# Pseudo-differential operators and symmetries

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# Part III

# Representation theory of compact groups

We might call the traditional topology and measure theory by the name "commutative geometry", referring to the commutative function algebras; "non-commutative geometry" would refer to the study of non-commutative algebras. Non-commutativity is the characteristic feature of this part.

Here we present the necessary material on the compact groups and their representations. The presentation is made gradually increasing the availability of topological and differentiable structures, thus tracing the development from the general compact groups to the linear Lie groups. Moreover, we present an additional material on the Hopf algebras joining together the material of this part to the analysis of algebras from Chapter D. Nevertheless, we tried to make the exposition self-contained, providing references to Part I when necessary. If the reader wants to gain more profound knowledge of Lie groups, Lie algebras and their representation, there are many excellent monographs available on different aspects of these theories at different levels, for example [9, 19, 20, 31, 36, 37, 38, 47, 48, 49, 50, 51, 58, 61, 64, 65, 73, 74, 85, 120, 124, 144, 145, 146, 151], to mention a few.

# Chapter 6

# Groups

## 6.1 Introduction

Loosely speaking, groups encode symmetries of (geometric) objects: if we consider a space $X$ with some specific structure (e.g. a Riemannian manifold), a *symmetry of* $X$ is a bijection $f : X \to X$ preserving the natural involved structure (e.g. the Riemannian metric) — here, the compositions and inversions of symmetries yield new symmetries. In a handful of assumptions, the concept of groups captures the essential properties of wide classes of symmetries, and provides powerful tools for related analysis.

Perhaps the first non-trivial group that the mankind encountered was the set $\mathbb{Z}$ of integers; with the usual addition $(x, y) \mapsto x + y$ and "inversion" $x \mapsto -x$ this is a basic example of a group. Intuitively, a group is a set $G$ that has two mappings $G \times G \to G$ and $G \to G$ generalising the properties of the integers in a simple and natural way.

We start by defining the groups, and we study the mappings preserving such structures, i.e. group homomorphisms. Of special interest are representations, that is those group homomorphisms that have values in groups of invertible linear operators on vector spaces. Representation theory is a key ingredient in the theory of groups.

In this framework we study analysis on compact groups, foremost measure theory and Fourier transform. Remarkably, for a compact group $G$ there exists a unique translation-invariant linear functional on $C(G)$ corresponding to a probability measure. We shall construct this so-called Haar measure, closely related to the Lebesgue measure of a Euclidean space. We shall also introduce Fourier series of functions on a group.

Groups having a smooth manifold structure (with smooth group operations) are called Lie groups, and their representation theory is especially interesting. Left-invariant first-order partial differential operators on such a group can be identified with left-invariant vector fields on the group, and the corresponding set called the Lie algebra is studied.

Finally, we introduce Hopf algebras and study the Gelfand theory related to them.

*Remark* 6.1.1 (**–morphisms**). If $X, Y$ are spaces with the same kind of algebraic structure, the set $\mathrm{Hom}(X,Y)$ of *homomorphisms* consists of mappings $f : X \to Y$ respecting the structure. Bijective homomorphisms are called *isomorphisms*. Homomorphisms $f : X \to X$ are called *endomorphisms* of $X$, and their set is denoted by $\mathrm{End}(X) := \mathrm{Hom}(X,X)$. Isomorphism-endomorphisms are called *automorphisms*, and their set is $\mathrm{Aut}(X) \subset \mathrm{End}(X)$. If there exist the zero-elements $0_X, 0_Y$ in respective algebraic structures $X, Y$, the *null space* or the *kernel* of $f \in \mathrm{Hom}(X,Y)$ is

$$\mathrm{Ker}(f) := \{x \in X : \ f(x) = 0_Y\}.$$

Sometimes algebraic structures might have, say, topology, and then the homomorphisms are typically required to be continuous. Hence, for instance, a homomorphism $f : X \to Y$ between Banach spaces $X, Y$ is usually assumed to be continuous and linear, denoted by $f \in \mathcal{L}(X,Y)$, unless otherwise mentioned; for short, let $\mathcal{L}(X) := \mathcal{L}(X,X)$. The assumptions in theorems etc. will still be explicitely stated.

**Conventions.** $\mathbb{N}$ is the set of positive integers
$\mathbb{Z}^+ = \mathbb{N}$,
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
$\mathbb{Z}$ is the set of integers,
$\mathbb{Q}$ the set of rational numbers,
$\mathbb{R}$ the set of real numbers,
$\mathbb{C}$ the set of complex numbers, and
$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

## 6.2 Groups without topology

We start with groups without complications, without assuming supplementary properties. This choice helps in understanding the purely algebraic ideas, and only later we will mingle groups with other structures, e.g. topology.

**Definition 6.2.1 (Groups).** A *group* consists of a set $G$ having an element $e = e_G \in G$ and endowed with mappings

$$\begin{aligned} ((x,y) \mapsto xy) &\quad : \quad G \times G \to G, \\ (x \mapsto x^{-1}) &\quad : \quad G \to G \end{aligned}$$

satisfying

$$\begin{aligned} &x(yz) = (xy)z, \\ &ex = x = xe, \\ &x\,x^{-1} = e = x^{-1}x, \end{aligned}$$

for all $x, y, z \in G$. We may freely write $xyz := x(yz) = (xy)z$; the element $e \in G$ is called the *neutral element*, and $x^{-1}$ is the *inverse* of $x \in G$. If the group operations are implicitely known, we may say that $G$ is a *group*. If $xy = yx$ for all $x, y \in G$ then $G$ is called *commutative* (or *Abelian*).

*Example.* Let us give some examples of groups:

1. **(Symmetric group).** Let $G = \{f : X \to X \mid f \text{ bijection}\}$, where $X \neq \emptyset$; this is a group with operations $(f, g) \mapsto f \circ g$, $f \mapsto f^{-1}$. This group $G$ of bijections on $X$ is called the *symmetric group of* $X$, and it is non-commutative whenever $|X| \geq 3$, where $|X|$ is the number of elements of $X$. The neutral element is $\mathrm{id}_X = (x \mapsto x) : X \to X$.

2. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are commutative groups with operations $(x, y) \mapsto x + y$, $x \mapsto -x$. The neutral element is 0 in each case.

3. Any vector space is a commutative group with operations $(x, y) \mapsto x + y$, $x \mapsto -x$; the neutral element is 0.

4. **(Automorphisms $\mathbf{Aut}(V)$).** Let $V$ be a vector space. The set $\mathrm{Aut}(V)$ of invertible linear operators $V \to V$ forms a group with operations $(A, B) \mapsto AB$, $A \mapsto A^{-1}$; this group is non-commutative when $\dim(V) \geq 2$. The neutral element is $I = (v \mapsto v) : V \to V$.

5. Sets $\mathbb{Q}^\times := \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ (more generally, invertible elements of a unital ring) form multiplicative groups with operations $(x, y) \mapsto xy$ (ordinary multiplication) and $x \mapsto x^{-1}$ (as usual). The neutral element is 1 in each case.

6. The set

$$\text{Aff}(V) = \{A_a = (v \mapsto Av + a) : V \to V \mid A \in \text{Aut}(V),\ a \in V\}$$

of affine mappings forms a group with operations defined to be $(A_a, B_b) \mapsto (AB)_{Ab+a}$, $A_a \mapsto (A^{-1})_{A^{-1}a}$; this group is non-commutative when $\dim(V) \geq 1$. The neutral element is $I_0$.

7. If $G$ and $H$ are groups then $G \times H$ has a natural group structure:

$$((g_1, h_1), (g_2, h_2)) \mapsto (g_1 h_1, g_2 h_2), \quad (g, h) \mapsto (g^{-1}, h^{-1}).$$

The neutral element is $e_{G\times H} := (e_G, e_H)$.

**Exercise 6.2.2.** Let $G$ be a group and $x, y \in G$. Prove:
(a) $(x^{-1})^{-1} = x$.
(b) If $xy = e$ then $y = x^{-1}$.
(c) $(xy)^{-1} = y^{-1}x^{-1}$.

**Definition 6.2.3 (Finite groups).** If a group has finitely many elements it is said to be *finite.*

*Example.* The symmetry group of a set consisting of $n$ elements is called *the permutation group of* $n$ *elements.* Such group is a finite group and has $n! = 1\cdot 2 \cdots n$ elements.

**Definition 6.2.4 (Notation).** Let $G$ be a group, $x \in A$, $A, B \subset G$ and $n \in \mathbb{Z}^+$. We denote

$$\begin{aligned}
xA &:= \{xa \mid a \in A\}, \\
Ax &:= \{ax \mid a \in A\}, \\
AB &:= \{ab \mid a \in A,\ b \in B\}, \\
A^0 &:= \{e\}, \\
A^{-1} &:= \left\{a^{-1} \mid a \in A\right\}, \\
A^{n+1} &:= A^n A, \\
A^{-n} &:= (A^n)^{-1}.
\end{aligned}$$

**Definition 6.2.5 (Subgroups $H < G$, and normal subgroups $H \lhd G$).** A set $H \subset G$ is a *subgroup* of a group $G$, denoted by $H < G$, if

$$e \in H, \quad xy \in H \quad \text{and} \quad x^{-1} \in H$$

for all $x, y \in H$ (hence $H$ is a group with the inherited operations). A subgroup $H < G$ is called *normal in* $G$ if

$$xH = Hx$$

for all $x \in G$; then we write $H \lhd G$.

*Remark* 6.2.6. With the inherited operations, a subgroup is a group. Normal subgroups are the well-behaving ones, as exemplified later in Proposition 6.2.16 and Theorem 6.2.20. In some books normal subgroups of $G$ are called *normal divisors* of $G$.

**Exercise 6.2.7.** Let $H < G$. Show that if $H \subset x^{-1}Hx$ for every $x \in G$, then $H \lhd G$.

**Exercise 6.2.8.** Let $H < G$. Show that $H \lhd G$ if and only if $H = x^{-1}Hx$ for every $x \in G$.

*Example.* Let us collect some instances and facts about subgroups:

1. We always have normal *trivial subgroups* $\{e\} \lhd G$ and $G \lhd G$. Subgroups of a commutative group are always normal.
2. **(Centre of a group).** The *centre* $Z(G) \lhd G$, where

$$Z(G) := \{z \in G \mid \forall x \in G: \; xz = zx\}.$$

   Thus, the centre is the collection of elements that commute with all elements of the group.
3. If $F < H$ and $G < H$ then $F \cap G < H$.
4. If $F < H$ and $G \lhd H$ then $FG < H$.
5. $\{I_a \mid a \in V\} \lhd \mathrm{Aff}(V)$.
6. The following two examples will be of crucial importance later so we formulate them as Remarks 6.2.9 and 6.2.10.

*Remark* 6.2.9 (**Groups GL**$(n,\mathbb{R})$**, O**$(n)$**, SO**$(n)$**)**. We have

$$\mathrm{SO}(n) < \mathrm{O}(n) < \mathrm{GL}(n,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{R}^n),$$

where the groups consist of real $n \times n$-matrices: $\mathrm{GL}(n,\mathbb{R})$ is the real *general linear* group consisting of invertible real matrices (i.e. determinant non-zero); $\mathrm{O}(n)$ is the *orthogonal* group, where the matrix columns (or rows) form an orthonormal basis for $\mathbb{R}^n$ (so that $A^T = A^{-1}$ for $A \in \mathrm{O}(n)$, $\det(A) \in \{-1,1\}$); $\mathrm{SO}(n)$ is the *special orthogonal* group, the group of rotation matrices of $\mathbb{R}^n$ around the origin, so that

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \; \det(A) = 1\}.$$

*Remark* 6.2.10 (**Groups GL**$(n,\mathbb{C})$**, U**$(n)$**, SU**$(n)$**)**. We have

$$\mathrm{SU}(n) < \mathrm{U}(n) < \mathrm{GL}(n,\mathbb{C}) \cong \mathrm{Aut}(\mathbb{C}^n),$$

where the groups consist of complex $n \times n$-matrices: $\mathrm{GL}(n,\mathbb{C})$ is the complex *general linear* group consisting of invertible complex matrices (i.e. determinant non-zero); $\mathrm{U}(n)$ is the *unitary* group, where the matrix columns (or rows) form an orthonormal basis for $\mathbb{C}^n$ (so that $A^* = A^{-1}$ for $A \in \mathrm{U}(n)$, $|\det(A)| = 1$); $\mathrm{SU}(n)$ is the *special unitary* group,

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \det(A) = 1\}.$$

*Remark* 6.2.11. The mapping $(z \mapsto (z)) : \mathbb{C} \to \mathbb{C}^{1\times 1}$ identifies complex numbers with complex $(1\times 1)$-matrices. Thereby the complex unit circle group $\{z \in \mathbb{C} : |z| = 1\}$ is identified with the group $\mathrm{U}(1)$.

**Definition 6.2.12** (**Right quotient** $G/H$)**.** Let $H < G$. Then

$$x \sim y \quad \Longleftrightarrow \quad xH = yH$$

defines an equivalence relation on $G$, as can be easily verified. The (*right*) *quotient of* $G$ *by* $H$ is the set

$$G/H = \{xH \mid x \in G\}.$$

Notice that $xH = yH$ if and only if $x^{-1}y \in H$.

Similarly, we can define

**Definition 6.2.13** (**Left quotient** $H\backslash G$)**.** Let $H < G$. Then

$$x \sim y \quad \Longleftrightarrow \quad Hx = Hy$$

defines an equivalence relation on $G$. The (*left*) *quotient of* $G$ *by* $H$ is the set

$$H\backslash G = \{Hx \mid x \in G\}.$$

Notice that $Hx = Hy$ if and only if $x^{-1}y \in H$.

*Remark* 6.2.14 (**Right for now**). We will deal mostly with the right quotient $G/H$ in Part III. However, we note that in Part IV we will actually need more the left quotient $H\backslash G$. It should be a simple exercise for the reader to translate all the results from "right" to "left". Indeed, simply replacing the side from which the subgroup acts from right to left, and changing all the words from "right" to "left" should do the job since the situation is completely symmetric. The reason for our change is that once we choose to identify the Lie algebras with the left-invariant vector fields in Part IV it leads to a more natural analysis of pseudo-differential operator on left quotients. However, because our intuition about division may be better suited to the notation $G/H$ we chose to explain the basic ideas for the right quotients, keeping in mind that the situation with the left quotients is completely symmetric.

*Remark* 6.2.15. It is often useful to identify the points $xH \in G/H$ with the sets $xH \subset G$. Also, for $A \subset G$ we naturally identify the sets

$$\begin{aligned} AH = \{ah : a \in A, h \in H\} &\quad \subset \quad G \quad \text{and} \\ \{aH : a \in A\} = \{\{ah : h \in H\} : a \in A\} &\quad \subset \quad G/H. \end{aligned}$$

This provides a nice way to treat the quotient $G/H$.

**Proposition 6.2.16 (When is $G/H$ a group?).** *Let $H \lhd G$ be normal. Then the quotient $G/H$ can be endowed with the group structure*

$$(xH, yH) \mapsto xyH, \quad xH \mapsto x^{-1}H.$$

*Proof.* The operations are well-defined mappings $(G/H) \times (G/H) \to G/H$ and $G/H \to G/H$, respectively, since

$$xHyH \overset{H \trianglelefteq G}{=} xyHH \overset{HH=H}{=} xyH,$$

and

$$(xH)^{-1} = H^{-1}x^{-1} \overset{H^{-1}=H}{=} Hx^{-1} \overset{H \trianglelefteq G}{=} x^{-1}H.$$

The group axioms follow, since by simple calculations we have

$$(xH)(yH)(zH) = xyzH,$$

$$(xH)(eH) = xH = (eH)(xH),$$
$$(x^{-1}H)(xH) = H = (xH)(x^{-1}H).$$

Notice that $e_{G/H} = e_G H = H$. □

**Definition 6.2.17 (Torus $\mathbb{T}^n$ as a quotient group).** The quotient $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ is called the (*flat*) *$n$-dimensional torus.*

**Definition 6.2.18 (Homomorphisms and isomorphisms).** Let $G, H$ be groups. A mapping $\phi : G \to H$ is called a *homomorphism* (or a *group homomorphism*), denoted by $\phi \in \mathrm{Hom}(G, H)$, if

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in G$. The *kernel* of $\phi \in \mathrm{Hom}(G, H)$ is

$$\mathrm{Ker}(\phi) := \{x \in G \mid \phi(x) = e_H\}.$$

A bijective homomorphism $\phi \in \mathrm{Hom}(G, H)$ is called an *isomorphism*, denoted by $\phi : G \cong H$.

*Remark* 6.2.19. Group homomorphisms are the natural mappings between groups, preserving the group operations. Notice especially that for a group homomorphism $\phi : G \to H$ it holds that

$$\phi(e_G) = e_H \quad \text{and} \quad \phi(x^{-1}) = \phi(x)^{-1}$$

for all $x \in G$.

*Example.* Examples of homomorphisms:

1. $(x \mapsto e_H) \in \mathrm{Hom}(G, H)$.

2. For $y \in G$, $(x \mapsto y^{-1}xy) \in \mathrm{Hom}(G, G)$.
3. If $H \lhd G$ then $x \mapsto xH$ is a surjective homomorphism $G \to G/H$.
4. For $x \in G$, $(n \mapsto x^n) \in \mathrm{Hom}(\mathbb{Z}, G)$.
5. If $\phi \in \mathrm{Hom}(F, G)$ and $\psi \in \mathrm{Hom}(G, H)$ then $\psi \circ \phi \in \mathrm{Hom}(F, H)$.
6. $\mathbb{T}^1 \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$.

**Theorem 6.2.20.** *Let $\phi \in \mathrm{Hom}(G, H)$ and $K = \mathrm{Ker}(\phi)$. Then:*

1. *$\phi(G) < H$.*
2. *$K \lhd G$.*
3. *$\psi(xK) := \phi(x)$ defines a group isomorphism $\psi : G/K \to \phi(G)$.*

*Thus we have the commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle x \mapsto xK}\downarrow & & \uparrow{\scriptstyle y \mapsto y} \\ G/K & \xrightarrow{\psi : G/K \cong \phi(G)} & \phi(G). \end{array}$$

*Proof.* Let $x, y \in G$. Now $\phi(G)$ is a subgroup of $H$, because

$$\begin{array}{rcl} e_H & = & \phi(e_G) \ \in \ \phi(G), \\ \phi(x)\phi(y) & = & \phi(xy) \ \in \ \phi(G), \\ \phi(x^{-1})\phi(x) & = & \phi(x^{-1}x) \ = \ \phi(e_G) \\ & = & e_H \\ & = & \ldots \ = \ \phi(x)\phi(x^{-1}); \end{array}$$

notice that $\phi(x)^{-1} = \phi(x^{-1})$. If $a, b \in \mathrm{Ker}(\phi)$ then

$$\begin{array}{rcl} \phi(e_G) & = & e_H, \\ \phi(ab) & = & \phi(a)\phi(b) \ = \ e_H e_H \ = \ e_H, \\ \phi(a^{-1}) & = & \phi(a)^{-1} \ = \ e_H^{-1} \ = \ e_H, \end{array}$$

so that $K = \mathrm{Ker}(\phi) < G$. If moreover $x \in G$ then

$$\phi(x^{-1}Kx) = \phi(x^{-1})\ \phi(K)\ \phi(x) = \phi(x)^{-1}\ \{e_H\}\ \phi(x) = \{e_H\},$$

meaning $x^{-1}Kx \subset K$. Thus $K \lhd G$ by Exercise 6.2.8. By Proposition 6.2.16, $G/K$ is a group (with the natural operations). Since $\phi(xa) = \phi(x)$ for every $a \in K$, $\psi = (xK \mapsto \phi(x)) : G/K \to \phi(G)$ is a well-defined surjection. Furthermore,

$$\psi(xyK) = \phi(xy) = \phi(x)\phi(y) = \psi(xK)\psi(yK),$$

thus $\psi \in \mathrm{Hom}(G/K, \phi(G))$. Finally,

$$\psi(xK) = \psi(yK) \iff \phi(x) = \phi(y) \iff x^{-1}y \in K \iff xK = yK,$$

so that $\psi$ is injective. □

**Exercise 6.2.21 (Universality of the permutation groups).** Let $G$ be a finite group. Show that there is a set $X$ with finitely many elements such that $G$ is isomorphic to a subgroup of the symmetric group of $X$.

## 6.3 Group actions and representations

Spaces can be studied by examining their symmetry groups. On the other hand, it is fruitful to investigate groups when they are acting as symmetries of some nicely structured spaces. Next we study actions of groups on sets. Especially interesting group actions are the linear actions on vector spaces, providing the machinery of linear algebra — this is the fundamental idea in the representation theory of groups.

**Definition 6.3.1 (Transitive actions).** An *action of a group $G$ on a set $M \neq \emptyset$* is a mapping

$$((x,p) \mapsto x \cdot p) : G \times M \to M,$$

for which

$$\begin{cases} x \cdot (y \cdot p) = (xy) \cdot p, \\ e \cdot p = p \end{cases}$$

for all $x, y \in G$ and $p \in M$; the action is *transitive* if

$$\forall p, q \in M \ \exists x \in G : \ x \cdot q = p.$$

If $M$ is a vector space and the mapping $p \mapsto x \cdot p$ is linear for each $x \in G$, the action is called *linear*.

*Remark* 6.3.2. To be precise, our *action* $G \times M \to M$ in Definition 6.3.1 should be called a *left action*, to make a difference to the *right actions* $M \times G \to M$, which are defined in the obvious way. When $G$ acts on $M$, it is useful to think $G$ as a (sub)group of symmetries of $M$. Transitivity means that $M$ is highly symmetric: there are enough symmetries to move any point to any other point.

