Lecture Notes for TCC Course “Geometric Analysis”

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This is a copy of the syllabus, advertising the course:

The main theme of the course will be proving the existence of solutions to partial differential equations over manifolds.

1. Review of finite-dimensional equations.
2. Elementary discussion of some model linear PDE.
3. General theory of elliptic differential operators over compact manifolds. Some connections with topology and differential geometry.
4. Sobolev inequalities.
5. The Implicit Function Theorem in Banach Spaces and applications to non-linear PDE .
6. Techniques of nonlinear PDE (continuity method, a priori estimates).with outline of some examples from differential geometry.

Some Books:

T. Aubin ”Some nonlinear equations in Riemannian geometry” Springer 1998 (An earlier version of this is ”Nonlinear analysis on manifolds: Monge-Ampere equations” Springer 1982.)

D. Gilbarg and N. S. Trudinger ”Elliptic partial differential equations of second order” Springer 2001

J. Jost ”Geometric analysis” Springer 2007

F. Warner ”Foundations of differentiable manifolds and Lie groups” Springer 1983

Contents

1 Introduction

The theme of the course is the description of solutions to partial differential equations defined over manifolds. The equations we will be mainly concerned with are those coming from differential geometry. By "describe" solutions we may mean for example: prove *existence theorems* or perhaps "count" the solutions. Later in the course we will talk about specific examples in more detail. Some prototypes we mention now, as motivation, are:

- A surface $S \subset \mathbf{R}^3$ has an induced Riemannian metric (first fundamental form). More generally we can consider an abstract 2-dimensional manifold with metric (M^2, g). The isometric embedding problem asks whether (M, g) can be realised as a surface in $\mathbf{R}^3$. In general the answer is no but if g has positive curvature then there always is such a representation (Weyl, Nirenberg). The statement asserts the existence of a solution to a PDE for a map $f : \Sigma \to \mathbf{R}^3$.

- Let X be a generic cubic surface in $\mathbf{C}^3$. Then a famous classical fact is that there are 27 lines contained in X. Similarly one can study lines, in a quintic 3-fold in $\mathbf{C}^4$. These are important in string theory and "Mirror symmetry". An important modern extension of these numerical geometry ideas involves extending the definition of "lines". In this extension the relevant objects are holomorphic curves which are solutions of a partial differential equation for a map $S^2 \to \overline{X}$. One needs a theory for "counting" solutions, with a count which is invariant under perturbations of the problem.

- The Plateau problem for minimal surfaces in $\mathbf{R}^3$.

- The eigenvalue equation $\Delta\phi = \lambda\phi$ for the Laplace operator on a Riemannian manifold.

- Problems involving curvature (Ricci curvature, scalar curvature...) of general Riemannian manifolds. These become PDE for the Riemannian metric.

In thinking about these kind of PDE problems it is useful to keep in mind things one knows in finite dimensions. Thus we might consider a map $f : \mathbf{R}^n \to \mathbf{R}^n$ and for given $y \in \mathbf{R}^n$ ask to "solve" the equation $f(x) = y$ for x (in the sense of proving existence, say). For example the "fundamental theorem of algebra" can be thought of in this way: any complex polynomial equation $p(z) = w$ has a solution z. Consider various approaches.

- **Variational**: seek to minimize the function $|f(z) - w|$ of z. Show that the minumum is attained and then, by a local analysis round the minimising point, show that the minimum must be 0.

- **Continuity method**: Obviously a solution exists for *some* $w = w_0$ and any other $w = w_1$ can be joined to w_0 by a path w_t. Now consider the 1-parameter family of equations $p(z_t) = w_t$ and seek to follow the solution from $t = 0$ to 1.

- **Topological**: Consider the winding number about w of $p(z)$ as z moves round a large circle. Show that this is non-zero and that this implies there must be a solution.

A common feature of these approaches is that, because $\mathbf{C}$ is not *compact*, one needs to consider the behaviour of $p(z)$ for large z.

Of course, in addition to an existence theorem we also have a "counting theorem": the number of solutions, counted with multiplicity, is equal to the degree of p.

To attack these finite-dimensional problems we have the tools of calculus. Near a point a function is approximated by its derivative, a linear map. Similarly the theory of linear differential operators is a crucial tool in nonlinear PDE. The first part of this course will discuss linear *elliptic* operators, mainly over compact manifolds.

Here, in the lectures, we recall the definition of a smooth manifold and a Riemannian metric. As the course progresses we will probably want to use other differential geometric objects (differential forms, bundles, spinors...), and we may say something about these, depending on time available.

2 The Poisson equation on a manifold

We want to consider the model equation $\Delta\phi = \rho$ with ρ given and ϕ to be found. Here Δ is the Laplace operator. We use the sign convention that in $\mathbf{R}^n$ this is

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2}.$$

So if one of f, g has compact support then integration by parts gives

$$\langle \Delta f, g\rangle = \langle \nabla f, \nabla g\rangle.$$

(Notation: The left hand side denotes the usual L^2 inner product of functions. The right hand side is

$$\langle \nabla f, \nabla g\rangle = \int_{\mathbf{R}^n} \nabla f . \nabla g,$$

where . is the dot product of vectors which we might also write as $(\nabla f, \nabla g)$.)

In the setting of a Riemannian manifold we have a Laplace operator Δ given by the local formula

$$\Delta = -\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x_i}\left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j}\right),$$

and more intrinsically by the analogue of the identity above. Notice that on a compact manifold M

$$\int_M \Delta\phi = 0,$$

so a necessary condition to solve our problem is that $\int_M \rho = 0$. The general result is

Theorem 1 *On a compact Riemannian manifold M if ρ is a smooth function of integral 0 then there is a smooth solution of the equation $\Delta\phi = \rho$, unique up to the addition of a constant.*

It will be a while before we can prove this in general.

2.1 Flat space

We begin with the case of the flat torus $T^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$. We use Fourier Series

$$\sum a_\nu e^{i\nu.x}$$

where ν runs over $\mathbf{Z}^n$. The Laplace operator is locally the usual operator on $\mathbf{R}^n$ and

$$\Delta e^{i\nu.x} = |\nu|^2 e^{i\nu.x}.$$

Ignoring analytical niceties, we solve our problem this way (which is probably familiar). We take the Fourier co-efficients ρ_ν of ρ so

$$\rho = \sum \rho_\nu e^{i\nu.x}.$$

Taking the operator inside the sum the equation becomes the condition for the Fourier coefficients ϕ_n of ϕ:

$$|\nu|^2 \phi_\nu = \rho_\nu.$$

The Fourier coefficient ρ_0 is proportional to the integral of ρ so we assume this vanishes. Then we can define $\phi_\nu = |\nu|^{-2}\rho_\nu$ for $\nu \neq 0$ and $\phi_0 = 0$, say to get a formal solution $\sum \phi_\nu e^{i\nu.x}$.

To make all this watertight we need to know some facts about Fourier Series.

- Given a smooth function f the Fourier co-efficients $f_\nu = (2\pi)^{-n} \int f e^{-i\nu.x}$ are rapidly decaying in ν. For any p,

  $$|f_\nu| \leq C_p |\nu|^{-p}.$$

  The sum $\sum f_\nu e^{i\nu.x}$ converges uniformly to f and similarly after differentiating any number of times in x.

- Conversely any rapidly decaying series a_ν appears as the Fourier series of a smooth function.

Thus we have a 1-1 correspondence between rapidly decaying series and smooth functions. Given this it is clear that our formal solution is valid. Notice that the solution ϕ has even more rapidly decaying Fourier series than ρ— the functionϕ is "smoother" than ρ. (This does not make precise sense at the moment but will do when we come to Sobolev spaces.)

Another case which may be familiar is for functions on $\mathbf{R}^n$. (Of course this is not compact.) For suitable ρ we can give an explicit solution using the Newton kernel. For $n \geq 3$ we have $K(x) = c|x|^{2-n}$ and when $n = 2$, $K(x) = -c \log |x|$. The solution is by the integral formula (convolution)

$$\phi(x) = \int K(x-y)\rho(y) dy,$$

which can also be written as

$$\int K(y)\rho(x-y).$$

Two basic facts are that for f smooth and of compact support (say) we have

$$K*(\Delta f)=f,$$

and

$$\Delta(K*f)=f$$

To prove the first assertion we may, by translation invariance, calculate at origin. Then the proof is a standard integration by parts over the complment of a small ball, followed by a limit as the radius of the ball shrinks to zero.

For the second part we examine the formula

$$(K*f)(x)=\int K(y)f(x-y)d\mu_y.$$

When we take the Laplacian with respect to x there is no problem in moving the differential operator inside the integral, since f is smooth and x does not appear inside the argument of K. Thus

$$\Delta(K*f)=\int K(y)\Delta_x f(x-y)d\mu_y,$$

where the notation means that we take the Laplacian with respect to x. But this is just the same as $K*\Delta f$, which is equal to f by the first part. (The point is that we can differentiate $f(x-y)$ with respect to either x or y.)

For any compact Riemannian manifold there is a "Green's function" $G(x,y)$ which solves the Poisson equation

$$\phi(x)=\int_M G(x,y)\rho(y)dy,$$

(although of course we haven't proved this yet) and has a singularity on the diagonal of the same nature as the Euclidean case. In some cases, such as the round 2-sphere one can find this explicitly.

A useful language for taking about this involves "distributions" (or generalised functions). We will not make much use of this so we just outline the definitions. A distribution is an element of the dual of the space of rapidly decaying smooth functions. That is if D is a distribution and f such a function we have a number $D(f)$. The delta function at 0 is the Distribution

$$\delta_0(f)=f(0).$$

Any locally integrable function S which grows slower than a power defines a distribution D_S:

$$D_S(f) = \int fS.$$

Differentiation of a distribution is motivated by integration by parts:

$$\left(\frac{\partial D}{\partial x_i}\right)(f) = -D(\frac{\partial f}{\partial x_i}).$$

The Newton Kernel is the solution of the equation $\Delta K = \delta_0$.

On the torus T^n the distributions correspond to formal Fourier series $\sum a_\nu e^{i\nu.x}$ where $|a_\nu| \leq C|\nu|^p$ for some p.

Note that we could also solve our problem on $\mathbf{R}^n$ in an analogous manner to the torus, using the Fourier transform. More on this later.

2.2 Hilbert space theory and weak solutions

Return to a general compact Riemannian manifold M and a function ρ of integral 0. A smooth function ϕ satisfies $\Delta\phi = \rho$ if and only if for all "test functions" f we have

$$\langle \nabla f, \nabla \phi \rangle = \langle f, \rho \rangle.$$

The advantage of this weak formulation is that it only involves one derivative of ϕ. Let C_0^∞ be the space of smooth function of integral 0. Define a norm on this space by

$$\|f\|_H^2 = \int_M |\nabla f|^2.$$

This is associated to an inner product $\langle\ ,\ \rangle_H$, making C_0^∞ a pre-Hilbert space. So we have a Hilbert space completion H. (Here it is crucial to realise that the norm $\|\ \|_H$ is *not* the same as the L^2-norm.)

Recall the Riesz representation theorem: any bounded linear map $\alpha : H \rightarrow \mathbf{R}$ is represented by the inner product: there is a unique $a \in H$ with $\alpha(f) = \langle a, f \rangle_H$ for all $f \in H$. For the proof see any standard Functional Analysis text.

The formula $\alpha_\rho(f) = \langle \rho, f \rangle$ defines a linear map from C_0^∞ to $\mathbf{R}$. *Suppose* we know that this extends to a bounded map on H. Then by the representation theorem there is a $\phi \in H$ such that

$$\langle \nabla \phi, \nabla f \rangle = \langle \rho, f \rangle,$$

for all f, which is the identity satisfied by a genuine smooth solution. We say ϕ is a weak solution of our equation. Thus we have two tasks:

- Prove that α_ρ extends to a bounded linear map on H.
- Prove that a weak solution $f \in H$ is in fact smooth.

But tackling these we make two remarks.

First, the proof of the Riesz theorem goes by considering the hyperplane $\{f : \alpha(f) = 1\}$ in H and showing that there it contains a point of minimal norm. Thus one considers a *minimizing sequence* f_i in the hypersurface and the trick is to show that this is Cauchy. Unwinding this, in our situation, we consider a minimising sequence ϕ_i for the Dirichlet integral

$$\int_M |\nabla\phi|^2$$

restricted to functions ϕ with

$$\int_M \phi = 0 \qquad \int_M \rho\phi = 1.$$

Second, the first item above follows if we establish the *Poincaré inequality*:

$$\int_M |\phi|^2 \leq C^2 \int_M |\nabla\phi|^2,$$

for all functions ϕ of integral 0. Conversely suppose we prove a little more than stated in the first item, that in fact

$$|\alpha_\rho(f)| \leq C\|\rho\|_{L^2}\|f\|_H.$$

for some fixed constant C, independent of ρ. Then taking $f = \rho$ we deduce the Poincaré inequality.

The best constant C in the Poincare inequality is an important geometric invariant of a compact Riemannian manifold. As we will see later, it is in the inverse of the smallest eigenvalue of the Laplace operator.

The foundation of our proof of the first item (α_ρ bounded) will be a result from calculus. Suppose Ω is a bounded, convex, open set in $\mathbf{R}^n$. (For our applications it suffices to consider the case of a ball). Let V be the volume of Ω and d be its diameter.

Theorem 2 *Let ψ be a smooth function on an open set containing the closure $\overline{\Omega}$ and let $\overline{\psi}$ denote the average*

$$\overline{\psi} = \frac{1}{V}\int_\Omega \psi d\mu,$$

where $d\mu$ is the standard Lebesgue measure on $\mathbf{R}^n$. Then for $x \in \Omega$ we have

$$|\psi(x) - \overline{\psi}| \leq \frac{d^n}{nV}\int_\Omega \frac{1}{|x-y|^{n-1}}|\nabla\psi(y)|d\mu_y.$$

(Here the notation is supposed to indicate that the variable of integration on the right hand side is $y \in \Omega$.)

