This notes are a supplement to the notes on the previous (2008) version of the course. Exercises for the 2010 version will be found at the end.

1 The Poisson equation $\Delta f = \rho$ on a compact manifold

This followed the earlier notes closely.

2 Applications to Riemann surfaces

This followed Chapter 8 and 10 of the notes “Riemann Surfaces” on this website.

3 General Theory of Linear Elliptic operators

Followed the earlier notes with the addition of the following discussion.

Let $D : \Gamma(E) \rightarrow \Gamma(E)$ be a first order self-adjoint elliptic operator over a manifold M. Then over $M \times \mathbf{R}$ (or $M \times S^1$) we have an operator $\delta = \frac{\partial}{\partial t} + D$ which is again elliptic, with adjoint $-\frac{\partial}{\partial t} + D$. Now suppose that $\delta : \Gamma(E) \rightarrow \Gamma(F)$ is an elliptic operator over a manifold N. We get a self-adjoint operator $\delta + \delta^* : \Gamma(E \oplus F) \rightarrow \Gamma(E \oplus F)$. Performing the construction above we get a new operator over $N \times \mathbf{R}$ or $N \times S^1$ where we now take s as co-ordinate on the extra factor. Compose with the algebraic operator which acts as i on E and $-i$ on F. This gives a self-adjoint operator

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \partial_s & \delta \\ \delta^* & \partial_s \end{pmatrix} = \begin{pmatrix} i\partial_s & i\delta \\ -i\delta^* & -i\partial_s \end{pmatrix}.$$

Thus starting with a self-adjoint operator in one dimension n we get another in dimension $n+2$ and we can repeat the process. Begin in dimension 1 with $D = i\partial_\theta$. In dimension 2 we get the Cauchy-Riemann operator $\overline{\partial} = \partial_t + i\partial_\theta$. In dimension 3 we get the Dirac operator

$$\begin{pmatrix} i\partial_s & -\partial_\theta + i\partial_t \\ \partial_\theta + i\partial_t & -i\partial_s \end{pmatrix}.$$

And in general this process generates the Dirac operator on $\mathbf{R}^n$ for all n. What is not obvious is that this operator commutes with an action of the (double cover of) the orthogonal group, and we get corresponding operators on Riemannian manifolds with “spin structures”.

4 The heat equation and Weyl's formula

Let M be a compact Riemannian n-manifold. From the spectral decomposition of the Laplacian we know that we have for $t \geq 0$ an operator $K_t : L^2 \to L^2$, defined by $K_t(\phi_\lambda) = e^{-\lambda t}\phi_\lambda$. We can take the Sobolev norms to be

$$\| \sum a_\lambda \phi_\lambda \|^2_{L^2_k} = \sum \lambda^k |a_\lambda|^2.$$

Then it is clear from our elliptic inequalities and Sobolev embedding that K_t is bounded on any L^2_k, with bound independent of t, and for $t > 0$ is given by a smooth kernel

$$k_t(x,y) = \sum e^{-\lambda t}\phi_\lambda(x)\phi_\lambda(y).$$

Remark We can also see that K_t is a *contraction* on C^0 using the maximum principle. But we do not use this.

We want to get an asymptotic description of k_t for small t. We know that on $\mathbf{R}^n$ the fundamental solution of the heat equation is $(4\pi t)^{-n/2}e^{-x^2/4t}$. Fix a point y_0 on M and work in geodesic coordinates around this point. Let r be the Riemannian distance to this point. Our first guess (dropping the factor) is

$$\Phi = t^{-n/2}e^{-r^2/4t}.$$

Computing we find that

$$(\partial_t + \Delta)\Phi = -\Phi \frac{1}{4t}(2n + \Delta r^2) = \frac{W\Phi}{t},$$

say. Note that $W = W(x)$ is smooth and vanishes at the origin (i.e at y_0).

The next step is to improve things by considering a multiple $\Psi = \beta\Phi$ where $\beta = \beta(x)$ is smooth, to be chosen suitably. We find that

$$(\partial_t + \Delta)\Psi = \left(\beta \frac{W}{t} + \Delta\beta - \frac{1}{2t} r\partial_r \beta\right)\Phi.$$

Where in geodesic coordinates x_i we have

$$r\partial_r = \sum x_i \frac{\partial}{\partial x_i}.$$

The key observation is that for any smooth function $G(x)$ which vanishes at the origin we can solve the equation $r\partial_r f = G$. In fact for our purposes we could just use polynomials. The we can use the fact that the monomials are eigenvectors for the operator:

$$r\partial_r(x_1^{n_1}x_2^{n_2}) = (n_1+n_2)x_1^{n_1}x_2^{n_2}.$$

Since W vanishes at the origin we can solve $r\partial_r f = 2W$ and then set $\beta = e^f$. Then $(\partial_t + \Delta)\Psi = Q_0\Psi$ where $Q_0 = Q_0(x)$ is smooth (in fact $Q_0 = \Delta\beta$).

Now we seek an asymptotic expansion

$$\Psi(1 + a_1(x)t + a_2(x)t + \ldots)$$

where the a_p are smooth. Suppose inductively that we have defined a_i for $i \leq p-1$ in such a way that $S_{p-1} = \psi(1 + a_1 t \ldots + t^{p-1}a_{p-1})$ has

$$(\partial_t + \Delta)S_{p-1} = Q_{p-1}(x)t^{p-1}\Psi,$$

for smooth Q_{p-1}. When $p = 1$ this is what we have achieved above. Now for $a = a(x)$

$$(\partial_t + \Delta)(t^p a\Psi) = at^p(\partial_t + \Delta)\Psi + pt^{p-1}a\Psi + t^p\Psi\Delta a + 2\nabla a.\nabla\beta\beta^{-1}\Psi + \frac{\Psi}{t}r\partial_r a.$$

This is

$$(\partial_t + \Delta)(t^p a\Psi) = \Psi t^p\left(H + \frac{1}{t}(r\partial_r a + pa)\right),$$

where $H = aQ_0 + \Delta a - 2\nabla a.\nabla\beta\beta^{-1}$. Thus H depends on a but for any smooth a we get a smooth H. Now for $p > 0$ we can solve the equation $(r\partial_r f + pf) = G$ for any G. So define a_p to be the solution of

$$(r\partial_r a + pa) = -Q_{p-1},$$

and then we continue the induction with Q_p the function H, defined by this a_p.

Fix $p > n/2$ and let $\tilde{k}_t(x) = (4\pi)^{-n/2}\Psi(1 + a_1 t + \ldots a_p t^p)$. The difference $\eta_t(x) = \tilde{k}_t(x) - k_t(x, y_0)$ has $(\partial_t + \Delta)\eta = \rho_t$ where $|\rho_t| \leq C$. One sees easily that $\eta_t \to 0$ as $t \to 0$ in the sense of distributions. Thus $\eta_t = \sum \eta_\lambda(t)\phi_\lambda$ where $\eta_\lambda(t) \to 0$ as $t \to 0$. Then

$$(\frac{d}{dt} + \lambda)\eta_\lambda = \rho_\lambda$$

and

$$\eta_\lambda(t) = \int_0^t \rho_\lambda(s)e^{\lambda(s-t)}ds.$$

That is

$$\eta_t = \int_0^t K_{t-s}(\rho_s)ds.$$

(This is *Duhamel's formula.*) Now one can either use the fact that K_t is a contraction on C^0 or argue as follows. It is easy to compute that the L^2_k norm of $e^{-x^2/t}$ is $O(t^{n/4-k/2})$. Choose k the smallest integer such that $k - n/2 > 0$. Thus $k - n/2$ is at most 1 and the norm above is at most $O(t^{-1/2})$. Using the fact that K_t is uniformjly bounded in operator norm on L^2_k we have

$$\|\eta_t\|_{L^2_k} \leq C\int_0^t s^{-1/2}ds \leq Ct^{1/2}.$$

Now the Sobolev embedding $L^2_k \to C^0$ implies that $|\eta_t| \leq Ct^{1/2}$. In other words

$$K_t(x, y_0) = (4\pi t)^{-n/2}\Psi(1 + \ldots + a_p t^p) + O(t^{1/2}).$$

Clearly the same argument applies with error $O(t^q)$, for any q, if we take enough terms in the expansion, and everything is uniform in the point y_0.