*Example.* Let us present some examples of actions:

1. On a vector space $V$, the group $\mathrm{Aut}(V)$ acts linearly by $(A, v) \mapsto Av$.
2. If $\phi \in \mathrm{Hom}(G, H)$ then $G$ acts on $H$ by $(x, y) \mapsto \phi(x)y$. Especially, $G$ acts on $G$ transitively by $(x, y) \mapsto xy$.
3. The rotation group $\mathrm{SO}(n)$ acts transitively on the sphere $\mathbb{S}^{n-1} := \{x = (x_j)_{j=1}^n \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ by $(A, x) \mapsto Ax$.

4. If $H < G$ and $((x,p) \mapsto x \cdot p) : G \times M \to M$ is an action then the restriction $((x,p) \mapsto x \cdot p) : H \times M \to M$ is an action.

**Definition 6.3.3 (Isotropy subgroup).** Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be an action. The *isotropy subgroup* of $q \in M$ is

$$G_q := \{x \in G \mid x \cdot q = q\}.$$

That is, $G_q \subset G$ contains those symmetries that fix the point $q \in M$.

**Theorem 6.3.4.** *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a transitive action. Let $q \in M$. Then the isotropy subgroup $G_q$ is a subgroup for which*

$$f_q := (xG_q \mapsto x \cdot q) : G/G_q \to M$$

*is a bijection.*

*Remark* 6.3.5. If $G_q \lhd G$ then $G/G_q$ is a group; otherwise the quotient is just a set. Notice also that the choice of $q \in M$ here is essentially irrelevant.

*Example.* Let $G = \mathrm{SO}(3)$, $M = \mathbb{S}^2$, and $q \in \mathbb{S}^2$ be the north pole (i.e. $q = (0,0,1) \in \mathbb{R}^3$). Then $G_q < \mathrm{SO}(3)$ consists of the rotations around the vertical axis (passing through the north and south poles). Since $\mathrm{SO}(3)$ acts transitively on $\mathbb{S}^2$, we get a bijection $\mathrm{SO}(3)/G_q \to \mathbb{S}^2$. The reader may think how $A \in \mathrm{SO}(3)$ moves the north pole $q \in \mathbb{S}^2$ to $Aq \in \mathbb{S}^2$.

*Proof of Theorem 6.3.4.* Let $a, b \in G_q$. Then

$$\begin{aligned}
&e \cdot q = q, \\
&(ab) \cdot q = a \cdot (b \cdot q) = a \cdot q = q, \\
&a^{-1} \cdot q = a^{-1} \cdot (a \cdot q) = (a^{-1}a) \cdot q = e \cdot q = q,
\end{aligned}$$

so that $G_q < G$. Let $x, y \in G$. Since

$$(xa) \cdot q = x \cdot (a \cdot q) = x \cdot q,$$

$f = (xG_q \mapsto x \cdot q) : G/G_q \to M$ is a well-defined mapping. If $x \cdot q = y \cdot q$ then

$$(x^{-1}y) \cdot q = x^{-1} \cdot (y \cdot q) = x^{-1} \cdot (x \cdot q) = (x^{-1}x) \cdot q = e \cdot q = q,$$

i.e. $x^{-1}y \in G_q$, that is $xG_q = yG_q$; hence $f$ is injective. Take $p \in M$. By transitivity, there exists $x \in G$ such that $x \cdot q = p$. Thereby $f(xG_q) = x \cdot q = p$, i.e. $f$ is surjective. □

*Remark* 6.3.6. If an action $((x,p) \mapsto x \cdot p) : G \times M \to M$ is not transitive, it is often reasonable to study only the *orbit* of $q \in M$, defined by

$$G \cdot q := \{x \cdot q \mid x \in G\}.$$

Now

$$((x, p) \mapsto x \cdot p) : G \times (G \cdot q) \to (G \cdot q)$$

is transitive, and $(x \cdot q \mapsto xG_q) : G \cdot q \to G/G_q$ is a bijection. Notice that either $G \cdot p = G \cdot q$ or $(G \cdot p) \cap (G \cdot q) = \emptyset$; thus the action of $G$ cuts $M$ into a disjoint union of "slices" (orbits).

**Definition 6.3.7** (**Unitary groups).** Let $(v, w) \mapsto \langle v, w\rangle_{\mathcal{H}}$ be the inner product of a complex vector space $\mathcal{H}$. Recall that the adjoint $A^* \in \mathrm{Aut}(\mathcal{H})$ of $A \in \mathrm{Aut}(\mathcal{H})$ is defined by

$$\langle A^* v, w\rangle_{\mathcal{H}} := \langle v, Aw\rangle_{\mathcal{H}}.$$

The *unitary group* of $\mathcal{H}$ is

$$\mathcal{U}(\mathcal{H}) := \{A \in \mathrm{Aut}(\mathcal{H}) \mid \forall v, w \in \mathcal{H} :\ \langle Av, Aw\rangle_{\mathcal{H}} = \langle v, w\rangle_{\mathcal{H}}\},$$

i.e. $\mathcal{U}(\mathcal{H})$ contains the unitary linear bijections $\mathcal{H} \to \mathcal{H}$. Clearly $A^* = A^{-1}$ for $A \in \mathcal{U}(\mathcal{H})$. The *unitary matrix group* for $\mathbb{C}^n$ is

$$\mathrm{U}(n) := \left\{A = (a_{ij})_{i,j=1}^n \in \mathrm{GL}(n, \mathbb{C}) \mid A^* = A^{-1}\right\},$$

see Remark 6.2.10; here $A^* = (\overline{a_{ji}})_{i,j=1}^n = A^{-1}$, i.e.

$$\sum_{k=1}^n \overline{a_{ki}} a_{kj} = \delta_{ij}.$$

**Definition 6.3.8** (**Representations).** A *representation of a group $G$ on a vector space $V$* is any $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$; the *dimension* of $\phi$ is $\dim(\phi) := \dim(V)$. A representation $\psi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ is called a *unitary representation*, and $\psi \in \mathrm{Hom}(G, \mathrm{U}(n))$ is called a *unitary matrix representation*.

*Remark* 6.3.9. The main idea here is that we can study a group $G$ by using linear algebraic tools via representations $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$.

*Remark* 6.3.10. There is a bijective correspondence between the representations of $G$ on $V$ and linear actions of $G$ on $V$. Indeed, if
$\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ then

$$((x, v) \mapsto \phi(x)v) : G \times V \to V$$

is an action of $G$ on $V$. Conversely, if $((x, v) \mapsto x \cdot v) : G \times V \to V$ is a linear action then

$$(x \mapsto (v \mapsto x \cdot v)) \in \mathrm{Hom}(G, \mathrm{Aut}(V))$$

is a representation of $G$ on $V$.

*Example.* Let us give some examples of representations:

1. If $G < \mathrm{Aut}(V)$ then $(A \mapsto A) \in \mathrm{Hom}(G, \mathrm{Aut}(V))$.

2. If $G < \mathcal{U}(\mathcal{H})$ then $(A \mapsto A) \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$.
3. There is always the trivial representation $(x \mapsto I) \in \mathrm{Hom}(G, \mathrm{Aut}(V))$.
4. **(Representations $\pi_L$ and $\pi_R$).** Let $\mathcal{F}(G) = \mathbb{C}^G$, i.e. the complex vector space of functions $f : G \to \mathbb{C}$. Let us define left and right regular representations $\pi_L, \pi_R \in \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{F}(G)))$ by
$$\begin{aligned}(\pi_L(y)f)(x) &:= f(y^{-1}x),\\ (\pi_R(y)f)(x) &:= f(xy)\end{aligned}$$
for all $x, y \in G$.
5. Let us identify the complex $(1 \times 1)$-matrices with the complex numbers by the mapping $((z) \mapsto z) : \mathbb{C}^{1\times 1} \to \mathbb{C}$. Then $\mathrm{U}(1)$ is identified with the unit circle $\{z \in \mathbb{C} :\ |z| = 1\}$, and $(x \mapsto \mathrm{e}^{\mathrm{i}x\cdot\xi}) \in \mathrm{Hom}(\mathbb{R}^n, \mathrm{U}(1))$ for all $\xi \in \mathbb{R}^n$.
6. Analogously, $(x \mapsto \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}) \in \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathrm{U}(1))$ for all $\xi \in \mathbb{Z}^n$.
7. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(W))$, where $V, W$ are vector spaces over the same field. Then
$$\phi \oplus \psi = (x \mapsto \phi(x) \oplus \psi(x)) \in \mathrm{Hom}(G, \mathrm{Aut}(V \oplus W)),$$
$$\phi \otimes \psi|_G = (x \mapsto \phi(x) \otimes \psi(x)) \in \mathrm{Hom}(G, \mathrm{Aut}(V \otimes W)),$$
where $V \oplus W$ is the direct sum and $V \otimes W$ is the tensor product space.
8. If $\phi = (x \mapsto (\phi(x)_{ij})_{i,j=1}^n) \in \mathrm{Hom}(G, \mathrm{GL}(n, \mathbb{C}))$ then the conjugate $\overline{\phi} = (x \mapsto (\overline{\phi(x)_{ij}})_{i,j=1}^n) \in \mathrm{Hom}(G, \mathrm{GL}(n, \mathbb{C}))$.

**Definition 6.3.11** (**Invariant subspaces and irreducible representations).** Let $V$ be a vector space and $A \in \mathrm{End}(V)$. A subspace $W \subset V$ is called *$A$-invariant* if
$$AW \subset W,$$
where $AW = \{Aw :\ w \in W\}$. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$. A subspace $W \subset V$ is called *$\phi$-invariant* if $W$ is $\phi(x)$-invariant for all $x \in G$ (abbreviated $\phi(G)W \subset W$); moreover, $\phi$ is *irreducible* if the only $\phi$-invariant subspaces are the *trivial subspaces* $\{0\}$ and $V$.

*Remark* 6.3.12 (**Restricted representations**). If $W \subset V$ is $\phi$-invariant for $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$, we may define the *restricted representation* $\phi|_W \in \mathrm{Hom}(G, \mathrm{Aut}(W))$ by $\phi|_W(x)w := \phi(x)w$. If $\phi$ is unitary then its restriction is also unitary.

**Lemma 6.3.13.** *Let $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$. Let $W \subset \mathcal{H}$ be a $\phi$-invariant subspace. Then its orthocomplement*
$$W^\perp = \{v \in \mathcal{H} \mid \forall w \in W :\ \langle v, w\rangle_{\mathcal{H}} = 0\}$$
*is also $\phi$-invariant.*

*Proof.* If $x \in G$, $v \in W^\perp$ and $w \in W$ then

$$\langle \phi(x)v, w\rangle_{\mathcal{H}} = \langle v, \phi(x)^* w\rangle_{\mathcal{H}} = \langle v, \phi(x)^{-1} w\rangle_{\mathcal{H}} = \langle v, \phi(x^{-1})w\rangle_{\mathcal{H}} = 0,$$

meaning that $\phi(x)v \in W^\perp$. □

**Definition 6.3.14** (**Direct sums).** Let $V$ be an inner product space and let $\{V_j\}_{j\in J}$ be some family of its mutually orthogonal subspaces (i.e. $\langle v_i, v_j\rangle_V = 0$ if $v_i \in V_i$, $v_j \in V_j$ and $i \neq j$). The (*algebraic*) *direct sum* of $\{V_j\}_{j\in J}$ is the subspace

$$W = \bigoplus_{j\in J} V_j := \operatorname{span} \bigcup_{j\in J} V_j.$$

If $A_j \in \operatorname{End}(V_j)$ then let us define

$$A = \bigoplus_{j\in J} A_j \in \operatorname{End}(W)$$

by $Av := A_j v$ for all $j \in J$ and $v \in V_j$. If $\phi_j \in \operatorname{Hom}(G, \operatorname{Aut}(V_j))$ then we define

$$\phi = \bigoplus_{j\in J} \phi_j \in \operatorname{Hom}(G, \operatorname{Aut}(W))$$

by $\phi|_{V_j} = \phi_j$ for all $j \in J$, i.e. $\phi(x) := \bigoplus_{j\in J} \phi_j(x)$ for all $x \in G$.

*Remark* 6.3.15. In a sense, irreducible representations are the building blocks of representations. Given a representation of a group, a fundamental task is to find its invariant subspaces, and describe the representation as a direct sum of irreducible representations. To reach this goal, we often have to assume some extra conditions, e.g. of algebraic or topological nature.

**Theorem 6.3.16** (**Reducing finite-dimensional representations).** *Let $\phi \in \operatorname{Hom}(G, \mathcal{U}(\mathcal{H}))$ be finite-dimensional. Then $\phi$ is a direct sum of irreducible unitary representations.*

*Proof (by induction).* The claim is true for $\dim(\mathcal{H}) = 1$, since then the only subspaces of $\mathcal{H}$ are the trivial ones. Suppose the claim is true for representations of dimension $n$ or less. Suppose $\dim(\mathcal{H}) = n+1$. If $\phi$ is irreducible, there is nothing to prove. Hence assume that there exists a non-trivial $\phi$-invariant subspace $W \subset \mathcal{H}$. Then also the orthocomplement $W^\perp$ is $\phi$-invariant by Lemma 6.3.13. Due to the $\phi$-invariance of the subspaces $W$ and $W^\perp$, we may define restricted representations $\phi|_W \in \operatorname{Hom}(G, \mathcal{U}(W))$ and $\phi|_{W^\perp} \in \operatorname{Hom}(G, \mathcal{U}(W^\perp))$. Hence $\mathcal{H} = W \oplus W^\perp$ and $\phi = \phi|_W \oplus \phi|_{W^\perp}$. Moreover, $\dim(W) \leq n$ and $\dim(W^\perp) \leq n$; the proof is complete, since unitary representations up to dimension $n$ are direct sums of irreducible unitary representations by the induction hypothesis. □

*Remark* 6.3.17. By Theorem 6.3.16, finite-dimensional unitary representations can be decomposed nicely. More precisely, if $\phi \in \operatorname{Hom}(G, \mathcal{U}(\mathcal{H}))$ is finite-dimensional then

$$\mathcal{H} = \bigoplus_{j=1}^{k} W_j, \quad \phi = \bigoplus_{j=1}^{k} \phi|_{W_j},$$

where each $\phi|_{W_j} \in \mathrm{Hom}(G, \mathcal{U}(W_j))$ is irreducible.

**Definition 6.3.18 (Equivalent representations).** A linear mapping $A : V \to W$ is an *intertwining operator* between representations $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut(W)})$, denoted by $A \in \mathrm{Hom}(\phi, \psi)$, if

$$A\phi(x) = \psi(x)A$$

for all $x \in G$, i.e. if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi(x)} & V \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ W & \xrightarrow{\psi(x)} & W \end{array}$$

commutes for every $x \in G$. If $A \in \mathrm{Hom}(\phi, \psi)$ is invertible then $\phi$ and $\psi$ are said to be *equivalent*, denoted by $\phi \sim \psi$.

*Remark* 6.3.19. Always $0 \in \mathrm{Hom}(\phi, \psi)$, and $\mathrm{Hom}(\phi, \psi)$ is a vector space. Moreover, if $A \in \mathrm{Hom}(\phi, \psi)$ and $B \in \mathrm{Hom}(\psi, \xi)$ then $BA \in \mathrm{Hom}(\phi, \xi)$.

**Exercise 6.3.20.** Let $G$ be a finite group and let $\mathcal{F}(G)$ be the vector space of functions $f : G \to \mathbb{C}$. Let

$$\int_G f \, \mathrm{d}\mu_G := \frac{1}{|G|} \sum_{x \in G} f(x),$$

when $f \in \mathcal{F}(G)$. Let us endow $\mathcal{F}(G)$ with the inner product

$$\langle f, g \rangle_{L^2(\mu_G)} := \int_G f \, \overline{g} \, \mathrm{d}\mu_G.$$

Define $\pi_L, \pi_R : G \to \mathrm{Aut}(\mathcal{F}(G))$ by

$$(\pi_L(y) \ f)(x) := f(y^{-1}x),$$

$$(\pi_R(y) \ f)(x) := f(xy).$$

Show that $\pi_L$ and $\pi_R$ are equivalent unitary representations.

**Exercise 6.3.21.** Let $G$ be non-commutative and $|G| = 6$. Endow $\mathcal{F}(G)$ with the inner product given in Exercise 6.3.20. Find the $\pi_L$-invariant subspaces and give orthogonal bases for them.

**Exercise 6.3.22 (Torus $\mathbb{T}^n$).** Let us endow the $n$-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ with the quotient group structure and with the Lebesgue measure. Let $\pi_L, \pi_R : \mathbb{T}^n \to \mathcal{L}(L^2(\mathbb{T}^n))$ be defined by

$$(\pi_L(y) \ f)(x) := f(x - y),$$

$$(\pi_R(y)\ f)(x) := f(x+y)$$

for almost every $x \in \mathbb{T}^n$. Show that $\pi_L$ and $\pi_R$ are equivalent reducible unitary representations. Describe the minimal $\pi_L$- and $\pi_R$-invariant subspaces containing the function $x \mapsto \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}$, where $\xi \in \mathbb{Z}^n$.

*Remark* 6.3.23. One of the main results in the representation theory of groups is Schur's Lemma 6.3.25, according to which the intertwining space $\mathrm{Hom}(\phi,\phi)$ may be rather trivial. The most of the work for such a result is carried out in the proof of the following Proposition 6.3.24:

**Proposition 6.3.24.** *Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V_\phi))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(V_\psi))$ be irreducible. If $A \in \mathrm{Hom}(\phi,\psi)$ then either $A = 0$ or $A : V_\phi \to V_\psi$ is invertible.*

*Proof.* The image $AV_\phi \subset V_\psi$ of $A$ is $\psi$-invariant, because

$$\psi(G)\ AV_\phi = A\ \phi(G)V_\phi = AV_\phi,$$

so that either $AV_\phi = \{0\}$ or $AV_\phi = V_\psi$, as $\psi$ is irreducible. Hence either $A = 0$ or $A$ is a surjection.

The kernel $\mathrm{Ker}(A) = \{v \in V_\phi \mid Av = 0\}$ is $\phi$-invariant, since

$$A\ \phi(G)\ \mathrm{Ker}(A) = \psi(G)\ A\ \mathrm{Ker}(A) = \psi(G)\,\{0\} = \{0\}\,,$$

so that either $\mathrm{Ker}(A) = \{0\}$ or $\mathrm{Ker}(A) = V_\phi$, as $\phi$ is irreducible. Hence either $A$ is injective or $A = 0$.

Thus either $A = 0$ or $A$ is bijective. □

**Corollary 6.3.25. (Schur's Lemma (finite-dimensional [1905])).** *Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ be irreducible and finite-dimensional. Then $\mathrm{Hom}(\phi,\phi) = \mathbb{C}I = \{\lambda I \mid \lambda \in \mathbb{C}\}$.*

*Proof.* Let $A \in \mathrm{Hom}(\phi,\phi)$. The finite-dimensional linear operator $A$ has an eigenvalue $\lambda \in \mathbb{C}$, i.e. $\lambda I - A : V \to V$ is not invertible. On the other hand, $\lambda I - A \in \mathrm{Hom}(\phi,\phi)$, so that $\lambda I - A = 0$ by Proposition 6.3.24. □

**Corollary 6.3.26 (Representations of commutative groups).** *Let $G$ be a commutative group. Irreducible finite-dimensional representations of $G$ are one-dimensional.*

*Proof.* Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ be irreducible, $\dim(\phi) < \infty$. Due to the commutativity of $G$,

$$\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x)$$

for all $x, y \in G$, so that $\phi(G) \subset \mathrm{Hom}(\phi,\phi)$. By Schur's Lemma 6.3.25, $\mathrm{Hom}(\phi,\phi) = \mathbb{C}I$. Hence if $v \in V$ then

$$\phi(G)\mathrm{span}\{v\} = \mathrm{span}\{v\},$$

i.e. $\mathrm{span}\{v\}$ is $\phi$-invariant. Therefore either $v = 0$ or $\mathrm{span}\{v\} = V$. □

**Corollary 6.3.27.** *Let $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_\phi))$ and $\psi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_\psi))$ be finite-dimensional. Then $\phi \sim \psi$ if and only if there exists isometric isomorphism $B \in \mathrm{Hom}(\phi,\psi)$.*

*Remark* 6.3.28 (**Isometries**). An *isometry* $f : M \to N$ between metric spaces $(M, d_M)$ and $(N, d_N)$ satisfies $d_N(f(x), f(y)) = d_M(x,y)$ for all $x, y \in M$.

