# Conjectures in Kahler geometry

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Abstract. We state a general conjecture about the existence of Kahler metrics of constant scalar curvature, and discuss the background to the conjecture

## 1. The equations

In this article we discuss some well-known problems in Kahler geometry. The general theme is to ask whether a complex manifold admits a preferred Kahler metric, distinguished by some natural differential-geometric criterion. A paradigm is the well-known fact that any Riemann surface admits a metric of constant Gauss curvature. Much of the interest of the subject comes from the interplay between, on the one hand, the differential geometry of metrics, curvature tensors *etc.* and, on the other hand, the complex analytic or algebraic geometry of the manifold. This is, of course, a very large field and we make no attempt at an exhaustive account, but it seems proper to emphasise at the outset that many of these questions have been instigated by seminal work of Calabi.

Let $(V, \omega_0)$ be a compact Kahler manifold of complex dimension $n$. The Kahler forms in the class $[\omega_0]$ can be written in terms of a Kahler potential $\omega_\phi = \omega_0 + i\overline{\partial}\partial\phi$. In the case when $2\pi[\omega_0]$ is an integral class, $e^\phi$ has a geometrical interpretation as the change of metric on a holomorphic line bundle $L \to V$. The *Ricci form* $\rho = \rho_\phi$ is $-i$ times the curvature form of $K_V^{-1}$, with the metric induced by $\omega_\phi$, so $[\rho] = 2\pi c_1(V) \in H^2(V)$. In the 1950's, Calabi [**C1**] initiated the study of *Kahler-Einstein* metrics, with

$$\rho_\phi = \lambda\omega_\phi, \tag{1}$$

for constant $\lambda$. For these to exist we need the topological condition $2\pi c_1(V) = \lambda[\omega]$. When this condition holds we can write (by the $\overline{\partial}\partial$ Lemma)

$$\rho_0 - \lambda\omega_0 = i\overline{\partial}\partial f,$$

for some function $f$. The Kahler-Einstein equation becomes the second order, fully nonlinear, equation

$$(\omega_0 + i\overline{\partial}\partial\phi)^n = e^{f-\lambda\phi}\omega_0^n. \tag{2}$$

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More explicitly, in local co-ordinates $z_\alpha$ and in the case when the metric $\omega_0$ is Euclidean, the equation is

$$\det\left(\delta_{\alpha\beta} + \frac{\partial^2\phi}{\partial z_\alpha \partial \overline{z}_\beta}\right) = e^{f-\lambda\phi}. \tag{3}$$

This is a complex Monge-Ampere equation and the analysis is very much related to that of real Monge-Ampere equations of the general shape

$$\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = F(x,u), \tag{4}$$

where $u$ is a convex function on an open set in $\mathbf{R}^n$. There is a tremendous body of work on these real and complex Monge-Ampere equations. In the Kahler setting, the decisive contributions, dating back to the 1970's are due to Yau [**Y**] and Aubin[**A2**]. The conclusion is, roughly stated, that PDE techniques reduce the problem of finding a solution to that of finding a priori bounds for $\|\phi\|_{L^\infty}$. In the case when $\lambda < 0$ this bound is easily obtained from the maximum principle; in the case when $\lambda = 0$ the bound follows from a more sophisticated argument of Yau. This leads, of course, to the renowned Calabi-Yau metrics on manifolds with vanishing first Chern class. When $\lambda$ is positive, so in algebro-geometric language we are considering a *Fano manifold* $V$, the bound on $\|\phi\|_{L^\infty}$, and hence the existence of a Kahler-Einstein metric, may hold or may not, depending on more subtle properties of the geometry of the manifold $V$ and, in a long series of papers, Tian has made enormous progress towards understanding precisely when a solution exists. Notably, Tian made a general conjecture in [**T**], which we will return to in the next section.