The contribution of the 0 eigenvalue of the Laplacian to $k_t(x, y)$ is the constant $c = \mathrm{Vol}(M)^{-1}$. We have

$$\int_0^\infty k_t(x, y) - c\,dt = \sum_{\lambda > 0} \lambda^{-1}\phi_\lambda(x)\phi_\lambda(y)$$

and this is the Green's kernel $G(x, y)$. That is, the solution of $\Delta f = \rho$, when ρ has integral 0 and the solution is normalised to have integral zero, is

$$f(x) = \int_M G(x, y)\rho(y)dy.$$

Now set

$$\tilde{G}(x, y) = \int_0^1 k_t(x, y)dt.$$

Then G differs from $\tilde{G}$ by the addition of a smooth function on $M \times M$. We can approximate $\tilde{G}$ using our asymptotic expansion of k_t:

$$\int_0^1 e^{-r^2/4t} t^{p-n/2} dt = \left(\frac{r^2}{4}\right)^{p-n/2+1} \int_{r^2/4}^\infty e^{-u} u^{n/2-p-2} du.$$

If $p < n/2 - 1$ we can replace the lower limit by 0 since we only the change the result by a smooth function. Then we get a term in r^{2p-n+2}. If $p > n/2 - 1$ the contribution is smooth so we can discard it. If $p = n/2 - 1$ (and so n must be even) we get a $\log r$ term. The upshot is an asymptotic expansion of $G(x, y_0)$, for n odd;

$$b_0 r^{2-n} + b_1 r^{4-n} + \ldots b_m r^{-1} + O(1),$$

and for n even (and say > 2);

$$b_0 r^{2-n} + \ldots b_m r^{-2} + \beta \log r + O(1).$$

Where b_i are smooth functions which can be obtained from the asymptotic expansion of the heat kernel. We *cannot* go on to obtain the $O(1)$ terms by such local considerations.

4.1 The Weyl formula.

Let $N(\rho)$ be the number of eigenvalues less that or equal to ρ. Weyl's formula is

$$N(\rho) \sim \frac{1}{(4\pi)^{n/2}\Gamma(n/2+1)}\mathrm{Vol}(M)\rho^{n/2},$$

as $\rho \to \infty$. For example in the case of a torus this amounts to the asymptotic relation between the volume of an ellipsoid and the number of lattice points it contains, using the fact that the volume of the unit ball in $\mathbf{R}^n$ is $\pi^{n/2}/\Gamma(n/2+1)$.

The argument is to consider the integral $I(t) = \int_M k_t(x,x)dx$. This is the "trace" of the operator K_t,

$$I(t) = \sum_\lambda e^{-\lambda t}.$$

By our asymptotic formula we know that

$$I(t) \sim (4\pi t)^{-n/2}\mathrm{Vol}(M),$$

as $t \to 0$. To simplify notation suppose that $\mathrm{Vol}(M) = (4\pi)^{n/2}$ and write $n/2 = m$. What we need to show is that if we have a countable, locally finite, set of positive numbers λ and if $\sum e^{-\lambda t} \sim t^{-m}$ then $N(\rho) \sim (1/\Gamma(m+1))\rho^m$. This is done using *Tauberian theory.*

Let $d\nu$ be the counting measure defined by the set $\{\lambda\}$–a sum of δ-functions at these points. Set $v = t^{-1}$ so we can write

$$t^m I(t) = \int_{u=0}^{\infty} (u/v)^m e^{-(u/v)}\ u^{-m} d\nu(u).$$

which is

$$\int_{u=0}^{\infty} f(u/v)\ u^{-m} d\nu(u), \quad (1)$$

where $f(u) = u^m e^{-u}$. On the other hand writing $v = \rho$

$$v^{-m}N(v) = \int_{u=0}^{\infty} g(u/v)\ \ u^{-m} d\nu(u), \quad (2)$$

where $g(u) = u^m$ if $0 \leq u \leq 1$ and $g(u) = 0$ if $u > 1$. Now write $u = e^y$. Let $d\mu$ be the measure on $\mathbf{R}$ given by point masses e^{-my} at the points $y = \log\lambda$. Then (1) becomes

$$A(z) = \int_{-\infty}^{\infty} F(y-z)d\mu(y),$$

where $F(y) = e^{my}\exp(-e^y)$, while (2) becomes

$$B(z) = \int_{-\infty}^{\infty} G(y-z)d\mu(y),$$

with $G(y) = e^{my}$ if $y < 0$ and 0 otherwise. Here $z = \log v$. What we need to show is that if $A(z)$ tends to 1 as $z \to \infty$ then $B(z)$ tends to $1/\Gamma(m+1)$.

Consider the Fourier transform $\hat{F}(\xi)$. This can be written as

$$\hat{F}(\xi) = \int_0^\infty u^{m-1+i\xi} e^{-u} du,$$

which is $\Gamma(m + i\xi)$. In particular the integral of F is $\Gamma(m)$. The crucial fact however is that the Fourier Transform does not vanish for any $\xi \in \mathbf{R}$. Indeed the Γ function has no zeros in $\mathbf{C}$ since $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$. Now the integral of G is $1/m$ so the ratio of the inetgrals of F and G is $m\Gamma(m) = \Gamma(m+1)$ which is the factor we want. After rescaling, we can cast our question in the following form. We have a function F of (Lebesgue) integral 1 and with nowhere-vanishing Fourier transform. We have a measure μ such that $A(z)$ above is defined for all z, is bounded by a fixed constant for all z and tends to limit 1 as $z \to \infty$. For another function G we want to be able to deduce that $B(z)$ tends to a limit L where $L = \int_{-\infty}^{\infty} G(y)dy$. This is the content of Wiener's Tauberian Theorem, under suitable hypotheses on the functions F, G. The line of proof is

- The result is trivially true if G is a translate of F.
- The result is true if G is a finite linear combination of translates $G(y) = \sum_{i=1}^{N} m_i F(y - z_i)$.
- Now consider functions G with $\hat{G}$ of compact support. Then $\hat{G}/\hat{F}$ is a smooth function of compact support and so has an inverse Fourier Transform $M(y)$. By construction $G = M * F$ and this says that G is a "continuous linear combination" of translates of F,

  $$G(y) = \int M(z)F(y - z)dz.$$

  Approximating the integral by a sum we deduce what we want.
- Finally approximate our given G (under suitable hypotheses) by functions with Fourier transforms of compact support.

Given the idea, and some standard background in Fourier analysis, the details are fairly straightforward.

