STUDIA MATHEMATICA

Online First version

Hardy–Littlewood–Paley inequalities and Fourier multipliers on SU(2)

by

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Abstract. We prove noncommutative versions of Hardy–Littlewood and Paley inequalities relating a function and its Fourier coefficients on the group SU(2). We use it to obtain lower bounds for the L^p-L^q norms of Fourier multipliers on SU(2) for $1 < p \leq 2 \leq q < \infty$. In addition, we give upper bounds of a similar form, analogous to the known results on the torus, but now in the noncommutative setting of SU(2).

1. Introduction. Let $\mathbb{T}^n$ be the n-dimensional torus and let $1 < p \leq q < \infty$. A sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^n}$ of complex numbers is said to be a *multiplier of trigonometric Fourier series* from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$ if the operator

$$T_\lambda f(x) = \sum_{k\in\mathbb{Z}^n} \lambda_k \widehat{f}(k) e^{ikx}$$

is bounded from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$. We denote by $\mathbf{m}_p^q$ the set of such multipliers.

Many problems in harmonic analysis and partial differential equations can be reduced to the boundedness of multiplier transformations. There arises a natural question of finding sufficient conditions for $\lambda \in \mathbf{m}_p^p$. The topic of $\mathbf{m}_p^q$ multipliers has been extensively researched. Using methods such as Littlewood–Paley decomposition and Calderón–Zygmund theory, it is possible to prove Hörmander–Mihlin type theorems (see e.g. Mihlin [Mih57, Mih56], Hörmander [Hör60], and later works).

Multipliers have been analysed in a variety of different settings (see e.g. Gaudry [Gau66], Cowling [Cow74], Vretare [Vre74]). The literature on spectral multipliers is too rich to be reviewed here (see e.g. a recent paper

2010 *Mathematics Subject Classification*: Primary 43A85, 43A15; Secondary 35S05.

Key words and phrases: Fourier multipliers, Hardy–Littlewood inequality, Paley inequality, noncommutative harmonic analysis.

Received 16 October 2014; revised 21 April 2016.

Published online *.

DOI: 10.4064/sm8106-4-2016 © Instytut Matematyczny PAN, 2016

[CKS11] and references therein). The same is true for multipliers on locally compact abelian groups (see e.g. [Arh12]), or for Fourier or spectral multipliers on symmetric spaces (see e.g. [Ank90] or [CGM93], resp.). We refer to the above and to other papers for further references on the history of $\mathbf{m}_p^q$ multipliers on spaces of different types.

In this paper we are interested in Fourier multipliers on compact Lie groups, in which case the literature is much more sparse; below, we will make a more detailed review of the existing results. In this paper we will be investigating several questions in the model case of Fourier multipliers on the compact group SU(2). Although we will not explore them in this paper, there are links between multipliers on SU(2) and those on the Heisenberg group (see Ricci and Rubin [RR86]).

In general, most of the multiplier theorems imply that $\lambda \in \mathbf{m}_p^p$ for all $1 < p < \infty$ at once. Stein [Ste70] raised the question of finding more subtle sufficient conditions for a multiplier to belong to some $\mathbf{m}_p^p$, $p \neq 2$, without implying that it also belongs to all $\mathbf{m}_p^p$, $1 < p < \infty$. Nursultanov and Tleukhanova [NTl00] provided conditions on $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ to belong to $\mathbf{m}_p^q$ for $1 < p \leq 2 \leq q < \infty$. In particular, they established lower and upper bounds for the norms of $\lambda \in \mathbf{m}_p^q$ which depend on p and q. This provided a partial answer to Stein's question. Let us recall their result in the case $n = 1$:

THEOREM 1.1. *Let $1 < p \leq 2 \leq q < \infty$ and let M_0 denote the set of all finite arithmetic progressions in $\mathbb{Z}$. Then*

$$\sup_{Q\in M_0} \frac{1}{|Q|^{1+1/q-1/p}} \Big| \sum_{m\in Q} \lambda_m \Big| \lesssim \|T_\lambda\|_{L^p\to L^q} \lesssim \sup_{k\in\mathbb{N}} \frac{1}{k^{1+1/q-1/p}} \sum_{m=1}^{k} \lambda_m^*,$$

where λ_m^* is a non-increasing rearrangement of λ_m, and $|Q|$ is the number of elements in the arithmetic progression Q .

In this paper we study noncommutative versions of this and other related results. As a model case, we concentrate on Fourier multipliers between Lebesgue spaces on the group SU(2) of 2×2 unitary matrices with determinant one. Sufficient conditions for Fourier multipliers on SU(2) to be bounded on L^p-spaces have been analysed by Coifman–Weiss [CW71b] and Coifman–de Guzman [CdG71] (see also Chapter 5 in Coifman and Weiss' book [CW71a]), and are given in terms of the Clebsch–Gordan coefficients of representations of SU(2). A more general perspective was provided in [RW13] where conditions on Fourier multipliers to be bounded on L^p were obtained for general compact Lie groups, and Mihlin–Hörmander theorems on general compact Lie groups have been established in [RW15].

Results about spectral multipliers are better known, for functions of the Laplacian (N. Weiss [Wei72] or Coifman and Weiss [CW74]), or of the sub-Laplacian on SU(2) (Cowling and Sikora [CS01]). However, following [CW71b, CW71a, RW13, RW15], here we are rather interested in Fourier multipliers.

In this paper we obtain lower and upper estimates for the norms of Fourier multipliers acting between L^p and L^q spaces on SU(2). These estimates explicitly depend on the parameters p and q. Thus, this paper can be regarded as a contribution to Stein's question in the noncommutative setting of SU(2). At the same time we provide a noncommutative analogue of Theorem 1.1. Briefly, let A be the Fourier multiplier on SU(2) given by

$$\widehat{Af}(l) = \sigma_A(l)\widehat{f}(l) \quad \text{for } \sigma_A(l) \in \mathbb{C}^{(2l+1)\times(2l+1)},\ l \in \tfrac{1}{2}\mathbb{N}_0,$$

where we refer to Section 2 for definitions and notation related to Fourier analysis on SU(2). For such operators, in Theorem 3.1, for $1 < p \le 2 \le q < \infty$, we give two lower bounds, one of which is

$$\sup_{l\in\frac{1}{2}\mathbb{N}_0} \frac{1}{(2l+1)^{1+1/q-1/p}} \frac{1}{2l+1} |\operatorname{Tr}\sigma_A(l)| \lesssim \|A\|_{L^p(\mathrm{SU}(2))\to L^q(\mathrm{SU}(2))}. \tag{1.1}$$

A related upper bound

$$\|A\|_{L^p(\mathrm{SU}(2))\to L^q(\mathrm{SU}(2))} \lesssim \sup_{s>0} s\Big(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ \|\sigma_A(l)\|_{\mathrm{op}}\ge s}} (2l+1)^2\Big)^{1/p-1/q} \tag{1.2}$$

will be given in Theorem 4.1.

The proof of the lower bound is based on the new inequalities describing the relationship between the "size" of a function and the "size" of its Fourier transform. These inequalities can be viewed as a noncommutative SU(2)-version of the Hardy–Littlewood inequalities [HL27]. To explain this briefly, we recall that Hardy and Littlewood [HL27] showed that for $1 < p \le 2$ and $f \in L^p(\mathbb{T})$,

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p \le K\|f\|^p_{L^p(\mathbb{T})}, \tag{1.3}$$

and argued that this is a suitable extension of the Plancherel identity to L^p-spaces. Referring to Section 1 and to Theorem 2.1 for more details, our analogue for this is the inequality

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)(2l+1)^{5(p-2)/2}\|\widehat{f}(l)\|^p_{\mathtt{HS}} \le c\|f\|^p_{L^p(\mathrm{SU}(2))}, \qquad 1 < p \le 2, \tag{1.4}$$

which for $p = 2$ gives the ordinary Plancherel identity on SU(2) (see (2.1)). We refer to Theorem 2.2 for this statement and to Corollary 2.3 for its dual. For $p \ge 2$, necessary conditions for a function to belong to L^p are usually harder to obtain. In Theorem 2.8 we give such a result for $2 \leq p < \infty$:

$$(1.5)\qquad \sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2}\Big(\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ k\geq l}} \frac{1}{2k+1}|\mathrm{Tr}\,\widehat{f}(k)|\Big)^p \leq c\|f\|^p_{L^p(\mathrm{SU}(2))},\qquad 2\leq p<\infty.$$

In turn, this gives a noncommutative analogue to the known similar result on the circle (which we recall in Theorem 2.7). Similar to (1.1), the averaged trace appears also in (1.5)—it is the usual trace divided by the number of diagonal elements in the matrix.

Hörmander [Hör60] proved a Paley-type inequality for the Fourier transform on $\mathbb{R}^N$. In this paper we obtain an analogue of this inequality on SU(2).

The results on the group SU(2) are usually quite important since, in view of the resolved Poincaré conjecture, they provide information about corresponding transformations on general closed simply-connected three-dimensional manifolds (see [RT10] for a more detailed outline of such relations). In our context, they give explicit versions of known results on the circle $\mathbb{T}$ or on the torus $\mathbb{T}^n$, in the simplest noncommutative setting of SU(2).

At the same time, we note that some results of this paper can be extended to Fourier multipliers on general compact Lie groups. However, such analysis requires a more abstract approach, and will appear elsewhere.

The paper is organised as follows. In Section 2 we fix the notation for the representation theory of SU(2) and formulate estimates relating functions to their Fourier coefficients: the SU(2)-version of the Hardy–Littlewood and Paley inequalities and further extensions. In Section 3 we formulate and prove lower bounds for the operator norms of Fourier multipliers, and in Section 4 we establish upper bounds. Our proofs are based on the inequalities from Section 2. In Section 5 we complete the proofs of the results presented in the previous sections.

We shall use the symbol C to denote various positive constants, and $C_{p,q}$ for constants which may depend only on p and q. We shall write $x \lesssim y$ for the relation $|x| \leq C|y|$, and write $x \cong y$ if $x \lesssim y$ and $y \lesssim x$.

2. Hardy–Littlewood and Paley inequalities on SU(2).

The aim of this section is to discuss necessary conditions and sufficient conditions for the $L^p(\mathrm{SU}(2))$-integrability of a function by means of its Fourier coefficients. The main results of this section are Theorems 2.2, 2.4 and 2.8. They provide a noncommutative version of known results of this type on the circle $\mathbb{T}$. The proofs of most of the results of this section are given in Section 5.