To prove this there is no loss in supposing that the point x is the origin in $\mathbf{R}^n$ (applying a translation in $\mathbf{R}^n$) and that $\psi(0)$ is zero (changing ψ by addition of a constant). To simplify notation we will do the case $n = 2$ (the general case is essentially the same). We work in standard polar co-ordinates (r, θ) on the plane. Thus we can write

$$\overline{\psi} = \frac{1}{V} \int_0^{2\pi} \int_0^{R(\theta)} \psi(r, \theta) r dr d\theta,$$

where $R(\theta)$ is the length of the portion of the ray at angle θ lying in Ω. (Here we use the fact that Ω is convex.) Now if we introduce another radial variable ρ we can write, for each (r, θ)

$$\psi(r, \theta) = \int_0^r \frac{\partial \psi}{\partial \rho} d\rho,$$

using the fact that ψ vanishes at the origin. So now we have

$$\overline{\psi} = \frac{1}{V} \int_0^{2\pi} \int_0^{R(\theta)} \int_{\rho=0}^{r} \rho \frac{\partial \psi}{\partial \rho} r \frac{\partial \psi}{\partial \rho} d\rho dr d\theta.$$

We interchange the order of the r and ρ integrals, so

$$\overline{\psi} = \frac{1}{A} \int_0^{2\pi} \int_{\rho=0}^{R(\theta)} \left(\int_{r=\rho}^{R(\theta)} r dr \right) \frac{\partial \psi}{\partial \rho} dr d\theta.$$

The innermost integral is

$$\int_{r=\rho}^{R(\theta)} r dr = \frac{1}{2} (R(\theta)^2 - r^2)$$

which is positive and no larger than $\frac{R(\theta)^2}{2}$, while, by definition, $R(\theta) \leq d$. Thus

$$|\overline{\psi}| \leq \frac{d^2}{2V} \int_0^{2\pi} \int_0^{R(\theta)} \frac{1}{\rho} |\frac{\partial \psi}{\partial \rho}| \rho d\rho d\theta.$$

The modulus of the radial derivative $\frac{\partial \psi}{\partial \rho}$ is at most that of the full derivative $\nabla \psi$, so switching back to a co- ordinate free notation we have

$$|\overline{\psi}| \leq \frac{d^2}{2V} \int_\Omega \frac{1}{|y|} |\nabla \psi_y| d\mu_y,$$

as required.

Corollary 1 *Under the hypotheses above, for a constant $c(n, \Omega)$ which we could calculate,*

$$\int_\Omega |\psi(x) - \overline{\psi}|^2 d\mu_x \leq c \int_\Omega |\nabla \psi|^2 d\mu.$$

To prove this, and for later use, we recall the notion of the *convolution* of functions on $\mathbf{R}^n$. The convolution of functions f, g is defined by

$$(f * g)(x) = \int_{\mathbf{R}^n} f(y) g(x-y) d\mu_y,$$

The operation * is commutative and associative and if $\| \|_T$ is any translation-invariant norm on functions on $\mathbf{R}^n$ we have

$$\|f * g|_T \leq \|f\|_{L^1} \|g\|_T,$$

where $\|f\|_{L^1}$ is the usual L^1 norm

$$\|f\|_{L^1} = \int_{R^2} |f| d\mu.$$

In particular this holds when $\| \|_T$ is the L^2 norm

$$\|g\|_{L^2}^2 = \int_{\mathbf{R}^n} |g|^2 d\mu.$$

(Strictly we should specify what class of functions we are considering in the definition of the convolution, but this will be clear in the different contexts as they arise.)

To prove the corollary, we define

$$K(x) = \frac{d^n}{nV} \frac{1}{|x|^{n-1}} \text{ for } |x| < d,$$

and $K(x) = 0$ if $|x| \geq d$. This has a singularity at the origin but is nevertheless an integrable function so write c for its L^1 norm. Define a function g on $\mathbf{R}^2$ by

$$g(y) = |\nabla \psi|^2(y),$$

if $y \in \Omega$ and $g(y) = 0$ if $y \notin \Omega$. Then $K * g$ is a positive function on $\mathbf{R}^2$ and the Theorem above asserts that for all $x \in \Omega$,

$$|\psi(x) - \overline{\psi}| \leq |(K * g)(x)|.$$

It follows that

$$\int_\Omega |\psi(x) - \overline{\psi}|^2 d\mu_x \leq \|K * g\|_{L^2}^2 \leq \|K\|_{L^1}^2 \|g\|_{L^2}^2 \leq c^2 \|\nabla \psi\|_{L^2}^2,$$

as asserted.

What we have proved is a "local" Poincaré inequality, for functions on a convex set $\Omega \subset \mathbf{R}^n$. Now we want to transfer this to our Riemannian manifold. Fix a finite cover by co-ordinate charts U_α which map to balls in $\mathbf{R}^n$. It is elementary to prove that we can write $\rho = \sum \rho_\alpha$ where ρ_α is supported in U_α

and the integral of each ρ_α is zero. For example consider the case of two charts $M = U_1 \cup U_2$. Fix a *partition of unity*

$$1 = \chi_1 + \chi_2,$$

subordinate to this cover. So of course we have $\rho = \rho\chi_1 + \rho\chi_2$. The problem is that $\rho\chi_i$ may not have integral 0. So fix a function σ of integral 1 supported on the intersection $U_1 \cap U_2$. Let I be the integral of $\rho\chi_1$, which is minus the integral of $\rho\chi_2$, since ρ has integral 0, by hypothesis. Then the functions

$$\rho_1 = \rho\chi_1 - I\sigma \quad , \quad \rho_2 = \rho\chi_2 + I\sigma$$

do the job. The general case is similar (exercise). Further one can choose ρ_i so that

$$\|\rho_i\|_{L^2} \leq c\|\rho\|_{L^2},$$

for a fixed constant c. This means that we can reduce to the case when ρ is supported in a single coordinate chart in our manifold, which we identify with a bounded convex set Ω in $\mathbf{R}^n$. Now since the integral of ρ is zero we also have

$$\int_M \rho\phi d\mu_g = \int_\Omega \rho(\phi - \overline{\phi})d\mu_g,$$

where $d\mu_g$ is the Riemannian volume element and $\overline{\phi}$ is the average of ϕ over Ω with respect to ordinary Lebesgue measure. By the Cauchy-Schwartz inequality

$$|\int_\Omega \rho(\phi - \overline{\phi})d\mu| \leq \|\rho\|_{L^2(\Omega)}\|\phi - \overline{\phi}\|_{L^2(\Omega)}.$$

Here crucially we use the fact that the norms computed using the Riemannian structure of the Euclidean structure are uniformly equivalent. Using the Corollary above, we deduce that

$$|\alpha_\rho(\phi)| \leq C\|\rho\|_{L^2}\|\nabla\phi\|,$$

for suitable C.

Now we turn to the second item, the smoothness of a weak solution. We postpone the general case and treat the situation when the manifold is locally Euclidean, so the Laplace operator is locally just that on $\mathbf{R}^n$. In fact, this argument also works for any 2-dimensional Riemannian manifold provided we know that there are local "isothermal co-ordinates" in which the metric is $e^F(dx^2 + dy^2)$. This is the same as saying that (after perhaps passing to the oriented cover) the Riemannian manifold is naturally a *Riemann surface*. In that case the Poisson equation can be transformed locally into the standard equation on $\mathbf{R}^2$. The result we want to prove is essentially what is known as "Weyl's lemma" in Riemann surface theory.

Suppose that ϕ is an element of H which is a weak solution to our problem in the sense explained above. That is, we have a sequence of smooth functions, of integral 0, ϕ_i on M which is Cauchy with respect to the Dirichlet norm $\| \|_H$ and, for any ψ

$$\langle \phi_i, \psi \rangle_H \to \alpha_\rho(\psi),$$

as i tends to infinity. By the Poincaré inequality the sequence is Cauchy in L^2 so has an L^2 limit. So we can identify the limiting object ϕ in the abstract completion with an L^2 function. We need to show that ϕ is smooth. Since smoothness is a local property we can fix attention on a single co-ordinate chart.

Proposition 1 *Let Ω be a bounded open set in $\mathbf{R}^n$ and ρ be a smooth function on Ω. Suppose ϕ is an L^2 function on Ω with the property that for any smooth function χ of compact support in Ω*

$$\int_\Omega \Delta\chi \ \phi = \int_\Omega \chi\rho.$$

Then ϕ is smooth and satisfies the equation $\Delta\phi = \rho$.

The proof will involve a number of steps. The first step is to reduce to the case when ρ is zero. Since smoothness is a local property it suffices to prove that ϕ is smooth over any given interior set Ω', where we suppose that the ϵ neighbourhood of Ω' is contained in Ω. Then we can choose a ρ' equal to ρ on a neighbourhood of the closure of Ω' and of compact support in Ω. Suppose we can find some smooth solution ϕ' of the equation $\Delta\phi' = \rho'$ over Ω. Then $\psi = \phi - \phi'$ will be a weak solution of the equation $\Delta\psi = 0$ on Ω'. If we can prove that ψ is smooth then so will ϕ be.

To find the smooth solution ψ' we use the "Newton potential" $K(x)$ as discussed above. We take a function τ of compact support in Ω and equal to 1 on Ω'. Then $\psi' = K * (\tau\rho)$ satisfies $\Delta\phi' = \rho$ over Ω'

We have now reduced to the case when $\rho = 0$ so, changing notation, let us suppose that ϕ is a weak solution of $\Delta\phi = 0$ on Ω and seek to prove that ϕ is smooth on the interior domain Ω', with the ϵ-neighbourhood of Ω' contained in Ω. The argument now exploits the *mean value property* of smooth harmonic functions. This says that if ψ is a smooth harmonic function on a neighbourhood of a closed ball then the value of ψ at the centre of the ball is equal to the average value on the boundary. Fix a smooth function β on $\mathbf{R}$ with $\beta(r)$ constant for small r and vanishing for $r \geq \epsilon$ and such that

$$Vol(S^{n-1}). \int_0^\infty r^{n-1}\beta(r)dr = 1.$$

Now let B be the function $B(z) = \beta(|z|)$ on $\mathbf{R}^n$. Then B is smooth and has integral 1 over $\mathbf{R}^n$ (with respect to ordinary Lebesgue measure). Suppose first that ψ is a smooth harmonic function on a neighbourhood of the closed ϵ-ball centred at the origin. Then we have

$$\int_{\mathbf{R}^n} B(-z)\psi(z)d\mu_z = \int_0^\infty \int_{S^{n-1}} r^{n-1}\beta(r)\psi(r,\theta)d\theta dr = \psi(0),$$

where we have switched to "generalised polar coordinates" and used the mean value property. Now the integral above is just that defining the convolution $B*\psi$ at 0. By translation invariance we obtain the following

Proposition 2 *Let ψ be a smooth function on $\mathbf{R}^n$ and suppose that $\Delta\psi$ is supported in a compact set $J \subset \mathbf{R}^n$. Then $B*\psi - \psi$ vanishes outside the ϵ-neighbourhood of J.*

We see in particular from this that *if* our function ϕ on Ω is smooth we must have $B*\phi = \phi$ in Ω'. Conversely, for any L^2 function ϕ the convolution $B*\phi$ is smooth. So proving the smoothness of ϕ in Ω' is equivalent to establishing the identity $B*\phi = \phi$ in Ω'. To do this we proceed as follows. It suffices to show that for any smooth test function χ of compact support in Ω' we have

$$\langle \chi, \phi - B*\phi\rangle = 0,$$

where we are writing $\langle\ ,\ \rangle$ for the usual L^2 inner product"

$$\langle f, g\rangle = \int fgd\mu.$$

We use the fact that for any functions f, g, h in a suitable class

$$\langle f, g*h\rangle = \langle g*f, h\rangle.$$

This follows by straightforward re-arrangements of the integrals. We will not bother to spell out conditions on the functions involved, since the validity of the identity will be fairly obvious in our applications below.

Put $h = K*(\chi - B*\chi) = K*\chi - B*K*\chi$. Now $K*\chi$ is a smooth function on $\mathbf{R}^n$ and $\Delta(K*\chi) = \chi$ by the properties of the Newton potential. Thus $\Delta(K*\chi)$ vanishes outside the support of χ, and hence by the Proposition above $B*K*\chi$ equals $K*\chi$ outside the ϵ-neighbourhood of the support of χ. Thus h has compact support contained in Ω. So we can use h as a test function in the hypothesis that $\Delta\phi = 0$ weakly, i.e. we have

$$\langle \Delta h, \phi\rangle = 0.$$

But $\Delta h = \Delta(K*(\chi - B*\chi)) = \chi - B*\chi$ (since χ and $B*\chi$ have compact support). So we see that

$$\langle \chi - B*\chi, \phi\rangle = 0.$$

But applying the identity above again this gives

$$\langle \chi, \phi - B*\phi\rangle = 0,$$

as desired.

2.3 Compactness

We continue with our compact Riemannian manifold M. We have seen that there is a continuous inclusion $H \to L^2$. In fact more is true: the inclusion map is *compact*. Recall here that a map $T : X \to Y$ between Banach spaces is called compact if for any bounded sequence x_i the sequence Tx_i has a convergent subsequence in Y. It is a little more convenient to drop the restriction to functions of integral zero and define the *Sobolve space* L^2_1 to be the completion of the smooth functions with the norm

$$\|f\|^2_{L^2 1} = \int |\nabla f|^2 + |f|^2.$$

Then we have

Theorem 3 *The inclusion $L^2_1 \to L^2$ is compact.*

It is easy to show that this is equivalent to the compactness of $H \to L^2$.