*Proof of Corollary 6.3.27.* The "if"-part is trivial. Assume that $\phi \sim \psi$. Recall that there are direct sum decompositions

$$\phi = \bigoplus_{j=1}^{m} \phi_j, \quad \psi = \bigoplus_{k=1}^{n} \psi_k,$$

where $\phi_j, \psi_k$ are irreducible unitary representations on $\mathcal{H}_{\phi_j}, \mathcal{H}_{\psi_k}$, respectively. Now $n = m$, since $\phi \sim \psi$. Moreover, we may arrange the indices so that $\phi_j \sim \psi_j$ for each $j$. Choose invertible $A_j \in \mathrm{Hom}(\phi_j, \psi_j)$. Then $A_j^*$ is invertible, and $A_j^* \in \mathrm{Hom}(\psi_j, \phi_j)$: if $x \in G$, $v \in \mathcal{H}_{\phi_j}$ and $w \in \mathcal{H}_{\psi_j}$ then

$$\begin{aligned}
\langle A_j^* \psi_j(x) w, v\rangle_{\mathcal{H}_\phi} &= \langle w, \psi_j(x)^* A_j v\rangle_{\mathcal{H}_\psi} \\
&= \langle w, \psi_j(x^{-1}) A_j v\rangle_{\mathcal{H}_\psi} \\
&= \langle w, A_j \phi_j(x^{-1}) v\rangle_{\mathcal{H}_\psi} \\
&= \langle \phi_j(x^{-1})^* A_j^* w, v\rangle_{\mathcal{H}_\phi} \\
&= \langle \phi_j(x) A_j^* w, v\rangle_{\mathcal{H}_\phi}.
\end{aligned}$$

Thereby $A_j^* A_j \in \mathrm{Hom}(\phi_j, \phi_j)$ is invertible. By Schur's Lemma 6.3.25, $A_j^* A_j = \lambda_j I$, where $\lambda_j \neq 0$. Let $v \in \mathcal{H}_{\phi_j}$ such that $\|v\|_{\mathcal{H}_\phi} = 1$. Then

$$\lambda = \lambda \|v\|_{\mathcal{H}_\phi}^2 = \langle \lambda v, v\rangle_{\mathcal{H}_\phi} = \langle A_j^* A_j v, v\rangle_{\mathcal{H}_\phi} = \langle A_j v, A_j v\rangle_{\mathcal{H}_\psi} = \|A_j v\|_{\mathcal{H}_\psi}^2 > 0,$$

so that we may define $B_j := \lambda^{-1/2} A_j \in \mathrm{Hom}(\phi_j, \psi_j)$. Then the mapping $B_j : \mathcal{H}_{\phi_j} \to \mathcal{H}_{\psi_j}$ is an isometry, $B_j^* B_j = I$. Finally, define

$$B := \bigoplus_{j=1}^{m} B_j.$$

Clearly, $B : \mathcal{H}_\phi \to \mathcal{H}_\psi$ is an isometry, bijection, and $B \in \mathrm{Hom}(\phi,\psi)$. □

We have now dealt with groups in general. In the sequel, by specialising to certain classes of groups, we will obtain a fruitful ground for further results in representation theory.

# Chapter 7

# Topological groups

A topological group is a natural amalgam of topological spaces and groups: it is a Hausdorff space with continuous group operations. Topology adds a new flavour to the representation theory. Especially interesting are compact groups, where group-invariant probability measures exist. Moreover, nice-enough functions on a compact group have Fourier series expansions, which generalise the classical Fourier series of periodic functions.

## 7.1 Topological group

Next we marry topology to groups.

**Definition 7.1.1 (Topological groups).** A group and a topological space $G$ is called a *topological group* if $\{e\} \subset G$ is closed and if the group operations

$$\begin{aligned} ((x,y) \mapsto xy) &\quad : \quad G \times G \to G, \\ (x \mapsto x^{-1}) &\quad : \quad G \to G \end{aligned}$$

are continuous.

*Remark* 7.1.2. The reader may wonder why we assumed that $\{e\} \subset G$ is closed — actually, this condition is left out in some other definitions for a topological group. Notice that the good property brought by this assumption is that the topological groups become even Hausdorff spaces (see Exercise 7.1.3), which appeals to those who work in analysis.

*Example.* In the following, when not specified, the topologies and the group operations are the usual ones:

1. Any group $G$ endowed with the so-called discrete topology $\mathcal{P}(G) = \{U : U \subset G\}$ is a topological group.

2. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are topological groups when the group operation is the addition and the topology is as usual.
3. $\mathbb{Q}^\times$, $\mathbb{R}^\times$, $\mathbb{C}^\times$ are topological groups when the group operation is the multiplication and the topology is as usual.
4. Topological vector spaces are topological groups with vector addition: such a space is both a vector space and a topological Hausdorff space such that the vector space operations are continuous.
5. Let $X$ be a Banach space. The set $\mathrm{AUT}(X) := \mathrm{Aut}(X) \cap \mathcal{L}(X)$ of invertible bounded linear operators $X \to X$ forms a topological group with respect to the norm topology.
6. Subgroups of topological groups are topological groups.
7. If $G$ and $H$ are topological groups then $G \times H$ is a topological group. Actually, Cartesian products always preserve the topological group structure.

**Exercise 7.1.3.** Show that a topological group is a Hausdorff space.

**Lemma 7.1.4.** *Let $G$ be a topological group and $y \in G$. Then*

$$x \mapsto xy, \quad x \mapsto yx, \quad x \mapsto x^{-1}$$

*are homeomorphisms $G \to G$.*

*Proof.* The mapping

$$(x \mapsto xy) : G \overset{x \mapsto (x,y)}{\to} G \times G \overset{(a,b) \mapsto ab}{\to} G$$

is continuous as a composition of continuous mappings. Its inverse mapping $(x \mapsto xy^{-1}) : G \to G$ is also continuous; hence this is a homeomorphism. Similarly, $(x \mapsto yx) : G \to G$ is a homeomorphism. By definition, the group operation $(x \mapsto x^{-1}) : G \to G$ is continuous, and it is its own inverse. □

**Corollary 7.1.5.** *If $U \subset G$ is open and $S \subset G$ then $SU, US, U^{-1} \subset G$ are open.*

**Proposition 7.1.6.** *Let $G$ be a topological group. If $H < G$ then $\overline{H} < G$. If $H \lhd G$ then $\overline{H} \lhd G$.*

*Proof.* Let $H < G$. Trivially $e \in H \subset \overline{H}$. Now

$$\overline{H}\,\overline{H} \subset \overline{HH} = \overline{H},$$

where the inclusion is due to the continuity of the mapping $((x,y) \mapsto xy) : G \times G \to G$. The continuity of the inversion $(x \mapsto x^{-1}) : G \to G$ gives

$$\overline{H}^{-1} \subset \overline{H^{-1}} = \overline{H}.$$

Thus $\overline{H} < G$.

Let $H \triangleleft G$, $y \in G$. Then

$$y\overline{H} = \overline{yH} = \overline{Hy} = \overline{H}y;$$

notice how homeomorphisms $(x \mapsto yx), (x \mapsto xy) : G \to G$ were used. □

*Remark* 7.1.7. Let $H < G$ and $S \subset G$. For analysis on the quotient space $G/H$, let us recall Remark 6.2.15: the mapping $(x \mapsto xH) : G \to G/H$ identifies the sets

$$SH = \{sh : \ s \in S, \ h \in H\} \quad \subset \quad G,$$
$$\{sH : \ s \in S\} = \{\{sh : \ h \in H\} : \ s \in S\} \quad \subset \quad G/H.$$

**Definition 7.1.8** (**Quotient topology on** $G/H$)**.** Let $G$ be a topological group, $H < G$. The *quotient topology* of $G/H$ is

$$\tau_{G/H} := \{\{uH : \ u \in U\} : \ U \subset G \text{ open}\};$$

in other words, $\tau_{G/H}$ is the strongest (i.e. largest) topology for which the quotient map $(x \mapsto xH) : G \to G/H$ is continuous. If $U \subset G$ is open, we may identify sets $UH \subset G$ and $\{uH : u \in H\} \subset G/H$.

**Proposition 7.1.9.** *Let $G$ be a topological group and $H < G$. Then a function $f : G/H \to \mathbb{C}$ is continuous if and only if $(x \mapsto f(xH)) : G \to \mathbb{C}$ is continuous.*

*Proof.* If $f \in C(G/H)$ then $(x \mapsto f(xH)) \in C(G)$, since it is obtained by composing $f$ and the continuous quotient map $(x \mapsto xH) : G \to G/H$.

Now suppose $(x \mapsto f(xH)) \in C(G)$. Take open $V \subset \mathbb{C}$. Then $U := (x \mapsto f(xH))^{-1}(V) \subset G$ is open, so that $U' := \{uH : \ u \in U\} \subset G/H$ is open. Trivially, $f(U') = V$. Hence $f \in C(G/H)$. □

**Proposition 7.1.10** (**When is** $G/H$ **Hausdorff?**)**.** *Let $G$ be a topological group and $H < G$. Then $G/H$ is a Hausdorff space if and only if $H$ is closed.*

*Proof.* If $G/H$ is a Hausdorff space then $H = (x \mapsto xH)^{-1}(\{H\}) \subset G$ is closed, because the quotient map is continuous and $\{H\} \subset G/H$ is closed.

Next suppose $H$ is closed. Take $xH, yH \in G/H$ such that $xH \neq yH$. Then $S := ((a,b) \mapsto a^{-1}b)^{-1}(H) \subset G \times G$ is closed, since $H \subset G$ is closed and $((a,b) \mapsto a^{-1}b) : G \times G \to G$ is continuous. Now $(x,y) \notin S$. Take open sets $U \ni x$ and $V \ni y$ such that $(U \times V) \cap S = \emptyset$. Then the sets

$$\begin{aligned} U' &:= \{uH : \ u \in U\} \quad \subset \quad G/H, \\ V' &:= \{vH : \ v \in V\} \quad \subset \quad G/H \end{aligned}$$

are disjoint and open such that $xH \in U'$ and $yH \in V'$. Thus $G/H$ is Hausdorff. □

**Theorem 7.1.11** (**When is** $G/H$ **a topological group?**)**.** *Let $G$ be a topological group and $H \triangleleft G$. Then*

$$\begin{aligned} ((xH, yH) \mapsto xyH) &\quad : \quad (G/H) \times (G/H) \to G/H, \\ (xH \mapsto x^{-1}H) &\quad : \quad G/H \to G/H \end{aligned}$$

*are continuous. Moreover, $G/H$ is a topological group if and only if $H$ is closed.*

*Proof.* We know already that the operations in the theorem are well-defined group operations, because $H$ is normal in $G$. Recall Remark 7.1.7, how we may identify certain subsets of $G$ with subsets of $G/H$. Then a neighbourhood of the point $xyH \in G/H$ is of the form $UH$ for some open $U \subset G$, $U \ni xy$. Take open $U_1 \ni x$ and $U_2 \ni y$ such that $U_1 U_2 \subset U$. Then

$$(xH)(yH) \subset (U_1H)(U_2H) = U_1U_2H \subset UH,$$

so that $((xH, yH) \mapsto xyH) : (G/H) \times (G/H) \to G/H$ is continuous. A neighbourhood of the point $x^{-1}H \in G/H$ is of the form $VH$ for some open $V \subset G$, $V \ni x^{-1}$. But $V^{-1} \ni x$ is open, and $(V^{-1})^{-1} = V$, so that $(xH \mapsto x^{-1}H) : G/H \to G/H$ is continuous.

Notice that $e_{G/H} = H$. If $G/H$ is a topological group, then

$$H = (x \mapsto xH)^{-1}\left\{e_{G/H}\right\} \subset G$$

is closed. On the other hand, if $H \triangleleft G$ is closed then

$$(G/H) \setminus \{e_{G/H}\} \cong (G \setminus H)H \subset G$$

is open, i.e. $\{e_{G/H}\} \subset G/H$ is closed. □

**Definition 7.1.12 (Continuous homomorphisms).** Let $G_1, G_2$ be topological groups. Let

$$\mathrm{HOM}(G_1, G_2) := \mathrm{Hom}(G_1, G_2) \cap C(G_1, G_2),$$

i.e. the set of continuous homomorphisms $G_1 \to G_2$.

*Remark* 7.1.13. By Theorem 7.1.11, closed normal subgroups of $G$ correspond bijectively to continuous surjective homomorphisms from $G$ to some other topological group (up to isomorphism).

*Remark* 7.1.14. Let us recall some topological concepts: A topological space is *connected* if the only subsets which are both closed and open are the empty set and the whole space. A non-connected space is called *disconnected*. The *component* of a point $x$ in a topological space is the largest connected subset containing $x$.

**Proposition 7.1.15.** *Let $G$ be a topological group and $C_e \subset G$ the component of $e$. Then $C_e \triangleleft G$ is closed.*

*Proof.* Components are always closed, and $e \in C_e$ by definition. Since $C_e \subset G$ is connected, also $C_e \times C_e \subset G \times G$ and is connected. By the continuity of the group operations, $C_eC_e \subset G$ and $C_e^{-1} \subset G$ are connected. Since $e = ee \in C_eC_e$, we have $C_eC_e \subset C_e$. And since $e = e^{-1} \in C_e^{-1}$, also $C_e^{-1} \subset C_e$. Take $y \in G$. Then $y^{-1}C_ey \subset G$ is connected, by the continuity of $(x \mapsto y^{-1}xy) : G \to G$. Now $e = y^{-1}ey \in y^{-1}C_ey$, so that $y^{-1}C_ey \subset C_e$; $C_e$ is normal in $G$. □

**Proposition 7.1.16.** *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a continuous action of $G$ on $M$, and let $q \in M$. If $G_q$ and $G/G_q$ are connected then $G$ is connected.*

*Proof.* Suppose $G$ is disconnected and $G_q$ is connected. Then there are non-empty disjoint open sets $U, V \subset G$ such that $G = U \cup V$. The sets

$$U' := \{uG_q : u \in U\} \subset G/G_q,$$
$$V' := \{vG_q : v \in V\} \subset G/G_q$$

are non-empty and open, and $G/G_q = U' \cup V'$. Take $u \in U$ and $v \in V$. As a continuous image of a connected set, $uG_q = (x \mapsto ux)(G_q) \subset G$ is connected; moreover $u = ue \in uG_q$; thereby $uG_q \subset U$. In the same way we see that $vG_q \subset V$. Hence $U' \cap V' = \emptyset$, so that $G/G_q$ is disconnected. □

**Corollary 7.1.17 (When is a group connected?).** *If $G$ is a topological group, $H < G$ is connected and $G/H$ is connected then $G$ is connected.*

*Proof.* Using the notation of Proposition 7.1.16, let $M = G/H$, $q = H$ and $x \cdot p = xp$, so that $G_q = H$ and $G/G_q = G/H$. □

**Exercise 7.1.18 (Groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$ and $\mathbf{U}(n)$ are connected).** Show that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ are connected for every $n \in \mathbb{Z}^+$. How about $\mathrm{O}(n)$?

**Exercise 7.1.19 (Finiteness of connected components).** Prove that a compact topological group can have only finitely many connected components. Consequently, conclude that a discrete compact group is finite.

## 7.2 Representations of topological groups

**Definition 7.2.1 (Strongly continuous representations).** Let $G$ be a topological group and $\mathcal{H}$ be a Hilbert space. A representation $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ is *strongly continuous* if

$$(x \mapsto \phi(x)v) : G \to \mathcal{H}$$

is continuous for all $v \in \mathcal{H}$.

*Remark* 7.2.2. The strong continuity in Definition 7.2.1 means that the mapping $(x \mapsto \phi(x)) : G \to \mathcal{L}(\mathcal{H})$ is continuous, when $\mathcal{L}(\mathcal{H}) \supset \mathcal{U}(\mathcal{H})$ is endowed with the *strong operator topology*:

$$A_j \overset{\text{strongly}}{\to} A \overset{\text{definition}}{\Longleftrightarrow} \forall v \in \mathcal{H} : \ \|A_j v - Av\|_{\mathcal{H}} \to 0.$$

Why we should not endow $\mathcal{U}(\mathcal{H})$ with the operator norm topology (which is even stronger, i.e.a larger topology)? The reason is that there are interesting unitary representations, which are continuous in the strong operator topology, but not in the operator norm topology. This phenomenon is exemplified by Exercise 7.2.3:

**Exercise 7.2.3.** Let us define $\pi_L : \mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ by

$$(\pi_L(y)f)(x) := f(x - y)$$

for almost every $x \in \mathbb{R}^n$. Show that $\pi_L$ is strongly continuous, but not norm continuous.

**Definition 7.2.4** (**Topologically irreducible representations).** A strongly continuous $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is called *topologically irreducible* if the only closed $\phi$-invariant subspaces are the trivial ones $\{0\}$ and $\mathcal{H}$.

**Exercise 7.2.5.** Let $V$ be a topological vector space and let $W \subset V$ be an $A$-invariant subspace, where $A \in \mathrm{Aut}(V)$ is continuous. Show that the closure $\overline{W} \subset V$ is also $A$-invariant.

**Definition 7.2.6** (**Cyclic representations and cyclic vectors).** A strongly continuous $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is called a *cyclic representation* if

$$\mathrm{span}\ \phi(G)v \subset \mathcal{H}$$

is dense for some $v \in \mathcal{H}$; such $v$ is called a *cyclic vector*.

*Example.* If $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is topologically irreducible then any non-zero $v \in \mathcal{H}$ is cyclic. Indeed, if $V := \mathrm{span}\ \phi(G)v$ then $\phi(G)V \subset V$ and consequently $\phi(G)\overline{V} \subset \overline{V}$, so that $\overline{V}$ is $\phi$-invariant. If $v \neq 0$ then $\overline{V} = \mathcal{H}$, because of the topological irreducibility.

**Definition 7.2.7 (Representation as a direct sum).** A Hilbert space $\mathcal{H}$ is a *direct sum* of closed subspaces $(\mathcal{H}_j)_{j\in J}$, denoted by

$$\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$$

if the subspace family is pairwise orthogonal and the linear span of the set $\cup_{j\in J}\mathcal{H}_j$ is dense in $\mathcal{H}$. Then the vectors in $\mathcal{H}$ have a unique orthogonal series expansion, more precisely

$$\forall x \in \mathcal{H}\ \forall j \in J\ \exists! x_j \in \mathcal{H}_j:\ x = \sum_{j\in J} x_j, \quad \|x\|_{\mathcal{H}}^2 = \sum_{j\in J} \|x_j\|_{\mathcal{H}}^2 .$$

If $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ and each $\mathcal{H}_j$ is $\phi$-invariant then $\phi$ is said to be the *direct sum*

$$\phi = \bigoplus_{j\in J} \phi|_{\mathcal{H}_j}$$

where $\phi|_{\mathcal{H}_j} = (x \mapsto \phi(x)v) \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_j))$.

**Proposition 7.2.8** (**Decomposition of strongly continuous representations).** *Let* $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ *be strongly continuous. Then*

$$\phi = \bigoplus_{j\in J} \phi|_{\mathcal{H}_j},$$

*where each* $\phi|_{\mathcal{H}_j}$ *is cyclic.*

*Proof.* Let $\tilde{J}$ be the family of all closed $\phi$-invariant subspaces $V \subset \mathcal{H}$ for which $\phi|_V$ is cyclic. Let

$$S = \left\{ s \subset \tilde{J} \;\middle|\; \forall V, W \in s: \; V = W \text{ or } V \perp W \right\}.$$

It is easy to see that $\{\{0\}\} \in S$, so that $S \neq \emptyset$. Let us introduce a partial order on $S$ by inclusion:

$$s_1 \leq s_2 \overset{\text{definition}}{\Longleftrightarrow} s_1 \subset s_2.$$

The chains in $S$ have upper bounds: if $R \subset S$ is a chain then $r \leq \cup_{s \in R}\, s \in S$ for all $r \in R$. Therefore by **Zorn's Lemma**, there exists a maximal element $t \in S$. Let

$$V := \bigoplus_{W \in t} W.$$

To get a contradiction, suppose $V \neq \mathcal{H}$. Then there exists $v \in V^\perp \setminus \{0\}$. Since $\mathrm{span}(\phi(G)v)$ is $\phi$-invariant, its closure $W_0$ is also $\phi$-invariant (see Exercise 7.2.5). Clearly $W_0 \subset \overline{V^\perp} = V^\perp$, and $\phi|_{W_0}$ has cyclic vector $v$, yielding

$$s := t \cup \{W_0\} \in S,$$

where $t \leq s \not\leq t$. This contradicts the maximality of $t$; thus $V = \mathcal{H}$. □

**Exercise 7.2.9.** Fill in the details in the proof of Proposition 7.2.8.

**Exercise 7.2.10.** Assuming that $\mathcal{H}$ is separable, prove Proposition 7.2.8 by ordinary induction (without resorting to Zorn's Lemma).

## 7.3 Compact groups

**Definition 7.3.1 ((Locally) compact groups).** A topological group is a (*locally*) *compact group* if it is (locally) compact as a topological space.

*Remark* 7.3.2. We have the following properties:

1. Any group $G$ with the discrete topology is a locally compact group; then $G$ is a compact group if and only if it is finite.
2. $\mathbb{Q}, \mathbb{Q}^\times$ are not locally compact groups;
   $\mathbb{R}, \mathbb{R}^\times, \mathbb{C}, \mathbb{C}^\times$ are locally compact groups, but non-compact.
3. A normed vector space is a locally compact group if and only if it is finite-dimensional.
4. $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ are compact groups.
5. $\mathrm{GL}(n)$ is a locally compact group, but non-compact.
6. If $G, H$ are locally compact groups then $G \times H$ is a locally compact group.

7. If $\{G_j\}_{j\in J}$ is a family of compact groups then $\prod_{j\in J} G_j$ is a compact group.
8. If $G$ is a compact group and $H < G$ is closed then $H$ is a compact group.

