (X, D) is K-stable for $\lambda \in (\lambda_0, 1]$, where λ_0 corresponds to β_0.

4. Show that the set of cone angles for which a KE metric exists is open.

5. Now consider an increasing sequence of angles $\beta_i > \beta_0$ converging to $\beta \leq 1$. Suppose there are KE metrics ω_i with these angles. If the appropriate volume estimates etc. hold then we can run just the same argument as in Section 2. But now we are in the simpler situation envisaged in 4.3 above, of a sequence of varieties in the same $SL(N+1, \mathbf{C})$ orbit, so we do not run into the complications with "chains". If X is actually $\overline{K}$-stable we can use the Stoppa technique to conclude finally that in fact the sequence of metrics converges.

Remark 1

This method depends on the existence of a smooth divisor in $|-K_X|$. It seems that this holds for the vast majority of known examples, but if there are cases where it fails something else will be required.

Remark 2

One can think of this approach as a variant of the standard continuity method for the Kahler-Einstein equation, by regarding a cone singularity along D as a metric whose Ricci tensor contains a distributional component $\alpha[D]$. Likewise we get an algebro-geometric setting for the standard continuity method. We consider pairs (X, σ) where $X \subset \mathbf{CP}^N$ is a projective manifold and σ is a closed $(1,1)$-form on X. Then define the Chow weight by

$$\mathrm{Ch}(X, \sigma, A) = \int_X H d\mu_{FS} + \int_X H \omega_{FS}^{n-1} \wedge \sigma - C\mathrm{Tr}(A),$$

where as usual the constant C is chosen so that $\mathrm{Ch}(X, \sigma, I) = 0$ where I is the identity matrix. The same scheme outlined above applies assuming one can has appropriate results about solutions of the equation $\mathrm{Ric}(\omega) = t\omega + (1-t)\sigma$.

6 Conclusions

If the algebro-geometric arguments sketched in Section 4 stand up then we have achieved Goal 1. We fail to achieve Goal 2 because of the difficulty with "chains" described in Section 2. Of course if the cone singularity theory of Section 5 can be developed then that gives a route to the stronger Goal 3. On the other hand for concrete applications Goals 1 and 3 are perhaps more similar than they appear. In practice, working from the definition, it is very difficult to determine if a manifold is K-stable, so Goals 1 and 3 each assert that the set of "Kahler-Einstein points" is some Zariski open subset of moduli space, but without giving an effective way of describing the set exactly. The difficulty comes from the fact that the definition of K-stability involves arbitrarily large powers L^k. In this connection, it is worth pointing out that our proofs give in a sense rather *more* that what is asserted in Goals 1,2,3. This is because we detect the nonexistence of a Kahler-Einstein metric by studying linear systems $|k_0 L|$ for some "known" value k_0, which could be computed from the constants in the volume estimate. The actual value of k_0 one would obtain from our analytical arguments would probably be very large. On the other hand, one suspects that in reality, for simple specific examples, the algebro-geometric phenomena may be detected from some small value of k_0, say $k_0 = 1, 2$. If that were the case then one could hope to give a truly explicit description of the Kahler-Einstein set.

To sum up, there seem to be the three notable directions for further progress:

- Find a way around the difficulty with chains.
- Develop the cone-singularity theory.
- Find useful estimates on a "k_0" which will yield explicit results.

Another interesting question is whether in fact the limits W are in fact the same for large enough k. This should be related to the problem of endowing a Gromov-Haussdorf limit of Kahler-Einstien manifolds with a complex algebraic structure.