First, let us fix the notation concerning representations of the compact Lie group SU(2). There are different types of notation in the literature for the relevant objects; we follow the notation of Vilenkin [Vil68], as well as that in [RT10, RT13]. Let us identify $z = (z_1, z_2) \in \mathbb{C}^{1\times 2}$, and let $\mathbb{C}[z_1, z_2]$ be the space of two-variable polynomials $f\colon \mathbb{C}^2 \to \mathbb{C}$. Consider mappings

$$t^l\colon \mathrm{SU}(2) \to \mathrm{GL}(V_l), \qquad (t^l(u)f)(z) = f(zu),$$

where $l \in \frac{1}{2}\mathbb{N}_0$ is called the *quantum number*, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and where V_l is the $(2l+1)$-dimensional subspace of $\mathbb{C}[z_1, z_2]$ consisting of the homogeneous polynomials of order $2l \in \mathbb{N}_0$, i.e.

$$V_l = \Big\{ f \in \mathbb{C}[z_1, z_2]\colon f(z_1, z_2) = \sum_{k=0}^{2l} a_k z_1^k z_2^{2l-k},\ \{a_k\}_{k=0}^{2l} \subset \mathbb{C} \Big\}.$$

The unitary dual of SU(2) is

$$\widehat{\mathrm{SU}(2)} \cong \big\{ t^l \in \mathrm{Hom}(\mathrm{SU}(2), \mathrm{U}(2l+1))\colon l \in \tfrac{1}{2}\mathbb{N}_0 \big\},$$

where $\mathrm{U}(d) \subset \mathbb{C}^{d\times d}$ is the unitary matrix group, and the matrix components $t^l_{mn} \in C^\infty(\mathrm{SU}(2))$ can be written as products of exponentials and Legendre–Jacobi functions (see Vilenkin [Vil68]). It is also customary to let the indices m, n range from $-l$ to l, equi-spaced with step one. We define the Fourier transform on SU(2) by

$$\widehat{f}(l) := \int_{\mathrm{SU}(2)} f(u) t^l(u)^*\, du,$$

with the inverse Fourier transform (Fourier series) given by

$$f(u) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)\, \mathrm{Tr}\, \widehat{f}(l) t^l(u).$$

The Peter–Weyl theorem on SU(2) implies, in particular, that this pair of transforms are inverse to each other and that the Plancherel identity

$$\|f\|^2_{L^2(\mathrm{SU}(2))} = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \|\widehat{f}(l)\|^2_{\mathtt{HS}} =: \|\widehat{f}\|^2_{\ell^2(\widehat{\mathrm{SU}(2)})} \tag{2.1}$$

holds true for all $f \in L^2(\mathrm{SU}(2))$. Here $\|\widehat{f}(l)\|^2_{\mathtt{HS}} = \mathrm{Tr}\, \widehat{f}(l)\widehat{f}(l)^*$ denotes the Hilbert–Schmidt norm of matrices. For more details on the Fourier transform on SU(2) and on arbitrary compact Lie groups, and for subsequent Fourier and operator analysis, we refer to [RT10].

There are different ways to compare the "sizes" of f and $\widehat{f}$. Apart from the Plancherel identity (2.1), there are other important relations, such as the Hausdorff–Young or the Riesz–Fischer theorems. However, such estimates usually require the change of the exponent p in L^p-measurements of f and $\widehat{f}$. Our first results deal with comparing f and $\widehat{f}$ in the same scale of L^p-measurements. Let us indicate the background of this problem. Hardy and Littlewood [HL27, Theorems 10 and 11] proved the following generalisation of the Plancherel identity.

THEOREM 2.1 (Hardy–Littlewood [HL27]).

(1) *Let* $1 < p \leq 2$. *If* $f \in L^p(\mathbb{T})$, *then*

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p \leq K_p\|f\|^p_{L^p(\mathbb{T})}, \tag{2.2}$$

where K_p *is a constant which depends only on* p.

(2) *Let* $2 \leq p < \infty$. *If* $\{\widehat{f}(m)\}_{m\in\mathbb{Z}}$ *is a sequence of complex numbers such that*

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p < \infty, \tag{2.3}$$

then there is a function $f \in L^p(\mathbb{T})$ *with Fourier coefficients* $\widehat{f}(m)$, *and*

$$\|f\|^p_{L^p(\mathbb{T})} \leq K'_p \sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p.$$

Hewitt and Ross [HR74] generalised this theorem to all compact abelian groups. Now, we give an analogue of Theorem 2.1 in the noncommutative setting of the compact group SU(2).

THEOREM 2.2. *If* $1 < p \leq 2$ *and* $f \in L^p(\mathrm{SU}(2))$, *then*

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{5p/2-4}\|\widehat{f}(l)\|^p_{\mathtt{HS}} \leq c_p\|f\|^p_{L^p(\mathrm{SU}(2))}. \tag{2.4}$$

We can write this in the form more resembling the Plancherel identity:

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)(2l+1)^{5(p-2)/2}\|\widehat{f}(l)\|^p_{\mathtt{HS}} \leq c_p\|f\|^p_{L^p(\mathrm{SU}(2))}, \tag{2.5}$$

providing a link to both (2.2) and (2.1). By duality, we obtain

COROLLARY 2.3. *If* $2 \leq p < \infty$ *and* $\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{5p/2-4}\|\widehat{f}(l)\|^p_{\mathtt{HS}} < \infty$, *then* $f \in L^p(\mathrm{SU}(2))$ *and*

$$\|f\|^p_{L^p(\mathrm{SU}(2))} \leq c_p \sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{5p/2-4}\|\widehat{f}(l)\|^p_{\mathtt{HS}}. \tag{2.6}$$

For $p = 2$, both statements reduce to the Plancherel identity (2.1).

Hoŕmander [Hör60] proved a Paley-type inequality for the Fourier transform on $\mathbb{R}^N$. We now give an analogue of this inequality on SU(2).

THEOREM 2.4. *Let* $1 < p \leq 2$. *Suppose* $\{\sigma(l)\}_{l\in\frac{1}{2}\mathbb{N}_0}$ *is a sequence of complex matrices* $\sigma(l) \in \mathbb{C}^{(2l+1)\times(2l+1)}$ *such that*

$$K_\sigma := \sup_{s>0} s \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \|\sigma(l)\|_{\mathtt{op}}\geq s}} (2l+1)^2 < \infty. \tag{2.7}$$

Then

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{p(2/p-1/2)}\|\widehat{f}(l)\|_{\mathtt{HS}}^p\|\sigma(l)\|_{\mathtt{op}}^{2-p}\lesssim {K_\sigma}^{2-p}\|f\|^p_{L^p(\mathrm{SU}(2))}. \tag{2.8}$$

It will be useful to recall the spaces $\ell^p(\widehat{\mathrm{SU}(2)})$ on the discrete unitary dual $\widehat{\mathrm{SU}(2)}$. For general compact Lie groups these spaces have been introduced and studied in [RT10, Section 10.3]. In the particular case of SU(2), for a sequence of complex matrices $\sigma(l)\in\mathbb{C}^{(2l+1)\times(2l+1)}$ they can be defined by the finiteness of the norms

$$\|\sigma\|_{\ell^p(\widehat{\mathrm{SU}(2)})}:=\Big(\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{p(2/p-1/2)}\|\sigma(l)\|_{\mathtt{HS}}^p\Big)^{1/p},\qquad 1\leq p<\infty, \tag{2.9}$$

and

$$\|\sigma\|_{\ell^\infty(\widehat{\mathrm{SU}(2)})}:=\sup_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{-1/2}\|\sigma(l)\|_{\mathtt{HS}}. \tag{2.10}$$

Among other things, it was shown in [RT10, Section 10.3] that these spaces are interpolation spaces, they satisfy the duality property and, with $\sigma=\widehat{f}$, the Hausdorff–Young inequality holds:

$$\begin{aligned}\Big(\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{p'(2/p'-1/2)}\|\widehat{f}(l)\|_{\mathtt{HS}}^{p'}\Big)^{1/p'}&\equiv\|\widehat{f}\|_{\ell^{p'}(\widehat{\mathrm{SU}(2)})}\\ &\lesssim\|f\|_{L^p(\mathrm{SU}(2))},\qquad 1\leq p\leq 2.\end{aligned} \tag{2.11}$$

Further, we recall a result on interpolation of weighted spaces from [BL76]:

Theorem 2.5 (Interpolation of weighted spaces). *Let $d\mu_0(x)=\omega_0(x)d\mu(x)$, $d\mu_1(x)=\omega_1(x)d\mu(x)$, and write $L^p(\omega)=L^p(\omega d\mu)$ for the weight ω. Suppose that $0<p_0,p_1<\infty$. Then*

$$(L^{p_0}(\omega_0),L^{p_1}(\omega_1))_{\theta,p}=L^p(\omega),$$

where $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $\omega=w_0^{p(1-\theta)/p_0}w_1^{p\theta/p_1}$.

From this we obtain:

Corollary 2.6. *Let $1<p\leq b\leq p'<\infty$. If $\{\sigma(l)\}_{l\in\frac{1}{2}\mathbb{N}_0}$ satisfies condition (2.7) with constant K_σ, then*

$$\begin{aligned}&\Big(\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{b(2/b-1/2)}(\|\widehat{f}(l)\|_{\mathtt{HS}}\|\sigma(l)\|_{\mathtt{op}}^{1/b-1/p'})^b\Big)^{1/b}\\ &\qquad\qquad\lesssim(K_\sigma)^{1/b-1/p'}\|f\|_{L^p(\mathrm{SU}(2))}.\end{aligned} \tag{2.12}$$

This reduces to (2.11) when $b=p'$ and to (2.8) when $b=p$.

Proof. We consider a sublinear operator A which takes a function f to its Fourier transform $\widehat{f}(l)$ divided by $\sqrt{2l+1}$, i.e.

$$f \mapsto Af := \{\widehat{f}(l)/\sqrt{2l+1}\}_{l\in\frac{1}{2}\mathbb{N}_0},$$

where

$$\widehat{f}(l) = \int\limits_{\mathrm{SU}(2)} f(u)t^l(u)^*\,du \in \mathbb{C}^{(2l+1)\times(2l+1)}, \qquad l \in \tfrac{1}{2}\mathbb{N}_0.$$

The statement follows from Theorem 2.5 if we regard the left-hand sides of inequalities (2.8) and (2.11) as an $\|Af\|_{L^p}$-norm in a weighted sequence space over $\frac{1}{2}\mathbb{N}_0$ with the weights given by $w_0(l) = (2l+1)^2\|\sigma(l)\|_{\mathtt{op}}^{2-p}$ and $w_1(l) = (2l+1)^2$, $l \in \frac{1}{2}\mathbb{N}_0$. ∎

Coming back to the Hardy–Littlewood Theorem 2.1, we see that the convergence of the series (2.3) is a sufficient condition for f to belong to $L^p(\mathbb{T})$, for $p \geq 2$. However, this condition is not necessary. Hence, the question arises of finding necessary conditions for f to belong to L^p, or in other words, of finding lower estimates for $\|f\|_{L^p}$ in terms of the series of the form (2.3). Such a result on $L^p(\mathbb{T})$ was obtained by Nursultanov and can be stated as follows.