In the early 1980's, Calabi initiated another problem [**C2**]. His starting point was to consider the $L^2$ norm of the curvature tensor as a functional on the metrics and seek critical points, called *extremal Kahler metrics.* The Euler-Lagrange equations involve the *scalar curvature*

$$S = (\rho \wedge \omega^{n-1})/\omega^n.$$

The extremal condition is the equation

$$\overline{\partial}(\text{grad } S) = 0. \tag{5}$$

On the face of it this is a very intractable partial differential equation, combining the full nonlinearity of the Monge-Ampere operator, which is embedded in the definition of the curvature tensor, with high order: the equation being of order six in the derivatives of the Kahler potential $\phi$. Things are not, however, quiet as bad as they may seem. The extremal equation asserts that the vector field grad$S$ on $V$ is holomorphic so if, for example, there are no non-trivial holomorphic vector fields on $V$ the equation reduces to the *constant scalar curvature equation*

$$S = \sigma, \tag{6}$$

where the constant $\sigma$ is determined by $V$ through Chern-Weil theory. This reduction still leaves us with an equation of order four and, from the point of view of partial differential equations, the difficulty which permeates the theory is that one cannot directly apply the maximum principle to equations of this order. From the point of view of Riemannian geometry, the difficulty which permeates the theory is that control of the scalar curvature—in contrast to the Ricci tensor—does not give much control of the metric.

In the case when $[\rho] = \lambda[\omega]$ the constant scalar curvature and Kahler-Einstein conditions are equivalent. Of course this is a global phenomenon: locally the equations are quite different. Obviously Kahler-Einstein implies constant scalar curvature. Conversely, one has an identity

$$\overline{\partial} S = \partial^* \rho,$$

so if the scalar curvature is constant the Ricci form $\rho$ is *harmonic*. But $\lambda\omega$ is also a harmonic form so if $\rho$ and $\lambda\omega$ are in the same cohomology class they must be equal, by the uniqueness of harmonic representatives.

There are two parabolic evolution equations associated to these problems. The Ricci flow

$$\frac{\partial \omega}{\partial t} = \rho - \lambda \omega. \tag{7}$$

and the Calabi flow

$$\frac{\partial \omega}{\partial t} = i\overline{\partial}\partial S. \tag{8}$$

Starting with an extremal metric, the Calabi flow evolves the metric by diffeomeorphims (the one-parameter group generated by the vector field grad $S$): the geometry is essentially unchanged. The analogues of general extremal metrics (nonconstant scalar curvature) for the Ricci flow are the "Ricci solitions"

$$\rho - \lambda\omega = L_v \omega, \tag{9}$$

where $L_v$ is the Lie derivative along a holomorphic vector field $v$.

## 2. Conjectural picture

We present a precise algebro-geometric condition which we expect to be equivalent to the existence of a constant scalar curvature Kahler metric. This conjecture is formulated in [**D2**]; in the Kahler-Einstein/Fano case the conjecture is essentially the same as that made by Tian in [**T**]. An essential ingredient is the notion of the "Futaki invariant". Suppose $L \to V$ is a holomorphic line bundle with $c_1(L) = 2\pi[\omega_0]$ and with a hermitian metric whose induced connection has curvature $-i\omega$. Suppose we have a $\mathbf{C}^*$-action $\alpha$ on the pair $V, L$. Then we get a complex-valued function $H$ on $V$ by comparing the horizontal lift of the vector field generating the action on $V$ with that generating the action on $L$. In the case when $S^1 \subset \mathbf{C}^*$ acts by isometries $H$ is real valued and is just the Hamiltonian in the usual sense of symplectic geometry. The Futaki invariant of the $\mathbf{C}^*$-action is

$$\int_V (S - \sigma) H,$$

where $\sigma$ is the average value of the scalar curvature $S$ (a topological invariant). There is another, more algebro-geometric, way of describing this involving determinant lines. For large $k$ we consider the line

$$\Lambda^{max} H^0(V; L^k).$$

The $\mathbf{C}^*$-action on $(V, L)$ induces an action on this line, with some integer weight $w_k$. Let $d_k$ be the dimension of $H^0(V; L^k)$ and $F(k) = w_k/kd_k$. By standard theory, this has an expansion for large $k$:

$$F(k) = F_0 + F_1 k^{-1} + F_2 k^{-2} + \dots .$$

The equivariant Riemann-Roch formula shows that the Futaki invariant is just the co-efficient $F_1$ in this expansion. Turning things around, we can *define* the Futaki invariant to be $F_1$, the advantage being that this algebro-geometric point of view extends immediately to singular varieties, or indeed general schemes.