This result can be compared with the 'Ascoli-Arzela theorem: if we define the C^0, C^1 norms in the obvious way then $C^1 \to C^0$ is compact. (Actually this is a special case: the Ascoli-Arzela theorem deals with any "equicontinuous" set of functions: e.g. those satisfying a Hölder condition.)

Interesting example.

Consider the function defined to $f = \log |\log r|$ for small $r < 1/2$ where r is the radial co-ordinate in $\mathbf{R}^2$ extended by the obvious constant for $r \geq 1/2$. Choose any compact Riemannian manifold which contains an isometric copy of the unit disc and regard f as a function on this manifold. (Of course the manifold is completely artifical here, just included to fit in with our definitions.) Then $f \in L^2_1$ but is not bounded near the origin.

We prove the Theorem first in the case of the torus. The L^2_1 norm of $f = \sum a_\nu e^{i\nu . x}$ is the square root of

$$\sum_\nu (1 + |\nu|^2) |a_\nu|^2.$$

So we have to show that $a^{(i)}_\nu$ is a sequence of Fourier-coefficient sets with

$$\sum_\nu (1 + |\nu|^2) |a^{(i)}_\nu|^2 \leq C$$

then there is subsequence which converges in l^2. For each fixed ν the $a^{(i)}_\nu$ are obviously bounded so we can choose a convergent subsequence; by a diagonal argument we get a single subsequence which works for every ν. Without loss of generality we can suppose this is the original sequence. Now we claim that the sequence automatically converges in l^2 or equivalently is Cauchy. Given ϵ we choose a large R such that

$$\sum_{|\nu|>R} |a_\nu|^2 \leq \epsilon.$$

Now there are just a finite number of terms with $|\nu| \leq R$ and the argument should be clear.

Notice that the same applies if we take any norm on Fourier co-efficients

$$\|f\|_W^2 = \sum W(\nu)|a_\nu|^2,$$

with a positive weight function $W(\nu)$ which tends to infinity as $|\nu| \to \infty$. Such functions are roughly analogous to "moduli of continuity", as in Ascoli-Arzela.

Now for a general manifold cover by a finite number of co-ordinate patches in the familiar way and identify these patches with patches in the torus. Fix a partition of unity β_α subordinate to this cover. If f_i is bounded in L^2_1 then so are $\beta_\alpha f_i$, and we deduce what we want by considering these as functions on the torus.

Now return to the Poisson equation $\Delta\phi = \rho$. It follows easily from what we did that for each $\rho \in L^2$ of integral zero there is a unique L^2 solution $\phi = G\rho$, also of integral zero, where of course we interpret the equation in the weak sense. Further since

$$\|\nabla\phi\|^2 = \langle \Delta\phi, \phi \rangle \leq \|\phi\| \, \|\rho\|,$$

the Poincaré inquality implies that $G : L^2 \to L^2$ is a bounded operator. A corollary of the Theorem is that G is actually compact (Exercise.) There is a rather easy theory of compact self-adjoint operators on Hilbert spaces. This gives a complete orthonormal system of eigenfunctions ϕ_i in L^2 with $G\phi_i = \mu_i\phi_i$ and the eigenvalues μ_i tend to 0. leaving out the zero eigenspace, the inverses $\lambda_i = \mu_i^{-1}$ are the eigenvalues of the Laplace operator. (Although we have not yet shown that these eigenfunctions are actually smooth.)

Examples

- On the torus the eigenfunctions are just the $e^{i\nu x}$.
- On the 2-sphere the eigenfunctions are given by the spherical harmonics, as described in classical analysis and mathematical physics texts.

Remark

Notice that this derivation of the "spectrum"does not require any mention of smoothness. For example one can extend the notion of a Riemannian metric to situations where the g_{ij} are arbitrary measurable functions with, in the sense of quadratic forms,

$$0 < c \leq (g_{ij}) \leq C < \infty,$$

and the theory works perfectly well.

3 General theory of linear elliptic operators

We consider the general set-up of a pair of vector bundles E, F over a compact Riemannian manifold M, with metrics on the fibres. This defines the L^2 norm on sections of E, F. We consider a linear differential operator of order r,

$$D : \Gamma(E) \to \Gamma(F).$$

There is a formal adjoint $D^* : \Gamma(F) \to \Gamma(E)$ defined by the identity

$$\langle f, Dg \rangle = \langle D^* f, g \rangle.$$

We postpone for the moment the definition of what it means for D to be *elliptic*. Let us just say that D is elliptic if and only if D^* is and that the Laplace operator is a prototype. Now we can state the main theorem, write $\ker D$ for the smooth sections of E which satisfy the equation $Df = 0$, and similarly for D^*.

Theorem 4 *In this situation*

1. $\ker D, \ker D^*$ *are finite-dimensional vector spaces,*
2. *We can solve the equation* $Df = \rho$ *if and only if* ρ *is orthogonal to* $\ker D^*$.

The condition of being elliptic depends on the notion of the *symbol* of a differential operator. Let $D : \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order r (obvious definition). Then at each point $x \in M$ and for each cotangent vector $\xi \in T^*M$ there is a linear map

$$\sigma_\xi : E_x \to F_x.$$

One way to define this is by choosing bundle trivialisations so D is given by a matric of differential operators $D_{\alpha\beta}$ say. In turn each of these is, in local coordinates, a sum

$$D_{\alpha\beta} = \sum_I a_{\alpha,\beta,I} \frac{\partial}{\partial x^I},$$

where we use multi-index notation $I = (i_1, i_2, \dots i_n)$. Then we define σ by taking the terms of highest order $\sum i_\lambda = r$ and regarding the operators $\frac{\partial}{\partial x_i}$ as linear functions on the cotangent space so $\frac{\partial}{\partial X_I}$ is a polynomial function. Then one can check that this definition does not depend on the choice of trivialisations and local co-ordinates. A more invariant definition is to choose a section s of E and a function f on M, vanishing at x and with $df = \xi$ at x. Then we can define

$$\sigma_\xi(s(x)) = D(f^r s)(x).$$

Again one needs to check that this is does not depend on the choice of f, s.

Now an operator is elliptic if for each nonzero $\xi \in TM_x$ the linear map σ_ξ is an isomorphism from E_x to F_x. Note that this can only happen if E, F have the same rank. The symbol of the formal adjoint is $(-1)^r$ times the adjoint of the symbol so D is elliptic if and only if D^* is.

For example if $D = \Delta$ is the Laplace operator on functions then the σ_ξ is multiplication by $-|\xi|^2$, so Δ is elliptic.

Example: Hodge Theory

Here we assume knowledge of the bundles of differential forms and the exterior derivative

$$d : \Omega^p \to \Omega^{p+1}$$

with $d^2 = 0$. The cohomology $\ker d/\mathrm{Im} d$ is a topological invariant of the manifold (isomorphic to the singular cohomology). Using a Riemannian metric we have

$$d^* : \Omega^{p+1} \to \Omega^p,$$

and a Hodge Laplacian $\Delta = dd^* + d^*d$. This is obviously self-adjoint.

For example consider the Laplacian on 1-forms over flat space $\mathbf{R}^n$. If $\theta = f dx_1$ we have (writing f_i, f_{ij} for the partial derivatives of f):

$$d\theta = \sum_{i>1} f_i dx_i dx_1,$$

and

$$d^* d\theta = -\left(\sum_{i>1} f_{ii} dx_1 - f_{i1} dx_i \right).$$

While

$$d^*\theta = -f_1,$$

and

$$dd^*\theta = -\sum_i f_{i1} dx_i.$$

Cancelling terms we find that

$$\Delta\theta = -(\sum_i f_{ii}) dx_1.$$

In general, on flat space Δ acts on forms of all degrees as the ordinary scalar Laplacian on the individual co-efficients. On any Riemannian manifold the symbol of Δ at ξ is $-|\xi|^2$ times the identity. Thus Δ is elliptic.

Forms in the kernel of Δ are called harmonic forms. The Hodge theorem asserts that each cohomology class has a unique harmonic representative. Uniqueness is easy, the hard part is existence. To obtain this from our main theorem, assuming the fact that Δ is elliptic, we take any p-form ϕ. Then the projection $H\phi$ to the harmonic forms is also smooth and $\phi - H\phi$ is orthogonakl to $\ker\Delta = \ker\Delta^*$ so we can write

$$\phi - H\phi = \Delta f$$

for some f. Now suppose ϕ is closed, $d\phi = 0$. Then $d\Delta f = dd^*df = 0$ and taking the inner product with df we see that $df = 0$. So $\Delta f = dd^*f$ and $\phi = H\phi + dd^*f$ demonstrates the harmonic representative $H\phi$ in the cohomology class $[\phi]$.

Remark

The Hodge Theorem is easier to understand if one thinks of the harmonic forms as solutions of $d\phi = d^*\phi = 0$. The harmonic representative is the form which minimises the L^2 norm among all representatives in the cohomology class. One can avoid mentioning the second order operator Δ and this is preferable for some purposes.

Applications

1. Topology.

The Levi-Civita connection of a Riemannian metric defines a covariant derivative

$$\nabla : \Gamma T^*M \to \Gamma(T^*M \otimes T^*M).$$

then we have another operator

$$\nabla^*\nabla : \Gamma(T^*M) \to \Gamma(T^*M).$$

This differs from Δ by a curvature term, in fact (Bochner formula):

$$\Delta\theta = \nabla^*\nabla\theta + \mathrm{Ric}(\theta),$$

where Ric is the *Ricci tensor*. hence we have

Corollary 2 *If a compact manifold M has positive definite Ricci tensor then $H^1(M) = 0$.*

This is related to *Myer's Theorem*; proved in quite a different way using geodesics. Myer's theorem gives stronger information in the positive definite case, but the Bochner argument tells one more in the case when $Ric \geq 0$.

2. Lowest eigenvalue.

Suppose Ric $\geq c > 0$. Let f be a (nonconstant) eigenfunction of the laplace operator, $\Delta f = \lambda f$. Then if $\theta = df$ we have

$$\Delta\theta = \Delta df = d\Delta f = \lambda df = \lambda\theta.$$

Taking the L^2 inner product with θ in the Bochner formula we immediately see that $\lambda \geq c$. With a little more care we see that (Lichnerowicz)

$$\lambda \geq \frac{n}{n-1}c,$$

(since $|d^*\theta|^2 \geq \frac{1}{n}|\nabla\theta|^2$.) One can show further that equality is achived only in the case of the round sphere (Obata's Thheorem).

Example of a non-elliptic operator

Let κ be an irrational number and let A be the operator

$$A = -i\left(\frac{\partial}{\partial x_1} - \kappa\frac{\partial}{\partial x_2}\right),$$

acting on functions over the 2-torus. Then A is self-adjoint and acts as a multiplier operator on Fourier series, multiplying $\exp(i\nu x)$ by $n_1 - \kappa n_2$. Thus $\ker(A) = \ker(A^*)$ consists of the constant functions. If the integral of ρ is zero we can write down a formal solution to the equation $Af = \rho$ in the usual way, $f = \sum f_\nu e^{i\nu x}$ where

$$f_\nu = \frac{1}{\nu_1 - \kappa\nu_2}\rho_\nu.$$

Suppose κ can be approximated to infinite order by rationals, so there is are sequences p_i, q_i with

$$|\kappa - \frac{p_i}{q_i}| \leq \frac{1}{q_i^i}.$$

Then the formal solution f need not be smooth even if ρ is. For when $(\nu_1, \nu_2) = (p_i, q_i)$ we have

$$|\frac{1}{\nu_1 - \kappa\nu_2}| \geq c|\nu|^i.$$

Proof of the main theorem

Facts from functional analysis

Let E be a Banach space and $T : E \to E$ a bounded operator. Then

- If the operator norm of T is less than 1 then $1 + T$ is invertible.
- If T is compact then $\ker(1 + T)$ is finite dimensional and the image of $1 + T$ is a closed subspace of finite codimension.

For simplicity we consider an elliptic operator of order 1. We define Sobolev spaces L^2_k, then D is a bounded operator

$$D : L^2_k \to L^2_{k-1}.$$

The intersection of L^2_k over all k gives exactly the smooth functions (or sections) and the inclusions $L^2_k \to L^2_{k-1}$ are compact.

There are three main ideas/auxiliary results:

- The existence of a *parametrix*, a bounded operator

$$P : L^2 \to L^2_1,$$

such that

$$D \circ P = 1 + K$$

where K is compact.

- The fundamental *elliptic inequalities*

$$\|f\|_{L^2_k} \leq C(\|Df\|_{L^2_{k-1}} + \|f\|_{L^2}.$$

- *Elliptic regularity*: if $\rho \in L^2_{k-1}$ and f is a weak solution of the equation $Df = \rho$ then f is in L^2_k.

Remark The first item uses the *surjectivity* of the symbol and the other two the *injectivity*.

Suppose we have established these assertions. Then the image of $D : L^2_1 \to L^2$ contains the image of $1 + K$ and so is closed of finite codimension. It follows that this image is equal to its double orthogonal complement. The orthogonal complement of the image is by definition the set of weak solutions of the equation $D^*\rho = 0$. By elliptic regularity these are all smooth. So we see that for any smooth ρ which is orthogonal to $\ker D^*$ there is an L^2_1 solution f of the equation $Df = \rho$. Now elliptic regularity implies that f is actually smooth. The finite dimensionality of $\ker D$ follows from the elliptic inequality (using the criterion that the unit ball in a Banach space is compact iff the space is finite dimensional). One can also deduce the finite dimensionality from the above, replacing D by D^*.