THEOREM 2.7 ([Nur98a]). *If $2 < p < \infty$ and $f \in L^p(\mathbb{T})$, then*

$$\sum_{k=1}^{\infty} k^{p-2}\Big(\sup_{\substack{e\in M\\ |e|\geq k}} \frac{1}{|e|}\Big|\sum_{m\in e}\widehat{f}(m)\Big|\Big)^p \leq C\|f\|_{L^p(\mathbb{T})}^p, \tag{2.13}$$

where M is the set of all finite arithmetic progressions in $\mathbb{Z}$.

We now present a (noncommutative) version of this result on the group SU(2).

THEOREM 2.8. *If $2 < p < \infty$ and $f \in L^p(\mathrm{SU}(2))$, then*

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2}\Big(\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ k\geq l}} \frac{1}{2k+1}|\mathrm{Tr}\,\widehat{f}(k)|\Big)^p \leq c\|f\|_{L^p(\mathrm{SU}(2))}^p. \tag{2.14}$$

For completeness, we give a simple argument for Corollary 2.3.

Proof of Corollary 2.3. The application of the duality of L^p spaces yields

$$\|f\|_{L^p(\mathrm{SU}(2))} = \sup_{\substack{g\in L^{p'}\\ \|g\|_{L^{p'}=1}}} \Big|\int\limits_{\mathrm{SU}(2)} f(x)\overline{g(x)}\,dx\Big|.$$

Using Plancherel's identity (2.1), we get

$$\int\limits_{\mathrm{SU}(2)} f(x)\overline{g(x)}\,dx = \sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)\,\mathrm{Tr}\,\widehat{f}(l)\widehat{g}(l)^*.$$

It is easy to see that

$$2l+1=(2l+1)^{5/2-4/p+5/2-4/p'},$$

$$|\mathrm{Tr}\,\widehat{f}(l)\widehat{g}(l)^*|\le\|\widehat{f}(l)\|_{\mathtt{HS}}\|\widehat{g}(l)\|_{\mathtt{HS}}.$$

Using these inequalities, and applying the Hölder inequality, for any $g\in L^{p'}$ with $\|g\|_{L^{p'}}=1$ we have

$$\begin{aligned}\Big|\sum_{l\in\frac12\mathbb{N}_0}(2l+1)\,\mathrm{Tr}\,\widehat{f}(l)\widehat{g}(l)^*\Big| &\le\sum_{l\in\frac12\mathbb{N}_0}(2l+1)^{5/2-4/p}\|\widehat{f}(l)\|_{\mathtt{HS}}(2l+1)^{5/2-4/p'}\|\widehat{g}(l)\|_{\mathtt{HS}}\\ &\le\Big(\sum_{l\in\frac12\mathbb{N}_0}(2l+1)^{5p/2-4}\|\widehat{f}(l)\|_{\mathtt{HS}}^p\Big)^{1/p}\Big(\sum_{l\in\frac12\mathbb{N}_0}(2l+1)^{5p'/2-4}\|\widehat{g}(l)\|_{\mathtt{HS}}^{p'}\Big)^{1/p'}\\ &\le\Big(\sum_{l\in\frac12\mathbb{N}_0}(2l+1)^{5p/2-4}\|\widehat{f}(l)\|_{\mathtt{HS}}^p\Big)^{1/p}\|g\|_{L^{p'}},\end{aligned}$$

where we have used Theorem 2.2 in the last line. Thus, we have just proved that

$$\begin{aligned}\Big|\int_{\mathrm{SU}(2)}f(x)\overline{g(x)}\,dx\Big| &\le\Big|\sum_{l\in\frac12\mathbb{N}_0}(2l+1)\,\mathrm{Tr}\,\widehat{f}(l)\widehat{g}(l)^*\Big|\\ &\le\Big(\sum_{l\in\frac12\mathbb{N}_0}(2l+1)^{5p/2-4}\|\widehat{f}(l)\|_{\mathtt{HS}}^p\Big)^{1/p}\|g\|_{L^{p'}}.\end{aligned}$$

Taking the supremum over all $g\in L^{p'}(\mathrm{SU}(2))$, we get (2.6). This proves Corollary 2.3. ∎

3. Lower bounds for Fourier multipliers on SU(2)**.** Let A be a continuous linear operator from $C^\infty(\mathrm{SU}(2))$ to $\mathcal{D}'(\mathrm{SU}(2))$. Here we are concerned with left-invariant operators, which means that $A\circ\tau_g=\tau_g\circ A$ for the left-translation $\tau_g f(x)=f(g^{-1}x)$. Using the Schwartz kernel theorem and the Fourier inversion formula one can prove that every left-invariant continuous operator A can be written as a Fourier multiplier,

$$\widehat{Af}(l)=\sigma_A(l)\widehat{f}(l),$$

with *symbol* $\sigma_A(l)\in\mathbb{C}^{(2l+1)\times(2l+1)}$. It follows from the Fourier inversion formula that we can write this also as

$$Af(u)=\sum_{l\in\frac12\mathbb{N}_0}(2l+1)\,\mathrm{Tr}\,t^l(u)\sigma_A(l)\widehat{f}(l),\tag{3.1}$$

where

$$\sigma_A(l) = t^l(e)^* A t^l(e) = A t^l(e),$$

where e is the identity matrix in SU(2), and $(At^l)_{mk} = A(t^l_{mk})$ is defined componentwise for $-l \le m, k \le l$. We refer to operators in these equivalent forms as (noncommutative) Fourier multipliers. The class of these operators on SU(2) and their L^p-boundedness were investigated in [CW71b, CW71a], and on general compact Lie groups in [RW13]. In particular, these authors proved Hörmander–Mihlin type multiplier theorems in those settings, giving sufficient condition for the L^p-boundedness in terms of symbols. These conditions guarantee that the operator is of weak type $(1,1)$, which, combined with a simple L^2-boundedness statement, implies the boundedness on L^p for all $1 < p < \infty$.

For a general (non-invariant) operator A, its matrix symbol $\sigma_A(u,l)$ will also depend on u. Such quantization (3.1) has been consistently developed in [RT10] and [RT13]. We note that the L^p-boundedness results in [RW13] also cover such non-invariant operators.

For a noncommutative Fourier multiplier A we will write $A \in M_p^q(\mathrm{SU}(2))$ if A extends to a bounded operator from $L^p(\mathrm{SU}(2))$ to $L^q(\mathrm{SU}(2))$. We introduce a norm $\|\cdot\|$ on $M_p^q(\mathrm{SU}(2))$ by setting

$$\|A\|_{M_p^q} := \|A\|_{L^p \to L^q}.$$

Thus, we are concerned with the question of what assumptions on the symbol σ_A guarantee that $A \in M_p^q$. The sufficient conditions on σ_A for $A \in M_p^p$ were investigated in [RW13]. The aim of this section is to give a necessary condition on σ_A for $A \in M_p^q$, for $1 < p \le 2 \le q < \infty$.

Suppose that $1 < p \le 2 \le q < \infty$ and that $A\colon L^p(\mathrm{SU}(2)) \to L^q(\mathrm{SU}(2))$ is a Fourier multiplier. The Plancherel identity (2.1) implies that the operator A is bounded from $L^2(\mathrm{SU}(2))$ to $L^2(\mathrm{SU}(2))$ if and only if $\sup_l \|\sigma_A(l)\|_{\mathsf{op}} < \infty$. Various other function spaces on the unitary dual have been discussed in [RT10]. Following Stein, we search for more subtle conditions on the symbols of noncommutative Fourier multipliers ensuring their L^p-L^q boundedness, and we now prove a lower estimate which depends explicitly on p and q.

Theorem 3.1. *Let $1 < p \le 2 \le q < \infty$ and let A be a left-invariant operator on* SU(2) *such that $A \in M_p^q(\mathrm{SU}(2))$. Then*

$$\sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{\min_{n \in \{-l,\dots,+l\}} |\sigma_A(l)_{nn}|}{(2l+1)^{1/p'+1/q}} \lesssim \|A\|_{L^p(\mathrm{SU}(2)) \to L^q(\mathrm{SU}(2))}, \tag{3.2}$$

$$\sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{|\mathrm{Tr}\, \sigma_A(l)|}{(2l+1)^{1+1/p'+1/q}} \lesssim \|A\|_{L^p(\mathrm{SU}(2)) \to L^q(\mathrm{SU}(2))}. \tag{3.3}$$

One can see a similarity between (3.2), (3.3) and (1.1) as

$$\sup_{l\in\frac{1}{2}\mathbb{N}_0} \frac{1}{(2l+1)^{1/p'+1/q}} \frac{1}{2l+1} |\operatorname{Tr} \sigma_A(l)| \lesssim \|A\|_{L^p(\mathrm{SU}(2))\to L^q(\mathrm{SU}(2))}. \tag{3.4}$$

We also note that estimates (3.2) and (3.3) cannot be immediately compared because the trace in (3.3) depends on the signs of the diagonal entries of $\sigma_A(l)$.