Given $(V, L)$, we define a "test configuration" of exponent $r$ to consist of

(1) a scheme $\mathcal{V}$ with a line bundle $\mathcal{L} \to \mathcal{V}$;
(2) a map $\pi : \mathcal{V} \to \mathbf{C}$ with smooth fibres $V_t = \pi^{-1}(t)$ for non-zero $t$, such that $V_t$ is isomorphic to $V$ and the restriction of $\mathcal{L}$ to $L^r$;
(3) a $\mathbf{C}^*$-action on $\mathcal{L} \to \mathcal{V}$ covering the standard action on $\mathbf{C}$.

We define the Futaki invariant of such a configuration to be the invariant of the action on the central fibre $\pi^{-1}(0)$, (with the restriction of $\mathcal{L}$): noting that this may not be smooth. We say that the configuration is "destabilising" if the Futaki invariant is bigger than or equal to zero and, in the case of invariant zero the configuration is not a product $V \times \mathbf{C}$. Finally we say that $(V, L)$ is "K-stable" (Tian's terminology) if there are no destabilising configurations. Then our conjecture is:

Conjecture 1. *Suppose $(V, \omega_0)$ is a compact Kahler manifold and $[\omega_0] = 2\pi c_1(L)$. Then there is a metric of constant scalar curvature in the class $[\omega_0]$ if and only if $(V, L)$ is K-stable.*

The direct evidence for the truth of this conjecture is rather slim, but we will attempt to explain briefly why one might hope that it is true.

The first point to make is that "K-stability", as defined above, is related to the standard notion of "Hilbert-Mumford stability" in algebraic geometry. That is, we consider for fixed large $k$, the embedding $V \to \mathbf{CP}^N$ defined by the sections of $L^k$ which gives a point $[V, L]_k$ in the appropriate Hilbert scheme of subschemes of $\mathbf{CP}^N$. The group $SL(N+1, \mathbf{C})$ acts on this Hilbert scheme, with a natural linearisation, so we have a standard notion of Geometric Invariant Theory stability of $[V, L]_k$. Then K-stability of $(V, L)$ is closely related to the stability of $[V, L]_k$ for all sufficiently large $k$. (The notions are not quite the same: the distinction between them is analogous to the distinction between "Mumford stability" and "Gieseker stability" of vector bundles.)

The second point to make is that there is a "moment map" interpretation of the differential geometric set-up. This is explained in more detail in [**D1**] , although the main idea seems to be due originally to Fujiki [**F**]. For this, we change our point of view and instead of considering different metrics (i.e. symplectic forms) on a fixed complex manifold we fix a symplectic manifold $(M, \omega)$ and consider the set $\mathcal{J}$ of compatible complex structures on $M$. Thus a point $J$ in $\mathcal{J}$ gives the same data–a complex manifold with a Kahler metric–which we denoted previously by $(V, \omega)$. The group $\mathcal{G}$ of "exact" symplectomorphisms of $(M, \omega)$ acts on $\mathcal{J}$ and one finds that the map $J \mapsto S - \sigma$ is a *moment map* for the action. In this way, the moduli space of constant scalar curvature Kahler metrics appears as the standard symplectic quotient of $\mathcal{J}$. In such situations, one anticipates that the symplectic quotient will be identified with a complex quotient, involving the complexification of the relevant group. In the case at hand, the group $\mathcal{G}$ does not have a *bona fide* complexification but one can still identify infinite-dimensional submanifolds which play the role of the orbits of the complexification: these qare just the equivalence classes under the equivalence relation $J_1 \sim J_2$ if $(M, J_1)$ and $(M, J_2)$ are isomorphic as complex manifolds. With this identification, and modulo the detailed notion of stability, Conjecture 1 becomes the familiar statement that a stable orbit for the complexified group contains a zero of the moment map. Of course all of this is a formal picture and does not lead by itself to any kind of proof, in this infinite-dimensional setting. We do however get a helpful and detailed analogy with the better understood theory of Hermitian Yang-Mills connections. In this analogy the constant scalar curvature equation corresponds to the Hermitian Yang-Mills equation, for a connection $A$ on a holomorphic bundle over a *fixed* Kahler manifold,

$$F_A.\omega = \text{constant}.$$

The extremal equation $\overline{\partial}(\text{grad } S) = 0$ corresponds to the Yang-Mills equation

$$d_A^* F_A = 0,$$

whose solutions, in the framework of holomorphic bundles, are just direct sums of Hermitian-Yang-Mills connections.