Digression These ideas were applied by Wiener to the proof the *Prime number theorem* (PNT). Let $\pi(n)$ be the number of primes $\leq n$. The PNT states that $\pi(n) \sim n/\log$ as $n \to \infty$. Recall that Riemann's zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

This is defined initially for $\Re(s) > 1$ and then extended by analytic continuation with a pole at $s = 1$. Wiener operates with the function

$$\Lambda(n) = \log p \text{ if} n = p^k,$$

and $\Lambda(n) = 0$ otherwise. There is an identity, for $0 \leq x < 1$

$$\sum_{m=1}^{\infty} \log m x^m = \sum_{m=1}^{\infty} \Lambda(m) \frac{x^m}{1 - x^m}.$$

With some work, this leads to

$$\lim_{N\to\infty} \frac{1}{N} \sum \Lambda(n) h(n/N) = 1, \quad (1PNT)$$

where h is the function

$$h(x) = \frac{d}{dx}\left(\frac{x}{e^x - 1}\right).$$

On the other hand, with some more work, it can be shown that the PNT follows from the statement

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) = 1. \quad (2PNT).$$

So the problem is to show that (1PNT) implies (2PNT). But now we can write the expression on the left hand side of (1PNT) as

$$\lim_{N\to\infty} \int_0^\infty f(n/N) d\mu,$$

where μ is the sum of point masses $p^{-k} \log p$ at powers p^k and $f(x) = xh(x)$. The expression on the left hand side of (2PNT) is

$$\lim_{N\to\infty} \int_0^\infty g(n/N) d\mu,$$

where $g(x) = x$ for $x \leq 1$ and otherwise zero. So we are exactly in the situation considered before and the crucial thing we need is that the corresponding Fourier transform has no zeros. That is

$$\int_0^\infty \frac{d}{dx}\left(\frac{x}{e^x - 1}\right) x^{it} dx \neq 0,$$

for t real. The final ingredient in Wiener's argument is an identity

$$\int_0^\infty \frac{d}{dx}\left(\frac{x}{e^x - 1}\right) x^{it} dx = it\zeta(1 + it)\Gamma(1 + it).$$

So the information needed about the ζ-function is that it has no zeros on the line $\Re(s) = 1$.

5 Some techniques for nonlinear problems

The first part of this followed closely the older notes.

Weyl's Problem

Now we discuss a much harder problem but where the overall strategy is the same.

Theorem 1 *Any metric on S^2 with strictly positive curvature can be isometrically embedded in $\mathbf{R}^3$, and the embedding is unique up to Euclidean motions.*

We concentrate first on the existence. The strategy is

- Show that the set of metrics with positive curvature is connected.
- Show that the subset which can be embedded is open.
- Show that the subset is closed, by establishing *a priori* estimates.

The first item is easy. By the uniqueness of the complex structure on S^2 it suffices to consider metrics $e^{2f} g_0$ where g_0 is the round metric. Then use the path $e^{2tf} g_0$.

Openness is much more tricky and will be our main topic. We can set up teh problem in an obvious way: we have a space of embeddings $\iota : S^2 \to \mathbf{R}^3$ which is an open subset of $C^\infty(S^2; \mathbf{R}^3)$ and any ι gives an induced metric $g_\iota = \mathcal{F}(\iota)$ on S^2. We can compute the linearisation of $\mathcal{F}$. This is a map from vector fields defines along $S = \iota(S^2)$ to symmetric 2-tensors on S^2. We write the vector fields as the sum of tangential and normal components so we have a pair (v, ϕ) where v is a vector field on S^2 and ϕ is a function. Then

$$D\mathcal{F}(v, \phi) = \mathcal{L}_v g + \phi B,$$

where $g = g_\iota$, $\mathcal{L}$ is the Lie derivative and B is the *second fundamental form* of the embedding. The *good feature* is that this is linear operator is *surjective*, that is

Proposition 1 *For any variation δg there is a pair (v, ϕ) with $\mathcal{L}_v g + \phi B = \delta g$. Moreover the solution is unique modulo the infinitesimal isometries of $\mathbf{R}^3$.*

The *bad feature* is that this operator $D\mathcal{F}$ is not elliptic, since it involves no derivatives of ϕ. We postpone the proof of the proposition for the moment.

There is a more sophisticated implicit function theorem (Nash-Moser) which can handle problems of this kind, but we do not want to use that, so we seek another way of setting the problem up. We use classical surface theory. Given any metric g on S^2 and a symmetric 2-tensor B we get an immersion in $\mathbf{R}^3$ provided that

$$B \wedge B = K(g) \quad D_g B = 0.$$

Here $D_g : \Gamma(s^2) \to \Gamma(T^*)$ is a linear operator which can be written in local co-ordinates as

$$D_g(B_{ij}) = (\det g)\epsilon^{jk} B_{ij;k}.$$

So we can set up the problem in a different way, trying to show that if $K(g) > 0$ there is a solution B to these equations. The *good feature* is that this is an elliptic equation. The *bad feature* is that the associated linearised operator is not surjective, so we appear to have an obstruction to deforming a solution. We want an argument which combines the good features of each approach, but this will involve some more detailed differential geometry.

Let us return to the Proposition. Write $s^2 = s_0^2 \oplus \mathbf{R}$ so $B = B_0 + Hg$. The scalar component of $v \mapsto \mathcal{L}_v g$ is the divergence of vThe space s_0^2 can be identified with quadratic differentials or equally with $(0,1)$ forms with values in the complex tangent bundle. Then the s_0^2 component of the operator $v \mapsto L_v g$ is the $\overline{\partial}$-operator on the tangent bundle. If $\delta g = \sigma + fg$ the equation to be solved becomes

$$\overline{\partial} v + \left(\frac{B_0}{H}\right) \mathrm{div} v = \left(\sigma + f\frac{B_0}{H}\right).$$

So we have to show that $v \mapsto \overline{\partial} v + \left(\frac{B_0}{H}\right) \mathrm{div} v$ is surjective. This is an elliptic operator as one can check. (It clearly is so near to the round metric, since $\overline{\partial}$ is elliptic.)

There is a general simple criterion, due to Gromov, for showing that operators of this kind are surjective. Consider first the $\overline{\partial}$-operator. The adjoint can be identified with another $\overline{\partial}$-operator so the kernel can be interpreted as holomorphic sections of a line bundle. If this line bundle has negative degree then there can be no kernel, since all zeros have positive multiplicty. The point is that the same argument can be applied, with more work, to deformations of the $\overline{\partial}$-operator such as we have at hand. The relevant bundle is the square of the cotangent bundle (quadratic differentials) which has degree -4 so the criterion applies.

By deforming to the round metric we see that the index of this operator is 6. This is the same as the dimension of the Euclidean group, so there can be no more kernel and we get infinitesimal uniqueness. This is the classical rigidity of convex surfaces (and there is an alternative classical proof which uses more differential geometric arguments).

We can apply the same ideas to the other point of view. Amazingly, the component of D_g mapping from s_0^2 is again a $\overline{\partial}$-operator, now acting on quadratic differentials. We write $B = B_0 + fH$ and eliminate f so that our equation becomes a single equation for B_0

$$\overline{\partial} B_0 = d\sqrt{K + |B_0|^2}.$$

Again we check that this is elliptic which is the good feature but now the signs work against us, the linearisation has index -6 and there must be a cokernel of dimension 6 (there is no kernel by the same reasoning as before).

Now we begin the argument. Choose a 6-dimensional subspace E representing the cokernel. For all nearby metrics g' we can use the implicit function theorem to solve our problem "modulo E". That is we can solve

$$B' \wedge B' = K(g') \quad D_{g'} B' = \eta,$$

for some $\eta(g') \in E$, and the solution (among sufficiently small variations) is unique. We want to show that *in fact* $\eta(g') = 0$. Such an identity usually reflects some underlying geometric principle but in this case the underlying cause is rather mysterious, at least for the lecturer.