**Proposition 7.3.3.** *Let $((x,p)\mapsto x\cdot p) : G\times M \to M$ be a continuous action of a compact group $G$ on a Hausdorff space $M$. Let $q\in M$. Then the mapping*

$$f := (xG_q \mapsto x\cdot q) : G/G_q \to G\cdot q$$

*is a homeomorphism.*

*Proof.* We already know that $f$ is a well-defined bijection. We need to show that $f$ is continuous. An open subset of $G\cdot q$ is of the form $V\cap(G\cdot q)$, where $V\subset M$ is open. Since the action is continuous, also $(x\mapsto x\cdot q) : G\to M$ is continuous, so that $U := (x\mapsto x\cdot q)^{-1}(V)\subset G$ is open. Thereby

$$f^{-1}(V\cap(G\cdot q)) = \{xG_q : \ x\in U\}\subset G/G_q$$

is open. Thus $f$ is continuous. The space $G$ is compact and the quotient map $(x\mapsto xG_q) : G\to G/G_q$ is continuous and surjective, so that $G/G_q$ is compact. From general topology we know that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (see Proposition A.12.7). □

**Corollary 7.3.4.** *If $G$ is compact, $\phi\in \mathrm{HOM}(G,H)$ and $K = \mathrm{Ker}(\phi)$ then*

$$\psi := (xK\mapsto \phi(x)) \in \mathrm{HOM}(G/K, \phi(G))$$

*is a homeomorphism.*

*Proof.* Using the notation of Proposition 7.3.3, we have $M = H$, $q = e_H$, $x\cdot p = \phi(x)p$, so that $G_q = K$, $G/G_q = G/K$, $G\cdot q = \phi(G)$, $\psi = f$. □

*Remark* 7.3.5. What could happen if we drop the compactness assumption in Corollary 7.3.4? If $G$ and $H$ are Banach spaces, $\phi\in\mathcal{L}(G,H)$ is compact and $\dim(\phi(G)) = \infty$ then $\psi = (x+\mathrm{Ker}(\phi)\mapsto\phi(x)) : G/\mathrm{Ker}(\phi)\to\phi(G)$ is a bounded linear bijection, but $\psi^{-1}$ is not bounded! But if $\phi\in\mathcal{L}(G,H)$ is a bijection then $\phi^{-1}$ is bounded by the **Open Mapping Theorem**! (Theorem B.4.31)

**Definition 7.3.6** (**Uniform continuity on a topological group).** Let $G$ be a topological group. A function $f : G\to\mathbb{C}$ is *uniformly continuous* if for every $\varepsilon > 0$ there exists open $U\ni e$ such that

$$\forall x,y\in G : \ x^{-1}y\in U \ \Rightarrow \ |f(x)-f(y)| < \varepsilon.$$

**Exercise 7.3.7.** Under which circumstances a polynomial $p : \mathbb{R}\to\mathbb{C}$ is uniformly continuous? Show that if a continuous function $f : \mathbb{R}\to\mathbb{C}$ is periodic or vanishes outside a bounded set then it is uniformly continuous.

**Theorem 7.3.8.** *If $G$ is a compact group and $f\in C(G)$ then $f$ is uniformly continuous.*

*Proof.* Take $\varepsilon > 0$. Define the open disk $\mathbb{D}(z,r) := \{w \in \mathbb{C} : |w - z| < r\}$, where $z \in \mathbb{C}$, $r > 0$. Since $f$ is continuous, the set

$$V_x := f^{-1}(\mathbb{D}(f(x),\varepsilon)) \ni x$$

is open. Then $x^{-1}V_x \ni ee = e$ is open, so that there exist open sets $U_{1,x}, U_{2,x} \ni e$ such that $U_{1,x}U_{2,x} \subset x^{-1}V_x$, by the continuity of the group multiplication. Define $U_x := U_{1,x} \cap U_{2,x}$. Since $\{xU_x : x \in G\}$ is an open cover of the compact space $G$, there is a finite subcover $\{x_j U_{x_j}\}_{j=1}^n$. Now the set

$$U := \bigcap_{j=1}^n U_{x_j} \ni e$$

is open. Suppose $x, y \in G$ such that $x^{-1}y \in U$. There exists $k \in \{1, \ldots, n\}$ such that $x \in x_k U_{x_k}$, so that

$$x, y \in xU \subset x_k U_{x_k} U_{x_k} \subset x_k x_k^{-1} V_{x_k} = V_{x_k},$$

yielding

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_k)| + |f(x_k) - f(y)| \\ &< 2\varepsilon. \end{aligned}$$

□

**Exercise 7.3.9.** Let $G$ be a compact group, $x \in G$ and $A = \{x^n\}_{n=1}^\infty$. Show that $\overline{A} < G$.

## 7.4 Haar measure and integral

On a group, it would be natural to integrate with respect to measures that are invariant under the group operations: consider e.g. the Lebesgue integral on $\mathbb{R}^n$. However, it is not obvious whether there exist such invariant integrals in general. Next we will show that on a compact group there exists a unique probability functional, which corresponds to the so-called *Haar measure*.

**Definition 7.4.1** (**Positive functionals).** Let $X$ be a compact Hausdorff space and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $C(X,\mathbb{K})$ is a Banach space over $\mathbb{K}$ with the norm

$$f \mapsto \|f\|_{C(X,\mathbb{K})} := \max_{x \in X} |f(x)|.$$

Its dual $C(X,\mathbb{K})' = \mathcal{L}(C(X,\mathbb{K}),\mathbb{K})$ consists of the bounded linear functionals $C(X,\mathbb{K}) \to \mathbb{K}$, and is endowed with the Banach space norm

$$L \mapsto \|L\|_{C(X,\mathbb{K})'} := \sup_{f \in C(X,\mathbb{K}):\ \|f\|_{C(X,\mathbb{K})} \leq 1} |Lf|.$$

A functional $L : C(X,\mathbb{K}) \to \mathbb{C}$ is called *positive* if $Lf \geq 0$ whenever $f \geq 0$.

**Exercise 7.4.2.** Let $X$ be compact Hausdorff space. Show that a positive linear functional $L : C(X, \mathbb{R}) \to \mathbb{R}$ is bounded.

By the Riesz Representation Theorem (see Theorem C.4.65), if $L \in C(X, \mathbb{K})'$ is positive then there exists a unique positive Borel regular measure $\mu$ on $X$ such that

$$Lf = \int_X f \, \mathrm{d}\mu$$

for every $f \in C(X, \mathbb{K})$; moreover, $\mu(X) = \|L\|_{C(X,\mathbb{K})'}$. For short, $C(X) := C(X, \mathbb{C})$. Note that this is different from Chapter A (see e.g. Exercise A.6.6) where we wrote $C(X)$ for $C(X, \mathbb{R})$.

In the sequel, we shall construct a unique positive normalised translation-invariant measure on $G$. More precisely, we shall prove the following result:

**Theorem 7.4.3** (**Haar functional**). *Let $G$ be a compact group. There exists a unique positive linear functional* $\mathrm{Haar} \in C(G)'$ *such that*

$$\begin{aligned} \mathrm{Haar}(f) &= \mathrm{Haar}(x \mapsto f(yx)), \\ \mathrm{Haar}(\mathbf{1}) &= 1, \end{aligned}$$

*for all $y \in G$, where $\mathbf{1} = (x \mapsto 1) \in C(G)$. Moreover, this* Haar functional *satisfies*

$$\begin{aligned} \mathrm{Haar}(f) &= \mathrm{Haar}(x \mapsto f(xy)) \\ &= \mathrm{Haar}(x \mapsto f(x^{-1})). \end{aligned}$$

*Remark* 7.4.4 (**Haar measure and integral**). By the Riesz Representation Theorem (see Theorem C.4.65), the Haar functional begets a unique Borel regular probability measure $\mu_G$ such that

$$\mathrm{Haar}(f) = \int_G f \, \mathrm{d}\mu_G = \int_G f(x) \, \mathrm{d}\mu_G(x).$$

This $\mu_G$ is called the *Haar measure* of $G$. Often the Haar measure is implicitely assumed, and we may write e.g.

$$\int_G f(x) \, \mathrm{d}x := \int_G f \, \mathrm{d}\mu_G.$$

Obviously,

$$\begin{aligned} \int_G \mathbf{1} \, \mathrm{d}\mu_G &= \mu_G(G) \ = \ 1, \\ \int_G f(x) \, \mathrm{d}x &= \int_G f(yx) \, \mathrm{d}x \\ &= \int_G f(xy) \, \mathrm{d}x \\ &= \int_G f(x^{-1}) \, \mathrm{d}x. \end{aligned}$$

Thus the *Haar integral* $\mathrm{Haar}(f) = \int_G f(x)\ \mathrm{d}x$ can be thought of as the most natural average of $f \in C(G)$. In practical applications we can know usually only finitely many values of $f$, i.e. we are able to take only samples $\{f(x) :\ x \in S\}$ for a finite set $S \subset G$. Then a natural idea for approximating $\mathrm{Haar}(f)$ would be computing the finite sum

$$\sum_{x \in S} f(x)\ \alpha(x),$$

where sampling weights $\alpha(x) \geq 0$ satisfy $\sum_{x \in S} \alpha(x) = 1$. Of course, such a sum is not usually invariant under the group operations. The problem is to find clever choices for sampling sets and weights, some sort of almost uniformly distributed unit mass on $G$ is needed; for this end we shall introduce convolutions.

*Example.* If $G$ is finite then

$$\int_G f\ \mathrm{d}\mu_G = \frac{1}{|G|} \sum_{x \in G} f(x).$$

*Example* (**Haar measure on** $\mathbb{T}^n$). For $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$,

$$\int_{\mathbb{T}^n} f\ \mathrm{d}\mu_{\mathbb{T}^n} = \int_0^1 f(x + \mathbb{Z}^n)\ \mathrm{d}x,$$

i.e. integration with respect to the Lebesgue measure on $[0,1)^n$.

What follows is the preparation for the proof of Theorem 7.4.3.

**Definition 7.4.5** (**Sampling measures).** Let $G$ be a compact group. A function $\alpha : G \to [0,1]$ is a *sampling measure* on $G$, $\alpha \in \mathcal{SM}_G$, if

$$\mathrm{supp}(\alpha) := \mathrm{cl}\ \{a \in G :\ \alpha(a) \neq 0\} \quad \text{is finite and} \quad \sum_{a \in G} \alpha(a) = 1.$$

The set $\mathrm{supp}(\alpha) \subset G$ is called the *support* of $\alpha$. Since $\mathrm{supp}(\alpha)$ is finite we also have $\mathrm{supp}(\alpha) = \{a \in G :\ \alpha(a) \neq 0\}$ and, therefore, a sampling measure $\alpha \in \mathcal{SM}_G$ can be regarded as a finitely supported probability measure on $G$, satisfying

$$\int_G f\ \mathrm{d}\alpha = \check{\alpha} * f(e) = f * \check{\alpha}(e),$$

where $\check{\alpha}(a) := \alpha(a^{-1})$.

*Remark* 7.4.6. A sampling measure is nothing else but

$$\alpha = \sum_j \alpha_j \delta_{a_j},$$

where the sum is finite, $a_j \in G$, $\delta_{a_j}$ is the Dirac measure at $a_j$ (i.e. a probability measure supported at $a_j$), and $\sum_j \alpha_j = 1$.

**Definition 7.4.7 (Convolutions).** Let $\alpha, \beta \in \mathcal{SM}_G$ and $f \in C(G, \mathbb{K})$. The *convolutions*

$$\alpha * \beta, \alpha * f, f * \beta : G \to \mathbb{K}$$

are defined by

$$\begin{aligned} \alpha * \beta(b) &= \sum_{a \in G} \alpha(a) \beta(a^{-1} b), \\ \alpha * f(x) &= \sum_{a \in G} \alpha(a) f(a^{-1} x), \\ f * \beta(x) &= \sum_{b \in G} f(x b^{-1}) \beta(b). \end{aligned}$$

Notice that these summations are finite, as the sampling measures are supported on finite sets.

**Definition 7.4.8 (Semigroups and monoids).** A *semigroup* is a non-empty set $S$ with an operation $((r,s) \mapsto rs) : S \times S \to S$ satisfying $r(st) = (rs)t$ for all $r, s, t \in S$. A semigroup is *commutative* if $rs = sr$ for all $r, s \in S$. Moreover, if there exists $e \in S$ such that $es = se = s$ for all $s \in S$ then $S$ is called a *monoid*.

*Example.* $\mathbb{Z}^+ = \{n \in \mathbb{Z} : \; n > 0\}$ is a commutative monoid with respect to multiplication, and a commutative semigroup with respect addition. If $V$ is a vector space then $(\mathrm{End}(V), (A, B) \mapsto AB)$ is a monoid with $e = I$.

**Lemma 7.4.9.** *The structure* $(\mathcal{SM}_G, (\alpha, \beta) \mapsto \alpha * \beta)$ *is a monoid.*

**Exercise 7.4.10.** Prove Lemma 7.4.9. How is $\mathrm{supp}(\alpha * \beta)$ related to $\mathrm{supp}(\alpha)$ and $\mathrm{supp}(\beta)$? In which case is $\mathcal{SM}_G$ is a group? Show that $\mathcal{SM}_G$ is commutative if and only if $G$ is commutative.

**Lemma 7.4.11.** *If* $\alpha \in \mathcal{SM}_G$ *then* $(f \mapsto \alpha * f), (f \mapsto f * \alpha) \in \mathcal{L}(C(G, \mathbb{K}))$ *and*

$$\begin{aligned} \|\alpha * f\|_{C(G,\mathbb{K})} &\leq \|f\|_{C(G,\mathbb{K})}, \\ \|f * \alpha\|_{C(G,\mathbb{K})} &\leq \|f\|_{C(G,\mathbb{K})}. \end{aligned}$$

*Moreover,* $\alpha * \mathbf{1} = \mathbf{1} = \mathbf{1} * \alpha$.

*Proof.* Trivially, $\alpha * \mathbf{1} = \mathbf{1}$. Because $(x \mapsto a^{-1} x) : G \to G$ is a homeomorphism and the summation is finite, $\alpha * f \in C(G, \mathbb{K})$. Linearity of $f \mapsto \alpha * f$ is clear. Next,

$$|\alpha * f(x)| \leq \sum_{a \in G} \alpha(a) |f(a^{-1} x)| \leq \sum_{a \in G} \alpha(a) \|f\|_{C(G,\mathbb{K})} = \|f\|_{C(G,\mathbb{K})}.$$

Similar conclusions hold also for $f * \alpha$. □

**Definition 7.4.12.** Let $G$ be a compact group. Let us define a mapping $p_G : C(G, \mathbb{R}) \to \mathbb{R}$ by

$$p_G(f) := \max(f) - \min(f).$$

**Lemma 7.4.13.** *If $f \in C(G,\mathbb{R})$ and $\alpha \in \mathcal{SM}_G$ then*

$$\min(f) \le \min(\alpha * f) \le \max(\alpha * f) \le \max(f),$$

$$\min(f) \le \min(f * \alpha) \le \max(f * \alpha) \le \max(f),$$

*so that*

$$p_G(\alpha * f) \le p_G(f), \quad p_G(f * \alpha) \le p_G(f).$$

*Proof.* Now

$$\min(f) = \sum_{a\in G} \alpha(a)\ \min(f) \le \min_{x\in G} \sum_{a\in G} \alpha(a) f(a^{-1}x) = \min(\alpha * f),$$

$$\max(\alpha * f) = \max_{x\in G} \sum_{a\in G} \alpha(a) f(a^{-1}x) \le \sum_{a\in G} \alpha(a)\ \max(f) = \max(f),$$

and clearly $\min(\alpha * f) \le \max(\alpha * f)$. The proof for $f * \alpha$ is symmetric. □

**Exercise 7.4.14.** Show that $p_G$ is a bounded seminorm on $C(G,\mathbb{R})$.

**Proposition 7.4.15.** *Let $f \in C(G,\mathbb{R})$. For every $\varepsilon > 0$ there exist $\alpha, \beta \in \mathcal{SM}_G$ such that*

$$p_G(\alpha * f) < \varepsilon, \quad p_G(f * \beta) < \varepsilon.$$

*Remark* 7.4.16. This is the decisive stage in the construction of the Haar measure. The idea is that for a non-constant $f \in C(G)$ we can find sampling measures $\alpha, \beta$ that tame the oscillations of $f$ so that $\alpha * f$ and $f * \beta$ are almost constant functions. It will turn out that there exists a unique constant function $\mathrm{Haar}(f)\mathbf{1}$ approximated by the convolutions of the type $\alpha * f$ and $f * \beta$. In the sequel, notice how compactness is exploited!

*Proof.* Let $\varepsilon > 0$. By Theorem 7.3.8, a continuous function on a **compact group** is uniformly continuous. Thus there exists an open set $U \supset e$ such that $|f(x) - f(y)| < \varepsilon$, when $x^{-1}y \in U$. We notice easily that if $\gamma \in \mathcal{SM}_G$ then also $|\gamma * f(x) - \gamma * f(y)| < \varepsilon$, when $x^{-1}y \in U$:

$$\begin{aligned} |\gamma * f(x) - \gamma * f(y)| &= \left| \sum_{a\in G} \gamma(a) \left( f(a^{-1}x) - f(a^{-1}y) \right) \right| \\ &\le \sum_{a\in G} \gamma(a) \left| f(a^{-1}x) - f(a^{-1}y) \right| \\ &< \sum_{a\in G} \gamma(a)\ \varepsilon \\ &= \varepsilon. \end{aligned}$$

Now $\{xU:\ x \in G\}$ is an open cover of the **compact** space $G$, hence having a finite subcover $\{x_j U\}_{j=1}^n$. The set $S := \{x_i x_j^{-1}:\ 1 \le i,j \le n\}$ has $|S| \le n^2$ elements. Define $\gamma_1 \in \mathcal{SM}_G$ by

$$\gamma_1(a) = \begin{cases} |S|^{-1}, & \text{when } a \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\gamma_{k+1} := \gamma_k * \gamma_1 \in \mathcal{SM}_G$. Then

$$\begin{aligned} & p_G(\gamma_{k+1} * f) \\ = \ & \max(\gamma_{k+1} * f) - \min(\gamma_{k+1} * f) \\ \le \ & \max(\gamma_{k+1} * f) - \min(\gamma_k * f) \\ = \ & \frac{1}{|S|} \max_{x \in G} \sum_{a \in S} \gamma_k * f(a^{-1}x) - \min(\gamma_k * f) \\ \overset{(\star)}{<} \ & \frac{1}{|S|} \left[(|S|-1)\max(\gamma_k * f) + [\min(\gamma_k * f) + \varepsilon]\right] - \min(\gamma_k * f) \\ = \ & \frac{|S|-1}{|S|}\, p_G(\gamma_k * f) + \frac{1}{|S|}\varepsilon, \end{aligned}$$

where the last inequality $(\star)$ was obtained by estimating $|S|-1$ terms in the sum trivially, and finally the remaining term was estimated by recalling the uniform continuity of $\gamma_k * f$. Notice that $(p_G(\gamma_k * f))_{k=1}^\infty \subset \mathbb{R}$ is a non-increasing sequence of non-negative numbers. Thus there exists the limit $\delta := \lim_{k\to\infty} p_G(\gamma_k * f) \ge 0$, and

$$\delta \le \frac{|S|-1}{|S|}\delta + \frac{1}{|S|}\varepsilon,$$

so that $\delta \le \varepsilon$. Hence there exists $k_0$ such that, say, $p_G(\gamma_k * f) \le 2\varepsilon$ for every $k \ge k_0$. This proves the claim. □

**Exercise 7.4.17.** In the proof above, check the validity of inequality $(\star)$.

**Definition 7.4.18.** The following Corollary 7.4.19 defines the Haar functional Haar : $C(G,\mathbb{R}) \to \mathbb{R}$.

**Corollary 7.4.19 (What is the Haar functional Haar$(f)$?).** *For $f \in C(G,\mathbb{R})$ there exists a unique constant function* $\mathrm{Haar}(f)\mathbf{1}$ *that belongs to the closure of the set*

$$\{\alpha * f:\ \alpha \in \mathcal{SM}_G\} \subset C(G,\mathbb{R}).$$

*Moreover,* $\mathrm{Haar}(f)\mathbf{1}$ *is the unique constant function that belongs to the closure of the set*

$$\{f * \beta:\ \beta \in \mathcal{SM}_G\} \subset C(G,\mathbb{R}).$$

*Proof.* Pick any $\alpha_1 \in \mathcal{SM}_G$. Suppose we have chosen $\alpha_k \in \mathcal{SM}_G$. Let $\alpha_{k+1} := \gamma_k * \alpha_k$, where $\gamma_k \in \mathcal{SM}_G$ satisfies

$$p_G(\alpha_{k+1} * f) = p_G(\gamma_k * (\alpha_k * f)) < 2^{-k}.$$

Now

$$\min(\alpha_k * f) \leq \min(\alpha_{k+1} * f) \leq \max(\alpha_{k+1} * f) \leq \max(\alpha_k * f),$$

so that there exists

$$\lim_{k\to\infty} \min(\alpha_k * f) = \lim_{k\to\infty} \max(\alpha_k * f) =: c_1 \in \mathbb{R}.$$

In the same way we may construct a sequence $(\beta_k)_{k=1}^{\infty} \subset \mathcal{SM}_G$ such that

$$\lim_{k\to\infty} \min(f * \beta_k) = \lim_{k\to\infty} \max(f * \beta_k) =: c_2 \in \mathbb{R}.$$

But

$$\begin{array}{rcl} |c_1 - c_2| & = & \|c_1\mathbf{1} - c_2\mathbf{1}\|_{C(G,\mathbb{R})} \\ & = & \|(c_1\mathbf{1} - \alpha_k * f) * \beta_k + \alpha_k * (f * \beta_k - c_2\mathbf{1})\|_{C(G,\mathbb{R})} \\ & \leq & \|(c_1\mathbf{1} - \alpha_k * f) * \beta_k\|_{C(G,\mathbb{R})} + \|\alpha_k * (f * \beta_k - c_2\mathbf{1})\|_{C(G,\mathbb{R})} \\ & \leq & \|c_1\mathbf{1} - \alpha_k * f\|_{C(G,\mathbb{R})} + \|f * \beta_k - c_2\mathbf{1}\|_{C(G,\mathbb{R})} \\ & \xrightarrow[k\to\infty]{} & 0. \end{array}$$

Thus $c_1 = c_2 \in \mathbb{R}$ is unique, depending only on the function $f$. □

**Definition 7.4.20 (Haar functional of $f \in C(G,\mathbb{C})$).** The *Haar functional* of $f \in C(G)$ is

$$\mathrm{Haar}(f) := \mathrm{Haar}(\mathrm{Re}(f)) + \mathrm{i}\ \mathrm{Haar}(\mathrm{Im}(f)),$$

where $\mathrm{Re}(f), \mathrm{Im}(f)$ are the real and imaginary parts of $f$, respectively.