Proof of Theorem 3.1. In [GT80] it was proven that for any $l \in \frac{1}{2}\mathbb{N}_0$ there exists a basis for $t^l \in \widehat{\mathrm{SU}(2)}$ and a diagonal matrix coefficient t^l_{nn} (i.e. for some $-l \le n \le l$) such that

$$\|t^l_{nn}\|_{L^p(\mathrm{SU}(2))} \cong \frac{1}{(2l+1)^{1/p}}. \tag{3.5}$$

Now, we use this result to establish a lower bound for the norm of $A \in M^q_p(\mathrm{SU}(2))$. Fix $l_0 \in \frac{1}{2}\mathbb{N}_0$ and the corresponding diagonal element $t^{l_0}_{nn}$. We consider $f_{l_0}(g)$ whose matrix-valued Fourier coefficient

$$\widehat{f_{l_0}}(l) = \operatorname{diag}(0,\dots,1,0,\dots)\delta^l_{l_0} \tag{3.6}$$

has only one non-zero diagonal coefficient 1 at the nth diagonal entry. Then by the Fourier inversion formula, $f_{l_0}(g) = (2l_0+1)t^{l_0}_{nn}(g)$. By definition,

$$\begin{aligned}\|A\|_{L^p\to L^q} &= \sup_{f\neq 0} \frac{\|\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\operatorname{Tr} t^l(u)\sigma_A(l)\widehat{f}(l)\|_{L^q(\mathrm{SU}(2))}}{\|f\|_{L^p(\mathrm{SU}(2))}} \\ &\ge \frac{\|\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\operatorname{Tr} t^l(u)\sigma_A(l)\widehat{f_{l_0}}(l)\|_{L^q(\mathrm{SU}(2))}}{\|f_{l_0}\|_{L^p(\mathrm{SU}(2))}}.\end{aligned}$$

Invoking (3.6), we get

$$\|A\|_{L^p\to L^q} \gtrsim \frac{\|(2l_0+1)\operatorname{Tr} t^{l_0}(g)\sigma_A(l_0)\widehat{f_{l_0}}(l)\|_{L^q(\mathrm{SU}(2))}}{\|f_{l_0}\|_{L^p(\mathrm{SU}(2))}}.$$

Setting $h(g) := (2l_0+1)\operatorname{Tr} t^{l_0}(g)\sigma_A(l_0)\widehat{f_{l_0}}(l_0)$, we have $\widehat{h}(l) = 0$ for $l \neq l_0$, and $\widehat{h}(l_0) = \sigma_A(l_0)\widehat{f_{l_0}}(l_0)$. Consequently,

$$\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ k\ge l}} \frac{1}{2k+1}|\operatorname{Tr}\widehat{h}(k)| = \begin{cases} 0, & l > l_0,\\ \dfrac{1}{2l_0+1}|\sigma_A(l_0)_{nn}|, & 1 \le l \le l_0.\end{cases}$$

Using this, Theorem 2.8 and (3.5), we obtain

$$\|A\|_{L^p\to L^q} \gtrsim \frac{\left(\sum_{l=1}^{l_0}(2l+1)^{q-2}\left(\frac{1}{2l_0+1}|\sigma_A(l_0)_{nn}|\right)^q\right)^{1/q}}{(2l_0+1)^{1-1/p}},$$

where l_0 is an arbitrary fixed half-integer. Direct calculation now shows that

$$\begin{aligned}
&\frac{\left(\sum_{l=1}^{l_0}(2l+1)^{q-2}\left(\frac{1}{2l_0+1}|\sigma_A(l_0)_{nn}|\right)^q\right)^{1/q}}{(2l_0+1)^{1-1/p}} \\
&\qquad = \frac{1}{2l_0+1}|\sigma_A(l_0)_{nn}|\frac{(\sum_{l=1}^{l_0}(2l+1)^{q-2})^{1/q}}{(2l_0+1)^{1-1/p}} \\
&\qquad = \frac{1}{2l_0+1}|\sigma_A(l_0)_{nn}|\frac{(2l_0+1)^{1-1/q}}{(2l_0+1)^{1-1/p}} \cong \frac{|\sigma_A(l_0)_{nn}|}{(2l_0+1)^{1/p'+1/q}}.
\end{aligned}$$

Taking the infimum over all $n \in \{-l_0, -l_0+1, \ldots, l_0-1, l_0\}$ and then the supremum over all half-integers, we obtain

$$\|A\|_{L^p \to L^q} \gtrsim \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{\min_{n \in \{-l,\ldots,+l\}} |\sigma_A(l)_{nn}|}{(2l+1)^{1/p'+1/q}}.$$

This proves (3.2).

Now, we will prove estimate (3.3). Fix $l_0 \in \frac{1}{2}\mathbb{N}_0$ and consider $f_{l_0}(u) := (2l_0+1)\chi_{l_0}(u)$, where $\chi_{l_0}(u) = \operatorname{Tr} t^{l_0}(u)$ is the character of the representation t^{l_0}. Then, in particular,

$$\widehat{f_{l_0}}(l) = \begin{cases} I_{2l+1}, & l = l_0, \\ 0, & l \neq l_0, \end{cases} \tag{3.7}$$

where $I_{2l+1} \in \mathbb{C}^{(2l+1)\times(2l+1)}$ is the identity matrix. Using the Weyl character formula, we can write

$$\chi_{l_0}(u) = \sum_{k=-l_0}^{l_0} e^{ikt}, \quad \text{where} \quad u = v^{-1}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} v.$$

The value of $\chi_{l_0}(u)$ does not depend on v since characters are central. Further, the application of the Weyl integral formula yields

$$\begin{aligned}
\|f_{l_0}\|_{L^p(\mathrm{SU}(2))} &= (2l_0+1)\|\chi_{l_0}\|_{L^p(\mathrm{SU}(2))} \\
&= (2l_0+1)\left(\int_0^{2\pi}\left|\sum_{k=-l_0}^{l_0} e^{ikt}\right|^p 2\sin^2 t\,\frac{dt}{2\pi}\right)^{1/p}.
\end{aligned}$$

It is clear that

$$\left|e^{i(-l_0-1)t}\sum_{k=-l_0}^{l_0} e^{i(k+l_0+1)t}\right| = \left|\sum_{k=1}^{2l_0+1} e^{ikt}\right|.$$

Applying [Nur98a, Corollary 4] to the Dirichlet kernel $D_{2l_0+1}(t) := \sum_{k=1}^{2l_0+1} e^{ikt}$, we get

$$\|\chi_{l_0}\|_{L^p(\mathrm{SU}(2))} \lesssim \|D_{2l_0+1}\|_{L^p(0,2\pi)} \cong (2l_0+1)^{1-1/p}. \tag{3.8}$$

Just as before,

$$\|A\|_{L^p\to L^q} \geq \frac{\| \sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\operatorname{Tr} t^l(u)\sigma_A(l)\widehat{f_{l_0}}(l)\|_{L^q(\mathrm{SU}(2))}}{\|f_{l_0}\|_{L^p(\mathrm{SU}(2))}}.$$

From (3.7), we obtain

$$\|A\|_{L^p\to L^q} \gtrsim \frac{\|(2l_0+1)\operatorname{Tr} t^{l_0}(g)\sigma_A(l_0)\|_{L^q(\mathrm{SU}(2))}}{\|f_{l_0}\|_{L^p(\mathrm{SU}(2))}}.$$

Setting $h(g) := (2l_0+1)\operatorname{Tr} t^{l_0}(g)\sigma_A(l_0)$, we have $\widehat{h}(l) = 0$ for $l \neq l_0$, and $\widehat{h}(l_0) = \sigma_A(l_0)$. Consequently,

$$\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ k\geq l}} \frac{1}{2k+1}|\operatorname{Tr}\widehat{h}(k)| = \begin{cases} 0, & l > l_0,\\ \dfrac{1}{2l_0+1}|\operatorname{Tr}\sigma_A(l_0)|, & 1\leq l\leq l_0.\end{cases}$$

Using this and Theorem 2.8, we get

$$\|A\|_{L^p\to L^q} \gtrsim \frac{\left(\sum\limits_{l=1}^{l_0}(2l+1)^{q-2}\left(\dfrac{1}{2l_0+1}|\operatorname{Tr}\sigma_A(l_0)|\right)^q\right)^{1/q}}{(2l_0+1)(2l_0+1)^{1-1/p}},$$

where l_0 is an arbitrary fixed half-integer. Direct calculation shows that

$$\begin{aligned}
&\frac{\left(\sum\limits_{l=1}^{l_0}(2l+1)^{q-2}\left(\dfrac{1}{2l_0+1}|\operatorname{Tr}\sigma_A(l_0)|\right)^q\right)^{1/q}}{(2l_0+1)(2l_0+1)^{1-1/p}}\\
&\quad= \frac{1}{2l_0+1}|\operatorname{Tr}\sigma_A(l_0)|\frac{(\sum_{l=1}^{l_0}(2l+1)^{q-2})^{1/q}}{(2l_0+1)(2l_0+1)^{1-1/p}}\\
&\quad= \frac{1}{2l_0+1}|\operatorname{Tr}\sigma_A(l_0)|\frac{(2l_0+1)^{1-1/q}}{(2l_0+1)(2l_0+1)^{1-1/p}} \cong \frac{|\operatorname{Tr}\sigma_A(l_0)|}{(2l_0+1)^{1+1/p'+1/q}}.
\end{aligned}$$

Taking the supremum over all half-integers, we get

$$\|A\|_{L^p\to L^q} \gtrsim \sup_{l\in\frac{1}{2}\mathbb{N}_0} \frac{|\operatorname{Tr}\sigma_A(l)|}{(2l+1)^{1+1/p'+1/q}}.$$

This proves (3.3). ∎

4. Upper bounds for Fourier multipliers on SU(2)**.** In this section we give a noncommutative SU(2) analogue of the upper bound for Fourier multipliers, analogous to the one on the circle $\mathbb{T}$ in Theorem 1.1 (see also [Nur98b, NTi11] for the circle case).