Return to the infinitesimal problem with a variation δg. We know how to solve the problem in the formulation so we have a (v, ϕ) with $\mathcal{L}_v g + \phi B = \delta g$. We can compute the associated infinitesimal change in B,

$$\delta B = \mathcal{L}_v B + \nabla\nabla\phi + \phi B^2.$$

Now we can substitute this formula into the other formulation and by straight calculation show that

$$\delta(B \wedge B) = \delta(K_g), \delta(D_g(B)) = 0. \quad (**)$$

Of course it is clear on general grounds that this must be true (by an "infinitesimal deformation of surface theory") but the assertion is just a differential geometric identity which can be verified. For example one can derive and apply a formula

$$\delta\Omega_g = dD_g(\delta g),$$

for the curvature 2-form Ω_g; and, if $\delta g = \gamma$,

$$(\delta D_g)\,\beta = \epsilon^{ij}(\gamma_{ia;k} - \gamma_{ik;a})\beta_{aj} + (D_g\gamma)_a\beta_{ai} + D_g(\gamma_{ia}\beta_{aj}) - \gamma_{ia}(D_g\beta)_a,$$

for the variation in the operator D_g with respect to g. The second formula arises from the formula for the Christoffel symbols

$$\Gamma^j_{jk} = \frac{1}{2} g^{ja}(g_{aj,k} + g_{ak,j} - g_{jk,a}).$$

The first formula can be established by reducing to the two cases $\delta g = \mathcal{L}_w g$ and $\delta g = fg$.

The discussion of the identity (**) has been in the case of a genuine solution, i.e. $\eta = 0$. But even if g, B is not a solution we can make the same ansatz. We solve $\delta g = \mathcal{L}_v(g) + \phi B$ and define δB by the formula above. The identitry does not hold but it does modulio terms involving η. That is we get

$$\|\delta(D_g B)\|, \|\delta(B \wedge B - K_g)\| \leq C\|\eta\|,$$

(and it does not matter what norms we use since E is finite dimensional). Now join nearby metrics by a path so we have $\eta(t)$. The inequality above implies that

$$\|\frac{d\eta}{dt}\| \leq C\|\eta\|,$$

and it follows that if $\eta(0)$ vanishes then $\eta(t)$ is always zero. This establishes the openness part of the proof.

The *a priori* estimates use the maximum principle. We need an identity

$$(Hg_{ij} - B_{ij})H_{ij} = \Delta K + |\nabla B_0|^2 - |\nabla H|^2 + K(H^2 - 4K). \quad (***).$$

Assuming this we argue that at a point where H is maximal the LHS above is negative and $|\nabla H| = 0$ so we get a bound on H.

Now once H is bounded the whole second fundamental form is also. From this it is not hard to get estimates of all higher derivatives but we postponce this since we need to develop more standard theory.

The identity (***) can best be understood as a formula for $\Delta(B \wedge B)$. The quadratic form $B \wedge B$ is $|H|^2 - |B_0|^2$ hence

$$(1/2)\Delta K = \nabla^* \nabla B \wedge B + |\nabla H|^2 - |\nabla B_0|^2.$$

Now operate with the term $\nabla^*\nabla B$. In flat space $\mathbf{R}^2$ we have an identity of the shape

$$\nabla^*\nabla\beta = D^* D\beta + LD\beta + \nabla\nabla \mathrm{Tr}\beta,$$

for any symmetric 2-tensor β. Here L is the operator $v \mapsto L_v g$. On a curves surface we get a curvature term, schematically:

$$\nabla^*\nabla\beta = D^* D\beta + LD\beta + \nabla\nabla \mathrm{Tr}\beta + K * \beta.$$

Apply this to $\beta = B$ and use the fact that $D_g B = 0$ to get

$$\nabla^*\nabla B \wedge B = \nabla\nabla H \wedge B + K * B \wedge B,$$

and this gives the formula.

Finally we discuss uniqueness. This is another classical "rigidity" theorem, but we can also argue as follows. Consider first the case of the round metric g_0 with $K = 1$. Then $\Delta K = 0$ and the maximum principle applied to (***) shows that B_0 is identically zero, since $H^2 - 4k \geq 0$ with equality iff $B_0 = 0$. It follows that the embedding is standard. Now suppose there were two embeddings of some other metric g_1. Join this to g_0 by a path g_t. The argument above (local uniqueness under deformation) shows that we could deform both solutions for g_1 back to g_0: a contradiction.

6 Schauder and Calderon-Zygmund inequalities, Sobolev embedding theorems and Singular Integral operators

There is material on this in the older notes, but here we give more proofs.

The Sobolev spaces L^2_k are inadequate for many problems, especially involving nonlinear PDE. Thinking back to our treatment of the general theory what we see is that we want other norms such that on functions of compact support, say, we have $\|\nabla^l f\| \leq C\|Df\|$, where D is an elliptic operator of order l. There are two standard things we can use: Holder norms and Sobolev norms based on L^p for general p, rather than just L^2. The relevant estimates are the *Schauder* and *Calderon-Zygmund* inequalities, respectively.

6.1 The Schauder Theory

Notation Fix $\nu \in (0,1)$. For any function f on $\mathbf{R}^n$ we write

$$[f]_\nu = \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\nu},$$

where sup $= \infty$ is allowed.

Let G be the Green's operator on $\mathbf{R}^n$ with $n > 2$. This is given by the Newton kernel

$$G\rho(x) = \int G(x,y)\rho(y)dy.$$

Let $\mathcal{D}$ be one of the differential operators $\frac{\partial^2}{\partial x_i \partial x_j}$ and define $T = \mathcal{D} \circ G$. This is certainly defined on in the sense that if, say, ρ is a smooth function of compact support then $T\rho \in C^0$ is defined pointwise.

The basic Schauder estimate is

Theorem 2 *There is a constant C depending on ν, n such that for any smooth function ρ of compact support*

$$[T\rho]_\nu \leq C[\rho]_\nu.$$

Preliminaries

It follows from the definition that T commutes with dilation. That is, if we write $f_\lambda(x) = f(\lambda^{-1}x)$ then $(Tf)_\lambda = T(f_\lambda)$. Using this, and linearity, it suffices to prove the estimate

$$|T\rho(x_1) - T\rho(x_2)| \leq C,$$

for points x_1, x_2 with $|x_1 - x_2| = 1$ and for functions $\rho \in C^\infty_c(\mathbf{R}^n)$ with $[\rho]_\nu = 1$.

Fix a smooth function ψ supported in $\{|x| \leq 1\}$ and with $\Delta\psi(x) = 1$ when $|x| \leq \delta$. Let $\chi = \Delta\psi$, so $\chi = 1$ on the δ ball and $T\chi = D\psi$. Let $c_0 = [\chi]_\nu$ and $c_1 = [T\chi]_\nu$.

Now the proof has two main steps

Step 1 Reduction to the case when $\rho(x_1) = \rho(x_2) = 0$.

Clearly we can suppose that $x_1 = 0$. Let $\lambda = \max(\delta^{-1}, |\rho(x_2)|^{1/\nu})$ and define $\sigma_0 = \rho(x_2)\chi_\lambda$. Then $\sigma_0(0) = \sigma_0(x_2) = \rho(x_2)$ but, from the scaling behaviour,

$$[\sigma_0]_\nu \leq c_0 \quad [T\sigma_0]_\nu \leq c_1.$$

Now define

$$\sigma_1 = (\rho(0) - \rho(x_2))\chi.$$

Then $\sigma_1(0) = \rho(0) - \rho(x_2), \sigma_1(x_2) = 0$ while

$$[\sigma_1]_\nu \leq c_0, [T\sigma_1]_\nu \leq c_1,$$

where we have used the fact that $|\rho(x_2) - \rho(0)| \leq [\rho]_\nu = 1$, by our hypotheses.

Now set $\rho' = \rho - \sigma_0 - \sigma_1$. Then ρ' vanishes at 0 and x_2 and we have

$$[\rho']_\nu \leq [\rho]_\nu + 2c_0 = 1 + 2c_0 \quad , \quad [T\rho]_\nu \leq [T\rho']_\nu + 2c_1.$$

It suffices to prove that $[T\rho']_\nu \leq C[\rho']_\nu$ or, simplifying notation, to assume that we are in the case when $\rho(0) = \rho(x_2) = 0$.