Let us now reformulate Theorem 7.4.3:

**Theorem 7.4.21 (Haar).** *The Haar functional* $\mathrm{Haar} : C(G) \to \mathbb{C}$ *on a compact group $G$ is the unique positive linear functional satisfying*

$$\begin{array}{rcl} \mathrm{Haar}(\mathbf{1}) & = & 1, \\ \mathrm{Haar}(f) & = & \mathrm{Haar}(x \mapsto f(yx)), \end{array}$$

*for all $f \in C(G)$ and $y \in G$. Moreover,*

$$\mathrm{Haar}(f) = \mathrm{Haar}(x \mapsto f(xy)) = \mathrm{Haar}(x \mapsto f(x^{-1})).$$

*Proof.* By Definition 7.4.20 of Haar, it is enough to deal with real-valued functions here. From the construction, it is clear that

$$f \geq 0 \;\Rightarrow\; \mathrm{Haar}(f) \geq 0,$$

$$|\mathrm{Haar}(f)| \leq \|f\|_{C(G)},$$

$$\mathrm{Haar}(\lambda f) = \lambda\ \mathrm{Haar}(f),$$

$$\mathrm{Haar}(\mathbf{1}) = 1,$$

$$\mathrm{Haar}(f) = \mathrm{Haar}(x \mapsto f(yx)) = \mathrm{Haar}(x \mapsto f(xy)).$$

Choose $\alpha, \beta \in \mathcal{SM}_G$ such that

$$\begin{aligned} \|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} &< \varepsilon, \\ \|g * \beta - \mathrm{Haar}(g)\mathbf{1}\|_{C(G)} &< \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} & \|\alpha * (f+g) * \beta - (\mathrm{Haar}(f) + \mathrm{Haar}(g))\mathbf{1}\|_{C(G)} \\ = \;& \|(\alpha * f - \mathrm{Haar}(f)\mathbf{1}) * \beta + \alpha * (g * \beta - \mathrm{Haar}(g)\mathbf{1})\|_{C(G)} \\ \leq \;& \|(\alpha * f - \mathrm{Haar}(f)\mathbf{1}) * \beta\|_{C(G)} + \|\alpha * (g * \beta - \mathrm{Haar}(g)\mathbf{1})\|_{C(G)} \\ \leq \;& \|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} + \|g * \beta - \mathrm{Haar}(g)\mathbf{1}\|_{C(G)} \\ < \;& 2\varepsilon, \end{aligned}$$

so that $\mathrm{Haar}(f+g) = \mathrm{Haar}(f) + \mathrm{Haar}(g)$.

Suppose $L : C(G) \to \mathbb{C}$ is a positive linear functional such that $L(\mathbf{1}) = 1$ and $L(f) = L(x \mapsto f(yx))$ for all $f \in C(G)$ and $y \in G$. Let $f \in C(G)$, $\varepsilon > 0$ and $\alpha \in \mathcal{SM}_G$ be as above. Then

$$\begin{aligned} |L(f) - \mathrm{Haar}(f)| &= |L(\alpha * f - \mathrm{Haar}(f)\mathbf{1})| \\ &\leq \|L\|_{C(G)'}\ \|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} \\ &< \|L\|_{C(G)'}\ \varepsilon \end{aligned}$$

yields the uniqueness $L = \mathrm{Haar}$.

Finally, $(f \mapsto \mathrm{Haar}(x \mapsto f(x^{-1}))) : C(G) \to \mathbb{C}$ is a positive linear translation-invariant normalised functional, hence equaling to Haar by the uniqueness. □

**Exercise 7.4.22.** In the previous proof, many properties were declared clear, but the reader is encouraged to verify the claims.

**Definition 7.4.23** (**Spaces** $L^p(\mu_G)$)**.** For $1 \leq p < \infty$, the *Lebesgue-p-space* $L^p(\mu_G)$ on a topological group $G$ is a special case of the Lebesgue-$p$-space from Definition

C.4.6. Because the group is compact, by looking in local coordinates, we see from Exercise 1.3.33 that it is the completion of $C(G)$ with respect to the norm

$$f \mapsto \|f\|_{L^p(\mu_G)} := \left( \int_G |f|^p \, \mathrm{d}\mu_G \right)^{1/p}.$$

The space $L^\infty(\mu_G)$ is the usual Banach space of $\mu_G$-essentially bounded functions with the norm $f \mapsto \|f\|_{L^\infty(\mu_G)}$; on the closed subspace $C(G) \subset L^\infty(\mu_G)$ we have $\|f\|_{C(G)} = \|f\|_{L^\infty(\mu_G)}$. Notice that $L^p(\mu_G)$ is a Banach space, but it is a Hilbert space if and only if $p = 2$, having the inner product $(f, g) \mapsto \langle f, g \rangle_{L^2(\mu_G)}$ satisfying

$$\langle f, g \rangle_{L^2(\mu_G)} = \int_G f\overline{g} \, \mathrm{d}\mu_G$$

for $f, g \in C(G)$.

*Remark* 7.4.24. We have now seen that for a compact group $G$ there exists a unique translation-invariant probability functional on $C(G)$, the Haar functional! We also know that it is enough to demand only either left- or right-invariance, since one follows from the other. Moreover, the Haar functional is also inversion-invariant. It must be noted that an inversion-invariant probability functional on $C(G)$ is not necessarily translation-invariant: e.g. let us consider the Dirac point mass $\delta_e$ at $e \in G$, for which the functional

$$f \mapsto f(e) = \int_G f(x) \, \mathrm{d}\delta_e(x)$$

is inversion-invariant but clearly not translation-invariant (unless $G = \{e\}$). Next we observe that the Haar integral distinguishes continuous functions $f, g \in C(G)$ in the sense that if $\int_G |f - g| \, \mathrm{d}\mu_G = 0$ then $f = g$:

**Theorem 7.4.25.** *Let $G$ be a compact group and $f \in C(G)$. If $\int_G |f| \, \mathrm{d}\mu_G = 0$ then $f = 0$.*

*Proof.* The set $U := f^{-1}(\mathbb{C} \setminus \{0\}) \subset G$ is open, since $f$ is continuous and $\{0\} \subset \mathbb{C}$ is closed. Suppose $f \neq 0$. Then $U \neq \emptyset$, and $\{xU : x \in G\}$ is an open cover for $G$. By the compactness, there exists a subcover $\{x_j U\}_{j=1}^n$. Define $g \in C(G)$ by

$$g(x) := \sum_{j=1}^n \left| f(x_j^{-1} x) \right|.$$

Now $g(x) > 0$ for all $x \in G$, so that there exists $c := \min_{x \in G} g(x) > 0$ by the compactness. We use the normalisation, positivity and translation-invariance of $\mu_G$ to obtain

$$0 < c = \int_G c\mathbf{1} \, \mathrm{d}\mu_G \leq \int_G g \, \mathrm{d}\mu_G = n \int_G |f| \, \mathrm{d}\mu_G,$$

so that $0 < \int_G |f| \, \mathrm{d}\mu_G$. □

**Exercise 7.4.26.** Let $G, H$ be compact groups. Show that $\mu_{G\times H} = \mu_G \times \mu_H$ (i.e. the Haar measure of the product group is the product of the original Haar measures).

**Exercise 7.4.27.** Let $\mathcal{M}_G$ denote the $\sigma$-algebra of the Haar-measurable sets on the compact group $G$. Consider mappings $m, p_1, p_2 : G \times G \to G$, where

$$m(x,y) = xy, \quad p_1(x,y) = x, \quad p_2(x,y) = y.$$

Show that they are Haar measurable (that is, $(\mathcal{M}_{G\times G}, \mathcal{M}_G)$-measurable). Moreover, show that

$$\mu_G(E) = \mu_{G\times G}(m^{-1}(E)) = \mu_{G\times G}(p_1^{-1}(E)) = \mu_{G\times G}(p_2^{-1}(E)).$$

for all $E \in \mathcal{M}_G$.

### 7.4.1 Integration on quotient spaces

We have already noticed that the good subgroups of a topological group are the closed ones. Moreover, by now we know that a transitive action of a compact topological group $G$ on a Hausdorff space $X$ begets a homeomorphism $G/H \cong X$ of compact Hausdorff spaces, where $H$ is a closed subgroup of $G$; effectively, spaces $G/H$ and $X$ are the same. We are now about to show that for $X$ there exists a unique $G$-action-invariant probability functional on $C(X)$, which might be called the Haar functional of the action; the corresponding measure on $G/H$ will accordingly be denoted by $\mu_{G/H}$. We have seen that continuous functions on $G/H$ (and hence on $X$) can be interpreted as continuous right-$H$-translation-invariant functions on $G$, i.e. $f(xh) = f(x)$ for all $x \in G$ and $h \in H$. Next we show how $f \in C(G)$ casts a shadow $f_{G/H} \in C(G/H)$ in a natural way:

**Lemma 7.4.28** (**Projection $P_{G/H}$**). *Let $G$ be a compact group and $H < G$ closed. If $f \in C(G)$ then $P_{G/H} f \in C(G)$ and $f_{G/H} \in C(G/H)$, where*

$$f_{G/H}(xH) = P_{G/H} f(x) := \int_H f(xh)\ \mathrm{d}\mu_H(h).$$

*Furthermore, the projection $P_{G/H} : C(G) \to C(G)$ is bounded, more precisely $\|f_{G/H}\|_{C(G/H)} = \left\|P_{G/H} f\right\|_{C(G)} \le \|f\|_{C(G)}$.*

*Proof.* First, $H$ is a compact group having the Haar measure $\mu_H$. The integration in the definition is legitimate since $f_x := (h \mapsto f(xh)) \in C(H)$ for each $x \in G$. If $x \in G$ and $h_0 \in H$ then

$$P_{G/H} f(xh_0) = \int_H f_x(h_0 h)\ \mathrm{d}\mu_H(h) = \int_H f_x(h)\ \mathrm{d}\mu_H(h) = P_{G/H} f(x),$$

so that $f_{G/H} : G/H \to \mathbb{C}$. Next we prove the continuity. Let $\varepsilon > 0$. A continuous function on a compact group is uniformly continuous, so that for $f \in C(G)$ there exists an open $U \ni e$ such that

$$\forall x, y \in G : \ xy^{-1} \in U \ \Rightarrow \ |f(x) - f(y)| < \varepsilon$$

(apparently, this slightly deviates from our definition of the uniform continuity; however, this is clearly equivalent). Suppose $xy^{-1} \in U$. Then

$$\begin{aligned} |P_{G/H}f(x) - P_{G/H}f(y)| &= \left| \int_H f(xh) - f(yh) \, \mathrm{d}\mu_H(h) \right| \\ &\leq \int_H |f(xh) - f(yh)| \, \mathrm{d}\mu_H(h) \\ &< \varepsilon, \end{aligned}$$

so that $P_{G/H}f \in C(G)$ and $f_{G/H} \in C(G/H)$. Finally,

$$|P_{G/H}f(x)| \leq \int_H |f(xh)| \, \mathrm{d}\mu_H(h) \leq \int_H \|f\|_{C(G)} \, \mathrm{d}\mu_H(h) = \|f\|_{C(G)}.$$

□

**Exercise 7.4.29.** Show that the projection $P_{G/H} \in \mathcal{L}(C(G))$ extends uniquely to an orthogonal projection $P_{G/H} \in \mathcal{L}(L^2(\mu_G))$.

**Theorem 7.4.30 (Existence of action-invariant measure on quotient spaces).** *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a continuous transitive action of a compact group $G$ on a Hausdorff space $M$. Then there exists a unique Borel-regular probability measure $\mu_M$ on $M$ which is action-invariant in the sense that*

$$\int_M f_M \, \mathrm{d}\mu_M = \int_M f_M(x \cdot p) \, \mathrm{d}\mu_M(p)$$

*for all $f_M \in C(M)$ and $x \in G$.*

*Proof.* Given $q \in M$, we know that $M \cong G/G_q$. Hence it is enough to deal with $M = G/H$, where $H < G$ is closed and the action is $((x, yH) \mapsto xyH) : G \times G/H \to G/H$.

We first prove the existence of a $G$-action-invariant Borel regular probability measure $\mu_{G/H}$ on the compact Hausdorff space $G/H$. Define $\mathrm{Haar}_{G/H} : C(G/H) \to \mathbb{C}$ by

$$\mathrm{Haar}_{G/H}(f_{G/H}) := \int_G f_{G/H}(xH) \, \mathrm{d}\mu_G(x).$$

Notice that

$$\begin{aligned} \mathrm{Haar}_{G/H}(f_{G/H}) &= \int_G \int_H f(xh) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_G(x) \\ &\overset{\text{Fubini}}{=} \int_H \int_G f(xh) \, \mathrm{d}\mu_G(x) \, \mathrm{d}\mu_H(h) \\ &= \int_H \mathrm{Haar}_G(f) \, \mathrm{d}\mu_H \\ &= \mathrm{Haar}_G(f). \end{aligned}$$

It is clear that $\mathrm{Haar}_{G/H}$ is a bounded linear functional, and that

$$\mathrm{Haar}_{G/H}(\mathbf{1}_{G/H}) = \mathrm{Haar}_G(\mathbf{1}_G) = 1.$$

By the Riesz Representation Theorem (see Theorem C.4.65), there exists a unique Borel-regular probability measure $\mu_{G/H}$ on $G/H$ such that

$$\mathrm{Haar}_{G/H}(f_{G/H}) = \int_{G/H} f_{G/H} \, \mathrm{d}\mu_{G/H}$$

for all $f_{G/H} \in C(G/H)$. The action-invariance follows from the left-invariance of the functional $\mathrm{Haar}_G$: if $g(y) := f(xy)$ for all $y \in G$ then $g_{G/H}(yH) = f_{G/H}(xyH)$ and

$$\begin{aligned}
\mathrm{Haar}_{G/H}(y \mapsto f_{G/H}(xyH)) &= \mathrm{Haar}_{G/H}(g_{G/H}) \\
&= \mathrm{Haar}_G(g) \\
&= \mathrm{Haar}_G(f) \\
&= \mathrm{Haar}_{G/H}(f_{G/H}).
\end{aligned}$$

Next we shall prove the uniqueness part. Suppose $L : C(G/H) \to \mathbb{C}$ is an action-invariant bounded linear functional for which $L(\mathbf{1}_{G/H}) = 1$. Recall the mapping $(f \mapsto f_{G/H}) : C(G) \to C(G/H)$ in Lemma 7.4.28. Then

$$\tilde{L}(f) := L(f_{G/H})$$

defines a bounded linear functional $\tilde{L} : C(G) \to \mathbb{C}$ such that $\tilde{L}(\mathbf{1}_G) = 1$ and

$$\tilde{L}(y \mapsto f(xy)) = L(y \mapsto f_{G/H}(xyH)) = L(f_{G/H}) = \tilde{L}(f).$$

Hence $\tilde{L} = \mathrm{Haar}_G$ by Theorem 7.4.21. Consequently,

$$L(f_{G/H}) = \tilde{L}(f) = \mathrm{Haar}_G(f) = \mathrm{Haar}_{G/H}(f_{G/H}),$$

yielding $L = \mathrm{Haar}_{G/H}$. □

*Remark* 7.4.31. Let $G$ be a compact group and $H < G$ closed. From the proof of Theorem 7.4.30 we see that

$$\int_G f \, \mathrm{d}\mu_G = \int_{G/H} \int_H f(xh) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_{G/H}(xH)$$

for all $f \in C(G)$.

**Exercise 7.4.32.** Let $\omega_j(t) \in \mathrm{SO}(3)$ denote the rotation of $\mathbb{R}^3$ by angle $t \in \mathbb{R}$ around the $j$th coordinate axis, $j \in \{1, 2, 3\}$. Show that $x \in \mathrm{SO}(3)$ can be represented in the form

$$x = x(\phi, \theta, \psi) = \omega_3(\phi) \, \omega_2(\theta) \, \omega_3(\psi)$$

where $0 \leq \phi, \psi < 2\pi$ and $0 \leq \theta \leq \pi$.

**Exercise 7.4.33.** Let the group $G = \mathrm{SO}(3)$ act on the sphere $M = \mathbb{S}^2$ by rotations. Let $q = (0,0,1) \in M$, i.e. $q$ is the north pole. Show that $G_q = \{\omega_3(\psi) : \ 0 \leq \psi < 2\pi\}$. We know that the Lebesgue measure is rotation-invariant. Using the normalised angular part of the Lebesgue measure of $\mathbb{R}^3$, deduce that here

$$\int_G f \ \mathrm{d}\mu_G = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(x(\phi,\theta,\psi)) \ \sin(\theta) \ \mathrm{d}\psi \ \mathrm{d}\theta \ \mathrm{d}\phi,$$

i.e. $\mathrm{d}\mu_{\mathrm{SO}(3)} = \frac{1}{8\pi^2} \ \sin(\theta) \ \mathrm{d}\psi \ \mathrm{d}\theta \ \mathrm{d}\phi$.

We return to the example of SO(3) in Chapter 11.

## 7.5 Peter–Weyl decomposition of representations

In the sequel we apply the Haar integral in studying unitary representations of compact groups. The main result is the Peter–Weyl Theorem 7.5.14, leading to a natural Fourier series representation for functions on a compact group.

**Exercise 7.5.1.** Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{H}))$ be a representation of a compact group $G$ on a finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$. Construct a $G$-invariant inner product $((u,v) \mapsto \langle u,v\rangle_{\mathcal{H}}) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, that is

$$\langle \phi(x)u, \phi(x)v\rangle_{\mathcal{H}} = \langle u,v\rangle_{\mathcal{H}}$$

for all $x \in G$ and $u,v \in \mathcal{H}$. Notice that now the representation $\phi$ is unitary with respect to this inner product!

**Lemma 7.5.2.** *Let $G$ be a compact group and $\mathcal{H}$ be a Hilbert space with the inner product $(u,v) \mapsto \langle u,v\rangle_{\mathcal{H}}$. Let $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ be cyclic and $w \in \mathcal{H}$ a $\phi$-cyclic vector with $\|w\|_{\mathcal{H}} = 1$. Then*

$$\langle u,v\rangle_\phi := \int_G \langle \phi(x)u, w\rangle_{\mathcal{H}} \ \langle w, \phi(x)v\rangle_{\mathcal{H}} \ \mathrm{d}x$$

*defines an inner product $(u,v) \mapsto \langle u,v\rangle_\phi$ for $\mathcal{H}$. Moreover, $\phi$ is unitary also with respect to this new inner product, and $\|u\|_\phi \leq \|u\|_{\mathcal{H}}$ for all $u \in \mathcal{H}$, where $\|u\|_\phi^2 := \langle u,u\rangle_\phi$.*

*Proof.* Defining $f_u(x) := \langle \phi(x)u, w\rangle_{\mathcal{H}}$, we notice that $f_u \in C(G)$, because

$$\begin{aligned} |f_u(x) - f_u(y)| &= |\langle (\phi(x) - \phi(y))u, w\rangle_{\mathcal{H}}| \\ &\leq \|(\phi(x) - \phi(y))u\|_{\mathcal{H}} \ \|w\|_{\mathcal{H}} \\ &\xrightarrow[x \to y]{} 0 \end{aligned}$$

due to the strong continuity of $\phi$. Thereby $f_u \overline{f_v}$ is Haar integrable, justifying the definition of $\langle u,v\rangle_\phi$.

Let $\lambda \in \mathbb{C}$ and $t, u, v \in \mathcal{H}$. Then it is easy to verify that

$$\begin{aligned}
\langle \lambda u, v\rangle_\phi &= \lambda \langle u, v\rangle_\phi, \\
\langle t+u, v\rangle_\phi &= \langle t, v\rangle_\phi + \langle u, v\rangle_\phi, \\
\langle u, v\rangle_\phi &= \overline{\langle v, u\rangle_\phi}, \\
\|u\|_\phi^2 &= \int_G |f_u|^2 \, \mathrm{d}\mu_G \ \geq \ 0.
\end{aligned}$$

What if $0 = \|u\|_\phi^2 = \int_G |f_u|^2 \mathrm{d}\mu_G$? Then $f_u = 0$ by Theorem 7.4.25, i.e.

$$0 = \langle \phi(x)u, w\rangle_{\mathcal{H}} = \langle u, \phi(x^{-1})w\rangle_{\mathcal{H}}$$

for all $x \in G$. Since $w$ is a cyclic vector, $u = 0$ follows. Thus $(u, v) \mapsto \langle u, v\rangle_\phi$ is an inner product on $\mathcal{H}$.