7 Appendix 1: Review of results for Kahler-Einstein metrics

Under our hypotheses for one of the manifolds in question, the volume is fixed and we have an upper bound on the diameter (from Myers' Theorem. According to Croke [4], since the Ricci curvature is bounded below we have a uniform Sobolev inequality, see also [1] Chapter 5, Theorem 7. By the Bishop-Gromov comparison theorem we have upper and lower bounds on the volume of balls

$$0 < c_0 r^{2n} \leq \mathrm{Vol}(B_r) \leq c_1 r^{2n}.$$

Supplement 2 to Hypothesis V

Suppose B is a unit ball in a complete Riemannian manifold wit centre p and suppose $|\mathrm{Riem}| \leq 1$ on B. Then a direct geometric argument [5], Theorem 4.3, shows that there for each ϵ there is a δ such that if the volume of B exceeds ϵ then the injectivity radius at p exceeds δ. Now the statement follows, after rescaling, from the lower bound on the volume of balls.

Additional hypothesis 2

We reproduce some material from [2]. We work with a manifold M of dimension $2n$ with $n > 1$, of volume 1 and with Ricci curvature bounded below by a strictly positive constant. Cheng and Li show that in this situation a Sobolev-type inequality of the form

$$\|f\|_{L^2} \leq C \|\nabla f\|_{L^2}^{\alpha} \|f\|_{L^1}^{1-\alpha}, \qquad (*)$$

holds, with fixed C, for all functions f of integral 0 and with $\alpha = n/(n+1)$. Now let $K(x,y,t)$ be the heat kernel on the manifold, so $K > 0$ and

$$\int_M K(x,y,t)dy = 1,$$

for all y,t and $K(x,y,t) \to 1$ as $t \to \infty$. Set $H = K - 1$. We have identities

$$H(x,x,t) = \int_M H(x,y,t/2)^2 dy,$$

$$-\frac{\partial H}{\partial t}(x,x,t) = \int_M |\nabla_y H(x,y,t/2)|^2.$$

Applying (*) we get

$$\int_M H(x,y,t/2)^2 dy \leq C \left(\int_M |\nabla_y H(x,y,t/2)|^2 dy\right)^\alpha \left(\int_M |H(x,y,t/2)| dy\right)^{2(1-\alpha)}.$$

The conditions above on K imply that

$$\int_M |H(x,y,t)| \leq 2.$$

Using our identities we obtain

$$H(x,x,t) \leq C(-\frac{\partial H}{\partial t}(x,,x,t)^\alpha.$$

Integrating this differential inequality gives

$$H(x,x,t) \leq Ct^{-n},$$

for fixed C. Now writing

$$H(x,y,t) = \sum_\lambda e^{-\lambda t}\phi_\lambda(x)\phi_\lambda(y),$$

we see that

$$|H(x,y,t)| \leq \frac{1}{2}(H(x,x,t) + H(y,y,t)),$$

so $|H(x,y,t)| \leq Ct^{-n}$. Set

$$\tilde{G}(x,y) = \int_0^\infty H(x,y,t)dt.$$

Then $\tilde{G}$ is a Green's function on M but with a different normalisation from that considered in the statement in Section 2. By construction

$$\int_M G(x,y)dy = 0.$$

In terms of $\tilde{G}$, what we need is a lower bound $\tilde{G}(x,y) \geq -C$, for fixed C and all x, y. Now, since $n > 1$, we have

$$|\int_1^\infty H(x,y,t)dt| \leq C \int_1^\infty t^{-n} \leq C.$$

On the other hand since $K(x,y,t) \geq 0$ we have $H(x,y,t) \geq -1$ so

$$\int_0^1 H(x,y,t) \geq -1$$

which completes the proof.

Additional Hypothesis 3

For a holomorphic section s of L^k we have a Weitzenbock formula $\nabla^*\nabla s = ks$. Set $f = |s|$ so $-\Delta f \leq kf$ (in a weak sense). Now we apply the Moser iteration argument. Set $\sigma = n/(n-1)$ so we have a Sobolev inequality, for positive functions F

$$\left(\int F^{2\sigma}\right)^{1/\sigma} \leq C \left(\int |\nabla F|^2 + F^2\right).$$