Step 2 Integral estimates.

Let $K(x, y) = \mathcal{D}_x G(x, y)$ where the notation indicates that the derivatives are taken with respect to the x-variable. Then formal manipulation would suggest that

$$T\rho(x) = \int K(x, y)\rho(y)dy \quad (*).$$

However the kernel K is not locally integrable. In fact

$$|K(x, y)| = c|x - y|^{-n}$$

for some c. So this formula does not make sense as its stands. But if $\rho \in C_c^\infty(\mathbf{R}^n)$ vanishes at x then clearly the integral in (*) is well-defined.

Proposition 2 *If $\rho \in C_c^\infty(\mathbf{R}^n)$ has $\rho(x) = 0$ then the formula (*) holds.*

This is an exercise.

Now to complete the proof we have to estimate

$$\int (K(0, y) - K(x_2, y))\rho(y)dy,$$

when $\rho(0) = \rho(x_2) = 0$ and $[\rho]_\nu = 1$. First consider the region where $|y| \geq 2$, say. Clearly we have

$$|K(0,y) - K(x_2,y)| \leq c_3 |y|^{-(n+1)},$$

for some c_3. Since $|\rho(y)| \leq |y|^\nu$ we have

$$\int_{|y|\geq 2} |K(0,y) - K(x_2,y)||\rho(y)|dy \leq c_3 \int_2^\infty r^{-(n+1)} r^\nu r^{n-1} dr,$$

which is finite since $\nu < 1$.

Now we certainly have

$$\int_{|y|\leq 2} |K(0,y) - K(x_2,y)|\rho(y)dy \leq I_1 + I_2,$$

where

$$I_1 = \int_{|y|\leq 2} |K(0,y)\rho(y)|dy \ , \ I_2 = \int_{|y-x_2|\leq 3} |K(x_2,y)\rho(y)|dy.$$

Using the fact that $\rho(0) = 0$ we have

$$I_1 \leq c \int_0^2 r^{-n} r^\nu r^{n-1} dr,$$

which is finite since $\nu > 0$. Similarly for I_2, using the fact that $\rho(x_2) = 0$.

Using this it is straightforward to deduce that on a compact manifold we have

$$\|f\|_{C^{k+2,\nu}} \leq C(\|\Delta f\|_{C^{k,\nu}} + \|f\|_{C^0}),$$

and similarly for other elliptic operators.

6.2 Sobolev Inequalities

Recall some basics. On the functions on a measure space X we have L^p norms. Knowing these is related to knowing the *distribution function*. We set $\Omega(t) = \{x \in X : |F(x)| > t\}$ and $\mu_F(t) = \mathrm{Vol}(\Omega(t))$. Then

$$\|F\|_{L^1} = \int_0^\infty \mu_F(t) dt,$$

and

$$\|F\|_{L^p}^p = \int_0^\infty \mu_F(t^{1/p} dt = p \int_0^\infty \mu_F(t) t^{p-1} dt.$$

Now consider functions on $\mathbf{R}^n$. The basic facts are that if $F \in C_c^\infty(\mathbf{R}^n)$ then

$$\|f\|_{L^q} \leq C_{p,q} \|\nabla F\|_{L^p},$$

where $1 - n/p = -n/q$. If $1 - n/p > 0$ and if F is supported in the unit ball then the result holds $q = \infty$.

The proof in the case $q = \infty$ is easy by integrating ∇F along rays, using the fact that the function $r^{-(n-1)}$ is locally in L^s for $s < n/n - 1$ and $n/n - 1$ is the conjugate exponent to n (in the sense of Holder's inequality).

For the remainder we can easily reduce to the case $F \geq 0, p = 1, q = n/n-1$, by considering powers of $|F|$. So we want to show

$$\int F^{n/n-1} \leq C \left(\int |\nabla F| \right)^{n/n-1}.$$

This is equivalent to the isoperimetric inequality $\mathrm{Vol}(\Omega)^{1/n} \leq C\mathrm{Vol}(\partial\Omega)^{1/n-1}$ for domains $\Omega \subset \mathbf{R}^n$. The point is that if χ_Ω is the characteristic function of Ω one has

$$\|\nabla \chi_\Omega\|_{L^1} = \mathrm{Vol}(\partial\Omega),$$

where the left hand side is defined by a smoothing procedure. Then if

$$\Omega(t) = \{x \in \mathbf{R}^n : F \geq t\},$$

and $F_t = \chi_{\Omega(t)}$ we have

$$F = \int_0^\infty F_t dt,$$

so

$$\|F\|_{L^{n/n-1}} \leq \int_0^\infty \|F_t\|_{L^{n/n-1}} \leq C \int_0^\infty \mathrm{Vol}(\partial\Omega(t))dt.$$

The *co-area* formula is

$$\int_0^\infty \mathrm{Vol}(\partial\Omega_t)dt = \int_{\mathbf{R}^n} |\nabla F|.$$

There are many proofs of the isoperimetric/Sobolev inequalities.

6.3 Singular integral operators

We encountered above the operator $T = \mathcal{D} \circ G$ over $\mathbf{R}^n$ given formally by convolution by K where $|K(x)| = c|x|^{-n}$. More precisely, K is homogeneous of degree $-n$:

$$K(\lambda x) = \lambda^{-n} K(x).$$

Suppose that $\mathcal{D}$ is a derivative $\partial_i \partial_j$ with $i \neq j$. Then it is easy to see that K has the form $K(x) = c x_i x_j |x|^{-(n+2)}$. In particular the average value of K over the unit sphere is zero. Then we can define T as a singular integral operator

$$T(f)(x) = \lim_{\epsilon \to 0} \int_{|y| > epsilon} K(y) f(x-y) dy.$$

Similarly if $\mathcal{D}$ is a differential operator $\sum a_{ij}\partial_i\partial_j$ with $\sum a_{ii} = 0$. Since $\Delta G = 1$ we get a formula for $\mathcal{D} \circ G$ for all second order $\mathcal{D}$.

Such operators occur in other ways: the basic example is the Hilbert transform on $\mathbf{R}$

$$Hf(x) = \pi^{-1} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

Taking Fourier transforms they go over to multiplication operators by $\hat{K}$ which is homogeneous of degree 0. So T is bounded on L^2.

The central result (Calderon-Zygmund) is:

Theorem 3 *For each p with $1 < p < \infty$, T defines a bounded operator $T : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$.*

It is possible to prove this in many cases by a trick to reduce to the the Hilbert transform on $\mathbf{R}$–separating into radial and angular variables. This applies initially to the case when the kernel K is an odd function. But we can take composites to extend further. However the n-dimensional proof below is more fundamental and the ideas apply to other problems in PDE and analysis.

The strategy of this proof of Proposition 8 is to interpolate, using what we know about L^2. The first obstacle is that the result is not even true for the other extremes $p = 1, \infty$. For example the Hilbert transform takes a function with a "jump" discontinuity tone one with a logarithmic singularity. It is even more obvious that a singular integral operator T of the kind considered on $\mathbf{R}^n$ cannot map L^1 to L^1: for any function f, say of compact support, with

$$\int_{\mathbf{R}^n} f \neq 0$$

$|Tf|$ decays like $|x|^{-n}$ at infinity and so is not integrable. The way around this is to substitute "weak-type" bounds when $p = 1$. A map T is said to be of weak type $(1, 1)$ if

$$|\{x : |Tf(x)| > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

A simple form of the *Marcinkiewicz interpolation theorem* is

Theorem 4 *If T is of weak type (1,1) and bounded as a map from L^2 to L^2 then T defines a bounded map from L^p to L^p for $1 < p \leq 2$*

The proof involves only simple measure theory concepts and gives explicit bounds on the L^p operator norms.