The original norm dominates the $\phi$-norm, since

$$\begin{aligned}
\|u\|_\phi^2 &= \int_G |\langle \phi(x)u, w\rangle_{\mathcal{H}}|^2 \, \mathrm{d}x \\
&\leq \int_G \|\phi(x)u\|_{\mathcal{H}}^2 \ \|w\|_{\mathcal{H}}^2 \, \mathrm{d}x \\
&= \int_G \|u\|_{\mathcal{H}}^2 \, \mathrm{d}x \\
&= \|u\|_{\mathcal{H}}^2.
\end{aligned}$$

The $\phi$-unitarity of $\phi$ follows by

$$\begin{aligned}
\langle u, \phi(y)^* v\rangle_\phi &= \langle \phi(y)u, v\rangle_\phi \\
&= \int_G \langle \phi(xy)u, w\rangle_{\mathcal{H}} \ \langle w, \phi(x)v\rangle_{\mathcal{H}} \, \mathrm{d}x \\
&\overset{z=xy}{=} \int_G \langle \phi(z)u, w\rangle_{\mathcal{H}} \ \langle w, \phi(zy^{-1})v\rangle_{\mathcal{H}} \, \mathrm{d}z \\
&= \langle u, \phi(y)^{-1} v\rangle_\phi,
\end{aligned}$$

where we applied the translation invariance of the Haar integral. □

**Exercise 7.5.3.** Check the missing details in the proof of Lemma 7.5.2.

**Lemma 7.5.4.** *Let $\langle u, v\rangle_\phi$ be as above. Then*

$$\langle u, Av\rangle_{\mathcal{H}} := \langle u, v\rangle_\phi$$

*defines a compact self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$. Furthermore, $A$ is positive definite and $A \in \mathrm{Hom}(\phi, \phi)$.*

*Proof.* By Lemma 7.5.2, if $v \in \mathcal{H}$ then $F_v(u) := \langle u, v\rangle_\phi$ defines a linear functional $F_v : \mathcal{H} \to \mathbb{C}$, which is bounded in both norms, since

$$|F_v(u)| = |\langle u, v\rangle_\phi| \leq \|u\|_\phi \ \|v\|_\phi \leq \|u\|_{\mathcal{H}} \ \|v\|_\phi.$$

The Riesz Representation Theorem B.5.19 implies that $F_v$ is represented by a unique vector $A(v) \in \mathcal{H}$, i.e. $F_v(u) = \langle u, A(v)\rangle_{\mathcal{H}}$ for all $u \in \mathcal{H}$. Thus we have an operator $A : \mathcal{H} \to \mathcal{H}$, which is clearly linear. We obtain a bound $\|A\|_{\mathcal{L}(\mathcal{H})} \leq 1$ from

$$\|Av\|_{\mathcal{H}}^2 = \langle Av, Av\rangle_{\mathcal{H}} = \langle Av, v\rangle_\phi \leq \|Av\|_\phi \ \|v\|_\phi \leq \|Av\|_{\mathcal{H}} \ \|v\|_{\mathcal{H}}.$$

Self-adjointness follows from

$$\langle u, A^* v\rangle_{\mathcal{H}} = \langle Au, v\rangle_{\mathcal{H}} = \overline{\langle v, Au\rangle_{\mathcal{H}}} = \overline{\langle v, u\rangle_\phi} = \langle u, v\rangle_\phi = \langle u, Av\rangle_{\mathcal{H}}.$$

Moreover, $A$ is positive definite, because $\langle u, Au\rangle_{\mathcal{H}} = \langle u, u\rangle_\phi = \|u\|_\phi^2 \geq 0$, where $\|u\|_\phi = 0$ if and only if $u = 0$.

The property that $A \in \mathrm{Hom}(\phi, \phi)$ is seen from

$$\begin{aligned} \langle u, A\phi(y)v\rangle_{\mathcal{H}} &= \langle u, \phi(y)v\rangle_\phi \\ &= \langle \phi(y)^{-1}u, v\rangle_\phi \\ &= \langle \phi(y)^{-1}u, Av\rangle_{\mathcal{H}} \\ &= \langle u, \phi(y)Av\rangle_{\mathcal{H}}. \end{aligned}$$

Let $\mathbb{B} = \{u \in \mathcal{H} : \ \|u\|_{\mathcal{H}} \leq 1\}$, the closed unit ball of $\mathcal{H}$. To show that $A \in \mathcal{L}(\mathcal{H})$ is compact, we must show that $\overline{A(\mathbb{B})} \subset \mathcal{H}$ is a compact set. So take a sequence $(v_j)_{j=1}^\infty \subset A(\mathbb{B})$; we have to find a converging subsequence. Take a sequence $(u_j)_{j=1}^\infty \subset \mathbb{B}$ such that $Au_j = v_j$. By the Banach–Alaoglu Theorem B.5.30, the closed ball $\mathbb{B}$ is weakly compact: there exists a subsequence $(u_{j_k})_{k=1}^\infty$ such that $u_{j_k} \xrightarrow[k\to\infty]{} u \in \mathbb{B}$ weakly, i.e.

$$\langle u_{j_k}, v\rangle_{\mathcal{H}} \xrightarrow[k\to\infty]{} \langle u, v\rangle_{\mathcal{H}}$$

for all $v \in \mathcal{H}$. Then

$$\begin{aligned} \|v_{j_k} - Au\|_{\mathcal{H}}^2 &= \|A(u_{j_k} - u)\|_{\mathcal{H}}^2 \\ &= \langle A(u_{j_k} - u), u_{j_k} - u\rangle_\phi \\ &= \int_G g_k \ \mathrm{d}\mu_G \end{aligned}$$

where

$$g_k(x) := \langle \phi(x)A(u_{j_k} - u), w\rangle_{\mathcal{H}} \ \langle w, \phi(x)(u_{j_k} - u)\rangle_{\mathcal{H}}.$$

Let us show that $\int_G g_k \, \mathrm{d}\mu_G \to 0$ as $k \to \infty$. First, $g_k \in C(G)$ (hence $g_k$ is integrable) and for each $x \in G$

$$\begin{aligned} |g_k(x)| &= \left|\langle u_{j_k} - u, A^*\phi(x^{-1})w\rangle_{\mathcal{H}}\right| \left|\langle \phi(x^{-1})w, u_{j_k} - u\rangle_{\mathcal{H}}\right| \\ &\xrightarrow[k\to\infty]{} 0 \end{aligned}$$

by the weak convergence. Second,

$$\begin{aligned} |g_k(x)| &\leq \|\phi(x)\|^2_{\mathcal{L}(\mathcal{H})} \; \|A^*\|_{\mathcal{L}(\mathcal{H})} \; \|w\|^2_{\mathcal{H}} \; \|u_{j_k} - u\|^2_{\mathcal{H}} \\ &\leq 4, \end{aligned}$$

because $\|\phi(x)\|_{\mathcal{L}(\mathcal{H})} = 1$, $\|A\|_{\mathcal{L}(\mathcal{H})} = \|A^*\|_{\mathcal{L}(\mathcal{H})} \leq 1$, $\|w\|_{\mathcal{H}} = 1$ and $u_{j_k}, u \in \mathbb{B}$. Thus $\int_G g_k \, \mathrm{d}\mu_G \xrightarrow[k\to\infty]{} 0$ by the Lebesgue Dominated Convergence Theorem (see Theorem C.3.22). Equivalently, $v_{j_k} \xrightarrow[k\to\infty]{} Au \in A(\mathbb{B})$. We have shown that the set $\overline{A(\mathbb{B})} = A(\mathbb{B}) \subset \mathcal{H}$ is compact. □

**Theorem 7.5.5** (**Decomposition in finite dimensional representations).** *Let $G$ be a compact group and $\mathcal{H}$ a Hilbert space. Let $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ be strongly continuous. Then $\phi$ is a direct sum of finite-dimensional irreducible unitary representations.*

*Proof.* We know by Proposition 7.2.8 that $\phi$ is a direct sum of cyclic representations. Therefore it is enough to assume that $\phi$ itself is cyclic. The operator $A \in \mathrm{Hom}(\phi,\phi)$ in Lemma 7.5.4 is compact and self-adjoint. Hence by the Hilbert–Schmidt Spectral Theorem B.5.26 we have

$$\mathcal{H} = \bigoplus_{\lambda\in\sigma(A)} \mathrm{Ker}(\lambda I - A),$$

where $\dim(\mathrm{Ker}(\lambda I - A)) < \infty$ for each $\lambda \in \sigma(A)\backslash\{0\}$. This can be extended to $\lambda = 0$ as well by Lemma 7.5.4 and the definition of $A$. Since $A \in \mathrm{Hom}(\phi,\phi)$, the subspace $\mathrm{Ker}(\lambda I - A) \subset \mathcal{H}$ is $\phi$-invariant for each $\lambda$. Thereby

$$\phi = \bigoplus_{\lambda\in\sigma(A)} \phi|_{\mathrm{Ker}(\lambda I - A)},$$

where $\phi|_{\mathrm{Ker}(\lambda I - A)}$ is finite-dimensional and unitary for all $\lambda \in \sigma(A)$. The proof is concluded, since we know that a finite-dimensional unitary representation is a direct sum of irreducible unitary representations. □

**Corollary 7.5.6** (**Finite dimensionality of representations!).** *Strongly continuous irreducible unitary representations of compact groups are finite-dimensional.*

**Definition 7.5.7** (**Unitary dual $\widehat{G}$).** The (*unitary*) *dual* $\widehat{G}$ of a locally compact group $G$ is the set consisting of all equivalence classes of strongly continuous irreducible unitary representations of $G$ (for the definition of equivalent representations see Definition 6.3.18).

*Remark* 7.5.8 (**Continuity is enough**). For a compact group $G$, the set $\widehat{G}$ consists of the equivalence classes of *continuous* irreducible unitary representations (due to the finite-dimensionality), i.e.

$$\widehat{G} = \{[\phi] \mid \phi \text{ continuous irreducible unitary representation of } G\},$$

where $[\phi] = \{\psi \mid \psi \sim \phi\}$ is the equivalence class of $\phi$.

*Remark* 7.5.9 (**Duals $\widehat{\mathbb{R}^n}$ and $\widehat{\mathbb{T}^n}$**). It can be proven that

$$\widehat{\mathbb{R}^n} = \left\{[\mathrm{e}_\xi] \mid \xi \in \mathbb{R}^n,\ \mathrm{e}_\xi : \mathbb{R}^n \to \mathrm{U}(1),\ \mathrm{e}_\xi(x) := \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\right\}.$$

Noticing that $\mathrm{e}_\xi \mathrm{e}_\eta = \mathrm{e}_{\xi+\eta}$ and that $[\mathrm{e}_\xi] \neq [\mathrm{e}_\eta]$ for $\xi \neq \eta$, we may identify $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ as groups. Similarly, and in view of Theorem 3.1.17 and Remark 3.1.18, we have

$$\widehat{\mathbb{T}^n} = \left\{[\mathrm{e}_\xi] \mid \xi \in \mathbb{Z}^n,\ \mathrm{e}_\xi : \mathbb{R}^n \to \mathrm{U}(1),\ \mathrm{e}_\xi(x) := \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\right\},$$

so that $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$ as groups.

*Remark* 7.5.10 (**Pontryagin duality**). For a commutative locally compact group $G$ the unitary dual $\widehat{G}$ has a natural structure of a commutative locally compact group, and $\widehat{\widehat{G}} \cong G$; this is the so-called *Pontryagin duality*. For a compact non-commutative group $G$, the unitary dual $\widehat{G}$ is never a group, but still has a sort of weak algebraic structure; we do not consider this in the sequel.

*Remark* 7.5.11 (**Matrix representations**). Let $G$ be a compact group. For the equivalence class $\xi \in \widehat{G}$ there exists a unitary matrix representation $\phi \in \xi = [\phi]$. That is, we have a homomorphism $\phi = (\phi_{ij})_{i,j=1}^m : G \to \mathrm{U}(m)$, where functions $\phi_{ij} : G \to \mathbb{C}$ are continuous. We may find such a representation in the following way: if $\psi \in \xi$, $\psi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ and $\{e_j\}_{j=1}^m \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$, then we can define

$$\phi_{ij}(x) := \langle e_i, \psi(x) e_j\rangle_{\mathcal{H}}.$$

Next we present an $L^2$-orthogonality result for such functions $\phi_{ij} : G \to \mathbb{C}$.

**Lemma 7.5.12** (**Orthogonality of representations**). *Let $G$ be a compact group. Let $\xi, \eta \in \widehat{G}$, where $\xi \ni \phi = (\phi_{ij})_{i,j=1}^m \in \mathrm{Hom}(G, \mathrm{U}(m))$ and $\eta \ni \psi = (\psi_{kl})_{k,l=1}^n \in \mathrm{Hom}(G, \mathrm{U}(n))$. Then*

$$\langle \phi_{ij}, \psi_{kl}\rangle_{L^2(\mu_G)} = \begin{cases} 0, & \text{if } \xi \neq \eta, \\ \frac{1}{m}\ \delta_{ik}\delta_{jl}, & \text{if } \phi = \psi. \end{cases}$$

*Proof.* Fix $1 \le j \le m$ and $1 \le l \le n$. Define the matrix $E \in \mathbb{C}^{m\times n}$ by $E_{pq} := \delta_{pj}\delta_{lq}$ (i.e. the matrix elements of $E$ are zero except for the $(j,l)$-element, which is 1.) Define the matrix $A \in \mathbb{C}^{m\times n}$ by

$$A := \int_G \phi(y)\ E\ \psi(y^{-1})\ \mathrm{d}y.$$

Now $A \in \mathrm{Hom}(\psi, \phi)$, since

$$\begin{aligned}\phi(x)A &= \int_G \phi(xy)\ E\ \psi(y^{-1})\ \mathrm{d}y \\ &= \int_G \phi(z)\ E\ \psi(z^{-1}x)\ \mathrm{d}z \\ &= A\psi(x).\end{aligned}$$

Since $\phi, \psi$ are finite-dimensional irreducible unitary representations, Schur's Lemma 6.3.25 implies that

$$A = \begin{cases} 0, & \text{if } \phi \not\sim \psi, \\ \lambda I, & \text{if } \phi = \psi \end{cases}$$

for some $\lambda \in \mathbb{C}$. We notice that

$$\begin{aligned}A_{ik} &= \int_G \sum_{p=1}^{m} \sum_{q=1}^{n} \phi_{ip}(y)\ E_{pq}\ \psi_{qk}(y^{-1})\ \mathrm{d}y \\ &= \int_G \phi_{ij}(y)\ \overline{\psi_{kl}(y)}\ \mathrm{d}y \\ &= \langle \phi_{ij}, \psi_{kl} \rangle_{L^2(\mu_G)}.\end{aligned}$$

Now suppose $\phi = \psi$. Then $m = n$ and

$$\begin{aligned}\langle \phi_{kj}, \psi_{kl} \rangle_{L^2(\mu_G)} &= A_{kk} = \lambda = \frac{1}{m}\ \mathrm{Tr}(A) \\ &= \frac{1}{m} \int_G \mathrm{Tr}(\phi(y)\ E\ \phi(y^{-1}))\ \mathrm{d}y \\ &= \frac{1}{m} \int_G \mathrm{Tr}(E)\ \mathrm{d}y \\ &= \frac{1}{m}\delta_{jl},\end{aligned}$$

where we used the property $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ of the trace functional. The results can be collected from above. □

**Definition 7.5.13 (Left and right regular representations).** Let $G$ be a compact group. Its *left* and *right regular representations* $\pi_L, \pi_R : G \to \mathcal{U}(L^2(\mu_G))$ are defined, respectively, by

$$(\pi_L(y)\ f)(x) := f(y^{-1}x),$$

$$(\pi_R(y)\ f)(x) := f(xy)$$

for $\mu_G$-almost every $x \in G$.

The idea here is that $G$ is represented as a natural group of operators on the Hilbert space $L^2(\mu_G)$, enabling the use of functional analytic techniques in studying $G$. And now for a major result in representation theory:

**Theorem 7.5.14** (**Peter–Weyl Theorem, 1927).** *Let $G$ be a compact group. Then*

$$\mathcal{B} := \left\{ \sqrt{\dim(\phi)}\phi_{ij} \mid \phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)},\ [\phi] \in \widehat{G} \right\}$$

*is an orthonormal basis for $L^2(\mu_G)$. Let $\phi = (\phi_{ij})_{i,j=1}^{m}$, $\phi \in [\phi] \in \widehat{G}$. Then*

$$\mathcal{H}^{\phi}_{i,\cdot} := \operatorname{span}\{\phi_{ij} \mid 1 \le j \le m\} \subset L^2(\mu_G)$$

*is $\pi_R$-invariant and*

$$\phi \sim \pi_R|_{\mathcal{H}^{\phi}_{i,\cdot}},$$

$$L^2(\mu_G) = \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{m} \mathcal{H}^{\phi}_{i,\cdot},$$

$$\pi_R \sim \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{m} \phi.$$

*Remark* 7.5.15. Here $\bigoplus_{i=1}^{m} \phi := \phi\oplus\cdots\oplus\phi$, the $m$-fold direct sum of $\phi$; in literature, this is sometimes denoted even by $m\phi$.

*Remark* 7.5.16 (**Left Peter–Weyl**). We can formulate the Peter–Weyl Theorem 7.5.14 analogously for the left regular representation, as follows: Let $\phi = (\phi_{ij})_{i,j=1}^{m}$, where $\phi \in [\phi] \in \widehat{G}$. Then

$$\mathcal{H}^{\phi}_{\cdot,j} := \operatorname{span}\{\phi_{ij} \mid 1 \le i \le m\} \subset L^2(\mu_G)$$

is $\pi_L$-invariant and

$$\phi \sim \pi_L|_{\mathcal{H}^{\phi}_{\cdot,j}},$$

$$L^2(\mu_G) = \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{j=1}^{m} \mathcal{H}^{\phi}_{\cdot,j},$$

$$\pi_L \sim \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{j=1}^{m} \phi.$$

*Remark* 7.5.17 (**Peter–Weyl for $\mathbb{T}^n$**). Let $G = \mathbb{T}^n$. Recall from Remark 7.5.9 that

$$\widehat{\mathbb{T}^n} = \left\{ [\mathrm{e}_\xi] \mid \xi \in \mathbb{Z}^n,\ \mathrm{e}_\xi(x) = \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi} \right\}.$$

Now $\mathcal{B} = \{\mathrm{e}_\xi \mid \xi \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mu_{\mathbb{T}^n})$,

$$L^2(\mu_{\mathbb{T}^n}) = \bigoplus_{\xi \in \mathbb{Z}^n} \mathrm{span}\{\mathrm{e}_\xi\},$$

$$\pi_L \sim \bigoplus_{\xi \in \mathbb{Z}^n} \mathrm{e}_\xi \sim \pi_R.$$

Moreover, for $f \in L^2(\mu_{\mathbb{T}^n})$, we have

$$f = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi)\ \mathrm{e}_\xi,$$

where the Fourier coefficients $\widehat{f}(\xi)$ are calculated by

$$\widehat{f}(\xi) = \int_{\mathbb{T}^n} f\ \overline{\mathrm{e}_\xi}\ \mathrm{d}\mu_{\mathbb{T}^n} = \langle f, \mathrm{e}_\xi \rangle_{L^2(\mu_{\mathbb{T}^n})}.$$

Analogously, the Peter–Weyl Theorem 7.5.14 provides Fourier series expansions for $L^2$-functions on any compact group. We shall return to the *Fourier series* theme after the proof of the Peter–Weyl Theorem.

*Proof for the Peter–Weyl Theorem 7.5.14.* The $\pi_R$-invariance of $\mathcal{H}^\phi_{i,\cdot}$ follows due to

$$\pi_R(y)\phi_{ij}(x) = \phi_{ij}(xy) = \sum_{k=1}^{\dim(\phi)} \phi_{ik}(x)\phi_{kj}(y),$$

i.e. with $\lambda_k(y) = \phi_{kj}(y)$ we have

$$\pi_R(y)\phi_{ij} = \sum_{k=1}^{\dim(\phi)} \lambda_k(y)\ \phi_{ik} \in \mathrm{span}\{\phi_{ik}\}_{k=1}^{\dim(\phi)} = \mathcal{H}^\phi_{i,\cdot}.$$

If $\{e_j\}_{j=1}^{\dim(\phi)} \subset \mathbb{C}^{\dim(\phi)}$ is the standard orthonormal basis then

$$\phi(y)e_j = \sum_{k=1}^{\dim(\phi)} \phi_{kj}(y) e_k,$$

so that the equation

$$A \sum_{j=1}^{\dim(\phi)} \lambda_j e_j := \sum_{j=1}^{\dim(\phi)} \lambda_j \phi_{ij}$$

defines an intertwining isomorphism $A \in \mathrm{Hom}(\phi, \pi_R|_{\mathcal{H}^\phi_{i,\cdot}})$, i.e. $\phi \sim \pi_R|_{\mathcal{H}^\phi_{i,\cdot}}$.

By Lemma 7.5.12, the set $\mathcal{B} \subset L^2(\mu_G)$ is orthonormal. Let

$$\mathcal{H} := \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{\dim(\phi)} \mathcal{H}^{\phi}_{i,\cdot}.$$

We assume that $\mathcal{H} \neq L^2(\mu_G)$, and show that this leads to a contradiction (so that $\mathcal{H} = L^2(\mu_G)$ and $\mathcal{B}$ must be a basis). First, clearly $\mathcal{H}$ is $\pi_R$-invariant. By our assumption, $\mathcal{H}^\perp$ is a non-trivial $\pi_R$-invariant closed subspace. Since $\pi_R|_{\mathcal{H}^\perp}$ is a direct sum of irreducible unitary representations, there exists a non-trivial subspace $E \subset \mathcal{H}^\perp$ and a unitary matrix representation $\phi = (\phi_{ij})_{i,j=1}^m \in \mathrm{HOM}(G, \mathrm{U}(m))$ such that $\phi \sim \pi_R|_E$. The subspace $E$ has an orthonormal basis $\{f_j\}_{j=1}^m$ such that

$$\pi_R(y) f_j = \sum_{i=1}^m \phi_{ij}(y) f_i$$

for all $y \in G$ and $j \in \{1, \ldots, m\}$. Notice that $f_j \in L^2(\mu_G)$ has pointwise values perhaps only $\mu_G$-almost everywhere, so that

$$f_j(xy) = \sum_{i=1}^m \phi_{ij}(y) f_i(x)$$

may hold for only $\mu_G$-almost every $x \in G$. Let us define measurable sets

$$N(y) := \left\{ x \in G : \; f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\},$$

$$M(x) := \left\{ y \in G : \; f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\},$$

$$K := \left\{ (x,y) \in G \times G : \; f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\}.$$

By Exercise 7.4.27, we may utilise the Fubini Theorem to change the order of integration, to get

$$\begin{aligned}
\int_G \mu_G(M(x)) \, \mathrm{d}\mu_G(x) &= \mu_{G\times G}(K) \\
&= \int_G \mu_G(N(y)) \, \mathrm{d}\mu_G(y) \\
&= \int_G 0 \, \mathrm{d}\mu_G(y) \\
&= 0,
\end{aligned}$$

meaning that $\mu_G(M(x)) = 0$ for almost every $x \in G$. But it is enough to pick just one $x_0 \in G$ such that $\mu_G(M(x_0)) = 0$. Then

$$f_j(x_0 y) = \sum_{i=1}^{m} \phi_{ij}(y) f_i(x_0)$$

for $\mu_G$-almost every $y \in G$. If we denote $z := x_0 y$ then

$$\begin{aligned} f_j(z) &= \sum_{i=1}^{m} \phi_{ij}(x_0^{-1} z) f_i(x_0) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m} \phi_{ik}(x_0^{-1}) \phi_{kj}(z) f_i(x_0) \\ &= \sum_{k=1}^{m} \phi_{kj}(z) \sum_{i=1}^{m} \phi_{ik}(x_0^{-1}) f_i(x_0) \end{aligned}$$

for $\mu_G$-almost every $z \in G$. Hence

$$f_j \in \operatorname{span}\{\phi_{kj}\}_{k=1}^{m} \subset \bigoplus_{k=1}^{m} \mathcal{H}^{\phi}_{k,\cdot} \subset \mathcal{H}$$

for all $j \in \{1, \ldots, m\}$. Thereby

$$E = \operatorname{span}\{f_j\}_{j=1}^{m} \subset \mathcal{H};$$

at the same time $E \subset \mathcal{H}^{\perp}$, yielding $E = \{0\}$. This is a contradiction, since $E$ should be non-trivial. Hence $\mathcal{H} = L^2(\mu_G)$ and $\mathcal{B}$ is a basis. □

**Exercise 7.5.18.** Check the details of the proof of the Peter–Weyl Theorem. In particular, pay attention to verify the conditions for applying the Fubini Theorem.