We will concentrate on the proof of the first item.

Consider first a constant co-efficient operator D_0 acting on sections of trivial bundles over the torus T^n, or equivalently acting on periodic functions on $\mathbf{R}^n$. Then the action on Fourier coefficients is just

$$D_0(\sum a_\nu e^{i\nu x}) = i \sum \sigma_\nu(a_\nu) e^{i\nu x}.$$

So, just as for the Laplacian, the kernels of D_0, D_0^* are just given by the constant sections. Orthogonal to the constants we have an inverse operator G given by the action of σ_ν^{-1} on the Fourier coefficients. The crucial point of course is that for $\nu \neq 0$ this inverse exists, by the definition of ellipticity. Moreover, since σ_ν is linear in ν we have

$$|\sigma_\nu^{-1}| \leq c|\nu|^{-1}.$$

This means that G is a bounded map $L^2_{k-1} \to L^2_k$.

The constants are a nuisance which we can avoid in various ways. One way is to consider a fixed *nonintegral* vector $\underline{\alpha} = (\alpha_i)$ and functions f on $\mathbf{R}^n$ which satisfy

$$f(x + \underline{m}) = e^{i\underline{\alpha}\cdot\underline{m}} f(x),$$

for $\underline{m} \in \mathbf{Z}^n$. These behave just like periodic functions for our purposes and have a Fourier expansion:

$$f = \sum_\nu a_\nu e^{i(\nu+\alpha)x}.$$

The operator D_0 acts by $\sigma_{\nu+\alpha}$ on the Fourier coefficents and these are all invertible, so D_0 is now an *invertible* operator

$$D_0 : L^2_{k,\underline{\alpha}} \to L^2_{k-1,\underline{\alpha}},$$

where the meaning of the notation should, I hope, be clear.

Now consider a linear operator with periodic coefficients which is a small perturbation of D_0, say $D = D_0 + \delta$ where δ again has first order and the coefficients are sufficiently small (in fact, small in L^∞ will suffice for the moment). This means that $\delta : L^2_1 \to L^2$ is small in operator norm, so D is also invertible as a map from $L^2_{1,\alpha}$ to L^2_α. We have an inverse Q say, which is *compact* when regarded as a map from L^2_α to itself.

Next go to an operator D over a general compact manifold M. We fix charts U_λ over which we have bundle trivialisations. The next crucial idea is that when we rescale a small ball about a point to unit size the operator approaches a constant coefficient operator of order 1, determined by the symbol. This is much the same as the fact in Riemannian geometry then when we rescale a small ball to unit size the Riemannian metric looks very close to the Euclidean metric. Using this idea we arrange our trivialisations so that over each (rescaled) U_λ the operator is very close to the constant coefficient model. Moreover we can think of the U_λ as being embedded well inside the standard hypercube in $\mathbf{R}^n$ and extend the coefficients to be periodic on $\mathbf{R}^n$. Choose a smaller ball $V_\lambda \subset\subset U_\lambda$ and a cut off function γ equal to 1 on V_λ and supported in U_λ. If ρ is supported in V_λ then $\gamma Q(\rho)$ can be viewed as section of E over M and

$$D(\gamma Q\rho) = \gamma DQ\rho + (\nabla\gamma) * Q\rho,$$

where $*$ denotes some algebraic operation. But by construction,

$$\gamma DQ\rho = \rho.$$

So if we write the operator $\rho \mapsto \gamma Q(\rho)$ as P_λ we have

$$DP_\lambda\rho = \rho + K_\lambda\rho,$$

for sections ρ supported on V_λ, and K_λ is compact.

Next we choose a partition of unity $1 = \sum \beta_\lambda$ subordinate to a cover by sets V_λ, and we define

$$P\rho = \sum P_\lambda(\beta_\lambda \rho).$$

Then

$$DP(\rho) = \sum DP_\lambda \rho = \sum \beta_\lambda \rho + \sum K_\lambda(\beta_\lambda \rho)$$

and this has the form $\rho + K\rho$ where K is compact. Thus we have constructed our parametrix.

Remark There are other ways of doing this. for example if D is teh laplace operator we can define a parametrix to be the integral operator with kernel function

$$P(x, y) = c\sigma d(x, y)^{2-n}$$

(assuming $n > 2$) . Here $d(x, y)$ is the Riemannian distance between x, y, c is the inverse of the volume of S^{n-1} and σ is a cut-off function, equal to 1 in a neighbourhood of the diagonal.

The proofs of the other two items follow in a similar way, reducing to an operator which is a small perturbation of a constant coefficient operator.

4 Further topics

4.1 Other function spaces

One can consider many other different classes of functions on a compact manifold. We just mention

- For $1 \leq p < \infty$ and integer $k \geq 0$ the space L^p_k (the completion of the smooth functions in the norm given by the L^p norm of the derivatives up to order k).
- For $0 \leq \alpha \leq 1$ the space $C_{k,\alpha}$ of functions whose derivatives up to order order k are Holder continuous with exponent α (when $\alpha = 0$ we can interpret this just as saying that the derivatives of order k are continuous.)

We can ask

- what is the relation between these spaces?
- how do elliptic operators behave?

The first question is answered by the various *Sobolev embedding theorems.* To each of these spaces we can attach a *scaling weight.* We consider compactly supported functions f on $\mathbf{R}^n$ and the transformation of the highest term under the rescaling $f_\epsilon(x) = f(\epsilon x)$. This is $k - \frac{p}{n}$ for L^p_k and $k + \alpha$ for $C_{k,\alpha}$. Then we have natural inclusions

- $$L^p_k \to L^q_l$$
  if $k \geq l$ and $k - \frac{n}{p} \geq l - \frac{n}{q}$,
- $$L^p_k \to C_{l+\alpha}$$
  if $k - \frac{p}{n} \geq l + \alpha$.

and these embeddings are *compact* if strict inequality holds.

To give an idea of the proofs consider the embedding $L^p_1 \to C_0$ for $p > n$. This is easily deduced from an inequality, for all smooth functions f of compact support on the unit ball in $\mathbf{R}^n$

$$|f(0)| \leq C \|\nabla f\|_{L^p}.$$

This is straightforward to prove with the same kind of argument that we used for the Poincaré inequality in Section 2. We work in generalised polar co-ordinates and write

$$f(0) = c \int \left(\frac{\partial f}{\partial r} \frac{1}{r^{n-1}} \right) r^{n-1} dr d\theta,$$

where c is the inverse of the volume of S^{n-1}. Now use the fact that $|\frac{\partial f}{\partial r}| \leq |\nabla f|$ and that the function $r^{-(n-1)}$ is in L^q over the unit ball for any $q > \frac{n}{n-1}$. Then Holders inequality gives what we need.

For another model case consider the embedding $L^1_1 \to L^q$, where $q = n/(n-1)$. This can be derived from a basic inequality, for functions f of compact support on $\mathbf{R}^n$,

$$\|f\|_{L^q} \leq C \|\nabla f\|_{L^1}. \quad (*)$$

This is closely related to the *isoperimetric inequality.* For any bounded domain Ω with (say) smooth boundary we have

$$\mathrm{Vol}(\Omega)^{1/n} \leq c_n \mathrm{Vol}(\partial \Omega)^{1/(n-1)},$$

and in fact the *best* constant c_n is determined by saying that equality holds for a ball. In one direction, if we assume (*) is known then we can prove the isoperimetric inequality (with some constant) by considering smooth approximations to the characteristic function of a domain Ω. In the other direction it is possible to deduce (*) from the isoperimetric inequality using the *co-area formula.* For a positive function f write $\Omega_a = f^{-1}(a, \infty)$. Then

$$\|\nabla f\|_{L^1} = \int_0^\infty \mathrm{Vol}(\partial \Omega_a) da.$$

The best constants in the Sobolev embeddings are interesting invariants of a compact Riemannian manifold M of dimension n. For a function f on M and $p \geq 1$ write

$$\|f\|^*_{L^p} = \inf_{\alpha \in \mathbf{R}} \|f - \alpha\|_{L^p}.$$

The first eigenvalue λ of the Laplacian is given by

$$\sqrt{\lambda} = \inf \frac{\|\nabla f\|_{L^2}}{\|f\|^*_{L^2}}.$$

The *Sobolev constant* S is defined by

$$S^{1/n} = \inf \frac{\|\nabla f\|_{L^1}}{\|f\|^*_{L^{n/n-1}}}.$$

Now consider codimension 1 submanifolds $\Sigma \subset M$ dividing M into two regions Ω_1, Ω_2 with $\mathrm{Vol}(\Omega_1) \leq \mathrm{Vol}(\Omega_2)$. The *Cheeger constant* h is defined to be

$$h = \inf_{\Sigma} \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Omega_1)},$$

and the *isoperimetric constant* I is defined to be

$$I = \inf_{\Sigma} \frac{\mathrm{Vol}(\Sigma)^n}{\mathrm{Vol}(\Omega_1)^{n-1}}.$$

Then it is known that

$$\lambda \geq \frac{h^2}{4},$$

and

$$I \leq S \leq 2I.$$

We saw above that a positive lower bound on the Ricci curvature gives a bound on λ. There is a large body of results relating the Ricci curvature to the Sobolev and isoperimetric constants. A good source for this is the book *Eigenvalues in Riemannian geometry* by I. Chavel.

4.2 Elliptic operators

The basic fact is that elliptic operators behave well on the function spaces L^p_k so long as $1 < p < \infty$ and $C_{k,\alpha}$ so long as $0 < \alpha < 1$. This amounts to establishing inequalities, for an operator D of order r over a compact manifold,

$$\|f\|_{L^p_{k+r}} \leq C(\|Df\|_{L^p_k} + \|f\|_{L^p}),$$

(the *Calderon-Zygmund Theory*) and

$$\|f\|_{C_{k+r,\alpha}} \leq C(\|Df\|_{C_{k,\alpha}} + \|f\|_{C_{0,\alpha}}),$$

(the *Schauder Theory*).

As usual we can reduce the problem to a model case of constant co-efficient operators; here it will be more convenient to work over $\mathbf{R}^n$, rather than the torus. Consider for example the case when D is the Laplace operator, inverted by convolution with the Newton potential. Formally if $\Delta\phi = \rho$ then a second derivative $\nabla_i \nabla_j \phi$ is given by the convolution of ρ with a function, $k = k_{ij}$ say, which is homogeneous of degree $-n$. Thus k is not integrable around the origin so the integral involved in the convolution is not on the face of it well-defined. It has to be interpreted as a *singular integral operator*. The basic example is the Hilbert transform on functions of one variable

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy.$$

Such operators correspond under the Fourier transform to multiplier operators with a multiplier function $M(\xi)$ which is homogeneous of degree 0. For example the Hilbert transform corresponds to the multiplier $i\text{sgn}(\xi)$. The basic fact is that these operators are bounded on the spaces L^p $(1 < p < \infty)$ and $C_{,\alpha}$ $0 < \alpha < 1$. A good source for this is the book of E. Stein *Singular integral operators and differentiability properties of functions.*

Sketch of application: the *measurable Riemann mapping theorem.*

This is an extension of the assertion that any two dimensional Riemannian manifold is locally conformal to Euclidean space (mentioned in Section 2). The problem can be expressed in terms of a a complex-valued function μ on $\mathbf{C}$ with $|\mu(z)| \leq k < 1$ for some fixed k. Then we want to find a homeomorphism f from $\mathbf{C}$ to $\mathbf{C}$ solving the Beltrami equation

$$\overline{\partial} f + \mu \partial f = 0,$$

where $\overline{\partial} = \frac{\partial}{\partial \overline{z}}, \partial = \frac{\partial}{\partial z}$. The assertion is that, without any continuity or smoothness assumption on μ, this can be done (where the derivatives of f will be interpreted in a suitable sense). We consider the case when μ has compact support–the general case can be reduced to this by a trick. The idea is to write $f(z) = z + g(z)$ so we have to solve the equation

$$\overline{\partial} g + \mu \partial g = -\mu.$$

Let Q be the inverse to $\overline{\partial}$ and write $g = Q\rho$. So we have to solve

$$\rho + \mu \partial \circ Q(\rho) = \mu.$$

We work with $\rho \in L^p$. The operator norm of $\partial \circ Q$ on L^2 is equal to 1 (in fact Q is an isometry of L^2) and it follows that when p is close enough to 2 the norm is strictly less than k^{-1}. We fix such a $p > 2$. Then the operator norm of $\partial \circ Q$ is less than 1 and $(1 + \partial \circ Q)$ is invertible, so we have an L^p solution ρ. Now $g = Q\rho$ has derivative in L^0, and this implies that g is continuous, since $p > 2$. This is used to show that f is in fact a homeomorphism. Full details can be found in the book *Quasiconformal mappings* by L. Ahlfors.

4.3 Index theory

Let U, V be Banach spaces and $T : U \to V$ a bounded operator. The operator is called Fredholm if $\ker T$ is finite dimensional and the image of T is closed of finite codimension. Then we define the index of T to be

$$\text{index } T = \dim \ker T - \dim \operatorname{coker} T.$$

If U, V are finite-dimensional then this is simply $\dim U - \dim V$. In general the space of Fredholm operators is open, in the operator norm topology, and the index is constant on each connected component. It is often useful to think of it formally as the difference of the dimensions of U, V even when these are infinite.