THEOREM 4.1. *If* $1 < p \le 2 \le q < \infty$ *and* A *is a left-invariant operator on* $\mathrm{SU}(2)$*, then*

$$\|A\|_{L^p(\mathrm{SU}(2))\to L^q(\mathrm{SU}(2))} \lesssim \sup_{s>0} s\Big(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \|\sigma_A(l)\|_{\mathsf{op}}>s}} (2l+1)^2\Big)^{1/p-1/q}. \tag{4.1}$$

Proof. Since A is a left-invariant operator, it acts on f via multipication of $\widehat{f}$ by the symbol σ_A,

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \tag{4.2}$$

where

$$\sigma_A(\pi) = \pi(x)^* A\pi(x)|_{x=e}.$$

Let us first assume that $p \le q'$. Since $q' \le 2$, for $f \in C^\infty(\mathrm{SU}(2))$ the Hausdorff–Young inequality gives

$$\begin{aligned}\|Af\|_{L^q(\mathrm{SU}(2))} &\le \|\widehat{Af}\|_{\ell^{q'}(\widehat{\mathrm{SU}(2)})} = \|\sigma_A\widehat{f}\|_{\ell^{q'}(\widehat{\mathrm{SU}(2)})}\\ &= \Big(\sum_{l\in\widehat{\mathrm{SU}(2)}} (2l+1)^{2-q'/2}\|\sigma_A(l)\widehat{f}(l)\|_{\mathtt{HS}}^{q'}\Big)^{1/q'}\\ &\le \Big(\sum_{l\in\widehat{\mathrm{SU}(2)}} (2l+1)^{2-q'/2}\|\sigma_A(l)\|_{\mathsf{op}}^{q'}\|\widehat{f}(l)\|_{\mathtt{HS}}^{q'}\Big)^{1/q'}.\end{aligned} \tag{4.3}$$

The case $q' \le (p')'$ can be reduced to the case $p \le q'$ as follows. The application of Theorem 4.2 below with $G = \mathrm{SU}(2)$ and μ the Haar measure on $\mathrm{SU}(2)$ yields

$$\|A\|_{L^p(\mathrm{SU}(2))\to L^q(\mathrm{SU}(2))} = \|A^*\|_{L^{q'}(\mathrm{SU}(2))\to L^{p'}(\mathrm{SU}(2))}. \tag{4.4}$$

We have

$$\sigma_{A^*}(l) = \sigma_A^*(l), \quad l \in \tfrac{1}{2}\mathbb{N}_0, \tag{4.5}$$

and $\|\sigma_{A^*}(l)\|_{\mathsf{op}} = \|\sigma_A(l)\|_{\mathsf{op}}$. Now, we are in a position to apply Corollary 2.6. Set $1/r = 1/p - 1/q$. We observe that with $\sigma(t^l) := \|\sigma_A(t^l)\|_{\mathsf{op}}^r I_{2l+1}$, $l \in \frac{1}{2}\mathbb{N}_0$ and $b = q'$, the assumptions of Corollary 2.6 are satisfied and we obtain

$$\begin{aligned}&\Big(\sum_{l\in\widehat{\mathrm{SU}(2)}} (2l+1)^{2-q'/2}\|\sigma_A(l)\|_{\mathsf{op}}^{q'}\|\widehat{f}(l)\|_{\mathtt{HS}}^{q'}\Big)^{1/q'}\\ &\quad\lesssim \Big(\sup_{s>0} s\sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma(t^l)\|_{\mathsf{op}}^r>s}} (2l+1)^2\Big)^{1/r}\|f\|_{L^p(\mathrm{SU}(2))}, \quad f \in L^p(\mathrm{SU}(2)),\end{aligned} \tag{4.6}$$

in view of $1/q' - 1/p' = 1/p - 1/q = 1/r$. Thus, for $1 < p \le 2 \le q < \infty$,

$$\|Af\|_{L^q(\mathrm{SU}(2))} \lesssim \Bigg(\sup_{s>0} s \sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma(t^l)\|^r_{\mathsf{op}}>s}} (2l+1)^2\Bigg)^{1/r} \|f\|_{L^p(\mathrm{SU}(2))}. \tag{4.7}$$

Further, it can be easily checked that

$$\Bigg(\sup_{s>0} s \sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma(t^l)\|^r_{\mathsf{op}}>s}} (2l+1)^2\Bigg)^{1/r} = \Bigg(\sup_{s>0} s \sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma_A(t^l)\|_{\mathsf{op}}>s^{1/r}}} (2l+1)^2\Bigg)^{1/r}$$
$$= \Bigg(\sup_{s>0} s^r \sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma_A(t^l)\|_{\mathsf{op}}>s}} (2l+1)^2\Bigg)^{1/r} = \sup_{s>0} s\Bigg(\sum_{\substack{t^l\in\widehat{\mathrm{SU}(2)}\\ \|\sigma_A(t^l)\|_{\mathsf{op}}>s}} (2l+1)^2\Bigg)^{1/r}.$$

This completes the proof of Theorem 4.1. ■

For completeness, we give a short proof of Theorem 4.2 used above.

THEOREM 4.2. *Let (X,μ) be a measure space and $1<p,q<\infty$. Then*

$$\|A\|_{L^p(X,\mu)\to L^q(X,\mu)} = \|A^*\|_{L^{q'}(X,\mu)\to L^{p'}(X,\mu)}, \tag{4.8}$$

where $A^*\colon L^{q'}(X,\mu)\to L^{p'}(X,\mu)$ is the adjoint of A.

Proof. Let $f\in L^p\cap L^2$ and $g\in L^{q'}\cap L^2$. By the Hölder inequality,

$$\begin{aligned}|(Af,g)_{L^2}| = |(A^*g,f)_{L^2}| &\le \|A^*g\|_{L^{p'}}\|f\|_{L^p}\\ &\le \|A^*\|_{L^{q'}\to L^{p'}}\|g\|_{L^{q'}}\|f\|_{L^p}.\end{aligned} \tag{4.9}$$

Thus,

$$\|A\|_{L^p\to L^q} \le \|A^*\|_{L^{q'}\to L^{p'}}. \tag{4.10}$$

Analogously,

$$\|A^*\|_{L^{q'}\to L^{p'}} \le \|A\|_{L^p\to L^q}. \tag{4.11}$$

The combination of (4.10) and (4.11) yields

$$\|A\|_{L^p\to L^q} = \|A^*\|_{L^{q'}\to L^{p'}}. \ \blacksquare$$

5. Proofs of theorems from Section 2

Proof of Theorem 2.4. Let μ give measure $\|\sigma(t^l)\|^2_{\mathsf{op}}(2l+1)^2$, $l\in\frac{1}{2}\mathbb{N}_0$, to the set consisting of the single point $\{t^l\}$, $t^l\in\widehat{\mathrm{SU}(2)}$, and measure zero to every set which does not contain any of these points, i.e.

$$\mu\{t^l\} := \|\sigma(t^l)\|^2_{\mathsf{op}}(2l+1)^2.$$

We define $L^p(\widehat{\mathrm{SU}(2)},\mu)$, $1\le p<\infty$, as the space of complex (or real) sequences $a = \{a_l\}_{l \in \frac{1}{2}\mathbb{N}_0}$ such that

$$\|a\|_{L^p(\widehat{\mathrm{SU}(2)},\mu)} := \Big(\sum_{l \in \frac{1}{2}\mathbb{N}_0} |a_l|^p \|\sigma(t^l)\|^2_{\mathtt{op}} (2l+1)^2 \Big)^{1/p} < \infty. \tag{5.1}$$

We will show that the sublinear operator

$$A\colon L^p(\mathrm{SU}(2)) \ni f \mapsto Af = \left\{ \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} \right\}_{t^l \in \widehat{\mathrm{SU}(2)}} \in L^p(\widehat{\mathrm{SU}(2)}, \mu)$$

is well-defined and bounded from $L^p(\mathrm{SU}(2))$ to $L^p(\widehat{\mathrm{SU}(2)}, \mu)$ for $1 < p \le 2$. In other words, we claim that

$$\begin{aligned} \|Af\|_{L^p(\widehat{\mathrm{SU}(2)},\mu)} &= \Bigg(\sum_{t^l \in \widehat{\mathrm{SU}(2)}} \left(\frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} \right)^p \|\sigma(t^l)\|^2_{\mathtt{op}} (2l+1)^2 \Bigg)^{1/p} \\ &\lesssim K_\sigma^{(2-p)/p} \|f\|_{L^p(\mathrm{SU}(2))}, \end{aligned} \tag{5.2}$$

which would give (2.8) and where we have set

$$K_\sigma := \sup_{s>0} s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \ge s}} (2l+1)^2.$$

We will show that A is of strong type $(2,2)$ and of weak type $(1,1)$. For definition and discussions we refer to Section 6 where we give definitions of weak type, and we formulate and prove the Marcinkiewicz interpolation theorem 6.1. More precisely, with the distribution function ν as in Theorem 6.1, we show that

$$\nu_{\widehat{\mathrm{SU}(2)}}(y; Af) \le \left(\frac{M_2 \|f\|_{L^2(\mathrm{SU}(2))}}{y} \right)^2 \quad \text{with norm } M_2 = 1, \tag{5.3}$$

$$\nu_{\widehat{\mathrm{SU}(2)}}(y; Af) \le \frac{M_1 \|f\|_{L^1(\mathrm{SU}(2))}}{y} \quad \text{with norm } M_1 = K_\sigma. \tag{5.4}$$

Then (5.2) follows from Theorem 6.1.

Now, to show (5.3), using Plancherel's identity (2.1), we get

$$\begin{aligned} y^2 \nu_{\widehat{\mathrm{SU}(2)}}(y; Af) &\le \|Af\|^2_{L^p(\widehat{\mathrm{SU}(2)},\mu)} \\ &:= \sum_{t^l \in \widehat{\mathrm{SU}(2)}} \left(\frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} \right)^2 \|\sigma(t^l)\|^2_{\mathtt{op}} (2l+1)^2 \\ &= \sum_{t^l \in \widehat{\mathrm{SU}(2)}} (2l+1) \|\widehat{f}(t^l)\|^2_{\mathtt{HS}} = \|\widehat{f}\|^2_{\ell^2(\widehat{\mathrm{SU}(2)})} = \|f\|^2_{L^2(\mathrm{SU}(2))}. \end{aligned}$$

Thus, A is of strong type $(2,2)$ with norm $M_2 \le 1$. Further, we show that A is of weak type $(1,1)$ with norm $M_1 = C$; more precisely, we show that

$$\nu_{\widehat{\mathrm{SU}(2)}}\left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\} \lesssim K_\sigma \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}. \tag{5.5}$$

The left-hand side here is the weighted sum $\sum \|\sigma(t^l)\|^2_{\mathtt{op}}(2l+1)^2$ taken over those $t^l \in \widehat{\mathrm{SU}(2)}$ for which $\|\widehat{f}(t^l)\|_{\mathtt{HS}}/(\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}) > y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(t^l)\|_{\mathtt{HS}} \le \sqrt{2l+1}\,\|f\|_{L^1(\mathrm{SU}(2))}.$$

Therefore,

$$y < \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} \le \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{\mathtt{op}}}.$$

Hence

$$\left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\} \subset \left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\}$$

for any $y > 0$. Consequently,

$$\mu\left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\} \le \mu\left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\}.$$

Setting $v := \|f\|_{L^1(\mathrm{SU}(2))}/y$, we get

$$\mu\left\{t^l \in \widehat{\mathrm{SU}(2)} \colon \frac{\|\widehat{f}(t^l)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(t^l)\|_{\mathtt{op}}} > y\right\} \le \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \le v}} \|\sigma(t^l)\|^2_{\mathtt{op}}(2l+1)^2. \tag{5.6}$$