Given $p > 1$ we have

$$\int f^p - \Delta f \leq k \int f^{p+1}$$

Write $p = 2q - 1$, integrating by parts and rearrange to get

$$\int |\nabla f^q|^2 \leq \frac{kq^2}{2q-1} \int f^{2q}.$$

Now set $F = f^q$ in the Sobolev inequality and derive

$$\left(\int f^{2q\sigma}\right)^{1/\sigma} \leq C \left(\frac{kq^2}{2q-1} + 1\right) \int f^{2q}.$$

If $g = f^2$ this can be written as

$$\|g\|_{L^{q\sigma}} \leq \left(C(\frac{kq^2}{2q-1} + 1)\right)^{1/q} \|g\|_{L^q}.$$

Let $I_\nu = \|g\|_{L^{\sigma^\nu}}$, so

$$I_{\nu+1} \leq \left(C(\frac{k\sigma^{2\nu}}{2\sigma^\nu - 1} + 1)\right)^{1/\sigma^\nu} I_\nu.$$

Then

$$\|g\|_{L^\infty} = \lim_{\nu\to\infty} I_\nu \leq \prod_{\nu=0}^{\infty} \left(C(\frac{k\sigma^{2\nu}}{2\sigma^{\nu-1}} + 1)\right)^{1/\sigma^\nu} I_0.$$

Straightforward estimates give

$$\|g\|_{L^\infty} \leq Ck^{(1+\sigma^{-1}+\sigma^{-2}+\ldots)} I_0,$$

and $1 + \sigma^{-1} + \sigma^{-2} + \ldots = n$ since $\sigma = n/n - 1$. This is the desired result.

8 Appendix 2: Outline proof of Theorem 1

This amounts to a precis of relevant parts of Lu's paper [6], and papers by Tian, Ruan and others. Generalising the $O(\ ), o(\ )$ notation we write $\epsilon(k)$ for any term which is bounded by Ck^{-r} for all r. Essentially this means exponentially decaying terms, invisible in any asymptotic expansion in inverse powers of k.

Note that there will be a missing factor of $(2\pi)^n$ in our discussion.

To bring out the main point consider first a case when the metric is actually *flat* in the embedded unit ball $B \subset Z$. We then identify this with the standard ball in $\mathbf{C}^n$. We fix an identification of the fibre of L over the origin with $\mathbf{C}$. Taking the line bundle L^k we rescale the metric so we work in a large ball $B_{\sqrt{k}} \subset \mathbf{C}^n$, and we are operating with the standard holomorphic Hermitian line bundle with curvature $(\sqrt{-1}/2) \sum dz_i d\overline{z}_i$. Over this ball we have a standard holomorphic section of σ_0 with $|\sigma_0| = \exp(-|z|^2/4)$. We cut of this section near the boundary of the ball to get a compactly-supported section σ_1 which is approximately holomorphic in that

$$\|\overline{\partial}\sigma_1\| = \epsilon(k).$$

We can use a spherically symmetric cut-off function to do this, so σ_1 is $U(n)$-invariant in an obvious sense. We also have $\|\sigma_1\| = 1 + \epsilon(k)$ and evidently $\sigma_1(0) = 1$.

Now transplant σ_1 to a section of L^k over our manifold Z. We work with the rescaled metric so that Z has large volume $O(k^n)$. We project σ_1 to the space of holomorphic sections using the standard Hormander technique which we now recall. Since the Ricci curvature of the original manifold is bounded that of the rescaled manifold is $O(k^{-1})$. The Weitzenbock formula on L^k-valued $(0,1)$ forms (for the rescaled metric) takes the shape

$$\Delta_{\overline{\partial}} = \frac{1}{2}\nabla^*\nabla + \mathrm{Ric} + 1,$$

so once k is large enough (depending only on the original bound on the Ricci curvature) we have $\Delta_{\overline{\partial}} \geq \frac{1}{2}$ say, and the inverse operator G has L^2-operator norm at most 2. Now

$$\sigma_2 = \sigma_1 - \overline{\partial}^* G \overline{\partial}\sigma_1$$

is a holomorphic section. From the identity

$$\|\overline{\partial}^* G\overline{\partial}\sigma_1\|^2 = \langle G\overline{\partial}\sigma_1, \sigma_1\rangle,$$

we get

$$\|\sigma_2 - \sigma_1\|^2 \leq 2\|\sigma_1\|\|\overline{\partial}\sigma_1\| = \epsilon(k).$$