## 7.6 Fourier series and trigonometric polynomials

The classical Fourier series express a periodic function as an infinite sum of elementary waves that behave well under translations. This can be viewed as a special case of a more general phenomenon: a function on a compact group admits an analogous series expansion, thanks to the Peter–Weyl Theorem 7.5.14. We start by discussing the trigonometric polynomials because they play an important role as finite linear combinations of the basis elements of $L^2(\mu_G)$ provided by the Peter–Weyl theorem.

**Definition 7.6.1 (Trigonometric polynomials on groups).** Let $G$ be a compact group and

$$\mathcal{B} := \left\{ \sqrt{\dim(\phi)}\phi_{ij} \mid \phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)},\ [\phi] \in \widehat{G} \right\}$$

as in the Peter–Weyl Theorem 7.5.14. The space of *trigonometric polynomials* on $G$ is

$$\mathrm{TrigPol}(G) = \mathrm{span}(\mathcal{B}).$$

For instance, $f \in \mathrm{TrigPol}(\mathbb{T}^n)$ is of the form

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) \mathrm{e}^{\mathrm{i}2\pi x \cdot \xi},$$

where $\widehat{f}(\xi) \neq 0$ for only finitely many $\xi \in \mathbb{Z}^n$, see Remark 3.1.26. In the case of the torus the following density statement was already verified in the proof of Theorem 3.1.20.

**Theorem 7.6.2 (Density I).** $\mathrm{TrigPol}(G)$ *is a dense subalgebra of* $C(G)$.

*Proof.* It is enough to verify that $\mathrm{TrigPol}(G)$ is an involutive subalgebra of $C(G)$, because the Stone–Weierstrass Theorem A.14.4 provides then the density. We already know that $\mathrm{TrigPol}(G)$ is a subspace of $C(G)$.

First, $\phi = (x \mapsto (1)) \in \mathrm{Hom}(G, \mathrm{U}(1))$ is a continuous irreducible unitary representation, so that $\mathbf{1} = (x \mapsto 1) \in C(G)$ belongs to $\mathcal{B} \subset \mathrm{TrigPol}(G)$.

Let $[\phi] \in \widehat{G}$, $\phi = (\phi_{ij})_{i,j=1}^m$. Then $[\overline{\phi}] \in \widehat{G}$, where $\overline{\phi} = (\overline{\phi_{ij}})_{i,j=1}^m$, as it is easy to verify. Thereby we get the involutivity: $\overline{f} \in \mathrm{TrigPol}(G)$ whenever $f \in \mathrm{TrigPol}(G)$.

Let $[\psi] \in \widehat{G}$, $\psi = (\psi_{kl})_{k,l=1}^n$. Then $\phi \otimes \psi|_G = (x \mapsto \phi(x) \otimes \psi(x)) \in \mathrm{Hom}(G, \mathcal{U}(\mathbb{C}^m \otimes \mathbb{C}^n))$. Let $\{e_i\}_{i=1}^m \subset \mathbb{C}^m$ and $\{f_k\}_{k=1}^n \subset \mathbb{C}^n$ be orthonormal bases. Then $\{e_i \otimes f_k \mid 1 \leq i \leq m,\ 1 \leq k \leq n\}$ is an orthonormal basis for $\mathbb{C}^m \otimes \mathbb{C}^n$, and the $(ik)(jl)$-matrix element of $\phi \otimes \psi|_G$ is calculated by

$$\begin{aligned} (\phi \otimes \psi|_G)_{(ik)(jl)}(x) &= \langle (\phi \otimes \psi|_G)(x)(e_j \otimes f_l), e_i \otimes f_k \rangle_{\mathbb{C}^m \otimes \mathbb{C}^n} \\ &= \langle \phi(x) e_j, e_i \rangle_{\mathbb{C}^m} \ \langle \psi(x) f_l, f_k \rangle_{\mathbb{C}^n} \\ &= \phi_{ij}(x) \psi_{kl}(x). \end{aligned}$$

Hence $\phi_{ij}\psi_{kl}$ is a matrix element of $\phi \otimes \psi|_G$. Representation $\phi \otimes \psi|_G$ can be decomposed as a finite direct sum of irreducible unitary representations. Hence the matrix elements of $\phi \otimes \psi|_G$ can be written as linear combinations of elements of $\mathcal{B}$. Thus $\phi_{ij}\psi_{kl} \in \mathrm{TrigPol}(G)$, so that $fg \in \mathrm{TrigPol}(G)$ for all $f, g \in \mathrm{TrigPol}(G)$. □

**Corollary 7.6.3 (Density II).** $\mathrm{TrigPol}(G)$ *is dense in* $L^2(\mu_G)$.

*Remark* 7.6.4. Notice that we did not need the Peter–Weyl Theorem 7.5.14 to show that $\mathrm{TrigPol(G)} \subset L^2(\mu_G)$ is dense. Therefore this density provides another proof for the Peter–Weyl Theorem 7.5.14.

*Remark* 7.6.5. By now, we have encountered plenty of translation- and inversion-invariant function spaces on $G$. For instance, $\mathrm{TrigPol}(G)$, $C(G)$ and $L^p(G)$, and more: namely, if $[\phi] \in \widehat{G}$, $\phi = (\phi_{ij})_{i,j=1}^m$, then

$$\pi_L(y)\phi_{i_0 j_0}, \pi_R(y)\phi_{i_0 j_0} \in \mathrm{span}\{\phi_{ij}\}_{i,j=1}^m$$

for all $y \in G$ (and inversion-invariance is clear!).

**Exercise 7.6.6.** Prove that $f \in C(G)$ is a trigonometric polynomial if and only if

$$\dim\left(\operatorname{span}\{\pi_R(y)f:\ y \in G\}\right) < \infty.$$

As a direct consequence of knowing the basis of $L^2(\mu_G)$ by the Peter–Weyl theorem, we obtain:

**Corollary 7.6.7** (**Fourier series and Plancherel (matrix form)).** *On a compact group $G$, a Fourier series presentation of $f \in L^2(\mu_G)$ is given by*

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} \langle f, \phi_{ij}\rangle_{L^2(\mu_G)}\ \phi_{ij}, \tag{7.1}$$

*where we pick just one unitary matrix representation $\phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)}$ from each equivalence class $[\phi] \in \widehat{G}$. Moreover, there is the* Plancherel identity (*sometimes called the* Parseval identity)

$$\|f\|_{L^2(\mu_G)} = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} |\langle f, \phi_{ij}\rangle_{L^2(\mu_G)}|^2. \tag{7.2}$$

*Remark* 7.6.8. In $L^2(\mu_G)$, also clearly

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} \langle f, \overline{\phi_{ij}}\rangle_{L^2(\mu_G)}\ \overline{\phi_{ij}}.$$

A nice thing about the Fourier series is that the basis functions $\phi_{ij}$ and $\overline{\phi_{ij}}$ are well-behaved under translations and inversions.

**Definition 7.6.9** (**Fourier coefficients and Fourier transform).** Let $G$ be a compact group, $f \in L^1(\mu_G)$ and $\phi = (\phi_{ij})_{i,j=1}^m$, $[\phi] \in \widehat{G}$. The *$\phi$-Fourier coefficient of $f$* is

$$\widehat{f}(\phi) := \int_G f(x)\ \phi(x)^*\ \mathrm{d}x \in \mathbb{C}^{m\times m},$$

where the integration of the matrix-valued function is element-wise. The matrix-valued function $\widehat{f}$ is called the *Fourier transform* of $f \in L^1(\mu_G)$. We note that this definition immediately extends also to $L^2(\mu_G)$ in view of $L^2(\mu_G) \subset L^1(\mu_G)$, which follows e.g. by the Hölder's inequality from the compactness of $G$.

**Corollary 7.6.10** (**Fourier series and Plancherel).** *On a compact group $G$, a Fourier series presentation of $f \in L^2(\mu_G)$ is given by*

$$f(x) = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(\widehat{f}(\phi)\ \phi(x)\right) \tag{7.3}$$

*converging for $\mu_G$-almost every $x \in G$, as well as in $L^2(\mu_G)$. The Plancherel identity takes the form*

$$\|f\|^2_{L^2(\mu_G)} = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(\widehat{f}(\phi)\ \widehat{f}(\phi)^*\right).$$

*If $f, g \in L^2(G)$, then we also have the Parseval's identity*

$$(f,g)_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\Big(\widehat{f}(\phi)\ \widehat{g}(\phi)^*\Big) = (\widehat{f}(\phi), \widehat{g}(\phi))_{L^2(\widehat{G})},$$

*with $L^2(\widehat{G})$ as in Definition 7.6.11.*

*Proof.* Now

$$\widehat{f}(\phi)_{ij} = \int_G f(x)\ \left(\phi(x)^*\right)_{ij}\ \mathrm{d}\mu_G(x) = \int_G f(x)\ \overline{\phi_{ji}(x)}\ \mathrm{d}x = \langle f, \phi_{ji}\rangle_{L^2(\mu_G)},$$

so that

$$\begin{aligned}
\mathrm{Tr}\left(\widehat{f}(\phi)\ \phi(x)\right) &= \sum_{i=1}^{\dim(\phi)} \left(\widehat{f}(\phi)\ \phi(x)\right)_{ii} \\
&= \sum_{i,j=1}^{\dim(\phi)} \widehat{f}(\phi)_{ij}\ \phi_{ji}(x) \\
&= \sum_{i,j=1}^{\dim(\phi)} \langle f, \phi_{ji}\rangle_{L^2(\mu_G)}\ \phi_{ji}(x).
\end{aligned}$$

Hence (7.3) follows from (7.1). Finally, if $A = (A_{kl})_{k,l=1}^m \in \mathbb{C}^{m\times m}$ then

$$\|A\|^2_{HS} := \mathrm{Tr}(A^*A) = \sum_{k,l=1}^m |A_{kl}|^2,$$

completing the proof, if we take $A = \widehat{f}(\phi)$ and use (7.2). The details of the proof of the Parceval's identity will be given in Proposition 10.3.17. □

The convergence in $L^2(\mu_G)$ is automatic, see Theorem B.5.32.

**Definition 7.6.11** (**Hilbert space** $L^2(\widehat{G})$)**.** Let $G$ be a compact group. Let $L^2(\widehat{G})$ be the space containing the mappings

$$F : \widehat{G} \to \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m}$$

satisfying $F([\phi]) \in \mathbb{C}^{\dim(\phi)\times\dim(\phi)}$ such that

$$\sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \|F([\phi])\|^2_{\mathbb{C}^{\dim(\phi)\times\dim(\phi)}} < \infty.$$

Then $L^2(\widehat{G})$ is a Hilbert space with the inner product

$$\langle E, F\rangle_{L^2(\widehat{G})} := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \langle E([\phi]), F([\phi])\rangle_{\mathbb{C}^{\dim(\phi)\times\dim(\phi)}}.$$

**Exercise 7.6.12.** Verify that $L^2(\widehat{G})$ is indeed a Hilbert space.

**Theorem 7.6.13** (**Fourier transform is an isometry** $L^2(\mu_G) \to L^2(\widehat{G})$**).** *Let $G$ be a compact group. The Fourier transform $f \mapsto \widehat{f}$ defines a surjective isometry $L^2(\mu_G) \to L^2(\widehat{G})$.*

*Proof.* Let us choose one unitary matrix representation $\phi$ from each $[\phi] \in \widehat{G}$. If we define $F([\phi]) := \widehat{f}(\phi)$ then $F \in L^2(\widehat{G})$, and $f \mapsto F$ is isometric by the Plancherel equality.

Now take any $F \in L^2(\widehat{G})$. We have to show that $F([\phi]) = \widehat{f}(\phi)$ for some $f \in L^2(\mu_G)$, where $\phi \in [\phi] \in \widehat{G}$. Define

$$f(x) := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(F([\phi])\ \phi(x)\right)$$

for $\mu_G$-almost every $x \in G$. This can be done, since

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} F([\phi])_{ij}\ \phi_{ji}$$

belongs to $L^2(\mu_G)$ by

$$\|f\|^2_{L^2(\mu_G)} = \|F\|^2_{L^2(\widehat{G})} < \infty.$$

Clearly $\widehat{f}(\phi) = F([\phi])$, so that the Fourier transform is surjective. □

We will return to a more detailed analysis of the space $L^2(\widehat{G})$ and of other spaces of functions and distributions on the unitary dual $\widehat{G}$ in Section 10.3.

## 7.7 Convolutions

For functions $f$ and $g$ on a group, their convolution $f * g$ can be thought as a modulation of one with the other. More precisely, the Fourier coefficients of $f * g$ are the pointwise products of the Fourier coefficients of $f$ and $g$, as presented in Proposition 7.7.5.

**Definition 7.7.1 (Convolutions on compact groups).** Let $G$ be a compact group, and let $f \in L^1(\mu_G)$ and $g \in C(G)$ (or $f \in C(G)$ and $g \in L^1(\mu_G)$). The *convolution* $f * g : G \to \mathbb{C}$ is defined by

$$f * g(x) := \int_G f(y)\ g(y^{-1}x)\ \mathrm{d}y.$$

*Remark* 7.7.2. Now $f * g \in C(G)$. Indeed, due to the uniform continuity, for each $\varepsilon > 0$ there exists open $U \ni e$ such that $|g(x) - g(z)| < \varepsilon$ when $z^{-1}x \in U$. Thereby

$$\begin{aligned} |f * g(x) - f * g(z)| &\leq \int_G |f(y)|\ |g(y^{-1}x) - g(y^{-1}z)|\ \mathrm{d}y \\ &\leq \|f\|_{L^1(\mu_G)}\ \varepsilon, \end{aligned}$$

when $z^{-1}x \in U$. Furthermore, the linear mapping $g \mapsto f * g$ satisfies

$$\begin{aligned} \|f * g\|_{C(G)} &\leq \|f\|_{L^1(\mu_G)}\ \|g\|_{C(G)}, \\ \|f * g\|_{C(G)} &\leq \|f\|_{C(G)}\ \|g\|_{L^1(\mu_G)}, \\ \|f * g\|_{L^1(\mu_G)} &\leq \|f\|_{L^1(\mu_G)}\ \|g\|_{L^1(\mu_G)}. \end{aligned}$$

Hence we can consider $g \mapsto f * g$ as a bounded operator on $C(G)$ and $L^1(\mu_G)$; of course, we have symmetrical results for $g \mapsto g * f$.

It is also easy to show other $L^p$-boundedness results, like

$$\|f * g\|_{L^\infty(\mu_G)} \leq \|f\|_{L^2(\mu_G)}\ \|g\|_{L^2(\mu_G)}$$

and so on. Notice that the convolution product is commutative if and only if $G$ is commutative.

**Proposition 7.7.3.** *Let $f, g, h \in L^1(\mu_G)$. Then $f * g \in L^1(\mu_G)$,*

$$\|f * g\|_{L^1(\mu_G)} \leq \|f\|_{L^1(\mu_G)}\ \|g\|_{L^1(\mu_G)},$$

*and $f * g(x) = \int_G f(y^{-1})\ g(yx)\ \mathrm{d}y$ for almost every $x \in G$. Moreover, for $\mu_G$-almost every $x \in G$,*

$$\begin{aligned} f * g(x) &= \int_G f(xy^{-1})\ g(y)\ \mathrm{d}y \\ &= \int_G f(y^{-1})\ g(yx)\ \mathrm{d}y \\ &= \int_G f(xy)\ g(y^{-1})\ \mathrm{d}y. \end{aligned}$$

*The convolution product is also associative: $f * (g * h) = (f * g) * h$.*

**Exercise 7.7.4.** Prove Proposition 7.7.3.

**Proposition 7.7.5.** *For $f, g \in L^1(\mu_G)$ it holds $\widehat{f * g}(\phi) = \widehat{g}(\phi)\ \widehat{f}(\phi)$.*

*Proof.* It is enough to assume that $f, g \in C(G)$. Then

$$\begin{aligned}
\widehat{f * g}(\phi) &= \int_G f * g(x)\ \phi(x)^*\ \mathrm{d}x \\
&= \int_G \int_G f(y)\ g(y^{-1}x)\ \mathrm{d}y\ \phi(x)^*\ \mathrm{d}x \\
&= \int_G \int_G g(y^{-1}x)\ \phi(y^{-1}x)^*\ \mathrm{d}x\ f(y)\ \phi(y)^*\ \mathrm{d}y \\
&= \int_G g(z)\ \phi(z)^*\ \mathrm{d}z \int_G f(y)\ \phi(y)^*\ \mathrm{d}y \\
&= \widehat{g}(\phi)\ \widehat{f}(\phi),
\end{aligned}$$

completing the proof. □

*Remark* 7.7.6. There are plenty of other interesting results concerning the Fourier transform and convolutions on compact groups. For instance, one can study approximate identities for $L^1(\mu_G)$ and prove that the Fourier transform $f \mapsto \widehat{f}$ is injective on $L^1(\mu_G)$.

## 7.8 Characters

Loosely speaking, a character is the trace of a representation, and it contains all the essential information about the corresponding representation.

**Definition 7.8.1 (Characters).** Let $\phi : G \to \mathrm{Aut}(\mathcal{H})$ be a representation of a group $G$ on a finite-dimensional Hilbert space $\mathcal{H}$. The *character* of $\phi$ is the function $\chi_\phi : G \to \mathbb{C}$ defined by

$$\chi_\phi(x) := \mathrm{Tr}\left(\phi(x)\right).$$

*Remark* 7.8.2 (**Purpose of characters**). Notice that here $G$ is just any group, and that the character does not depend on the choice of the basis of $\mathcal{H}$. It turns out that on a compact group, characters provide a way of recognising equivalence of representations: namely, for finite-dimensional unitary representations, $\phi \sim \psi$ if and only if $\chi_\phi = \chi_\psi$, as we shall see.

**Proposition 7.8.3 (Properties of characters).** *Let $\phi, \psi$ be finite-dimensional representations of a group $G$. Then the following hold:*

(1) *If $\phi \sim \psi$ then $\chi_\phi = \chi_\psi$.*
(2) $\chi_\phi(xyx^{-1}) = \chi_\phi(y)$ *for all $x, y \in G$.*
(3) $\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi$.
(4) $\chi_{\phi\otimes\psi|_G} = \chi_\phi\ \chi_\psi$.
(5) $\chi_\phi(e) = \dim(\phi)$.

*Proof.* The results follow from the properties of the trace functionals, see e.g. Subsection B.5.1. □

*Remark* 7.8.4. Since the character depends only on the equivalence class of a representation, we may define $\chi_{[\phi]} := \chi_\phi$, where $[\phi]$ is the equivalence class of $\phi$.

**Proposition 7.8.5 (Orthonormality of characters).** *Let $G$ be a compact group and $\xi, \eta \in \widehat{G}$. Then*

$$\langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} = \begin{cases} 1 & \text{if } \xi = \eta, \\ 0 & \text{if } \xi \not\sim \eta. \end{cases}$$

*Proof.* Let $\phi = (\phi_{ij})_{i,j=1}^m \in \xi$ and $\psi = (\psi_{kl})_{k,l=1}^n \in \eta$. Then

$$\begin{aligned} \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} &= \sum_{j=1}^m \sum_{k=1}^n \langle \phi_{jj}, \psi_{kk} \rangle_{L^2(\mu_G)} \\ &= \begin{cases} 0 & \text{if } \phi \not\sim \psi, \\ 1 & \text{if } \phi = \psi \end{cases} \end{aligned}$$

by Lemma 7.5.12. □

**Theorem 7.8.6 (Irreducibility and equivalence characterisations).** *Let $\phi, \psi$ be finite-dimensional continuous unitary representations of a compact group $G$. Then $\phi$ is irreducible if and only if $\|\chi_\phi\|_{L^2(\mu_G)} = 1$. Moreover, $\phi \sim \psi$ if and only if $\chi_\phi = \chi_\psi$.*

*Proof.* We already know the "only if"-parts of the proof. So suppose $\phi$ is a finite-dimensional unitary representation. Then

$$\phi \sim \bigoplus_{[\xi] \in \widehat{G}} \bigoplus_{j=1}^{m_{[\xi]}} \xi,$$

where $m_{[\xi]} \in \mathbb{N}$ is non-zero for only finitely many $[\xi] \in \widehat{G}$, and with the convention that the empty sum gives zero. Then

$$\chi_\phi = \sum_{[\xi] \in \widehat{G}} m_{[\xi]} \chi_\xi,$$

and if $[\eta] \in \widehat{G}$ then

$$\langle \chi_\phi, \chi_\eta \rangle_{L^2(\mu_G)} = \sum_{[\xi] \in \widehat{G}} m_{[\xi]} \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} = m_{[\eta]}.$$

This implies that the multiplicities $m_{[\xi]} \in \mathbb{N}$ can be uniquely obtained by knowing only $\chi_\phi$; hence if $\chi_\phi = \chi_\psi$ then $\phi \sim \psi$. Moreover,

$$\begin{aligned}\|\chi_\phi\|^2_{L^2(\mu_G)} &= \langle \chi_\phi, \chi_\phi \rangle_{L^2(\mu_G)} \\ &= \sum_{[\xi],[\eta]\in\widehat{G}} m_{[\xi]}\ m_{[\eta]}\ \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} \\ &= \sum_{[\xi]\in\widehat{G}} m^2_{[\xi]},\end{aligned}$$

so that $\phi$ is irreducible if and only if $\|\chi_\phi\|_{L^2(\mu_G)} = 1$. □

**Exercise 7.8.7.** If $f \in L^2(\mu_G)$ then

$$f = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ f * \chi_\xi = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \ \mathrm{Tr}\Big(\xi(x)\ \widehat{f}(\xi)\Big).$$

Thus, the projection of $f \in L^2(G)$ to $\mathcal{H}^\xi$ is given by $f \mapsto f * \chi_\xi$. The solution of this exercise can be found in Corollary 10.11.6.