If $D : \Gamma(E) \to \Gamma(F)$ is an elliptic differential operator of order r over a compact manifold then it defines a Fredholm operator $L^2_{k+r} \to L^2_k$ for any k and the index does not depend on k. The index is just the difference of the dimensions of the kernels of D, D^*, which are made up of smooth sections. If D varies in a continuous family of such operators then the index does not change. For example the operator might depend on the choice of a Riemannian metric, but the index is independent of this choice. The index is a homotopy invariant of the data consisting of the bundles E, F over M and the symbol of D. The Atiyah-Singer index theorem gives the a general formula for the index in terms of algebraic topology. For a simple example consider the operator

$$d + d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}},$$

over an even dimensional manifold. Since $(d + d^*)^2 = \Delta$ the Hodge theorem identifies the index with the alternating sum of the Betti numbers

$$\text{index } d + d^* = \sum (-1)^p \dim H^p(M).$$

4.4 The Dirac operator and connections on complex line bundles

These are the last pieces of differential geometry we will need. More details will be given in the lectures. The theory applies in every dimension but we emphasise dimension 4.

A spin structure on an oriented Riemannian 4-manifold M is given by a pair of quaternionic line bundles S^+, S^- over M and an isomorphism γ from TM to the bundle of quaternion-linear maps from S^+ to S^-, compatible with the algebraic structure around. We usually regard S^+, S^- as rank 2 complex vector bundles. The algebraic compatability can be expressed by saying that at each point we can choose standard bases such that

$$\gamma(e_0) = 1, \gamma(e_i) = \sigma_i, \ i = 1, 2, 3$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & -0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Dirac operator on flat space is

$$\frac{\partial}{\partial x_0} + \sum \sigma_i \frac{\partial}{\partial x_i}.$$

Over the Riemannian manifold M the Levi-Civita connection defines a Dirac operator

$$D : \Gamma(S^+) \to \Gamma(S^-).$$

This is an elliptic operator. The Atiyah-Singer index theorem, in this case, is

$$\text{index } D = \frac{1}{8}\tau(M),$$

where $\tau(M)$ is the *signature* of M. It follows from the definitions that the kernels of D, D^* are quaternionic vector spaces so even dimensional as complex vector spaces. Hence we deduce *Rohlin's Theorem*, that the signature of a spin 4-manifold is divisible by 16.

The *Lichnerowicz formula* is

$$D^* D = \nabla^* \nabla + \frac{R}{4},$$

where R is the scalar curvature. If $R > 0$ then the kernels of D, D^* are trivial and the index is 0, hence the signature must be zero.

Examples

- The complex projective plane, with its standard metric, has positive scalar curvature and non-zero signature, but is not spin.
- Let M be a complex K3 surface. This is is spin and has signature 16. We see that any Riemannian metric with $R \geq 0$ must have $R = 0$ and admit a covariant constant section of S^+. This occurs exactly for the *Calabi-Yau* metrics on M.

Now consider a unitary connection A on a complex line bundle $L \to M$. In a local trivialisation this is expressed by a 1-form $A = \sum A_\alpha dx_\alpha$. The curvature is the 2-form $F = dA$. We can couple the Dirac operator to the connection to get an operator

$$D_A : \Gamma(S^+ \otimes L) \to \Gamma(S^- \otimes L).$$

Over flat space, in a local trivialisation, this is given by replacing derivatives $\frac{\partial}{\partial x_\alpha}$ by $\frac{\partial}{\partial x_\alpha} + iA_\alpha$. The index of D_A is

$$\frac{\tau(M)}{8} + \frac{1}{2}c_1(L)^2.$$

The Lichnerowicz formula becomes

$$D_A^* D_A \psi = \nabla^* \nabla \psi + \frac{R}{4}\psi + F * \psi,$$

where $*$ denotes a natural algebraic operator.

Application (Gromov and Lawson)

Suppose there is a Riemannian metric on the torus T^4 with $R > 0$ everywhere. Fix a line bundle $L \to T^4$ with $c_1(L)^2 \neq 0$. For any $n > 0$ we can find a covering $p : T^4 \to T^4$ such $p^*(L)$ has an nth. root

$$p^*(L) = \tilde{L}^{\otimes n}.$$

The curvature of $\tilde{L}$ is n^{-1} times the curvature of L. Thus after taking a covering we can find a line bundle with $c_1(\tilde{L})^2 \neq 0$ but with curvature $|\tilde{F}| << \min R$. This gives a contradiction. So the 4-torus has no metric of strictly positive scalar curvature.

5 Nonlinear equations: perturbative theory

5.1 General notions

We should begin by saying what we mean by a nonlinear partial differential equation. We need to steer a middle course between the concrete and the abstract. Consider first the bundle $\mathbf{R} \oplus T^*M$ over a manifold M. This can be viewed as the space of "1-jets" of functions on M. That is, for each p ain M we define an equivalence relation on functions by saying $f \sim g$ if f and g agree to first order at p. (Of course we have to check that this notion does not depend on a choice of local co-ordinates.) More generally, if $X \to M$ is a differentiable fibre bundle we can define, for each $p \in M$ and integer $k \geq 0$ the the space of k-jets of sections of X at p. The set of all k jets forms another bundle $J_k(X) \to M$. Any section σ of X defines a section $j^k(\sigma)$ of $J_k(X)$. Now a partial differential equation for sections of X is specified by some subset $E \subset J_k(X)$: the solutions are just sections σ such that $j_k(\sigma)$ lies in E. To be more specific we should probably require that E is a submanifold and in fact that $E \to M$ is a differentiable fibre bundle.

To make this more concrete, consider the case when $M = \mathbf{R}^n$(or an open subset of $\mathbf{R}^n$) and X is a trivial bundle, with fibre $Y \subset \mathbf{R}^m$. Then sections of

X are vector-valued functions $\underline{u}$, constrained to lie in Y. We suppose that the subset E is cut out as the zeros of smooth functions g_α. Then a kth. order PDE is a system of equations

$$g_\alpha(\underline{x}, \underline{u}, \nabla \underline{u}, \nabla\nabla \underline{u}, \dots, \nabla^k \underline{u}) = 0,$$

just as we always thought.

Given a PDE and a solution $\underline{u}_0$ there is a well-defined linearisation $L_{\underline{u}_0}$, which is a linear differentiable operator, mapping between sections of appropriate vector bundles. To take a concrete point of view, we consider sections $\underline{u} = \underline{u}_0 + tf$ and the equation $L_{\underline{u}_0}(f) = 0$ is the condition that $\underline{u}$ satisfies the equation to first order in t. Then we say that the PDE is elliptic *at* u_0 if $L_{\underline{u}_0}$ is a linear elliptic operator.

The simplest kind of example are equations which differ from an elliptic equation by arbitrary lower order terms.

$$\Delta u + u^3 = 0$$

Then at a solution u_0 the linearisation is

$$L(f) = \Delta f + 3u_0^2 f.$$

$$\Delta u + u|\nabla u|^2 = 0.$$

The linearisation at u_0 is

$$L(f) = \Delta f + |\nabla u_0|^2 f + 2u_0(\nabla u_0, \nabla f).$$

More complicated are equations such as the minimal surface equation

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{1}{1+|\nabla u|^2}\frac{\partial u}{\partial x_i}\right) = 0.$$

The linearisation is again a Laplace type operator. Or the *Monge-Ampère* equation:

$$\det(\frac{\partial^2 u}{\partial x_i \partial x_j} = 1.$$

The linearisation at a solution u_0 is

$$\sum u_0^{ij} f_{ij} = 0,$$

where u_0^{ij} is the inverse of the Hessian matrix of second derivatives. This is elliptic if u_0 is a *convex* function.

5.2 The inverse function theorem in Banach spaces

Let U, V be Banach spaces and $\Omega \subset U$ an open set containing 0. We have a notion of a C^1 map $\mathcal{F} : \Omega \to V$. Suppose $\mathcal{F}$ is such, with $\mathcal{F}(0) = 0$ and has a derivative L at 0 which is an isomorphism from U to V. Then the inverse function theorem asserts that for each small v in V there is a unique small solution u to the equation $\mathcal{F}(u) = v$, and u varies in a C^1-fashion with v.

For the proof we may as well suppose that $U = V$ and L is the identity. So we can write

$$\mathcal{F}(u) = u + \eta(u),$$

where

$$\|\eta(u) - \eta(u')\| \leq c\|u - u'\|,$$

where c can be made arbitrarily small if $\|u\|, \|u'\|$ are small enough. We write our equation as

$$\mathcal{G}(u) = u,$$

where

$$\mathcal{G}(u) = v - \eta(u).$$

Then we use the contraction mapping theorem. We start with $u_1 = v$ and set

$$u_n = \mathcal{G}(u_{n-1}),$$

to generate a sequence $u_1, u_2, u_3, \ldots$. One checks that if v is sufficiently small then $\|u_n - |u_{n-1}\|$ decays exponentially with n. Thus the sequence is Cauchy and converges in U to the desired solution.

Suppose now that $\mathcal{F}_\infty$ is defined by some partial differential operator of order r, so that the equation $\mathcal{F}_\infty(u) = v$ is a PDE. There is then an easy general principle that $\mathcal{F}_\infty$ defines a smooth Banach map $\mathcal{F}$ on Sobolev spaces $L^2_k, L^p_k, C_{k,\alpha}$ *once k is sufficiently large*. Suppose also that the linearisation L is elliptic. Then provided that the kernel of L and its adjoint are both trivial we know that L is invertible, and we can fit into the framework above. So for any small $v \in L^2_{k-r}$ we have a small solution $u \in L^2_k$ to the equation $\mathcal{F}(u) = v$. If k is large it is straightfowrward to use elliptic estimates and "bootstrapping" to prove that, if v is actually smooth, then so is u.

Remark. We are being very vague about this, but the proofs—if one is allowed to take k large— are truly very easy. We will work through some examples later. The interesting problems arise when we only know that v is small in some particular sense–say in L^{19}. Then the questions may become delicate and challenging.

Example Consider the equation

$$\Delta f + \sinh f = \rho,$$

for functions f, ρ over a compact n-manifold M, We set this up as a map

$$\mathcal{F}(f) = \Delta f + \sinh f,$$

with

$$\mathcal{F} : L^2_k \to L^2_{k-2},$$

and $k > n/2$. The linearised operator at 0 is

$$L(f) = \Delta f + f,$$

which is invertible. So for all small ρ (in L^2_{k-2} norm) there is a unique small solution f.

It is useful to see explicitly what is going on—unravelling the proof of the inverse function theorem. Let T be the inverse operator to $\Delta + 1$. We define a sequence f_n by

$$f_1 = T\rho,$$

$$f_2 = f_1 + T\left(T\rho - sinh T\rho\right), \ldots$$

and find our solution as the limit of the f_n. Then we can note:

- This is a definite procedure, independent of any function space machinery.
- One could use just the same procedure to find a solution numerically.
- The basic idea (contraction mapping) was also one of the foundations of the linear theory, In practise one might combine the two by using an approximate inverse to $\Delta + 1$.

The theory extends easily to families of equations. Suppose we have some family of nonlinear equations $\mathcal{F}_t(u) = v_t$, depending on a parameter $t \in \mathbf{R}$. If we have a solution u_0 at $t = 0$ and if the linearisation at that solution is invertible then for small t there is a unique nearby solution to the equation with parameter t.

Example

Let q be some given smooth function on our compact manifold M and consider the equation

$$\Delta f + tq|\nabla f|^2 + f = 1$$

When $t = 0$ there is a solution $f = 1$ and the linearisation is $\Delta + 1$ which is invertible. So for small parameter values t there is a unique solution f close to 1.

5.3 Fredholm maps: finite-dimensional reduction

A useful abstract notion is a *Fredholm* map $\mathcal{F} : \Omega \to V$, where $\Omega \subset U$ as before and the derivative $D\mathcal{F} : U \to V$ at each point in Ω is Fredhom, of index d. Let $S_y = \mathcal{F}^{-1}(y)$. We say that y is a regular value if $D\mathcal{F}$ is surjective at all points in S_y. In that case $d \geq 0$ and S_y is a manifold of dimension d. To see this consider a point $x \in S_y$ and let $H = \ker D\mathcal{F}_x$. Choose a complementary subspace H' (this can always be done when H is finite-dimensional). For small $h \in H$ we seek a solution of the equation $\mathcal{F}(x') = y$ where $x' = h + h'$, for $h' \in H'$. The inverse function theorem tells us that there is a unique small solution h'. This gives a local chart on S_y.

Now suppose that y is not a regular value so we have an $x \in S_y$ where the derivative is not surjective. Let I be the image of $D\mathcal{F}_x$ and choose a (finite-dimensional) complementary subspace I'. Let $\mathcal{F}_*$ be the projection of $\mathcal{F} - y$ to I. Then, in a small neighbourhood of x, the discussion above implies that $\mathcal{F}_*^{-1}(0)$ is a manifold and we have a chart

$$B \to \mathcal{F}_*^{-1}(0)$$

where B is a small ball in H. Let $\mathcal{F}_{**}$ be the projection of $\mathcal{F} - y$ to I'. By definition S_y is the intersection $\mathcal{F}_*^{-1}(0) \cap \mathcal{F}_{**}^{-1}(0)$. Composing with the chart above the restriction of $\mathcal{F}_{**}$ to $\mathcal{F}_*^{-1}(0)$ gives a smooth map

$$f : B \to I'$$

and a neighbourhood of x in S_y is identified with $f^{-1}(0)$. In short the solutions of the infinite-dimensional equation $\mathcal{F}(x) = y$ are locally modelled on the finite dimensional equation $f(h) = 0$.

In many situations we can show further that the map f is real analytic and the set S_y has an intrinsic "structure sheaf". One can also prove an infinite-dimensional form of Sard's Theorem (Smale): "generic" values y are regular. (Where "generic" can often mean open, dense.)

Here we can see again the way in which the index d behaves as the difference of the dimensions of U, V.

5.4 Infinite-dimensional manifolds

Often the space in which we have to work is not (or not naturally) an open subset of a Banach space but a "Banach manifold". We will only outline this because the theory will not be needed in the examples we treat in detail.