Go back to the unit ball (say) in $\mathbf{C}^n$. Here the difference $\sigma_2 - \sigma_1$ is holomorphic so the L^2 bound above gives a pointwise bound and

$$|\sigma_2(0) - 1| = \epsilon(k).$$

Now let τ be any section over Z which vanishes at the origin. The inner product $\langle \sigma_1, \tau \rangle$ is an integral over the ball $B_{\sqrt{k}}$ and this obviously vanishes by symmetry (considering the Taylor series of τ in our given trivialisation). Finally, for convenience, set $\sigma = \sigma_2 / \|\sigma_2\|$. What we have achieved is a section σ with the three properties

1. $\|\sigma\| = 1$;
2. $\sigma(0) = 1 + \epsilon(k)$;
3. $\langle \sigma, \tau \rangle = \epsilon(k) \|\tau\|$ if $\tau(0) = 0$.

No more analytical input is required. Let η be the section representing evaluation at 0, i.e.

$$\langle \tau, \eta \rangle = \tau(0)$$

for all τ. By definition the Bergman function at 0 is $\rho_k(0) = \|\eta\|^2$. What we need is

Lemma 7 *Let η, σ be two elements of a Hilbert space such that*

1. $\|\sigma\| = 1$*;*
2. $\langle \sigma, \eta \rangle = 1 + \epsilon(k)$*;*
3. $\langle \sigma, \tau \rangle = \epsilon(k) \|\tau\|$ *for any* τ *with* $\langle \tau, \eta \rangle = 0$*.*

Then $\|\eta\|^2 = 1 + \epsilon(k)$*.*

The proof is an elementary exercise (which takes place in the plane spanned by σ, η.) In our case the three hypotheses are re-statements of the properties above and we conclude that, in this flat situation, $\rho_k(0) = 1 + \epsilon(k)$. The point we want to bring out is that the only place in which the geometry of the manifold Z away from the given ball enters into the argument is through the bound on the Ricci curvature.

Now we go on to the general case. By the same argument as for the Lemma above if we produce a holomorphic section σ with $\|\sigma\| = 1$,

$$\sigma(0) = 1 + Ak^{-1} + O(k^{-2}),$$

and

$$|\langle \sigma, \tau \rangle| \leq Ck^{-1} \|\tau\|,$$

for all holomorphic sections τ vanishing at the origin, then $\rho_k(0) = (1 + Ak^{-1})^2 + O(k^{-2})$. Consider holomorphic co-ordinates w_a centred on the given point in the manifold. A Kahler potential ϕ has a Taylor series which we can obviously suppose begins as

$$\phi(w) = \sum_a w_a \overline{w}_a + O(w^3)$$

Write the cubic term schematically as $(3,0)+(2,1)+(1,2)+(0,3)$ in terms of the degree in $w_a, \overline{w}_a$. By a change of co-ordinates of the form

$$\tilde{w}_a = w_a + C_{abc} w_b w_c$$

we can reduce to the case when the $(2,1)$ and $(1,2)$ terms (which are complex conjugate) vanish. Then by adding the real part of a holomorphic quartic function to the Kahler potential we can remove the $(3,0)$ and $(0,3)$ terms. Similarly we can remove all the quartic terms in the Taylor expansion except for $(2,2)$ and all the quintic terms except for $(2,3)+(3,2)$. So we can suppose that

$$\phi = \sum w_a \overline{w}_a + \sum P_{abcd} w_a w_b \overline{w}_c \overline{w}_d + (\sum Q_{abcde} w_a w_b w_c \overline{w}_d \overline{w}_e + \text{complexconjugate}) + O(w^6).$$