Now consider an operator T defined by a kernel

$$(Tf)(x) = \int_{\mathbf{R}^n} k(y) f(x-y) dy.$$

We may even suppose k is smooth and of compact support initially if we like. For example if we take

$$k_\delta(x) = \pi^{-1}\frac{x}{x^2+\delta^2},$$

we can approximate the Hilbert transform by taking $\delta \to 0$. The crucial thing is to get estimates which are independent of δ.

Theorem 5 *Suppose the operator T above with kernel k satisfies*

- $\|Tf\|_{L^2} \leq c_1\|f\|_{L^2}$
- $|(\nabla k)(x)| \leq \frac{c_2}{|x-y|^{n+1}}$;

then

$$|\{x : |Tf(x)| < \alpha\}| \leq \frac{C}{\alpha}\|f\|_{L^1},$$

where the constant C depends only on c_1, c_2.

Corollary 1 *Such an operator defines a bounded map on L^p for all $p > 1$. Moreover the L^p operator norm can bounded above by an explicit expression involving only c_1, c_2.*

To obtain the the corollary one uses the interpolation theorem to handle the range $1 < p \leq 2$ and a duality argument to handle the range $2 \leq p < \infty$.

From this Corollary one easily deduces the main Calderon-Zygmund result, by approximating the singular kernel by smooth ones. (In fact the possible presence of a singularity on the diagonal is irrelevant in the proof.) o

As motivation let us prove our result under an extra hypothesis. Suppose that for each $\alpha > 0$ the set $\Omega_\alpha = \{|f| > \alpha\}$ is contained in a cube of volume comparable to that of Ω_α (more generally we could assume that Ω_α is covered by a fixed number of such cubes). the measure of $\{|Tf| > \alpha\}$ is bounded by the measure of the doubled cube plus that of the part of the set outside the doubled cube, and the latter is $O(\alpha^{-1})\|f\|_{L^1}$ since k decays as $|x|^{-n}$.

What we really need is something a little stronger.

Lemma 1 *Let $\Box$ be a cube in $\mathbf{R}^n$ of side-length r, and let $\Box^*$ be the cube with the same centre and side length $2r$. Then if β is a function supported in $\Box$ with $\int_\Box \beta = 0$ we have*

$$\|T\beta\|_{L^1(\mathbf{R}^n\setminus\Box^*)} \leq c\|\beta|_{L^1},$$

where the constant C depends only on c_2 (and not on the scale r).

This is quite easy.

Now the proof uses the famous Calderon-Zygmund cube decomposition of a function. Suppose initially that f is continuous and has compact support. Given α we find a large cube containing the support of f such that the average value of $|f|$ on the cube is less than α. Then we subdivide this cube into 2^n smaller ones. The rule is that we stop subdividing any cube once the average value of $|f|$ exceeds α. In this way we get a collection of disjoint cubes Q_i such that the average value of f on each Q_i lies between α and $2^n\alpha$ and $|f| \leq \alpha$ outside $\bigcup Q_i$. Now write $f = g + b$ where $b = \sum b_i$, each b_i is supported in Q_i and on Q_i,

$$b_i = f - \frac{1}{|Q_i|}\int_{Q_i} f.$$

The proof now follows by applying the Lemma to each b_i, the L^2 hypothesis to g, and piecing together what we know. The crucial thing is that the average value of $|f|$ on any cube Q_i is bounded above and below by multiples of α.

6.4 Application to the measureable Riemann mapping theorem

Let μ be a function on $\mathbf{C}$ with $|\mu| \leq k \leq 1$ and consider the operator $\overline{\partial}_\mu = \overline{\partial} + \mu\partial$. If μ is smooth we can think of it as defining an almost complex structure J_μ on $\mathbf{C}$ and $\overline{\partial}_\mu$ is the $\overline{\partial}$-operator of this structure. A holomorphic function, in this structure is a solution of $\overline{\partial}_\mu f = 0$. The MRMT says that this is still the case if $\mu \in L^\infty$ with $|\mu| \leq k < 1$. Moreover the structure J_μ is equivalent to the standard one: there is a solution f of the equation which gives a homeomorphism from $\mathbf{C}$ to $\mathbf{C}$.

To prove this we first assume that μ has compact support. Write $f = z + \phi$ so the equation to be solved is $\overline{\partial}_\mu \phi = -\mu$. Let S be the inverse of $\overline{\partial}$ which is given by the Cauchy kernel z^{-1} and $T = \partial \circ S$. We seek a solution $\phi = S\rho$ so the equation is

$$\rho + \mu T\rho = -\mu.$$

We have $T : L^p \to L^p$. When $p = 2$ the identity

$$\int |\partial g|^2 - |\overline{\partial} g|^2 = 0,$$

for compactly supported smooth g shows that $T : L^2 \to L^2$ is an isometry. It follows from the proof of the CZ theorem that when p is close to 2 the operator norm is close to 1. Fix $p > 2$ such that $\|T\rho\|_{L^p} \leq (1-\delta)k^{-1}\|\rho\|_{L^p}$ for all ρ and some fixed $\delta > 0$. Then μT has operator norm less than 1 and we can solve the equation by the usual Neumann series. We get a solution ϕ with $\nabla\phi$ in L^p.

Now $L^p_1 \to C^\nu$ with $\nu = 1 - 2/p$. (This is where it is crucial to take $p > 2$.) So we get an estimate

$$|f(x) - f(y)| \leq |x - y| + C\|\mu\|_{L^p}|x - y|^\nu,$$

where C depends only on k.

Now suppose for the moment that μ is smooth. Then we know that f is a diffeomorphism and we can consider the inverse diffeomorphism $g = f^{-1}$. This satisfies an equation $\overline{\partial}_{\mu'} g = 0$ where $|\mu'| < k$. So we get the same estimates for g and hence

$$|f(x) - f(y)| \geq C \min(|x - y|, |x - y|^{1/\nu}).$$

Now remove the smoothness hypothesis on μ. Now we cannot approximate a general μ in L^∞ norm (the space is not separable). But we can choose smooth μ_i with $\mu_i \to \mu$ in L^N for all N. Then it is easy to show that the solutions ϕ_i converge to ϕ in L^p_1 and hence in $C^{,\nu}$. Since the above estimates for f_i are uniform they hold also for f and we see that f is a homeomorphism from $\mathbf{C}$ to $\mathbf{C}$.

Finally there is a trick to remove the assumption that μ has compact support, but we will not go into this.

An application of this result is the Bers simultaneous uniformisation theorem. Let Σ, Σ' be compact Riemann surfaces of genus ≥ 2 and $h : \Sigma \to \Sigma'$ a diffeomorphism. Then there is a "hyperbolic cobordism" between Σ, Σ' inducing h on homotopy. This was used by Thurston to prove the existence of hyperbolic structures on certain compact 3-manifolds. (To be more accurate we should distingush between a Riemann surface and its complex conjugate.)

To prove the result we pull-back the complex structure on Σ' using h so we have two complex structures on Σ. We know that Σ is a quotient of the disc by a group Γ. Take this disc to be $D_\infty = \{|z| > 1\}$ in the Riemann sphere. Now Γ acts also on the unit disc D_0 and we use the other complex structure to define a Γ-invariant μ supported in D_0. The MRMT says that the Riemann sphere with this structure μ is equivalent to the standard sphere. So we get $\Gamma \subset PSL(2, \mathbf{C})$ and S^2 is $D^*_0 \cup D^*_\infty \cup C$ where C is a topological Γ-invariant circle. The Riemann surfaces Σ, Σ' are the quotients of D^*_0, D^*_∞ by Γ and the hyperbolic 3-manifold is the quotient of the 3-ball. (One has to show that Γ acts freely on the ball.)

