Spectral shift function of the Schrödinger operator in the large coupling constant limit

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1. Introduction. Let $H_0$ and $H$ be selfadjoint operators in a Hilbert space. If the difference $H - H_0$ is a trace class operator, then there exists a function $\xi \in L^1(\mathbb{R})$ such that the following trace formula due to I. M. Lifshitz and M. G. Krein holds true:

$$\text{Tr}(\phi(H) - \phi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)\phi'(\lambda)d\lambda, \quad \forall \phi \in C_{0}^\infty(\mathbb{R}).$$

The function $\xi(\lambda) = \xi(\lambda; H, H_0)$ is called the spectral shift function for the pair $H_0, H$. A detailed exposition of the spectral shift function theory can be found in the book [9]; see also the survey [3].

Let us consider the (selfadjoint) Laplace operator $\Delta$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Define the operator $H_0 = h(-\Delta)$, where the function $h : [0, +\infty) \to \mathbb{R}$ satisfies

$$h \in C^2(\mathbb{R}), \quad h(0) = 0, \quad h'(r) > 0 \quad \forall r > 0,$$

there exists the limit $\lim_{r \to +\infty} r^{-m}h(r) = h_\infty > 0, \quad m > 0$. (1)

Next, let the perturbation $V$ be the operator of multiplication by a real valued potential $V(x)$, which satisfies the estimate

$$|V(x)| \leq \frac{C}{(1 + |x|)^l}, \quad l > d.$$ (2)

Let $H = H_0 + V$. The most important case is the Schrödinger operator, which corresponds to the choice $h(r) = r$. However, we consider a fairly wide class of functions $h$ in order to demonstrate the dependence of our results on the symbol of the (pseudo)differential operator $H_0$. 

1
Although the difference $H - H_0$ is not of the trace class, condition (2) ensures that the difference of sufficiently high powers of the resolvents of $H$ and $H_0$ is of the trace class. This allows one to define the spectral shift function $\xi(\lambda; H, H_0)$ on the basis of the invariance principle (cf. [3]).

Various results about the high energy ($\lambda \to +\infty$) or semiclassical ($h(r) = h_\infty r$, $h_\infty \to 0$) asymptotic behaviour of the spectral shift function $\xi(\lambda; H_0 + \alpha V, H_0)$ are known. In the present paper we address the question of the asymptotic behaviour of the spectral shift function $\xi(\lambda; H_0 + \alpha V, H_0)$ in the large coupling constant limit: $\alpha \to +\infty$.

2. Results. It turns out that the asymptotic behaviour of the spectral shift function depends heavily on the sign of the potential $V$. For non-positive potentials one has

**Theorem 1** Let $h$ satisfy conditions (1). Let $H_0 = h(-\Delta)$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Assume that the potential $V \leq 0$ satisfies estimate (2) for $l > \max\{d, 2m\}$. Then for almost all $\lambda \in \mathbb{R}$ the following asymptotic formula holds true:

\[
\xi(\lambda; H_0 + \alpha V, H_0) = -\alpha^{d/(2m)} C_1 (1 + o(1)), \quad \alpha \to +\infty,
\]

\[
C_1 = (2\pi)^{-d} h_\infty^{-d/(2m)} \text{vol}\{x \in \mathbb{R}^d \mid |x| < 1\} \int_{\mathbb{R}^d} |V(x)|^{d/(2m)} \, dx.
\]

(3)

Let us now discuss the case of non-negative potentials. In this case one has to consider potentials with power asymptotics at infinity. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Assume that for some non-negative function $\Psi \in C(\mathbb{S}^{d-1})$ one has

\[
\sup_{\omega \in \mathbb{S}^{d-1}} |V(\rho \omega) - \Psi(\omega)\rho^{-l}| = o(\rho^{-d}), \quad \rho \to \infty.
\]

(4)

**Theorem 2** Let $h$ satisfy conditions (1). Let $H_0 = h(-\Delta)$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Assume that the potential $V \geq 0$ is bounded and satisfies the condition (4) with some function
\[ \Psi \in C(S^{d-1}), \Psi \geq 0, \text{ and some } l \geq d. \text{ Then for all } \lambda > 0 \text{ the following asymptotic formula holds true:} \]

\[
\xi(\lambda; H_0 + \alpha V, H_0) = \alpha^{d/l} C_2 (1 + o(1)), \quad \alpha \to +\infty,
\]

\[
C_2 = (2\pi)^{-d} d^{-1} \int_{\|p\|^2 < \lambda} (\lambda - h(\|p\|^2))^{-d/l} dp \int_{S^{d-1}} \Psi^{d/l}(\hat{x}) d\hat{x}.
\] (5)

Theorem 1 with \( h(r) = r \) has been proven by the first author in [5]. The case of general \( h \) can be easily dealt with by combining the techniques of [5] and [6]. Theorem 2 is a joint result of the authors [6].

The main ingredients of the proof of Theorem 2 are a representation for the spectral shift function from [4], the asymptotic formula for the spectrum of pseudodifferential operators \([1, 2]\), the variational quotients technique, and several facts about the boundary value problems for elliptic pseudodifferential equations. All the difficulties of the proof appear already in the case \( h(r) = r \); the generalization to the case of arbitrary \( h \) is not difficult.

3. Discussion. 1. For \( \lambda < 0 \) under the hypothesis of Theorem 1 the spectral shift function \( \xi(\lambda) \) is the negative of the number of eigenvalues of the operator \( H_0 + \alpha V \) in the interval \( (-\infty, \lambda) \). Therefore, for \( \lambda < 0, V \leq 0 \) and \( h(r) = r \), formula (3) turns into the well known Weyl asymptotic formula for the counting function for the spectrum of the Schrödinger operator (cf. [7, Theorem XIII.80]).

2. It is clear from (3) that the order of the leading term of the asymptotics of the spectral shift function depends on the growth order of the symbol \( h \) at infinity but does not depend of the potential \( V \). For \( V \geq 0 \) the situation is opposite and the roles of the coordinate and momentum variables are reversed.

3. Asymptotic coefficients \( C_1 \) and \( C_2 \) can be interpreted in terms of the phase space volume. Indeed, one readily checks that

\[
C_1 = \lim_{\alpha \to +\infty} \alpha^{-d/(2m)} \text{vol}\{(x, p) \in \mathbb{R}^{2d} \mid h(\|p\|^2) + \alpha V(x) < \lambda < h(\|p\|^2)\},
\]

\[
C_2 = \lim_{\alpha \to +\infty} \alpha^{-d/l} \text{vol}\{(x, p) \in \mathbb{R}^{2d} \mid h(\|p\|^2) < \lambda < h(\|p\|^2) + \alpha V(x)\}.
\]
4. The paper [5] contains a statement (Theorem 1.7) about potentials \(V\) of a variable sign. More precise results can be obtained by combining the techniques of papers [5], [6] and [8]. Let us note briefly that in the case of a potential \(V\) of variable sign and \(l \neq 2m\) the leading term of the asymptotics of the spectral shift function is \(Ca^\nu\), where \(\nu = \max\{d/(2m), d/l\}\), and constant \(C\) can be explicitly expressed in terms of the potential \(V\).

5. Finally we note that condition (1) can be relaxed in several directions.

**Literature**


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