Problem sheet 8

(a) Eliminate the arbitrary functions from the following to obtain first order partial differential equations for $u$: (i) $u = f(x + y)$
(ii) $u = f(xy)$
(iii) $u = x^n f(y/x)$.

(b) Solve the first order PDE

$$yu_x + xu_y = 0$$

subject to i) $u = y$ on $x = 0$ and ii) $u = \cos x$ on $x^2 + y^2 = 1$.

(c) Consider the first order linear PDE

$$(x - y)y^2 u_x + (y - x)x^2 u_y = (x^2 + y^2)u$$

and show that the general solution is of the form

$$F\left(x^3 + y^3, \frac{u}{(x-y)}\right) = 0$$

for arbitrary $F$.

(d) Solve the initial value problem

$$u_t + \exp(x) u_x = 0, \quad u(x, 0) = x.$$ 

(e) Consider the linear first order PDE

$$tv_x + xv_t = cv$$

determine the characteristics and show that the general solution has the form

$$v(x, t) = (x + t)^c F(x^2 - t^2).$$
(a) This example is going “backwards” and shows how the arbitrary function naturally appears. (i) \( u = f(x+y) \). We take \( u_x = f'(x+y), u_y = f'(x+y) \) and so \( u_x - u_y = 0 \).
(ii) \( u = f(xy) \). We take \( u_x = yf'(xy), u_y = xf'(xy) \) so \( xu_x - yu_y = 0 \).
(iii) \( u = x^n f(y/x) \) so \( u_x = nx^{n-1} f(y/x) - x^{n-2} yf'(y/x) \) and \( u_y = x^{n-1} f'(y/x) \). If we look at \( u_x \) we notice that \( u_x = nu/x - yu_y/x \) and so \( xu_x + yu_y = nu \)

(b) A linear first order homogeneous PDE. The characteristic equations
\[
\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0}
\]
so \( u = \text{constant} \) and \( y^2 - x^2 = \text{constant} \). The latter defines the characteristic equations. Since \( y^2 - x^2 \) is an integral of the PDE we have that
\[ u(x, y) = F(y^2 - x^2) \]
as our general solution.

1) If \( u(0, y) = y \) then \( y = F(y^2) \) and so \( u = \sqrt{y^2 - x^2} \). This solution is only valid in the sectors \( y > x \) and \( y < -x \). If we sketch the characteristics that intersect the \( y \) axis then we see that they only continue into those regions.

2) If \( u = \cos x \) on the circle \( x^2 + y^2 = 1 \) then
\[ \cos x = F(1 - 2x^2) \]
we have to invert this relation so we set \( t = 1 - 2x^2 \) and see that
\[ x = \sqrt{(1 - t)/2}, \quad F(t) = \cos(\sqrt{(1 - t)/2}) \]
therefore the solution is
\[ u(x, y) = \cos \left[ \left( \frac{1}{2} (1 + x^2 - y^2) \right)^{\frac{1}{2}} \right]. \]

(c) From the characteristic equations we obtain
\[ \frac{dx}{y^2} = -\frac{dy}{x^2} \]
and so \( x^3 + y^3 = \text{constant} = I_1 \). Following the hint
\[ \frac{d(x-y)}{(x-y)} = \frac{du}{u} \]
and so \( u/(x-y) = \text{constant} = I_2 \). Thus
\[ F \left( x^3 + y^3, \frac{u}{x-y} \right) = 0 \]
as required.

(d) This is a straightforward example:
\[ \frac{dt}{1} = e^{-x} dx = \frac{du}{0} \]
which gives \( u = f(t + e^{-x}) \) for arbitrary \( f \). The initial condition gives \( x = f(e^{-x}) \), set \( z = e^{-x} \) and we get \( f(z) = -\log z \) so
\[ u = -\log(t + e^{-x}). \]
(e) The characteristic equations are
\[
\frac{dx}{ds} = t, \quad \frac{dt}{ds} = x, \quad \frac{dv}{ds} = cv
\]
from which
\[
xdx = tdt, \quad x^2 - t^2 = \text{constant}
\]
and
\[
\frac{dv}{cv} = \frac{d(x + t)}{(x + t)}
\]
and thus \( v = A(x + t)^c \) for some constant \( A \). Putting this together we get
\[
v(x, t) = (x + t)^c F(x^2 - t^2).
\]