Another kind of application is to nonlinear PDE. Suppose that f satisfies an equation $\overline{\partial} f + \mu(f)\partial f = 0$, where $|\mu(f)| \leq k < 1$ if $|f| \leq C$. For example we might have

$$\mu(f) = \frac{|f|}{\sqrt{1 + |f|^2}}.$$

Suppose we know a solution f has $|f| \leq C$. Let β be a cut-off function and $g = \beta f$. Then

$$\overline{\partial} g + \mu(f)\partial g$$

is bounded in L^∞. By arguing as above we get an L^p bound on ∇g, hence ∇f, for close to 1. Thus we get a Holder bound on f. A similar argument can be applied to the equations $D_g B = 0, B \wedge B = K$ arising in the isometric embedding problem.

7 Tubular ends and gluing theorems

7.1 Linear theory on manifolds with tubular ends

We begin with a cylinder $M \times \mathbf{R}$ and consider a translation invariant elliptic operator D. Specifically we consider either $D_1 = \frac{\partial}{\partial t} + A$ where A is first order self-adjoint elliptic on M or $D_2 = -\frac{\partial^2}{\partial t^2} + \Delta_M + c$ where c is constant. In the first case a model example is $M = S^1$ and $A = i\frac{\partial}{\partial\theta} + c$.

In the case of D_1 the basic fact is that $D_1 : L^p_k \to L^p_{k-1}$ is an isomorphism provided that A has trivial kernel. To see this consider an orthonormal basis of eigenfunctions ϕ_λ for A. Given $\rho = \sum b_\lambda(t)\phi_\lambda$ we solve $D_1 f = \rho$ by reducing to a system of ODE's. Thus $f = \sum a_\lambda(t)\phi_\lambda$ where

$$\frac{d}{dt} a_\lambda + \lambda a_\lambda = b_\lambda.$$

Provided $\lambda \neq 0$ we get a solution with $\|a_\lambda\|_{L^2} \leq C\|b_\lambda\|_{L^2}$ for a fixed C independent of λ. It is straightforward to develop the theory from there on. Note that in the bad case when we do have a zero eigenvalue the operator is not Fredholm. This can be seen by considering

$$\frac{d}{dt} : L^2_1(\mathbf{R}) \to L^2(\mathbf{R}).$$

The Sobolev inclusions between the spaces L^p_k on the cylinder behave well. We have $L^p_k \subset L^q_l$ if $p < q, k > l, k - n/p \geq l - n/q$. The reason is that if $p < q$ the L^p norm is "stronger" with regard to decay properties. To be more precise, consider $n = 4$ and the embedding $L^2_1 \subset L^4$. For any compact 4-manifold-with-boundary Ω one has

$$\left(\int_\Omega f^4\right)^{1/2} \leq C \int_\Omega |\nabla f|^2 + |f|^2.$$

This can be proved by considering the "double" of Ω. Now take $\Omega = M \times [0,1]$ and consider the decomposition of $M \times \mathbf{R}$ into "bands" Ω_i. For f on $M \times \mathbf{R}$ write

$$a_i = \int_{\Omega_i} |\nabla f|^2 + |f|^2,$$

Then

$$\int_{M\times\mathbf{R}} |f|^4 \leq C^2 \sum a_i^2 \leq C^2 \left(\sum a_i\right)^2 = C^2 \|f\|^4_{L^2_1}.$$

There is an even simpler argument to show that $L^p_k \subset C^0$ if $k - n/p > 0$.

We can consider weighted Sobolev spaces. This is exactly the same as conjugating our operator by $e^{\alpha t}$ which changes A to $A + \alpha$. Since the spectrum is discrete, we can always choose α to arrive in the "good" case.

Similarly for D_2 we have a straightforward theory if $c > 0$.

Now we consider a noncompact manifold X with a cylindrical end modelled on $M \times \mathbf{R}^+$ and an elliptic operator which has the given form over the end. Provided the spectral condition on the cross-section is satisfied we get a Fredholm theory in L^p_k spaces–the cokernel is the L^2-complement of the kernel of the formal adjoint etc.

Suppose we have two such manifolds X_1, X_2 with the "same" end. For $T > 0$ we can glue them to get a compact manifold X_T with a neck of length T. A basic fact is that if the operator D is surjective on each side X_1, X_2 then on X_T, for $T >> 0$, the operator D has a uniformly bounded right inverse. That is, on X_T we can solve $Df = \rho$ with $\|f \leq C\|\rho\|$ with C independent of $T >> 0$.

This allows us to make certain "gluing constructions" for nonlinear problems. A well known example is that of holomorphic curves. Here there are some complications since A has a kernel given by the constants so one has to use weighted spaces, or some other device.

7.2 Calabi-Yau metrics

We illustrate these ideas by constructing Calabi-Yau (Ricci flat, Kahler) metrics on certain K3 surfaces.

The differential geometric background if that if Z is a complex m-manifold with a nowhere-vanishing holomorphic m-form χ and if Ω is a Kahler metric on Z with $\Omega^m = \chi \wedge \overline{\chi}$ then Ω is Ricci flat. If Ω_0 is some Kahler form we seek Ω in the form $\Omega_0 + \mathcal{D}\phi$ where $\mathcal{D} = 2i\overline{\partial}\partial$. The resulting PDE for ϕ is, in local co-ordinates,

$$\det\left(G_0 + \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j}\right) = |\chi|^2,$$

where G_0 is the matrix of Ω_0.

There are two building blocks in the construction

- $X = T^4 / \pm 1$ is an orbifold with 16 singular points. We consider the flat metric ω_X on X
- Y is the resolution of the singularity $\mathbf{C}^2 / \pm 1$. All we need to know about Y is that outside a compact set it is identified with $\mathbf{C}^2 / \pm 1$. There is an asymptotically flat, Ricci flat, *Eguchi-Hanson metric* on Y.

Really we will need 16 copies of Y, one for each singular point, but to simplify language we will speak as though there is just one. .

First we review the Eguchi-Hanson metric. Work on $\mathbf{C}^2$ minus the origin and consider a Kahler metric $\mathcal{D}(\psi)$ where $\psi = F(\rho)$ with $\rho = |z_1|^2 + |z_2|^2$. We have a holomorphic form $dz_1 dz_2$. Our CY equation becomes $(F')^2 - \rho F' F'' = 1$ and a solution (for the derivative) is

$$F' = \sqrt{1 + \rho^{-2}},$$

from which we can integrate to find F. All we really need to know is that $F = \rho + G(\rho)$ where $G(\rho) = O(\rho^{-1})$ as $\rho \to \infty$. When we take the quotient by ± 1 and resolve the singularity to get the complex manifold Y we find that this gives a smooth Ricci-flat metric Ω_Y. But the internal structure of Y is irrelevant for our purposes. We think of Y as having a fixed identification with a subset of $\mathbf{C}^2/\pm 1$ outside a compact set, and consider a function ρ_Y on Y which agrees with the given one outside this set. Write $r_Y = \sqrt{\rho_Y}$.

Given $R >> 0$ we construct another Kahler metric on Y in the form

$$\omega_{R,Y} = \omega_Y + cD(\beta_R G(\rho_Y),$$

where β_R is a cut-off function of "scale" $R^{1/2}$. Thus $\nabla \beta_R = O(R^{-1/2})$ etc. Now $\Omega_{R,W}$ agrees with the flat metric on $r_Y > 2R^{1/2}$ say and

$$\Omega_{R,Y}^2 = (1+\eta)^{-1}\Omega_Y^2,$$

where $|\eta| \leq R^{-2}$ and η is supported in the annulus $\sqrt{R} < r_Y < 2\sqrt{R}$.