Spaces of maps Let M, N be compact manifolds and consider the space of smooth maps from M to N. Let f be a point in this space. There is a vector bundle $f^*TN \to M$. A useful way to think of this is as the restriction of the "vertical part" or the tangent bundle of $M \times N$ to the graph of f. A neighbourhood of f in the space of maps can be identified (not canonically) with a neighbourhood of 0 in the space of sections of $f^*TN \to M$. To see one way of doing this, fix a Riemannian metric on N so for each $y \in N$ we have an exponential map

$$e_y : TN_y \to N.$$

Then we can map a section σ of f^*TN to the map which takes $x \in M$ to $e_{f(x)}(\sigma)$. For sufficiently large k this extends to Sobolev completions, so we get a space of L^2_k maps from M to N which is a Banach manifold, locally modelled on the L^2_k sections of f^*TN.

WARNING. It might seem most natural to try to model a neighbourhood of the identity $M \to M$ on the vector fields by using the time 1 map of teh flow generated by a vector field. This is different to what we are doing above, and does not work (there are diffeomorphisms arbitrarily close to the identity which do not arise as such time 1 maps).

Quotients

Consider the equation that a Riemannian metric on a 2-manifold M has constant Gauss curvature. Counting dimensions shows immediately that this cannot be elliptic. However this equation is invariant under the action of the group of diffeomorphisms of M. What we are really interested in is the space of solutions modulo this action. For simplicity we just consider an open set Ω of metrics which have no isometries. Then one can show that the quotient $\Omega/\text{Diff}M$ (after taking Sobolev completions) is an infinite dimensional manifold. To do this we consider *slices* for the action. For example the derivative of the action at a given metric g is a differential operator

$$L : \Gamma(TM) \to \Gamma(s^2T^*M).$$

We consider the subset of metrics $g + \gamma$ where $L^*\gamma = 0$. To show that this gives a slice we need to solve a PDE for a diffeomorphism close to the identity. The constant curvature equation for $g + \gamma$, together with the constraint $L^*\gamma = 0$ forms an elliptic system, except that one needs to modify the definitions lightly because the equations have different orders.

Similar remarks apply to other problems invariant under a group action, such as the minimal submanifold equations for parametrised submanifolds.

In either these two kinds of examples the equations one ends up studying are usually for a zero of a section of a vector bundle over a Banach manifold. For example the diffeomorphism group of M acts on the functions on M so we get a vector bundle

$$\Omega \times_{\text{Diff}M} C^\infty M \to \Omega/\text{Diff}M.$$

WE HAVE NOW COMPLETED OUR DISCUSSION OF PRELIMINARY MATERIAL AND CAN GET ON WITH THE MAIN BUSINESS: THE STUDY OF EXAMPLES.

6 Surfaces of constant Gauss curvature

In this section we use the continuity method to give a simple proof of the fact that any 2-dimensional Riemannian manifold, of negative Euler characteristic, is conformally equivalent to a constant curvature manifold.

Let g_0 be some metric on the compact M^2 with Gauss curvature K_0. Consider the conformal metric $g = e^{2u}g_0$. The Gauss curvature of g is

$$K = e^{-u}(K_0 - \Delta u).$$

Notice that

$$\int_M e^u K d\mu_0 = \int_M K_0 d\mu_0.$$

This is essentially the Gauss-Bonnet theorem since $e^u d\mu_0$ is the area form of the metric g. A simple preliminary observation is that we can choose u so that K has the same sign as the Euler characteristic. So we may as well suppose that this is true for K_0. We deal with the negative case, so $K_0 = -\kappa$ say, where $\kappa > 0$. We want to solve the equation $K = -1$ which is

$$\Delta u + e^u = \kappa.$$

For $t \in [0,1]$ set $\kappa_t = (1-t) + t\kappa$ and consider the equation

$$\Delta u + e^u = \kappa_t, \qquad (*)$$

for $u = u_t$. When $t = 0$ there is a solution $u = 0$, and we want a solution when $t = 1$. If for any t we have a solution the operator appearing in the linearisation is

$$Lf = \Delta f + e^u f,$$

and this is invertible. So there is a solution for parameter values t' close to t. Thus the problem comes down to proving *a priori bounds* on the solutions.

These bounds are easy to obtain using the *maximum principle*. At a point x where a function u achieves its maximum we have $\Delta u \geq 0$. For a solution of (*) we get

$$e^{u_{\max}} \leq \kappa_t(x),$$

which gives an upper bound on $u_{\max}$. Slightly more subtly we get a lower bound on $u_{\min}$, using the fact that $\kappa_t \geq c > 0$. (Of course this is the reason for our preliminary step.) Now e^u is bounded in L^2 so using the equation (*) and the elliptic estimates for Δ we get an L^2_2 bound on u. This gives an L^2_2 bound on e^u and then an L^2_4 bound on u, and so on.

Remarks

- There are more subtle forms of the maximum principle but we do not need these here.
- Clearly the same argument applies to a wide variety of other equations, provided the signs work in the favourable way.

- The case of positive Euler characteristic is much harder from this point of view. Surprisingly the complex analysis proof is *simpler* in this case. The action of the conformal group of the 2-sphere shows that it is impossible to obtain *a priori* bounds in this case. (See also the remark at the end of Section 8.)

7 The Seiberg-Witten equations, enumerative theories

The set-up here will involve an oriented Riemannian 4-manifold M with spin structure and connections on a complex line bundle $L \to M$. We will assume that M is simply connected, but this is really unnecessary.

More on spinors

Let V be a 2-dimensional complex vector space with Hermitian metric and a trivialisation $\Lambda^2 V = \mathbf{C}$. It is the same to say that V is a 1-dimensional quaterionic vector space, with metric. The symmetry group of V is a copy of $SU(2)$ which we call $SU(V)$, with Lie algebra $\mathfrak{su}(V)$. This is a 3-dimensional real Euclidean vector space. We have structure maps

$$\rho : \mathfrak{su}(V) \to \mathrm{End}(V),$$

$$\tau : V \to \mathfrak{su}(V),$$

where $\tau(v)$ is the trace-free part of ivv^*. We normalise so that

$$(i\rho(\tau(v))(v), v) = \frac{1}{2}|v|^4. \quad (*)$$

Now on an oriented Riemannian 4-manifold we have a decomposition of the 2-forms into self-dual and ant-self-dual parts

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

This is compatible with the Laplace operator so the harmonic forms decompose similarly and we have $b_2 = b^+ \oplus b^-$ where $b^+ - b^-$ is the signature.

A spin structure is given by bundles S^+, S^- and these are related to $\Lambda^\pm$ in just the way above, so we have

$$\rho : \Lambda^+ \to \mathrm{End}\ S^+,$$

etc. If A is a connection on L with curvature F_A the Lichnerowicz formula on S^+ is

$$D_A^* D_A \phi = \nabla_A^* \nabla_A \phi + \frac{R}{4}\phi + i\rho(F_A^+)(\phi).$$

The *Seiberg-Witten equations* are for a pair consisting of a connection A and positive spinor field ϕ are

$$D_A\phi = 0 \quad , F_A^+ = \tau(\phi).$$

There is an infinite-dimensional symmetry group given by the action of the $Maps(M, S^1)$ on $L \to M$. This acts freely on the set where $\phi \neq 0$. Solutions with $\phi = 0$ are called reducible solutions. They can only occur if there is an anti-self-dual harmonic form representing $c_1(L)$. It is easy to handle the group action in this case. We fix some connection A_0 and consider $A = A_0 + a$ where $d^*a = 0$. (This is the Coulomb gauge condition from classical electromagnetism.) The equations are then

$$D_{A_0+a}\phi = 0 \;\; d^+a = \tau(\phi) - F_{A_0}^+ \quad d^*a = 0.$$

Let $\tilde{\mathcal{M}}$ be the set of solutions (a, ϕ) to this equation. There is some residual symmetry given by multiplying ϕ by a constant element of S^1. This gives an S^1 action on $\tilde{\mathcal{M}}$ which is free if there are no reducible solutions. The quotient $\tilde{\mathcal{M}}/S^1$ is the "moduli space" of equivalence classes of solutions to the Seieberg-Witten equation.

The linearised operator is equal, modulo lower order terms, to the direct sum of

$$D_{A_0} : \Gamma(S^+ \to \Gamma(S^-),$$
$$d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+.$$

These are both elliptic. The total index, regarding the vector spaces as *real* is

$$d = c_1(L)^2 + \frac{1}{4}(b^+ - b^-) + (b_1 - b_+ - 1).$$

The magic of the theory is that the moduli space $\mathcal{M}$ is *compact*. To see this we apply the Lichnerowicz formula, the basic inequality

$$\Delta|\phi|^2 \leq 2(\phi, \nabla^*\nabla\phi),$$

and the maximum principle. We see then that

$$\max|\phi|^4 \leq c\max|\phi|^2,$$

where c is the maximum value of $-R/4$. Substituting into the equation we get an L^∞ bound, and hence an L^2 bound, on F_A^+, thus on d^+a. Since $d^* \oplus d^+$ has trivial kernel, this leads to an L_1^2 bound on A and from there on it is easy to bootstrap to obtain bounds on all higher derivatives. In shorthand we can go

$$\phi \in L^\infty, A \in L^2 \Rightarrow D_{A_0}\phi \in L^2 \Rightarrow \nabla\phi \in L^2(\&\phi \in L^\infty) \Rightarrow \nabla(\tau(\phi)) \in L^2 \Rightarrow \nabla F^+ \in L^2 \Rightarrow A \in L_2^2, \ldots$$

Now suppose that $b^+ > 0$ and that $c_1(L)$ is not zero.. Then it is not hard to show that there for generic Riemannian metrics g on M the harmonic form representing $c_1(L)$ is not anti-self-dual: thus there are no reducible solutions. There may be points in the moduli space where the linearised operator has non-trivial cokernel but it is easy to see that this cokernel is generated by Ω^+. For a fixed $\eta \in \Omega^+$ we can perturb the equations to

$$F_A^+ = \tau(\phi) + \eta.$$

For generic small perturbations η the corresponding moduli space $\mathcal{M}_\eta$ is a smooth manifold of dimension d (More precisely, $(0, \eta)$ is a regular value for the relevant map.) The compactness proof works equally well for the perturbed equations.

Remark There is a slight complication here because the operator $d^* : \Omega^1 \to \Omega^0$ always has cokernel, given by the constant functions. But this ties in with the S^1 action, so in the end we get the right formula for the dimension.

Suppose we have arranged the topological input so that $d = 0$. Then $\mathcal{M}_\eta$ is a finite set of points. We count these modulo 2 to get an element $n(\eta, g) \in \{0, 1\}$. If we make another choice (η', g') then an argument with generic 1-parameter families gives a compact cobordism between the two moduli spaces, *provided that* $b^+ > 1$. (Otherwise we might hit reducible solutions in the family.) Under this assumption, then, we see that $n(\eta, g) = n(\eta', g')$. We get an invariant in $\{0, 1\}$ of the data consisting of

- a compact, simply-connected, oriented differentiable 4-manifold with $w_2 = 0$ and and $b^+ > 1$
- a nonzero class $c \in H^2(M)$ such that $4c^2 = 3b^+ - b^- + 1$

These have very nontrivial applications in 4-manifold theory.

The theory can be extended in various ways

- If we discuss *orientations* we can attach a sign to each point and get an integer-valued invariant.
- If we take $d > 0$ we could try to use the *cobordism class* of the moduli space, or its *homology class* in the infinite-dimensional manifold of all pairs modulo equivalence.
- By considering the whole Seiberg-Witten *map* we get *Bauer-Furuta* invariants which lie, roughly speaking, in the stable homotopy groups of spheres.

  A good reference for the Seiberg-Witten theory is *J. Morgan: The Seiberg-Witten equations and applications to the topology of smooth 4-manifolds: Princeton UP*

7.1 Holomorphic curves and harmonic maps

Another important "enumerative" theory is given by considering a compact Riemann surface M and a symplectic manifold N with a choice of almost complex structure J. Then we can talk about maps $u : M \to N$ whose derivative is complex linear. This is an elliptic equation in $\mathrm{Maps}(M, N)$. The interesting feature, compared with the Seiberg-Witten theory, is that the moduli spaces are not generally compact. But there is a partial compactness theory which is sufficient to define certain *Gromov-Witten* invariants. We will just discuss some of the PDE aspects of this.

Example Take $M = N = S^2 = \mathbf{C} \cup \{\infty\}$ and the holomorphic maps given by rational functions

$$u(z) = \sum \frac{a_i}{z - \zeta_i}.$$

We may consider, for example, a family with fixed ζ_i and $a_1 = a_1^{(\alpha)} \to 0$. Away from ζ_1 these converge to a map of degree one less. To analyse the behaviour around ζ_1 suppose (without loss of generality) that $z_1 = 0$ and make a sequence of co-ordinate changes

$$w_\alpha = \frac{z}{a_1^{(\alpha)}}.$$

In these rescaled co-ordinates the maps converge, on compact subsets of $\mathbf{C}$.

The energy of any map $u : M \to N$ is

$$\int_M |du|^2.$$

Note that this is *conformally invariant* when M is 2-dimensional. Holomorphic maps minimise energy in their homotopy class (their energy is given by a simple topological formula). The part of the theory we want to discuss applies equally well to harmonic maps with bounded energy. (A map is harmonic if it is an extremum of the energy functional.) We will discuss this first for manifolds M of any dimension n.

Proposition 3 *There are constants C, r_0, ϵ_0 with the following property. Suppose we have a harmonic map $U : M \to N$ that $B \subset M$ is a ball of radius $r \leq r_0$ and there is an $\epsilon \leq \epsilon_0$ such that for ALL other balls $B_\rho \subset B$ of radius ρ we have*

$$\int_{B_\rho} |du|^2 \leq \epsilon \rho^{n-2}.$$

Then $|du|$ at the centre of the ball B is bounded by $C\epsilon r^{-2}$.