Next we rescale co-ordinates writing $w_a = k^{-1/2} z_a$ and setting $\Phi(z) = k\phi(w)$. Thus

$$\Phi(z) = |z|^2 + k^{-1} P(z) + k^{-3/2} Q(z) + O(k^{-2}),$$

in an obvious notation. We work over a ball $|z| \leq R$ where we can take R to be a very small multiple of $k^{1/4}$ so that $\Phi(z) - |z|^2, k^{-1}P(z), k^{-3/2}Q(z)$ are all very small over the ball. The volume form $(\sqrt{-1}\partial\overline{\partial}\Phi)^n$ in these co-ordinate can be written

$$J = 1 + k^{-1} p(z) + k^{-3/2} q(z) + O(k^{-2}),$$

where for example

$$p(z) = 4 \sum_{a,b,c} P_{abcb} z_a \overline{z}_c.$$

The choice of a Kahler potential precisely corresponds to the choice of a local trivialisation of our line bundle and hence a local holomorphic section σ_0 with $|\sigma_0|^2 = e^{-\Phi}$. Just as before we can modify σ_0 to get a global holomorphic section and this only introduced terms $\epsilon(k)$ which we can ignore. Regard σ_0 as a discontinuous section of the line bundle over the whole manifold, extending by zero outside our ball. We want to show first that

$$|\langle \sigma_0, \tau \rangle| \leq C k^{-1},$$

for all holomorphic sections vanishing at the origin. Second we want to find a number a such that $\|\sigma_0\|^2 = 1 + ak^{-1} + O(k^{-2})$. Then we will have established what we need with $A = -a/2$ (since $\sigma_0(0) = 1$ by construction).

Now

$$|\langle \sigma_0, \tau \rangle| \leq C k^{-1} \int_{|z| \leq R} (1 + |z|^4) e^{-|z|^2} \tau(z),$$

where $\tau(z)$ is the representative of τ in our local trivialisation. Here we use the fact that

$$\int_{|z| \leq R} \tau(z) e^{-|z|^2} = 0$$

when $\tau(0) = 0$. We obtain the desired estimate using the Cauchy-Schwartz inequality in the weighted norm. So it just remains to compute $\|\sigma_0\|^2$ which is

$$\int_{|z|\leq R} e^{-|z|^2}(1+k^{-1}P(z)+k^{-3/2}Q(z)+O(k^{-2})(1+k^{-1}p(z)+k^{-3/2}q(z)+O(k^{-2}).$$

Here we have used the Taylor series to expand the exponential term $e^{-\Phi}$ and we have skipped over some rather routine estimates.

If $z^I, \overline{z}^J$ are any monomials such that

$$\int_{|z|\leq R} z^I \overline{Z}^J e^{-|z|^2}$$

is non zero then we must have $|I| = |J|$. (To see this, consider the action of multiplication by $e^{\imath\theta}$.) It follows that the integrals appearing in the $k^{-3/2}$ terms above vanish. Extending the range of integration introduces errors $\epsilon(k)$ so we get

$$\|\sigma_0\|^2 = \int_{\mathbf{C}^n} \left(1 + k^{-1}(P(z) + p(z))\right) e^{-|z|^2} + O(k^{-2}).$$

The integral here is straightforward to calculate. We can also argue as follows. The tensor P_{abcd} is, from an invariant point of view, an element of $s^2(V)\otimes s^2(V)^*$ where V is the cotangent space. The symmetric power $s^2(V)$ is an irreducible representation of $U(n)$ so there is, up to a multiple, just one $U(n)$-invariant contraction $s^2(V) \otimes s^2(V)^* \to \mathbf{C}$. This is given by

$$c(P) = \sum_{a,b} P_{abab}.$$

It is clear that the scalar curvature and the integrals appearing in the $O(k^{-1})$ term above are both invariant contractions of P hence multiples of $c(P)$. This argument shows that the co-efficient we are after must be some universal multiple of the scalar curvature and of course we can identify the multiple from the Hirzebruch-Riemann-Roch formula.

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