We claim that

$$\sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \le v}} \|\sigma(t^l)\|^2_{\mathtt{op}}(2l+1)^2 \lesssim K_\sigma v. \tag{5.7}$$

In fact,

$$\sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \le v}} \|\sigma(t^l)\|^2_{\mathtt{op}}(2l+1)^2 = \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \le v}} (2l+1)^2 \int_0^{\|\sigma(t^l)\|^2_{\mathtt{op}}} d\tau.$$

We can interchange summation and integration to get

$$\sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathtt{op}} \le v}} (2l+1) \int_0^{\|\sigma(t^l)\|^2_{\mathtt{op}}} d\tau = \int_0^{v^2} d\tau \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \tau^{1/2} \le \|\sigma(t^l)\|_{\mathtt{op}} \le v}} (2l+1)^2.$$

Further, the substitution $\tau = s^2$ yields

$$\int_0^{v^2} d\tau \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \tau^{1/2} \le \|\sigma(t^l)\|_{\mathrm{op}} \le v}} (2l+1)^2 = 2\int_0^v s\,ds \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ s \le \|\sigma(t^l)\|_{\mathrm{op}} \le v}} (2l+1)^2$$
$$\le 2\int_0^v s\,ds \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ s \le \|\sigma(t^l)\|_{\mathrm{op}}}} (2l+1)^2.$$

Since

$$s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ s \le \|\sigma(t^l)\|_{\mathrm{op}}}} (2l+1)^2 \le \sup_{s>0} s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ s \le \|\sigma(t^l)\|_{\mathrm{op}}}} (2l+1)^2 = K_\sigma$$

is finite by the definition of K_σ, we have

$$2\int_0^v s\,ds \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ s \le \|\sigma(t^l)\|_{\mathrm{op}}}} (2l+1)^2 \lesssim K_\sigma v.$$

This proves (5.7). We have just proved inequalities (5.3), (5.4). Then by the Marcinkiewicz interpolation theorem (Theorem 6.1) with $p_1 = 1$, $p_2 = 2$ and $1/p = 1 - \theta + \theta/2$ we obtain

$$\bigg(\sum_{l \in \frac{1}{2}\mathbb{N}_0} \bigg(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{2l+1}\,\|\sigma(\pi)\|_{\mathtt{op}}}\bigg)^p \|\sigma(\pi)\|_{\mathtt{op}}^2 (2l+1)^2\bigg)^{1/p}$$
$$= \|Af\|_{L^p(\widehat{\mathrm{SU}(2)},\mu)} \lesssim K_\sigma^{(2-p)/p} \|f\|_{L^p(\mathrm{SU}(2))}.$$

This completes the proof of Theorem 2.4. ∎

Now we prove the Hardy–Littlewood type inequality given in Theorem 2.2.

Proof of Theorem 2.2. Let ν give measure $1/(2l+1)^4$ to the set consisting of the single point l, where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and measure zero to every set which does not contain any of these points. We will show that the sublinear operator

$$Tf := \{(2l+1)^{5/2} \|\widehat{f}(l)\|_{\mathtt{HS}}\}_{l \in \frac{1}{2}\mathbb{N}_0}$$

is well-defined and bounded from $L^p(\mathrm{SU}(2))$ to $L^p\big(\frac{1}{2}\mathbb{N}_0, \nu\big)$ for $1 < p \le 2$, with

$$\|Tf\|_{L^p(\widehat{\mathrm{SU}(2)},\nu)} = \Big(\sum_{l \in \frac{1}{2}\mathbb{N}_0} \big((2l+1)^{5/2} \|\widehat{f}(l)\|_{\mathtt{HS}}\big)^p \cdot (2l+1)^{-4}\Big)^{1/p}.$$

This will prove Theorem 2.2.

We first show that T is of strong type $(2,2)$ and weak type $(1,1)$. Using Plancherel's identity (2.1), we get

$$\begin{aligned}\|Tf\|^2_{L^p(\widehat{\mathrm{SU}(2)},\nu)} &= \sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)^{5p/2-4}\|\widehat{f}(l)\|^2_{\mathtt{HS}} = \sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\|\widehat{f}(l)\|^2_{\mathtt{HS}}\\ &= \|\widehat{f}\|^2_{\ell^2(\widehat{\mathrm{SU}(2)})} = \|f\|^2_{L^2(\mathrm{SU}(2))}.\end{aligned}$$

Thus, T is of strong type $(2,2)$.

Further, we will show that T is of weak type $(1,1)$, more precisely,

$$\nu\big\{l\in\tfrac{1}{2}\mathbb{N}_0\colon (2l+1)^{5/2}\|\widehat{f}(l)\|_{\mathtt{HS}}>y\big\}\le\frac{4}{3}\,\frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}. \tag{5.8}$$

The left-hand side here is the sum $\sum 1/(2l+1)^4$ taken over those $l\in\frac{1}{2}\mathbb{N}_0$ for which $(2l+1)^{5/2}\|\widehat{f}(l)\|_{\mathtt{HS}}>y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(l)\|_{\mathtt{HS}}\le\sqrt{2l+1}\,\|f\|_{L^1(\mathrm{SU}(2))}.$$

Therefore,

$$y<(2l+1)^{5/2}\|\widehat{f}(l)\|_{\mathtt{HS}}\le(2l+1)^{5/2+1/2}\|f\|_{L^1(\mathrm{SU}(2))}.$$

Hence

$$\big\{l\in\tfrac{1}{2}\mathbb{N}_0\colon (2l+1)^{5/2}\|\widehat{f}(l)\|_{\mathtt{HS}}>y\big\}\subset\left\{l\in\tfrac{1}{2}\mathbb{N}_0\colon 2l+1>\left(\frac{y}{\|f\|_{L^1}}\right)^{1/3}\right\}$$

for any $y>0$. Consequently,

$$\nu\big\{l\in\tfrac{1}{2}\mathbb{N}_0\colon (2l+1)^{5/2}\|\widehat{f}(l)\|_{\mathtt{HS}}>y\big\}\le\nu\left\{l\in\tfrac{1}{2}\mathbb{N}_0\colon 2l+1>\left(\frac{y}{\|f\|_{L^1}}\right)^{1/3}\right\}.$$

We set $w:=(y/\|f\|_{L^1(\mathrm{SU}(2))})^{1/3}$. Now, we estimate $\nu\big\{l\in\frac{1}{2}\mathbb{N}_0\colon 2l+1>w\big\}$. By definition, we have

$$\nu\left\{l\in\tfrac{1}{2}\mathbb{N}_0\colon 2l+1>\left(\frac{y}{\|f\|_{L^1}}\right)^{1/3}\right\}=\sum_{n>w}^{\infty}\frac{1}{n^4}.$$

In order to estimate this series, we introduce the following lemma.

Lemma 5.1. *Let $\beta>1$ and $w>0$. Then*

$$\sum_{n>w}^{\infty}\frac{1}{n^\beta}\le\begin{cases}\dfrac{\beta}{\beta-1}, & w\le 1,\\[2mm] \dfrac{1}{\beta-1}\dfrac{1}{w^{\beta-1}}, & w>1.\end{cases} \tag{5.9}$$

The proof is rather straightforward. Now, suppose $w\le 1$. Applying this lemma with $\beta=4$, we get

$$\sum_{n>w}^{\infty}\frac{1}{n^4}\le\frac{4}{3}.$$

Since $1 \leq 1/w^3$, we obtain

$$\sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \leq \frac{4}{3}\frac{1}{w^3}.$$

Recalling that $w = (y/\|f\|_{L^1(\mathrm{SU}(2))})^{1/3}$, we finally obtain

$$\nu\Big\{l \in \tfrac{1}{2}\mathbb{N}_0 \colon 2l+1 > \Big(\frac{y}{\|f\|_{L^1}}\Big)^{1/3}\Big\} = \sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{4}{3}\,\frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}.$$

Now, if $w > 1$, then

$$\sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{1}{3}\frac{1}{w^3} = \frac{4}{3}\,\frac{\|f\|_{L^1}}{y}.$$

Finally,

$$\nu\Big\{l \in \tfrac{1}{2}\mathbb{N}_0 \colon 2l+1 > \Big(\frac{y}{\|f\|_{L^1}}\Big)^{1/3}\Big\} \leq \frac{4}{3}\,\frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}.$$

This proves (5.8).

By the Marcinkiewicz interpolation theorem 6.1 with $p_1 = 1$, $p_2 = 2$, we obtain

$$\Big(\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{5p/2-4}\|\widehat{f}(l)\|_{\mathtt{HS}}^p\Big)^{1/p} = \|Tf\|_{L^p(\widehat{\mathrm{SU}(2)},\nu)} \leq c_p\|f\|_{L^p(\mathrm{SU}(2))}.$$

This completes the proof of Theorem 2.2. ∎

Proof of Theorem 2.8. We first simplify the expression for $\operatorname{Tr}\widehat{f}(k)$. By definition, we have

$$\widehat{f}(k) = \int_{\mathrm{SU}(2)} f(u)T^k(u)^*\,du, \qquad k \in \tfrac{1}{2}\mathbb{N}_0,$$

where T^k is a finite-dimensional representation of $\widehat{\mathrm{SU}(2)}$ as in Section 2. Hence

$$\operatorname{Tr}\widehat{f}(k) = \int_{\mathrm{SU}(2)} f(u)\overline{\chi_k(u)}\,du, \tag{5.10}$$

where $\chi_k(u) = \operatorname{Tr}T^k(u)$, $k \in \frac{1}{2}\mathbb{N}_0$, where we have changed the notation from t^k to T^k to avoid confusion with the notation that follows. The characters $\chi_k(u)$ are constant on the conjugacy classes of SU(2) and we follow [Vil68] to describe these classes explicitly.

It is well known from linear algebra that any unitary unimodular matrix u can be written in the form $u = u_1\delta u_1^{-1}$, where $u_1 \in \mathrm{SU}(2)$ and δ is the diagonal matrix

$$\delta = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \tag{5.11}$$

where $\lambda = e^{it/2}$ and $1/\lambda = e^{-it/2}$ are the eigenvalues of u. Moreover, among the matrices equivalent to u there is only one other diagonal matrix, namely, the matrix δ' obtained from δ by interchanging the diagonal elements.

Hence, classes of conjugate elements in SU(2) are determined by one parameter t, varying in $-2\pi \leq t \leq 2\pi$, where t and $-t$ give the same class. Therefore, we can regard the characters $\chi_k(u)$ as functions of one variable t, which ranges from 0 to 2π.