We postpone the proof for the moment. When $n = 2$ the condition is satisfied if

$$\int_B |du|^2 \leq \epsilon.$$

Suppose $u_\alpha : M \to N$ is a sequence of harmonic maps with bounded energy, say at most E. Then an elementary covering argument shows that, after passing to a subsequence, (which we ignore in the notation) there are

-
- a finite number of points $p_1, \dots p_k$, with $k \leq 2E/\epsilon$,
- a cover of $M \setminus \{p_1, \dots, p_k\}$ by a countable collection of balls B_i such that

$$\int_{2B_i} |du_\alpha|^2 \leq \epsilon$$

  for all i, α.

It follows that we have uniform bounds on the derivatives over compact subsets of $M \setminus \{p_1, \dots, p_k\}$ and this easily gives C^∞ convergence over these sets. The upshot, after more work, is that the general picture is much the same as that in the case of rational functions above.

To prove the proposition we use a differential inequality for the "energy density" $|du|^2$. A harmonic map from a domain in $\mathbf{R}^n$ to $\mathbf{R}^p$ is just a vector-valued function whose components are harmonic. In particular, all components of the derivative are again harmonic. In the general case we can regard the derivative du as a section of the vector bundle $T^*M \otimes u^*TN$ over M. Withe respect to the the natural covariant derivative on this bundle we have

$$\nabla^*\nabla(du) = K_3(du, du, du) + K_1(du)$$

where $K_3(\ ,\ ), K_1$ are respectively trilinear and linear maps determined by the curvature tensors of M and N. This means that, writing $f = |du|^2$, we have

$$\Delta f \leq C f^2 + C' f,$$

for constants C, C' depending on M, N.

To explain the proof we simplify slightly and suppose that B is a ball of radius r in $\mathbf{R}^n$ and that $C = 1, C' = 0$. Recall the *mean value property* for harmonic functions on open sets in $\mathbf{R}^n$. The same proof shows that if $\Delta g \leq 0$ on some ball then the value of g at the centre of the ball is no greater then the average over the ball. More generally if $\Delta g \leq 2n\kappa$ over over a ball centred at 0 then $g(0)$ does not exceed the average of $g + \kappa|x|^2$ over the ball. Let f be a positive function on B with

$$\Delta f \leq f^2.$$

Suppose that the integral of f over any interior ball of radius ρ is bounded by $\epsilon \rho^{n-2}$. Set

$$M = \max_{x \in B} f(x) d(x)^2,$$

where $d(x) = r - |x|$ is the distance from x to to the boundary of the ball. Let x_0 be a point where the maximum is attained, let $\rho \leq d(x_0)/2$ and let B_ρ be the ball of radius ρ centred at x_0. Write $E = f(x_0$. Then

$$f(x) \leq 4E$$

for all $x \in B_\rho$ so, on this ball,

$$\Delta f \leq 4E^2.$$

The mean value inequality gives

$$E = f(x_0) \leq \frac{1}{\mathrm{Vol}(B_\rho)} \left(\int_{B_\rho} f + \int_{B_\rho} \frac{4E^2}{n} |x - x_0|^2 \right).$$

Calculating the last integral and using the hypothesis we get

$$E \leq c \left(\epsilon \rho^{-2} + E^2 \rho^2 \right),$$

for an explicitly computable constant c. Thus

$$E\rho^2 \leq c(\epsilon + (E\rho^2)^2).$$

Consider the equation, for a real number t,

$$t = c(\epsilon + t^2).$$

When ϵ is sufficiently small this has two roots, a "small root" t_0 which is approximately $c\epsilon$ and a "large root"t_1 which is approximately c^{-1}. So if $t \leq c(\epsilon + t^2)$ we have either $t \leq t_0$ or $t > t_1$. We apply this to the quantity $E\rho^2$ above. When $\rho = 0$ the first alternative clearly holds and by continuity it must persist for all $\rho \leq d(x_0)/2$. Taking $\rho = d(x_0)/2$ we get $M \leq c'\epsilon$. Finally, by the definition of M, we have $f(0) \leq Mr^{-2}$.

A good reference for this material is *McDuff and Salamon: J-holomorphic curves and symplectic topology, AMS.*

8 The Yamabe problem

This is the higher-dimensional version of the problem discussed in Section 6 above. We have a compact Riemannian manifold (M, g) of dimension $n > 2$ and we seek a conformal metric $\tilde{g} = e^{2f} g$ such that the scalar curvature $\tilde{R}$ of $\tilde{g}$ is constant. This can be viewed as a variational problem. Without loss of generality suppose that the volume of (M, g) is 1 and consider the functional

$$I(\tilde{g}) = \frac{n-2}{4(n-1)} \int \tilde{R} d\tilde{\mu},$$

restricted to volume 1 conformal metrics. Then we will see that the constant scalar curvature equation is the corresponding Euler-Lagrange equation. In fact this is part of a more general picture: the condition $\delta I = 0$ among all metrics of volume 1 (not in a fixed conformal class) is the Einstein equation. (The factor $n-2/4(n-1)$ is included here to simplify things later.)

The main result (of Yamabe, Trudinger, Aubin, Schoen...) is

Theorem 5 *There is a volume* 1 *metric in the conformal class which minimises* I.

Remark This might be misleading in regard to the more general problem of Einstein metrics. There is never a minimum of I over all metrics of volume 1. This is distantly related to the work of Perelman on the Ricci flow.

To set up the problem we begin with the equation

$$\tilde{R} = e^{-2f}\left(R + 2(n-1)\Delta f + (n-1)(n-2)|\nabla f|^2\right).$$

It is convenient to write

$$e^{2f} = u^{4/(n-2)}$$

and then manipulation gives

$$\tilde{R} = u^{-(n+2)/(n-2)}\left(\frac{4(n-1)}{(n-2)}\Delta u + Ru\right).$$

In this notation

$$I(\tilde{g}) = \int u^{2n/(n-2)} u^{-(n+2)/(n-2)}\left(\Delta u + \frac{n-2}{4(n-1)}Ru\right),$$

which is just

$$I = \int u\Delta u + \frac{n-2}{4(n-1)}Ru^2,$$

or in turn

$$I(u) = |\nabla u|^2 + \frac{n-2}{4(n-1)}Ru^2.$$

The volume is the integral of $u^{2n/(n-2)}$ and the Euler-Lagrange equation (with Lagrange multiplier) is

$$\Delta u + \frac{n-2}{4(n-1)}Ru = \lambda u^{n+2/n-2},$$

which is the constant scalar curvature equation, with $\tilde{R} = (4(n-1)/(n-2))\lambda$.

Let L_g be the operator

$$L_g f = \Delta f + \frac{(n-2)}{4(n-1)}Rf.$$

This is *conformally invariant* in the sense that

$$L_{\Omega^2 g} f = \Omega^{-(n+2)/2} L_g \left(\Omega^{(n-2)/2} f\right).$$

This statement is a formal consequence of the identity we have above

$$\mathrm{Scal}(u^{4/n-2} g) = u^{-(n+2)/(n-2)} 4\frac{n-1}{n-2} L_g(u),$$

when one considers a product $u_1 u_2$.

The whole theory revolves around the connection with Sobolev embeddings. The exponent $p = 2n/n-2$ is critical for the embedding $L^2_1 \to L^p$. Suppose we changed the problem by fixing p' with $2 \leq p' < p$ and seeking to minimise $I(u)$ subject to the constraints $u > 0$ and $\|u\|_{L_{p'}} = 1$. Then the inclusion $L^2_1 \to L^{p'}$ is *compact*. It follows that a minimising sequence has a subsequence which converges weakly in L^2_1 and strongly in $L^{p'}$. So the constraint on the $L^{p'}$ norm is preserved in the limit. Crucially, the limit is not identically zero. We get a nontrivial weak solution $u \in L^2_1$ of the modified Euler-Lagrange equation, written in the weak form

$$\int \nabla u . \nabla \chi + \frac{n-2}{4(n-1)} R u \chi = \kappa \int u^{p'-1} \chi,$$

for all smooth test functions χ. To begin with we know that $u \geq 0$ (and strictly u is only defined almost everywhere). However one can show by straightforward bootstrapping arguments that u is actually C^2. (We will do a more delicate argument below.) Now a sharper version of the *maximum principle* shows that u is strictly positive.

Proposition 4 *Suppose $f \geq 0$ and satisfies $\Delta f \geq Cf$. Then if f vanishes at any point it is actually identically zero.*

One of the way of proving this is to write $F(\rho)$ for the integral of F over the ball of radius ρ centred at the zero. For simplicity we work in Euclidean space. Then F satisfies a differential inequality $\mathcal{L}(F) \leq CF$ where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = \rho^{n-1} \frac{d}{d\rho} \left(\frac{1}{\rho^{n-1}} \frac{d}{d\rho} \right).$$

Since f and its first derivatives vanishes at the origin we have $F = O(\rho^{n+2})$ for small ρ. We consider for comparison the function $G(\rho) = \rho^{n+1}$. Then $\mathcal{L}G = c\rho^{n-1}$ so $\mathcal{L}G \geq CG$ for $\rho \leq \rho_0$ say. Choose ϵ so that $F(\rho_0) = \epsilon G(\rho_0)$ and suppose $\epsilon > 0$. Then $F - \epsilon G$ has a strictly negative minimum in $(0, \rho_0$ and this gives a contradiction.

The **essential difficulty** in trying to carry this through at the critical exponent problem is that there is no reason why the weak limit should be non-trivial. (Indeed we can see an example of this if we use the conformal transformations of S^n.)

Consider the case when g is the round metric on $S^n = \mathbf{R}^n \cup \{\infty\}$. Metrics in the conformal class can be represented by functions u on $\mathbf{R}^n$ which decay suitably at ∞. The standard round metric is defined by

$$U = C\left(\frac{1}{1+|x|^2}\right)^{n/2-1},$$

where C is a constant chosen so that the integral of U^p is 1. The variational problem is simply to minimise $\|\nabla u\|_{L^2}$ subject to the constraint $\|u\|_{L^p} = 1$. Let μ_0 be the best constant in the Sobolev inequality

$$\mu_0 = \inf \frac{\|\nabla \chi\|_{L^2}^2}{\|\chi\|_{L^p}^2}.$$

Initially we could suppose that χ runs over functions of compact support on $\mathbf{R}^n$, but it is the same to work in a suitable completion of this space, containing U, and functions with the same decay behaviour as U. Then we see that μ_0 is the infimum of the functional I, for metrics conformal to the round sphere. In fact one can show using a symmetrisation argument that U realises this infimum.

Returning to a general manifold (M, g) we write the infimum of I as μ. Then the proof of the theorem has two main components:

- Show that if $\mu < \mu_0$ then there is a smooth minimiser.
- Show that, if (M, g) is not conformal to the round sphere then, $\mu < \mu_0$.

From now on we take $n = 4$, to simplify the notation. To show the first item we first prove

Proposition 5 *Suppose u is a smooth positive solution of an equation $\Delta u = u^3 - Fu$ on a 4-manifold, for some fixed smooth function F. There are ϵ_0, r_0, C such that if B is a ball of radius $r \leq r_0$ and*

$$\int_B u^4 = \epsilon < \epsilon_0$$

then the value of u at the centre of the ball is bounded by $C\epsilon^{1/4} r^{-1}$.

The proof is similar to that of Proposition 3 in the previous section.

This means that we get *a priori* estimates on all derivatives of a solution to our problem, provided we know that every point is the centre of a ball of *fixed size* over which the integral of u^4 is less than ϵ.

The other step is to establish this property, assuming that $\mu < \mu_0$. For simplicity we suppose we work around a point where the original metric is flat. Then we have

$$\Delta u = \mu u^3,$$

where $\mu < \mu_0$. Let β be a cut-off function, to be specified later. We multiply by $\beta^2 u$ and integrate by parts.

$$\int \nabla(\beta^2 u).\nabla u = \mu \int \beta^2 u^4.$$

Now

$$\nabla(\beta^2 u).\nabla u = |\nabla(\beta u)|^2 - |\nabla \beta|^2 u^2.$$

So

$$\int |\nabla(\beta u)|^2 = \mu \int \beta^2 u^4 + \int |\nabla\beta|^2 u^2.$$

Applying the Sobolev inequality to the function βu (viewed as a function of compact support on $\mathbf{R}^4$) we get

$$\mu_0 \|\beta u\|_{L^4}^2 \leq \mu \int \beta^2 u^4 + \int |\nabla \beta|^2 u^2.$$

Now use Cauchy-Schwartz to estimate the two terms on the RHS:

$$\mu_0 \|\beta u\|_{L^4}^2 \leq \mu \|\beta u\|_{L^4}^2 \|u\|_{L^4}^2 + \|\nabla\beta\|_{L^4}^2 \|u\|_{L^4}^2.$$

Now recall that the L^4 norm of u is fixed to be 1. We get

$$(\mu_0 - \mu)\|\beta u\|_{L^4}^2 \leq \|\nabla \beta\|_{L^4}^2.$$

We choose β to be supported on a ball of radius r and equal to 1 on a concentric ball of radius $r' < r$. If we make r' very small we can choose β so that the L^4 norm of $\nabla\beta$ is as small as we please. (This is essentially the fact that L^4_1 does *not* embed in L^∞.) Then we can make the integral of u^4 on the smaller ball as small as we please.

These arguments give us *a priori* bounds on all derivatives of a solution, assuming that $\mu < \mu_0$. With small adaptations they prove the existence of a smooth minimiser. For example the arguments give uniform bounds on solutions of the deformed problem for exponents $p' < p$, then we can take the limit as $p' \to p$.