Similarly on X we consider a positive function ρ_X equal to $|z_1|^2 + |z_2|^2$ near the singular point, and write $r_X = \sqrt{\rho_X}$.

We want to use the tubular ends machinery and for this we use *conformal* changes. Suppose in general that Z, Ω is a Kahler surface and that V is a fixed positive function on Z. Then $\Theta = V^{-1}\Omega$ is a Hermitian metric. Define

$$Qf = V^{1/2}\mathcal{D}V^{-1/2}f,$$

and $\Box f = (Q(f) \wedge \Theta)/\Theta^2$. What is important to note is that Q is unchanged if we multiply V by a constant. A short calculation shows that

$$\Box f = \Delta_\Theta f + Wf$$

where $W = V^{3/2}\Delta_{\text{Kahler}} V^{-1/2}$ and Δ_Θ, mean the Laplacians with respect to the two metrics. In the case $Z = \mathbf{C}^2 \setminus \{0\}$ and $V = \rho$ the conformal metric Θ is the cylinder $S^3 \times \mathbf{R}$ and the function W is 1.

Now on Y we take $V_Y = \rho_Y$ and on $X \setminus$ singularities we take $V_X = \rho_X$. Rescaling the metric $\omega_{R,Y}$ by V_Y^{-1} we get a metric which is a cylinder on $\rho_Y > 4R$. This is isometric to the rescaling $V_X^{-1}\omega_X$. We glue the cylindrical ends so that the sphere $r_Y = 2\sqrt{R}$ in Y maps to the sphere $r_X = 2/\sqrt{R}$ in X. The resulting manifold is Z. It has a nowhere vanishing holomorphic form. We have a hermitian metric Θ on Z containing a long cylinder. There is also a Kahler metric $\omega = V\Theta$ where $V = V_X$ on the "X part" and $V = V_Y R^{-2}$ on the "Y part"

Now we return to the PDE. We seek ϕ on Z such that

$$(\omega + \mathcal{D}\phi)^2 = (1+\eta)\omega^2.$$

Write $\phi = V^{1/2} f$ so the equation is

$$(\Theta + V^{-3/2}Q(f))^2 = (1+\eta)\Theta^2,$$

which is

$$\Box f + V^{-3/2}Q(f)^2 = \eta V^{3/2}.$$

The problem is that the coefficient $V^{-3/2}$ of the quadratic term can be very large. Its maximal value M is $O(R^3)$. Write $f = M^{-1}g$, then the equation becomes

$$\Box g + vQg^2 = MV^{3/2}\eta,$$

where $v = V^{-3/2}M^{-1} \leq 1$. Now the problem has been transferred to the right hand side. On the support of η, $V^{3/2}$ is $O(R^{-3/2})$ and $|\eta|$ is $O(R^{-2})$. So the right hand side is $O(R^{-1/2})$ (and supported on a band in the cylinder of fixed width, so the derivatives have the same order of magnitude, working in the cylindrical metric).

Now the nonlinear problem is reduced to linear theory. On the cylinder $\Box = \Delta + 1$ so we have a Fredholm theory in any L^p_k. Suppose $\Box f = 0$ on Y and f tends to zero at infinity. Then $\Delta_{\text{Kahler}}(r_Y^{-1} f) = 0$ and $r_Y^{-1} f$ tends to zero at infinity, so by the maximum principle $f = 0$. Similarly if $\Box f = 0$ on X then $\phi = r_X^{-1} f$ is harmonic with respect to the flat metric and $\phi = o(r^{-1})$. Since the fundamental solution of the Laplacian in four dimensions is $O(r^{-2})$ the only possibility is that ϕ is constant.

Thus we do have a small complication, since $\Box$ has a one dimensional cokernel on the X part. But we should expect this since we *cannot* solve our equation for all η; there is a constraint that the volume of the manifold is fixed. To handle this we introduce an auxiliary function σ of non-zero integral, supported inside X and solve the equation

$$\Box g + vQg^2 = \eta MV^{3/2} + \lambda\sigma$$

for a pair (g, λ) where $\lambda \in \mathbf{R}$. *A posteriori* we find that $\lambda = 0$, by considering the volume.

Questions

For those who wish to be assessed on the course: please send solutions to a selection of the problems to me at s.donaldson@imperial.ac.uk by April 19th. Solutions written in tex are preferred, but any other form is acceptable. As a guide, reasonable attempts at about 4-5 problems should get a good mark.

1. Find a formula for the Green's Function on the round sphere S^n.

2. Find a family of functions f_ρ on $\mathbf{R}^n$, for $\rho < 1$, with the following properties

- $f_\rho(x) = 1$ if $|x| \leq \rho$;
- $f_\rho(x) = 0$ if $|x| > 1$;
- $\|\nabla f_\rho\|_{L^n} \to 0$ as $\rho \to 0$.

3. Let E be a vector bundle over a compact Riemannian manifold M with a metric on the fibres. A *covariant derivative* on E is a map ∇ from sections of E to sections of $E \otimes T^*M$ such that

$$\nabla(fs) = f\nabla s + df \otimes s.$$

Give a characterisation of covariant derivatives in terms of their symbol.

A covariant derivative is *compatible with the fibre metric* if for any two sections s_1, s_2

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle.$$

Show that in this case the "Kato inequality" holds:

$$|\nabla|s|\ | \leq |\nabla s|.$$

(You may restrict attention here to points where s does not vanish, although in fact the inequality holds everywhere, with a suitable interpretation.) Now let F be another bundle over M and $\sigma : T^*M \otimes E \to F$ be a bundle map such that the composite $D = \sigma \circ \nabla$ is an elliptic operator. Show that there is a constant $k < 1$ such that for all sections s with $Ds = 0$ we have

$$|\nabla|s|\ | \leq k|\nabla s|,$$

(again, at points where s does not vanish).

4. Let M be an compact oriented Riemannian manifold. Show that the operator

$$d^* \oplus d : \Omega^{\text{even}} \to \Omega^{\text{odd}}$$

is elliptic. (The notation means the direct sum of all even/odd forms respectively.) Identify the kernel and cokernel of $d + d^*$ in terms of harmonic forms.

5. Let F be a smooth function on a compact Riemannian 2-manifold M and let I be the functional

$$I(u) = \int_M |\nabla u|^2 + (u - F)^4 d\mu.$$

Show that there is a function u which minimises I and which satisfies the equation $\Delta u + 4(u - F)^3 = 0$. (You could use the continuity method, or a direct calculus of variations approach.)

6. Suppose K is a smooth function on $\mathbf{R}^2 \setminus \{0\}$ with $K(\lambda x) = \lambda^{-2} K(x)$ for $\lambda > 0$ and $K(-x) = -K(x)$ and let T be the singular integral operator defined by K. Assuming known that the Hilbert transform is bounded as an operator $L^p(\mathbf{R}) \to L^p(\mathbf{R})$, show that $T : L^p(\mathbf{R}^2) \to L^p(\mathbf{R}^2)$ is bounded.

7. Prove "Lemma 1" on page 17 of these notes (used in the proof of the Calderon-Zygmund Theorem).

8. Let K be a smooth function on $\mathbf{R}^2$ and with $K \geq \epsilon > 0$. Let B be a function with values in 2×2 symmetric matrices which satisfies the equations $\det B = K$ and

$$\frac{\partial B_{ij}}{\partial x_k} = \frac{\partial B_{ik}}{\partial x_j}.$$

Suppose that $|B| = \sqrt{\sum B_{ij}^2} \leq C$. Obtain an *a priori* Holder bound on B of the form

$$|B(x) - B(y)| \leq c|x - y|^\alpha$$

for all x, y with $|x - y| \leq 1$ (say) where $\alpha > 0$ and α, c depend only on C, ϵ.