Leaving aside these larger conceptual pictures, let us explain in a down-to-earth way why one might expect Conjecture 1 to be true. Let us imagine that we can solve the Calabi flow equation (8) with some arbitrary initial metric $\omega_0$. Then, roughly, the conjecture asserts that one of four things should happen in the limit as $t \to \infty$. (We are discussing this flow, here, mainly for expository purposes. One would expect similar phenomena to appear in other procedures, such as the continuity method. But it should be stressed that, in reality, there are very few rigorous results about this flow in complex dimension $n > 1$: even the long time existence has not been proved.)

(1) The flow converges, as $t \to \infty$, to the desired constant scalar curvature metric on $V$.
(2) The flow is asymptotic to a one-parameter family of extremal metrics on the same complex manifold $V$, evolving by diffeomeorphisms. Thus in this case $V$ admits an extremal metric. Transforming to the other setting, of a fixed symplectic form, the flow converges to a point in the equivalence class defined by $V$. In this case $V$ cannot be K-stable, since the diffeomorphisms arise from a $\mathbf{C}^*$-action on $V$ with non-trivial Futaki invariant and we get a destabilising configuration by taking $\mathcal{V} = V \times \mathbf{C}$ with this action.
(3) The manifold $V$ does not admit an extremal metric but the transformed flow $J_t$ on $\mathcal{J}$ converges. In this case the limit of the transformed flow lies in another equivalence class, corresponding to another complex structure $V'$ on the same underlying differentiable manifold. The manifold $V'$ admits an extremal metric. The original manifold $V$ is not K-stable because there is a destabilising configuration where $\mathcal{V}$ is *diffeomorphic* to $V \times \mathbf{C}$ but the central fibre has the different complex structure $V'$ ("jumping" of complex structure).
(4) The transformed flow $J_t$ on $\mathcal{J}$ does not converge to any complex structure on the given underlying manifold but some kind of singularities develop. However, one can still make sufficient sense of the limit of $J_t$ to extract a

scheme from it, and this scheme can be fitted in as the central fibre of a destabilising configuration, similar to case (3).

We stress again that this is more of a programme of what one might hope eventually to prove, rather than a summary of what is really known. In the Kahler-Einstein/Fano situation one can develop a parrallel programme (as sketched by Tian in [**T**]) for the Ricci flow (about which much more is known), where Ricci solitons take the place of extremal metrics. In any event we hope this brings out that point that one can approach two kinds of geometric questions, which on the face of it seem quite different.

(1) ALGEBRAIC GEOMETRY PROBLEM. Describe the possible destabilising configurations and in particular the nature of the singularities of the central fibre (e.g does one need schemes as opposed to varieties?).
(2) PDE/DIFFERENTIAL GEOMETRY PROBLEM. Describe the possible behaviour of the Calabi flow/Ricci flow (or other continuity methods), and the nature of the singularities that can develop.

The essence of Conjecture 1 is that these different questions should have the same answer.

## 3. Toric varieties and a toy model

One can make some progress towards the verification of Conjecture 1 in the case when $V$ is a toric variety [**D2**]. Such a variety corresponds to an integral polytope $P$ in $\mathbf{R}^n$ and the metric can be encoded in a convex function $u$ on $P$. The constant scalar curvature condition becomes the equation (due to Abreu [**A1**])

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -\sigma, \tag{10}$$

where $(u^{ij})$ is the inverse of the Hessian matrix $(u_{ij})$ of second derivatives of $u$. This formulation displays very well the way in which the equation is an analogue, of order 4, of the real Monge-Ampere equation (4). The equation is supplemented by boundary conditions which can be summarised by saying that the desired solution should be an absolute minimum of the functional

$$\mathcal{F}(u) = \int_P -\log\det(u_{ij}) + \mathcal{L}(u), \tag{11}$$

where

$$\mathcal{L}(u) = \int_{\partial P} u \, d\rho - \sigma \int_P u \, d\mu. \tag{12}$$