8.1 Conformal deformations and the Dirac equation

It remains to show that for any compact Riemannian manifold M, $\mu \leq \mu_0$ with strict inequality unless M is conformal to the standard metric on S^n. We will do this assuming that M is a spin manifold.

Consider the standard metric g_{S^n} on S^n, which realises μ_0. This metric can be written locally as $u_0^{4/n-2}\delta_{ij}$ where $u_0 = 1 + O(x^2)$ for small x. Given a small number ϵ we choose v_ϵ to be constant in $|x| \leq \epsilon$ and equal to u_0 if $|x| \geq 2\epsilon$. This gives another metric $g_{S^n,\epsilon}$ on S^n which is flat in an ϵ-ball. It is clear that we can choose the function so that

$$I(g_{S^n,\epsilon}) = I(g_0) + O(\epsilon^{n+2}) \quad , \quad Vol(g_{S^n,\epsilon}) = Vol(g_0) + O(\epsilon^{n+2}).$$

Now consider a point $p \in M$ and the non-compact manifold $M^* = M \setminus \{p\}$. For simplicity we suppose that the metric on M is conformally flat near p, but this is not essential. Suppose there is a conformal metric g_* on M^* which outside a compact set K is isometric to $\mathbf{R}^n \setminus B$ where B is the unit ball. Let $g_{*,\epsilon}$ be this same metric scaled by a factor ϵ^2, so the complement of K is isometric to $\mathbf{R}^n \setminus \epsilon B$. Then we can glue $g_{S^n,\epsilon}$ and $g_{*,\epsilon}$ to get a metric g_ϵ on the compact manifold M, in the given conformal class. Although M^* is not compact the scalar curvature has compact support so its integral is defined and clearly

$$I(g_{*,\epsilon}) = \epsilon^{n-2} I(g_*).$$

Thus

$$I(g_\epsilon) = I(g_{*,\epsilon}) + I(g_{S^n,\epsilon}) = I(g_{S^n}) + \epsilon^{n-2} I(g_*) + O(\epsilon^{n+2}).$$

Similarly

$$\mathrm{Vol}(g_\epsilon) = \mathrm{Vol}(S^n) + O(\epsilon^n).$$

Instead of working with volume 1 metrics it is obviously equivalent to consider the functional

$$\frac{I(g)}{\mathrm{Vol}(g)^{(n-2)/n}}.$$

We see that this is equal to

$$I(S^n) + \epsilon^{n-2} I(g_*) + O(\epsilon^n).$$

Thus it suffices to show that

Proposition 6 *We can find a metric g_* with $I(g_*) \leq 0$ and with strict equality if M_* is not conformal to $\mathbf{R}^n$.*

Let x be the Euclidean co-ordinate on the "end" of M^*. The metric g_* we seek has the form $\sum dx_i^2$ on $|x| > 1$. Suppose instead that we have a metric g_{**} of the form

$$(1 + f) \sum dx_i^2,$$

where $f = O(|x|^{1-n}), \nabla f = O(|x|^{-n})$. Suppose that the scalar curvature of g_{**} is ≤ 0 with strict inequality somewhere. For large ρ we introduce a cut-off function β_ρ equal to 1 on $|x| \leq \rho$ vanishing when $|x| > \rho+1$. Then the curvature of metric $g_{*,\rho}$, given on the end by

$$(1 + \beta_\rho f) \sum dx_i^2,$$

has compact support and it is easy to check that the integral of the scalar curvature is strictly negative, for large enough ρ. (In fact the hypotheses imply that the scalar curvature of g_{**} is integrable, so $I(g_{**})$ is well-defined and $I(g_{*,\rho}$ tends to $I(g_{**})$ as ρ tends to infinity.) Scaling by ρ^{-2} we then get a metric g_* of the form we need.

Now we introduce spinors. The basic fact we exploit is the *conformal invariance* of the Dirac operator. First, we change our notation slightly from the earlier chapters. There we called $D : \Gamma(S^+) \to \Gamma(S^-)$ the Dirac operator. Here it is better to take the total spin space $S = S^+ \oplus S^-$ and the Dirac operator $D : \Gamma(S) \to \Gamma(S)$, equal to $D \oplus D^*$ in the previous notation. Suppose we have the spin bundle $S \to M$ and Dirac operator D determined by a metric g. If $g' = e^{2f} g$ is a conformal metric we can keep the same spin bundle, with the same Hermitian norm, but change the structure map

$$\gamma : T^*M \to \mathrm{End}(S)$$

to $e^{-f}\gamma$. The basic fact we need is that the Dirac operator D' of g' is related to D by

$$D'(s) = e^{-(n+1)f/2} D(e^{(n-1)f/2} s).$$

In particular if $D\sigma = 0$ then $D'(e^{-(n-1)f/2}\sigma) = 0$.

We now consider two cases, on our manifold (M, g).

Case 1 There is a nontrivial solution σ of the Dirac equation $D\sigma = 0$ on M.

Suppose first that σ does not vanish anywhere on M. By making a conformal change as above we can suppose that $|\sigma| = 1$ pointwise on M. Then

$$0 = \Delta|\sigma|^2 \leq 2(\nabla^*\nabla\sigma, \sigma) = -\frac{R}{2},$$

by the Lichnerowicz formula. So we have a metric of scalar curvature ≤ 0 and this means that $\mu \leq 0$, so certainly $\mu < \mu_0$.

To handle the case when σ vanishes somewhere notice that the function u corresponding to the conformal change we are considering above is $|\sigma|^{2\alpha}$ where $\alpha = \frac{n-1}{2(n-2)}$. For $\epsilon > 0$ introduce the function $g_\epsilon(t)$, equal to t^α for $t \geq \epsilon$ and to $a + bt$ for $t < \epsilon$, where a, b are chosen so that the function is C^1. Thus $b = \alpha\epsilon^{\alpha-1}$. Let S be the set of regular values for the function $|\sigma|^2$, which is dense by Sard's

Theorem. We take $\epsilon \in S$ so $\Omega_\epsilon = \{x \in M : |\sigma|^2(x) \leq \epsilon$ is a domain with smooth boundary. We consider the C^1 function

$$u = g_\epsilon(|\sigma|^2)$$

on M. Then Δu is defined except on the measure zero set $\partial\Omega_\epsilon$ and the identity

$$\int_M u\Delta u = \int_M |\nabla u|^2,$$

holds. Outside Ω_ϵ the argument above applies to show that

$$\Delta u + \frac{n-1}{4(n-2)} Ru \leq 0.$$

Inside Ω_ϵ we have, using the fact that $D\sigma = 0$ and the Lichnerowicz formula

$$\Delta u = -b(2|\nabla\sigma|^2 + \frac{R}{4}|\sigma|^2) \leq C\epsilon^\alpha$$

It follows that

$$\int_M |\nabla u|^2 + \frac{n-1}{4(n-2)} Ru^2 \leq C\epsilon^\alpha$$

We can then approximate u in C^1 by smooth functions to see that the infimum of the functional is ≤ 0.

Case 2 The kernel of the Dirac operator on M is trivial.

This the more interesting case. We choose a unit spinor at $p \in M$ and let δ be the corresponding "δ-spinor". Since D is self-adjoint there is a solution σ of the equation

$$D\sigma = \delta.$$

In a local co-ordinate y around p this behaves as

$$|\sigma| = c|y|^{1-n} + O(1). \quad (*)$$

More precisely, since we are assuming the metric is Euclidean near p, we can write σ as a standard fundamental solution, homogeneous of order $1-n$ plus a smooth spinor-valued function. Now we apply the same argument to this spinor. If σ does not vanish we have a conformal metric with a solution σ' of the Dirac equation with $|\sigma'| = 1$ everywhere. This metric is

$$|\sigma|^{4/n-1} \sum dy_i^2 = (|y|^{1-n} + O(1))^{4/n-1} = |y|^{-4}(1 + O(|y|^{n-1}) \sum dy_i^2.$$

Take co-ordinates x_i related to y_i by the standard inversion

$$x_i = |y|^{-2} y_i.$$

Then in the x_i co-ordinates the metric is

$$(1 + O(|x|^{1-n})) \sum dx_i^2,$$

which is just of the kind we wanted. We take a coordinate x related to y by inversion. By the same argument as before, this metric has scalar curvature ≤ 0.

There are just two details to clear up.

- what if the spinor vanishes somewhere
- show that equality holds only if M^* is isometric to $\mathbf{R}^n$.

An argument like that we did above shows that if the spinor vanishes then either there we get a metric with I strictly negative or the function $u = |\sigma|^{2\alpha}$ is a weak solution of the linear equation $\Delta u + \frac{(n-1)R}{4(n-2)} u = 0$, around the zero set. This means that u is smooth and then the maximum principle argument shows that in fact it does not vanish anywhere, as before. Further in the rescaled metric we have a *parallel spinor field*. All of this began with the choice of a unit spinor at p. If we take another choice then we get another solution u' of the same linear equation, with the same growth rate at p and it follows from the maximum principle (and the conformal invariance of the operator) that $u = u'$. It follows that the spinor bundle of the rescaled metric is *flat*, and this easily implies that the metric is Euclidean.

Remarks

1. If $n = 2m$ or $2m + 1$ the *real* dimension of the spin bundle is 2^{m+1} which exceeds n. Thus a generic section of the spin bundle has no zeros. This reinforces the idea that the possible zeros should not play an essential role.

2. The same argument can be adapted to the two dimensional case. If M is an oriented surface and $p \in M$ then either there is a harmonic spinor over M or a harmonic spinor with a pole at p. The first only occurs when the Euler characteristic is negative. In the second case we immediately get a flat metric on the punctured surface when we rescale using the norm of the spinor field. The argument is a variant of the standard Riemann surface theory proof that a surface of positive Euler characteristic is isomorphic to the Riemann sphere.

3. In the literature on the Yamabe problem the proof is usually laid out rather differently. First one shows that if $\mu > 0$ then there is a metric in the conformal class of positive scalar curvature. Then one can solve the equation $L_g u = \delta_p$ and the maximum principle shows that $u > 0$. In co-ordinates y_i around p,

$$u = |y|^{2-n} + A + O(|y|).$$

Essentially the same argument as before shows that it suffices to establish that $A \geq 0$, with equality only when $M^* = \mathbf{R}^n$. After conformal change we get a complete metric with zero scalar curvature with the asymptotic behaviour

$$(1 + A|x|^{2-n} + \ldots) \sum dx_i^2.$$

Then one invokes the *positive mass theorem* which says that in this situation $A \geq 0$, with equality if and only if the manifold is Euclidean. One proof of this (due to Witten) uses spinors, essentially as above. But this requires a spin structure. There is another proof (due to Schoen and Yau) using minimal submanifolds which avoids spinors but is technically much harder, particularly in high dimensions. The existence of these two approaches is somewhat similar to the harmonic 1-forms/ geodesics proofs that $Ricci > 0$ implies $H^1 = 0$.

Good references for the Yamabe problem are the book *Nonlinear analysis on manifolds* of Aubin, and the article *The Yamabe problem* by Parker and Lee, Bulletin AMS 1987.

Questions

For those who wish to be assessed on the course: please send solutions to a selection of the problems to me at s.donaldson@imperial.ac.uk by January 5th. As a guide, reasonable attempts at about 4-5 problems should get a good mark.

1. Find a formula for the Green's Functions on the round sphere S^n.

2. Find a family of functions f_ρ on $\mathbf{R}^n$, for $\rho < 1$, with the following properties

- $f_\rho(x) = 1$ if $|x| \leq \rho$;
- $f_\rho(x) = 0$ if $|x| > 1$;
- $\|\nabla f_\rho\|_{L^n} \to 0$ as $\rho \to 0$.

3. Let E be a vector bundle over a compact Riemannian manifold M with a metric on the fibres. A *covariant derivative* on E is a map ∇ from sections of E to sections of $E \otimes T^*M$ such that

$$\nabla(fs) = f\nabla s + df \otimes s.$$

This is *compatible with the fibre metric* if for any two sections s_1, s_2

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle.$$

Show that in this case the "Kato inequality" holds:

$$|\nabla |s|\ | \leq |\nabla s|.$$

(You may restrict attention here to points where s does not vanish, although in fact the inequality holds everywhere, with a suitable interpretation.) Now let F be another bundle over M and $\sigma : T^*M \otimes E \to F$ be a bundle map such that the composite $D = \sigma \circ \nabla$ is an elliptic operator. Show that there is a constant $k < 1$ such that for all sections s with $Ds = 0$ we have

$$|\nabla |s|\ | \leq k|\nabla s|,$$

(again, at points where s does not vanish).

4. Let M be an compact oriented Riemannian 4-manifold. Show that the operator

$$d^* \oplus d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$$

is elliptic. Identify the kernel and cokernel in terms of harmonic forms.

5. Write out the details of the proof that a moduli space of solutions of the Seiberg-Witten equation, on a fixed line bundle over a compact Riemannian 4-manifold, is compact.

6. Prove Proposition 5 in Section 8 of the notes for the course.

7. Let F be a smooth function on a compact Riemannian 2-manifold M and let I be the functional

$$I(u) = \int_M |\nabla u|^2 + Fu^2 d\mu,$$

restricted to functions u with $\|u\|_{L^4} = 1$. Show that I is bounded below. If λ is the infimum of I, show that a suitable minimising sequence converges to a weak solution of the equation

$$\Delta u + Fu = \lambda u^3.$$

(That is, u is in L^2_1 and for all smooth test functions χ we have

$$\int_M (\nabla u, \nabla \chi) + Fu\chi - \lambda \chi u^3 = 0.)$$