The special unitary group SU(2) is isomorphic to the group of unit quaternions. Hence, the parameter t has a simple geometrical meaning: it is the angle of rotation which corresponds to the matrix u.

Let us now derive an explicit expression for $\chi_k(u)$ as function of t. It was shown e.g. in [RT10] that $T^k(\delta)$ is a diagonal matrix with e^{-int}, $-k \leq n \leq k$, on the diagonal.

Let $u = u_1 \delta u_1^{-1}$. Since characters are constant on conjugacy classes, we get

$$\chi_k(u) = \chi_k(\delta) = \operatorname{Tr} T^k(\delta) = \sum_{n=-k}^{k} e^{int}. \tag{5.12}$$

It is natural to express the invariant integral over SU(2) in (5.10) in new parameters, one of which is t.

Since SU(2) is diffeomorphic to the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ (see, e.g., [RT10]), with

$$\mathrm{SU}(2) \ni u = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \leftrightarrow \varphi(u) = x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3,$$

we have

$$\int_{\mathrm{SU}(2)} f(u)\chi_k(u)\, du = \int_{\mathbb{S}^3} f(x)\chi_k(x)\, dS, \tag{5.13}$$

where $f(x) := f(\varphi^{-1}(x))$ and $\chi_k(x) := \chi_k(\varphi^{-1}(x))$. In order to find an explicit formula for this integral over $\mathbb{S}^3$, we consider the parametrisation

$$\begin{aligned} x_1 &= \cos(t/2), \\ x_2 &= v, \\ x_3 &= \sqrt{\sin^2(t/2) - v^2} \cdot \cos h, \\ x_4 &= \sqrt{\sin^2(t/2) - v^2} \cdot \sin h, \qquad (t, v, h) \in D, \end{aligned}$$

where $D = \{(t, v, h) \in \mathbb{R}^3\colon |v| \leq \sin(t/2),\ 0 \leq t, h \leq 2\pi\}$.

The reader will have no difficulty showing that $dS = \sin(t/2)\,dt\,dv\,dh$. Therefore,

$$\int\limits_{\mathbb{S}^3} f(x)\chi_k(t)\,dS = \int\limits_D f(h,v,t)\chi_k(t)\sin(t/2)\,dh\,dv\,dt.$$

Combining this with (5.13), we get

$$\operatorname{Tr}\widehat{f}(k) = \int\limits_D f(h,v,t)\chi_k(t)\sin(t/2)\,dh\,dv\,dt.$$

Thus, we have expressed the invariant integral over SU(2) in the parameters t, v, h. An application of Fubini's theorem yields

$$\int\limits_D f(h,v,t)\chi_k(t)\sin(t/2)\,dh\,dv\,dt = \int\limits_0^{2\pi} \chi_k(t)\sin(t/2)\,dt \int\limits_{-\sin(t/2)}^{\sin(t/2)} dv \int\limits_0^{2\pi} f(h,v,t)\,dh.$$

Combining this with (5.12), we obtain

$$\operatorname{Tr}\widehat{f}(k) = \int\limits_0^{2\pi} dt \sum_{n=-k}^{k} e^{int}\sin(t/2) \int\limits_{-\sin(t/2)}^{\sin(t/2)} dv \int\limits_0^{2\pi} f(h,v,t)\,dh.$$

Interchanging summation and integration yield

$$\operatorname{Tr}\widehat{f}(k) = \sum_{n=-k}^{k} \int\limits_0^{2\pi} e^{int}\sin(t/2)\,dt \int\limits_{-\sin(t/2)}^{\sin(t/2)} dv \int\limits_0^{2\pi} f(h,v,t)\,dh.$$

By making the change of variables $t \mapsto 2t$, we get

$$\operatorname{Tr}\widehat{f}(k) = \sum_{n=-k}^{k} \int\limits_0^{\pi} e^{-i2nt}\cdot 2\sin t\,dt \int\limits_{-\sin t}^{\sin t} dv \int\limits_0^{2\pi} f(h,v,2t)\,dh. \tag{5.14}$$

Let us now apply Theorem 2.7 in $L^p(\mathbb{T})$. To do this we introduce some notation. Denote

$$F(t) := 2\sin t \int\limits_{-\sin t}^{\sin t} \int\limits_0^{2\pi} f(h,v,2t)\,dh\,dv, \qquad t \in (0,\pi).$$

We extend $F(t)$ periodically to $[0,2\pi)$, that is, $F(x+\pi) = F(x)$. Since $f(t,v,h)$ is integrable, the integrability of $F(t)$ follows immediately from Fubini's theorem. Thus $F(t)$ has a Fourier series representation

$$F(t) \sim \sum_{k\in\mathbb{Z}} \widehat{F}(k)e^{ikt},$$

where the Fourier coefficients are

$$\widehat{F}(k) = \frac{1}{2\pi} \int\limits_{[0,2\pi]} F(t) e^{-ikt}\, dt.$$

Let A_k be the $2k+1$-element arithmetic progression with difference 2 and initial term $-2k$, i.e.,

$$A_k = \{-2k, -2k+2, \dots, 2k\} = \{-2k+2j\}_{j=0}^{2k}.$$

Using this notation and (5.14), we get

$$\operatorname{Tr} \widehat{f}(k) = \sum_{n \in A_k} \widehat{F}(n). \tag{5.15}$$

Define

$$B = \{A_k\}_{k=1}^{\infty}.$$

As B is a subset of the set M of all finite arithmetic progressions, (5.15) yields

(5.16)
$$\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ 2k+1 \geq 2l+1}} \frac{1}{2k+1} |\operatorname{Tr} \widehat{f}(k)| \leq \sup_{\substack{e \in B \\ |e| \geq 2l+1}} \frac{1}{|e|} \Big| \sum_{i \in e} \widehat{F}(i) \Big| \leq \sup_{\substack{e \in M \\ |e| \geq 2l+1}} \frac{1}{|e|} \Big| \sum_{i \in e} \widehat{F}(i) \Big|.$$

Denote $m := 2l+1$. If l runs over $\frac{1}{2}\mathbb{N}_0$, then m runs over $\mathbb{N}$. Using (5.16), we get

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2} \Bigg(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ 2k+1 \geq 2l+1}} \frac{1}{2k+1} |\operatorname{Tr} \widehat{f}(k)| \Bigg)^p \leq \sum_{m \in \mathbb{N}} m^{p-2} \Bigg(\sup_{\substack{e \in M \\ |e| \geq m}} \frac{1}{|e|} \Big| \sum_{i \in e} \widehat{F}(i) \Big| \Bigg)^p. \tag{5.17}$$

Application of (2.13) yields

$$\sum_{m \in \mathbb{N}} m^{p-2} \Bigg(\sup_{\substack{e \in M \\ |e| \geq m}} \frac{1}{|e|} \Big| \sum_{i \in e} \widehat{F}(i) \Big| \Bigg)^p \leq c \|F\|^p_{L^p(0,2\pi)}. \tag{5.18}$$

Using the Hölder inequality, we obtain

$$\int\limits_0^{\pi} |F(t)|^p\, dt \lesssim \int\limits_0^{\pi} \sin t\, dt \int\limits_{-\sin t}^{\sin t} dv \int\limits_0^{2\pi} |f(h, v, 2t)|^p\, dh.$$

By making the change of variables $t \mapsto t/2$ in the right hand side integral, we get

$$\int\limits_0^{\pi} |F(t)|^p\, dt \lesssim \int\limits_0^{2\pi} \sin(t/2)\, dt \int\limits_{-\sin(t/2)}^{\sin(t/2)} dv \int\limits_0^{2\pi} |f(h, v, t)|^p\, dh.$$

Thus, we have proved that

$$\|F\|_{L^p(0,\pi)} \le c_p \|f\|_{L^p(\mathrm{SU}(2))}, \tag{5.19}$$

where c_p depends only on p. Combining (5.16), (5.17) and (5.19), we obtain

$$\sum_{m\in\mathbb{N}} m^{p-2}\Big(\sup_{\substack{k\in\frac{1}{2}\mathbb{N}_0\\ 2k+1\ge m}} 1/2k+1|\operatorname{Tr}\widehat{f}(k)|\Big)^p \le c\|f\|^p_{L^p(\mathrm{SU}(2))}.$$

This completes the proof of Theorem 2.8. ∎

6. Marcinkiewicz interpolation theorem. In this section we prove the Marcinkiewicz interpolation theorem for linear mappings between a compact group G and the space of matrix-valued sequences Σ that will be realised via

$$\Sigma := \big\{h = \{h(\pi)\}_{\pi\in\widehat{G}}, h(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}\big\}.$$

Thus, a linear mapping $A\colon \mathcal{D}'(G) \to \Sigma$ takes a function to a matrix valued sequence, i.e.

$$f \mapsto Af =: h = \{h(\pi)\}_{\pi\in\widehat{G}},$$

where

$$h(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}, \qquad \pi \in \widehat{G}.$$

We say that a linear operator A is of *strong type* (p,q) if for every $f \in L^p(G)$, we have $Af \in \ell^q(\widehat{G},\Sigma)$ and

$$\|Af\|_{\ell^q(\widehat{G},\Sigma)} \le M\|f\|_{L^p(G)},$$

where M is independent of f, and the space $\ell^q(\widehat{G},\Sigma)$ is defined by the norm

$$\|h\|_{\ell^q(\widehat{G},\Sigma)} := \Big(\sum_{\pi\in\widehat{G}} d_\pi^{p(2/p-1/2)}\|h(\pi)\|^p_{\mathrm{HS}}\Big)^{1/p} \tag{6.1}$$

(cf. (2.9)). The least M for which this is satisfied is taken to be the strong (p,q)-norm of the operator A.

Denote the distribution functions of f and h by $\mu_G(t;f)$ and $\nu_{\widehat{G}}(u;h)$, respectively, i.e.

$$\mu_G(x;f) := \int\limits_{\substack{u\in G\\ |f(u)|\ge x}} du, \qquad x>0, \tag{6.2}$$

$$\nu_{\widehat{G}}(y;h) := \sum_{\substack{\pi\in\widehat{G}\\ \|h(\pi)\|_{\mathrm{HS}}/\sqrt{d_\pi}\ge y}} d_\pi^2, \quad y>0. \tag{6.3}$$

Then

$$\|f\|^p_{L^p(G)} = \int_G |f(u)|^p\,du = p\int_0^\infty x^{p-1}\mu_G(x;f)\,dx,$$

$$\|h\|^q_{\ell^q(\widehat{G},\Sigma)} = \sum_{\pi\in\widehat{G}} d_\pi^2 \left(\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}}\right)^q = q\int_0^\infty u^{q-1}\nu_{\widehat{G}}(y;h)\,dy.$$

A linear operator $A\colon \mathcal{D}'(G)\to\Sigma$ satisfying

$$\nu_{\widehat{G}}(y;Af) \le \left(\frac{M}{y}\|f\|_{L^p(G)}\right)^q \tag{6.4}$$

is said to be of *weak type* (p,q); the least value of M in (6.4) is called the *weak* (p,q) *norm* of A.