Here $d\mu$ is Lebesgue measure on $P$ and $d\rho$ is a natural measure on $\partial P$. (Each codimension-1 face of $\partial P$ is defined by a linear form, which we can normalise to have coprime integer co-efficients. This linear form and the volume element on $\mathbf{R}^n$ induce a volume element on the face.) We wish to draw attention to one interesting point, which can be seen as a very small part of Conjecture 1. Suppose there is a non-trivial convex function $g$ on $P$ such that $\mathcal{L}(g) \leq 0$. Then one can show that there is $\mathcal{F}$ does not attain a minimum so there is no constant scalar curvature metric. Suppose on the other hand that $f$ is a *piecewise linear, rational* convex function. (That is, the maxiumum of a finite set of rational affine linear functions.) Then one can associate a canonical test configuration to $f$ and show that this is destabilising if $\mathcal{L}(f) \leq 0$. Thus we have

CONJECTURE 2. *If there is a non-trivial convex function $g$ on $P$ with $\mathcal{L}(g) \leq 0$ then there is a non-trivial piecewise-linear, rational convex function $f$ with $\mathcal{L}(f) \leq 0$.*

This is a problem of an elementary nature, which was solved in [**D2**] in the case when the dimension $n$ is 2, but which seems quite difficult in higher dimensions. (And one can also ask for a more conceptual proof than that in [**D2**] for dimension 2.) On the other hand if this Conjecture 2 is false then very likely the same is true for Conjecture 1: in that event one probably has to move outside algebraic geometry to capture the meaning of constant scalar curvature.

Even in dimension 2 the partial differential equation (10) is formidable. We can still see some interesting things if we go right down to dimension 1. Thus is this case $V$ is the Riemann sphere and $P$ is the interval $[-1,1]$ in $\mathbf{R}$. The equation (10) becomes

$$\frac{d^2}{dx^2}((u'')^{-1}) = -\sigma, \tag{13}$$

which one can readily solve explicitly. This is no surprise since we just get a description of the standard round metric on the 2-sphere. To make things more interesting we can consider the equation

$$\frac{d^2}{dx^2}((u'')^{-1}) = -A, \tag{14}$$

where $A$ is a function on $(-1,1)$. This equation has some geometric meaning, corresponding to a rotationally invariant metric on the sphere whose scalar curvature is a given function $A(h)$ of the Hamiltonian $h$ for the circle action. The boundary conditions we want are, in this case, $u'' \sim (1 \pm x)^{-1}$ as $x \to \pm 1$. But if we have a solution with

$$u''(x) \to \infty \tag{15}$$

as $x \to \pm 1$, these are equivalent to the normalisations

$$\int_{-1}^{1} A(x)dx = 1, \quad \int_{-1}^{1} xA(x)dx = 0, \tag{16}$$

which we suppose hold.

We now consider the linear functional

$$\mathcal{L}_A(u) = u(1) + u(-1) - \int_{-1}^{1} u(x)A(x)dx. \tag{17}$$

and

$$\mathcal{F}_A(u) = \int_{-1}^{1} -\log(u''(x))dx + \mathcal{L}_A(u). \tag{18}$$

Then we have

THEOREM 1. *There is a solution to equations (14), (15) if and only if $\mathcal{L}_A(f) > 0$ for all (non-affine) convex functions $g$ on $[-1,1]$. In this case the solution is an absolute minimum of the functional $\mathcal{F}_A$.*

To prove this we consider the function $\phi(x) = 1/u''(x)$. This should satisfy the equation $\phi'' = -A$ with $\phi \to 0$ at $\pm 1$. Thus the function $\phi$ is given via the usual Green's function

$$\phi(x) = \int_{-1}^{1} g_x(y) A(y) dy,$$

where $g_x(y)$ is a linear function of $y$ on the intervals $(-1, x)$ and $(x, 1)$, vanishing on the endpoints $\pm 1$ and with a negative jump in its derivative at $y = x$. Thus $-g_x(y)$ is a convex function on $[-1, 1]$ and

$$\mathcal{L}_A(-g_x) = \int_{-1}^{1} A(y) g_x(y) dy = \phi(x).$$

Thus our hypothesis ($\mathcal{L}_A(g) > 0$ on convex $g$) implies that the solution $\phi$ is positive throughout $(-1, 1)$ so we can form $\phi^{-1}$ and integrate twice to solve the equation

$$u'' = \phi^{-1}$$

thus finding the desired solution $u$. The converse is similar. The fact that the solution is an absolute minimum follows from the convexity of the functional $\mathcal{F}_A$.

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