Every operation of strong type (p,q) is also of weak type (p,q), since

$$y(\nu_{\widehat{G}}(y;Af))^{1/q} \le \|Af\|_{L^q(\widehat{G})} \le M\|f\|_{L^p(G)}.$$

THEOREM 6.1. *Let* $1\le p_1<p<p_2<\infty$. *Suppose that a linear operator* A *from* $\mathcal{D}'(G)$ *to* Σ *is simultaneously of weak types* (p_1,p_1) *and* (p_2,p_2), *with norms* M_1 *and* M_2, *respectively, i.e.*

$$\nu_{\widehat{G}}(y;Af) \le \left(\frac{M_1}{y}\|f\|_{L^{p_1}(G)}\right)^{p_1}, \tag{6.5}$$

$$\nu_{\widehat{G}}(y;Af) \le \left(\frac{M_2}{y}\|f\|_{L^{p_2}(G)}\right)^{p_2}. \tag{6.6}$$

Then for any $p\in(p_1,p_2)$ *the operator* A *is of strong type* (p,p) *and*

$$\|Af\|_{\ell^p(\widehat{G},\Sigma)} \le M_1^{1-\theta}M_2^{\theta}\|f\|_{L^p(G)}, \qquad 0<\theta<1, \tag{6.7}$$

where $\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$.

Proof. The proof is an adaptation of one in Zygmund [Zyg56] to our setting. Let $f\in L^p(G)$. By definition,

$$\|Af\|^p_{\ell^p(\widehat{G},\Sigma)} = \sum_{\pi\in\widehat{G}} d_\pi^2\left(\frac{\|Af(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}}\right)^p = \int_0^\infty px^{p-1}\nu_{\widehat{G}}(x;Af)\,dx. \tag{6.8}$$

For a fixed $z>0$ we consider the decomposition $f=f_1+f_2$, where $f_1=f$ whenever $|f|<z$, and $f_1=0$ otherwise; thus $|f_2|>z$ or else $f_2=0$. Since f is in $L^p(G)$, so are f_1 and f_2; it follows that f_1 is in $L^{p_1}(G)$ and f_2 is in $L^{p_2}(G)$. Hence Af_1 and Af_2 exist, by hypothesis, and so does $Af=A(f_1+f_2)$. It follows that

$$|f_1| = \min(|f|,z), \qquad |f| = |f_1|+|f_2|. \tag{6.9}$$

The inequality

$$\|A(f_1+f_2)(\pi)\|_{\mathtt{HS}} \le \|Af_1(\pi)\|_{\mathtt{HS}} + \|Af_2(\pi)\|_{\mathtt{HS}}, \qquad \pi\in\widehat{G},$$

yields

$$\left\{\pi \in \widehat{G}\colon \frac{\|Af(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}} \geq y\right\} \subset \left\{\pi \in \widehat{G}\colon \frac{\|Af_1(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}} \geq \frac{y}{2}\right\} \cup \left\{\pi \in \widehat{G}\colon \frac{\|Af_2(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}} \geq \frac{y}{2}\right\}.$$

Then applying assumptions (6.5) and (6.6) to f_1 and f_2, we obtain

$$\begin{aligned} \nu_{\widehat{G}}(y; Af) &\leq \nu_{\widehat{G}}(y/2; Af_1) + \nu_{\widehat{G}}(y/2; Af_2) \\ &\leq M_1^{p_1} y^{-p_1} \|f_1\|^{p_1}_{L^{p_1}(G)} + M_1^{p_2} y^{-p_2} \|f_2\|^{p_2}_{L^{p_2}(G)}. \end{aligned} \tag{6.10}$$

The right side depends on z and the main idea of the proof is to define z as a suitable monotone function of t, $z = z(t)$, to be determined later. By (6.9),

$$\begin{aligned} &\mu_G(t; f_1) = \mu_G(t; f) && \text{for } 0 < t \leq z, \\ &\mu_G(t; f_1) = 0 && \text{for } t > z, \\ &\mu_G(t; f_2) = \mu_G(t + z; f) && \text{for } t > 0. \end{aligned}$$

Here, the last equation is a consequence of the fact that wherever $f_2 \neq 0$ we must have $|f_1| = z$, and so the second equation of (6.9) takes the form $|f| = z + |f_2|$.

It follows from (6.10) that the integral in (6.8) is less than

$$\begin{aligned} &M_1^{p_1} \int_0^\infty y^{p-p_1-1} \Big\{\int_G |f_1(u)|^{p_1}\, du\Big\}^{p_1/p_1} dy \\ &\quad + M_2^{p_2} \int_0^\infty y^{p-p_2-1} \Big\{\int_G |f_2(u)|^{p_1}\, du\Big\}^{p_2/p_2} dy \\ &= M_1^{p_1} p_1 \int_0^\infty y^{p-p_1-1} \Big\{\int_0^z x^{p_1-1} \mu_G(x; f)\, dx\Big\}\, dt \\ &\quad + M_2^{p_2} p_2 \int_0^\infty y^{p-p_2-1} \Big\{\int_z^\infty (x - z)^{p_2-1} \mu_G(x; f)\, dx\Big\}\, dt. \end{aligned} \tag{6.11}$$

Set $z(y) = A/y$. Denote by I_1 and I_2 the last two double integrals. We change the order of integration in I_1:

$$\begin{aligned} I_1 &= \int_0^\infty t^{p-p_1-1} \Big\{\int_0^z u^{p_1-1} \mu_G(u; f)\, du\Big\}\, dt \\ &= \int_0^\infty x^{p_1-1} \mu_G(x; f) \Big\{\int_0^{Ax} y^{p-p_1-1}\, dy\Big\}\, dx \\ &= \frac{A^{p-p_1}}{p - p_1} \int_0^\infty x^{p_1-1+p-p_1} \mu_G(x; f)\, dx. \end{aligned} \tag{6.12}$$

Similarly, making the substitution $x - z \mapsto x$ and using (6.9) we see that

$$
\begin{aligned}
I_2 &= M_2^{p_2} p_2 \int_0^\infty y^{p-p_2-1} \Big\{ \int_z^\infty (x-z)^{p_2-1} \mu_G(x; f)\, dx \Big\}\, dy \\
&= M_2^{p_2} p_2 \int_0^\infty y^{p-p_2-1} \Big\{ \int_0^\infty x^{p_2-1} \mu_G(x+z; f)\, dx \Big\}\, dy \\
&= M_2^{p_2} p_2 \int_0^\infty y^{p-p_2-1} \Big\{ \int_0^\infty x^{p_2-1} \mu_G(x; f_2)\, dx \Big\}\, dy \\
&= M_2^{p_2} p_2 \int_0^\infty \Big\{ \int_0^\infty x^{p_2-1} \mu_G(x; f_2) y^{p-p_2-1}\, dy \Big\}\, dx \\
&= M_2^{p_2} p_2 \int_0^\infty \Big\{ \int_{Ax^{1/\xi}}^\infty x^{p_2-1} \mu_G(x; f_2) y^{p-p_2-1}\, dy \Big\}\, dx \\
&= M_2^{p_2} p_2 \int_0^\infty x^{p_2-1} \mu_G(x; f_2) \Big\{ \int_{Ax}^\infty y^{p-p_2-1}\, dy \Big\}\, dx \\
&= \frac{A^{p-p_2}}{p_2 - p} M_2^{p_2} p_2 \int_0^\infty x^{p_2-1+p-p_2} \mu_G(x; f_2)\, dx \\
&\le \frac{A^{p-p_2}}{p_2 - p} M_2^{p_2} p_2 \int_0^\infty x^{p_2-1+p-p_2} \mu_G(x; f)\, dx.
\end{aligned} \tag{6.13}
$$

Collecting (6.11)–(6.13) we see that the integral in (6.8) does not exceed

$$
M_1^{p_1} p_1 \frac{A^{p-p_1}}{p-p_1} \int_0^\infty x^{p-1} \mu_G(x; f)\, dx + M_2^{p_2} p_2 \frac{A^{p-p_2}}{p_2-p} \int_0^\infty x^{p-1} \mu_G(x; f_2)\, dx. \tag{6.14}
$$

Now, using the identity

$$
\int_0^\infty x^{p-1} \mu_G(x; f)\, dx = \int_G |f(u)|^p\, du = \|f\|^p_{L^p(G)}
$$

and inequalities (6.8) and (6.14) we get

$$
\|Af\|^p_{\ell^p(\widehat{G})} \le \left(M_1^{p_1} p_1 \frac{A^{p-p_1}}{p-p_1} + M_2^{p_2} p_2 \frac{A^{p-p_2}}{p_2 - p} \right)^p \|f\|^p_{\ell^p(\widehat{G})}.
$$

Next we set

$$
A = M_1^{\frac{p_1}{p_1 - p_2}} M_2^{\frac{p_2}{p_2 - p_1}}.
$$

A simple computation shows that

$$M_1^{p_1}A^{p-p_1} = M_2^{p_2}A^{p-p_2} = M_1^{\frac{p_1(p_2-p)}{p_2-p_1}} M_2^{\frac{p_2(p_1-p)}{p_1-p_2}}$$
$$= M_1^{1-\theta}M_2^{\theta}, \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Finally, we have

$$\|Af\|_{\ell^p(\widehat{G})} \le K_{p,p_1,p_2} M_1^{1-\theta}M_2^{\theta}\|f\|_{L^p(G)},$$

where

$$K_{p,p_1,p_2} = \left(\frac{p_1}{p-p_1} + \frac{p_2}{p_2-p}\right)^{1/p}. \ \blacksquare$$

Acknowledgments. The authors would like to thank Véronique Fischer for useful remarks.

The first and the second authors were supported by the MESRK grants 4080/GF, 3311/GF4, 4080/GF4 and the third author was supported by the EPSRC Grant EP/K039407/1. No new data was collected or generated during the course of the research.

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