We note that the restriction of the representation and the characters to the maximal torus of the group determine them completely:

**Theorem 7.8.8** (**Cartan's maximal torus theorem**). *Let $\mathbb{T}^n \hookrightarrow G$ be an injective group homomorphism with the largest possible $n$. Then two representations $\phi$ and $\psi$ of $G$ are equivalent if and only if their restrictions to $\mathbb{T}^n$ are equivalent. In particular, the restriction $\chi_\phi|_{\mathbb{T}^n}$ of $\chi_\phi$ to $\mathbb{T}^n$ determines the class $[\phi]$.*

*Remark* 7.8.9 (**Tensor products of representations**). According to Proposition 7.8.3, (4), we have $\chi_{\phi\otimes\psi|_G} = \chi_\phi\ \chi_\psi$ for any two finite-dimensional representations $\phi$ and $\psi$ of $G$. By Theorem 7.5.5 the representation $\phi\otimes\psi|_G = (x \mapsto \phi(x)\otimes\psi(x)) \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}_\phi \otimes \mathcal{H}_\psi))$ can be decomposed as a direct sum of irreducible unitary representations:

$$\phi \otimes \psi|_G = \bigoplus_{[\xi]\in\widehat{G}} \bigoplus_{1}^{m_{\phi,\psi}([\xi])} \xi,$$

where $m_{\phi,\psi}([\xi])$ is the multiplicity of $[\xi]$ in $\phi \otimes \psi|_G$, and only finitely many of $m_{\phi,\psi}([\xi])$ are non-zero in view of the finite dimensionality. We also have

$$\chi_\phi\ \chi_\psi = \chi_{\phi\otimes\psi|_G} = \sum_{[\xi]\in\widehat{G}} m_{\phi,\psi}([\xi])\ \chi_\xi$$

in view of Proposition 7.8.3, (3). The multiplicities $m_{\phi,\psi}([\xi])$ can be analysed using Theorem 7.8.8 because we have, in particular, $\chi_\phi|_{\mathbb{T}^n}\ \chi_\psi|_{\mathbb{T}^n} = \chi_{\phi\otimes\psi|_{\mathbb{T}^n}} = \sum_{[\xi]\in\widehat{G}} m_{\phi,\psi}([\xi])\ \chi_\xi|_{\mathbb{T}^n}$.

## 7.9 Induced representations

A group representation trivially gives a representation of its subgroup: if $H < G$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ then the restriction

$$\mathrm{Res}_H^G \psi := (h \mapsto \psi(h)) \in \mathrm{Hom}(H, \mathrm{Aut}(V)). \tag{7.4}$$

In the sequel, we show how a representation of a subgroup sometimes *induces* a representation for the whole group. This induction process has also plenty of nice properties. Induced representations were defined and studied by Ferdinand Georg Frobenius in 1898 for finite groups, and by George Mackey in 1949 for (most of the) locally compact groups.

The technical assumptions here are that $G$ is a *compact group*, $H < G$ is *closed* and $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$ is *strongly continuous*; then $\phi$ induces a strongly continuous unitary representation

$$\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}_\phi^G \mathcal{H})\right),$$

where the notation will be explained in the sequel. We start by a lengthy definition of the induced representation space $\mathrm{Ind}_\phi^G \mathcal{H}$.

*Remark* 7.9.1 (**Uniformly continuous Hilbert space valued mappings**). Since $G$ is a compact group, continuous mappings $G \to \mathcal{H}$ are *uniformly continuous* in the following sense: Let $f \in C(G, \mathcal{H})$ and $\varepsilon > 0$. Then there exists open $U \ni e$ such that $\|f(x) - f(y)\|_{\mathcal{H}} < \varepsilon$ when $xy^{-1} \in U$ (or $x^{-1}y \in U$); the proof of this fact is as in the scalar-valued case. We shall also need to integrate $\mathcal{H}$-valued functions in the weak sense: that is, we need the concept of the Pettis integral, the details of which can be found from exercises related to Definition B.3.28 (see also Remark 7.9.3).

**Proposition 7.9.2.** *If $f \in C(G, \mathcal{H})$ then $f_\phi \in C(G, \mathcal{H})$, where*

$$f_\phi(x) := \int_H \phi(h) f(xh) \, \mathrm{d}\mu_H(h), \tag{7.5}$$

*defined in the weak sense as the Pettis integral. Moreover, we have $f_\phi(xh) = \phi(h)^* f_\phi(x)$ for all $x \in G$ and $h \in H$.*

*Remark* 7.9.3 (**Pettis integral**). The weak (Pettis) integration in (7.5) means that for every $f \in C(G, \mathcal{H})$ there exists a unique $f_\phi \in C(G, \mathcal{H})$ such that for all $u \in \mathcal{H}' = \mathcal{H}$ we have

$$\langle u, f_\phi \rangle_{\mathcal{H}} = \int_H \langle u, \phi(h) f(xh) \rangle_{\mathcal{H}} \, \mathrm{d}\mu_H(h).$$

We denote this $f_\phi$ as weak integral (7.5). The Riesz Representation Theorem B.5.19 gives the correctness of this integral definition since $f_\phi$ is clearly a bounded linear functional acting on $u \in \mathcal{H}$. For a more general version of the Pettis integral we refer to Definition B.3.28.

*Proof of Proposition 7.9.2.* Let $\{e_j\}_{j\in J} \subset \mathcal{H}$ be an orthonormal basis. Then

$$f_\phi(x) = \sum_{j\in J} \langle f_\phi(x), e_j\rangle_{\mathcal{H}} e_j \in \mathcal{H}$$

is the unique vector defined by inner products

$$\langle f_\phi(x), e_j\rangle_{\mathcal{H}} = \int_H \langle \phi(h) f(xh), e_j\rangle_{\mathcal{H}} \ \mathrm{d}\mu_H(h).$$

It is easy to prove that the integrals here are sound, since

$$(h \mapsto \langle \phi(h) f(xh), e_j\rangle_{\mathcal{H}}) \in C(H)$$

because $f \in C(G, \mathcal{H})$ and $\phi$ is strongly continuous. If $h_0 \in H$ then

$$\begin{aligned} f_\phi(xh_0) &= \int_H \phi(h)\ f(xh_0h)\ \mathrm{d}\mu_H(h) \\ &= \int_H \phi(h_0^{-1}h) f(xh)\ \mathrm{d}\mu_H(h) \\ &= \phi(h_0)^* f_\phi(x). \end{aligned}$$

Take $\varepsilon > 0$. By the uniform continuity of $f \in C(G, \mathcal{H})$ mentioned in Remark 7.9.1, there exists an open set $U \ni e$ such that $\|f(a) - f(b)\|_{\mathcal{H}} < \varepsilon$ whenever $ab^{-1} \in U$. If $x \in yU$ then

$$\begin{aligned} \|f_\phi(x) - f_\phi(y)\|_{\mathcal{H}}^2 &= \left\| \int_H \phi(h)(f(xh) - f(yh))\ \mathrm{d}\mu_H(h) \right\|_{\mathcal{H}}^2 \\ &\le \left( \int_H \|f(xh) - f(yh)\|_{\mathcal{H}}\ \mathrm{d}\mu_H(h) \right)^2 \\ &\le \varepsilon^2, \end{aligned}$$

proving the continuity of $f_\phi$. □

**Lemma 7.9.4.** *If $f, g \in C(G, \mathcal{H})$ then $(xH \mapsto \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}) \in C(G/H)$.*

*Proof.* Let $x \in G$ and $h \in H$. Then

$$\begin{aligned} \langle f_\phi(xh), g_\phi(xh)\rangle_{\mathcal{H}} &= \langle \phi(h)^* f_\phi(x), \phi(h)^* g_\phi(x)\rangle_{\mathcal{H}} \\ &= \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}, \end{aligned}$$

so that $(xH \mapsto \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}) : G/H \to \mathbb{C}$ is well-defined. There exists a constant $C < \infty$ such that $\|f_\phi(y)\|_{\mathcal{H}}, \|g_\phi(x)\|_{\mathcal{H}} \le C$ because $G$ is compact and $f_\phi, g_\phi \in C(G, \mathcal{H})$. Thereby

$$\begin{aligned} & |\langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}} - \langle f_\phi(y), g_\phi(y)\rangle_{\mathcal{H}}| \\ \le\ & |\langle f_\phi(x) - f_\phi(y), g_\phi(x)\rangle_{\mathcal{H}}| + |\langle f_\phi(y), g_\phi(x) - g_\phi(y)\rangle_{\mathcal{H}}| \\ \le\ & C\left(\|f_\phi(x) - f_\phi(y)\|_{\mathcal{H}} + \|g_\phi(x) - g_\phi(y)\|_{\mathcal{H}}\right) \\ \xrightarrow[x\to y]{}\ & 0 \end{aligned}$$

by the continuities of $f_\phi$ and $g_\phi$. □

**Definition 7.9.5** (**Induced representation space** $\mathrm{Ind}_\phi^G \mathcal{H}$). Let us endow the vector space

$$\begin{aligned} C_\phi(G,\mathcal{H}) &:= \{f_\phi \mid f \in C(G,\mathcal{H})\} \\ &= \{e \in C(G,\mathcal{H}) \mid \forall x \in G\ \forall h \in H:\ e(xh) = \phi(h)^* e(x)\} \end{aligned}$$

with the inner product defined by

$$\langle f_\phi, g_\phi \rangle_{\mathrm{Ind}_\phi^G \mathcal{H}} := \int_{G/H} \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}\ \mathrm{d}\mu_{G/H}(xH).$$

Let $\mathrm{Ind}_\phi^G \mathcal{H}$ be the completion of $C_\phi(G,\mathcal{H})$ with respect to the corresponding norm

$$f_\phi \mapsto \|f_\phi\|_{\mathrm{Ind}_\phi^G \mathcal{H}} := \sqrt{\langle f_\phi, f_\phi \rangle_{\mathrm{Ind}_\phi^G \mathcal{H}}};$$

this Hilbert space is called the *induced representation space*.

*Remark* 7.9.6. If $\mathcal{H} \neq \{0\}$ then $\{0\} \neq C_\phi(G,\mathcal{H}) \subset \mathrm{Ind}_\phi^G \mathcal{H}$. Why? Let $0 \neq u \in \mathcal{H}$. Due to the strong continuity of $\phi$, we can choose open $U \subset G$ such that $e \in U$ and $\|(\phi(h)-\phi(e))u\|_{\mathcal{H}} < \|u\|_{\mathcal{H}}$ for all $h \in H \cap U$. Choose $w \in C(G)$ such that $w \geq 0$, $w|_{G\setminus U} = 0$ and $\int_H w(h)\ \mathrm{d}\mu_H(h) = 1$. Let $f(x) := w(x)u$ for all $x \in G$. Then

$$\begin{aligned} \|f_\phi(e) - u\|_{\mathcal{H}} &= \left\| \int_H w(h)\ (\phi(h)-\phi(e))u\ \mathrm{d}\mu_H(h) \right\|_{\mathcal{H}} \\ &= \int_H w(h)\ \|(\phi(h)-\phi(e))u\|_{\mathcal{H}}\ \mathrm{d}\mu_H(h) \\ &< \|u\|_{\mathcal{H}}, \end{aligned}$$

so that $f_\phi(e) \neq 0$, yielding $f_\phi \neq 0$.

**Theorem 7.9.7** (**Induced representations**). *If $x, y \in G$ and $f_\phi \in C_\phi(G,\mathcal{H})$, let*

$$\left(\mathrm{Ind}_H^G \phi(y) f_\phi\right)(x) := f_\phi(y^{-1}x).$$

*This begets a unique strongly continuous $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}_\phi^G \mathcal{H})\right)$, called the* representation of $G$ induced by $\phi$.

*Proof.* If $y \in G$ and $f_\phi \in C_\phi(G,\mathcal{H})$ then $\mathrm{Ind}_H^G \phi(y) f_\phi = g_\phi \in C_\phi(G,\mathcal{H})$, where $g \in C(G,\mathcal{H})$ is defined by $g(x) := f(y^{-1}x)$. Thus we have a linear mapping $\mathrm{Ind}_H^G \phi(y) : C_\phi(G,\mathcal{H}) \to C_\phi(G,\mathcal{H})$. Clearly

$$\mathrm{Ind}_H^G \phi(yz) f_\phi = \mathrm{Ind}_H^G \phi(y)\ \mathrm{Ind}_H^G \phi(z) f_\phi.$$

Hence $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathrm{Aut}(C_\phi(G,\mathcal{H}))\right)$.

If $f, g \in C(G, \mathcal{H})$ then

$$\begin{aligned}\left\langle \mathrm{Ind}_H^G \phi(y) f_\phi, g_\phi \right\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}} &= \int_{G/H} \langle f_\phi(y^{-1}x), g_\phi(x) \rangle_{\mathcal{H}} \ \mathrm{d}\mu_{G/H}(xH) \\ &= \int_{G/H} \langle f_\phi(z), g_\phi(yz) \rangle_{\mathcal{H}} \ \mathrm{d}\mu_{G/H}(zH) \\ &= \left\langle f_\phi, \mathrm{Ind}_H^G \phi(y)^{-1} g_\phi \right\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}};\end{aligned}$$

hence we have an extension $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}_\phi^G \mathcal{H})\right)$. Next we exploit the uniform continuity of $f \in C(G, \mathcal{H})$: Let $\varepsilon > 0$. Take an open set $U \ni e$ such that $\|f(a) - f(b)\|_{\mathcal{H}} < \varepsilon$ when $ab^{-1} \in U$. Thereby, if $y^{-1}z \in U$ then

$$\begin{aligned}&\left\| \left( \mathrm{Ind}_H^G \phi(y) - \mathrm{Ind}_H^G \phi(z) \right) f_\phi \right\|_{\mathrm{Ind}_\phi^G \mathcal{H}}^2 \\ = &\int_{G/H} \left\| f_\phi(y^{-1}x) - f_\phi(z^{-1}x) \right\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{G/H}(xH) \\ \leq &\ \varepsilon^2.\end{aligned}$$

This shows the strong continuity of the induced representation. □

*Remark* 7.9.8. In the sequel, some elementary properties of induced representations are deduced. Briefly: induced representations of equivalent representations are equivalent, and induction process can be taken in stages leading to the same result modulo equivalence.

**Proposition 7.9.9.** *Let $G$ be a compact group and $H < G$ a closed subgroup. Let $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}_\phi))$ and $\psi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}_\psi))$ be strongly continuous. If $\phi \sim \psi$ then $\mathrm{Ind}_H^G \phi \sim \mathrm{Ind}_H^G \psi$.*

*Proof.* Since $\phi \sim \psi$, there is an isometric isomorphism $A \in \mathrm{Hom}(\phi, \psi)$. Then

$$(Bf_\phi)(x) := A(f_\phi(x))$$

defines a linear mapping $B : C_\phi(G, \mathcal{H}_\phi) \to C_\psi(G, \mathcal{H}_\psi)$, because if $x \in G$ and $h \in H$ then

$$\begin{aligned}(Bf_\phi)(xh) &= A(f_\phi(xh)) \\ &= A(\phi(h)^* f_\phi(x)) \\ &= A(\phi(h)^* A^* A(f_\phi(x))) \\ &= A(A^* \psi(h)^* A(f_\phi(x))) \\ &= \psi(h)^* A(f_\phi(x)) \\ &= \psi(h)^* (Bf_\phi)(x).\end{aligned}$$

Furthermore, $B$ can be extended to a unique linear isometry $C : \mathrm{Ind}_\phi^G \mathcal{H}_\phi \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$, since

$$\begin{aligned}
\|Bf_\phi\|^2_{\mathrm{Ind}_\psi^G \mathcal{H}_\psi} &= \int_{G/H} \|(Bf_\phi)(x)\|^2_{\mathcal{H}_\psi} \, \mathrm{d}\mu_{G/H}(xH) \\
&= \int_{G/H} \|A(f_\phi(x))\|^2_{\mathcal{H}_\psi} \, \mathrm{d}\mu_{G/H}(xH) \\
&= \int_{G/H} \|f_\phi(x)\|^2_{\mathcal{H}_\phi} \, \mathrm{d}\mu_{G/H}(xH) \\
&= \|f_\phi\|^2_{\mathrm{Ind}_\phi^G \mathcal{H}_\phi}.
\end{aligned}$$

Next, $C$ is a surjection: if $F \in C_\psi(G, \mathcal{H}_\psi)$ then $(y \mapsto A^{-1}(F(y)) \in C_\phi(G, \mathcal{H}_\phi)$ and $\left(C\left(y \mapsto A^{-1}(F(y))\right)\right)(x) = AA^{-1}(F(x)) = F(x)$, and this is enough due to the density of $C_\psi(G, \mathcal{H}_\psi)$ in $\mathrm{Ind}_\phi^G \mathcal{H}$. Finally,

$$\begin{aligned}
(C\ \mathrm{Ind}_H^G \phi(y) f_\phi)(x) &= A(\mathrm{Ind}_H^G \phi(y) f_\phi(x)) \\
&= A(f_\phi(y^{-1}x)) \\
&= (Cf_\phi)(y^{-1}x) \\
&= (\mathrm{Ind}_H^G \phi(y) C f_\phi)(x),
\end{aligned}$$

so that $C \in \mathrm{Hom}\left(\mathrm{Ind}_H^G \phi, \mathrm{Ind}_H^G \psi\right)$ is an isometric isomorphism. □

**Corollary 7.9.10.** *Let $G$ be a compact group and $H < G$ closed. Let $\phi_1$ and $\phi_2$ be strongly continuous unitary representations of $H$. Then $\mathrm{Ind}_H^G(\phi_1 \oplus \phi_2) \sim \left(\mathrm{Ind}_H^G \phi_1\right) \oplus \left(\mathrm{Ind}_H^G \phi_2\right)$.*

**Exercise 7.9.11.** Prove Corollary 7.9.10.

**Corollary 7.9.12.** $\mathrm{Ind}_H^G \phi$ *is irreducible only if $\phi$ is irreducible.*

**Exercise 7.9.13.** Let $G_1, G_2$ be compact groups and $H_1 < G_1, H_2 < G_2$ be closed. Let $\phi_1, \phi_2$ are strongly continuous unitary representations of $H_1, H_2$, respectively. Show that

$$\mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\phi_1 \otimes \phi_2) \sim \left(\mathrm{Ind}_{H_1}^{G_1} \phi_1\right) \otimes \left(\mathrm{Ind}_{H_2}^{G_2} \phi_2\right).$$

**Theorem 7.9.14** (**Inducing representations in steps**). *Let $G$ be a compact group and $H < K < G$, where $H, K$ are closed. If $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$ is strongly continuous then $\mathrm{Ind}_H^G \phi \sim \mathrm{Ind}_K^G \mathrm{Ind}_H^K \phi$.*

*Proof.* In this proof, $x \in G$, $k, k_0 \in K$ and $h \in H$. Let $\psi := \mathrm{Ind}_H^K \phi$ and $\mathcal{H}_\psi := \mathrm{Ind}_\phi^K \mathcal{H}$. Let $f_\phi \in C_\phi(G, \mathcal{H})$. Since $(k \mapsto f_\phi(xk)) : K \to \mathcal{H}$ is continuous and $f_\phi(xkh) = \phi(h)^* f_\phi(xk)$, we obtain $(k \mapsto f_\phi(xk)) \in C_\phi(K, \mathcal{H}) \subset \mathcal{H}_\psi$. Let us define $f_\phi^K : G \to \mathcal{H}_\psi$ by

$$f_\phi^K(x) := (k \mapsto f_\phi(xk)).$$

If $x \in G$ and $k_0 \in K$ then

$$\begin{aligned} f_\phi^K(xk_0)(k) &= f_\phi(xk_0k) \\ &= f_\phi^K(x)(k_0k) \\ &= \left(\psi(k_0)^* f_\phi^K(x)\right)(k), \end{aligned}$$

i.e. $f_\phi^K(xk_0) = \psi(k_0)^* f_\phi^K(x)$. Let $\varepsilon > 0$. By the uniform continuity of $f_\phi$, take open $U \ni e$ such that $\|f_\phi(a) - f_\phi(b)\|_{\mathcal{H}} < \varepsilon$ if $ab^{-1} \in U$. Thereby if $xy^{-1} \in U$ then

$$\begin{aligned} \left\|f_\phi^K(x) - f_\phi^K(y)\right\|_{\mathcal{H}_\psi}^2 &= \int_{K/H} \left\|f_\phi^K(x)(k) - f_\phi^K(y)(k)\right\|_{\mathcal{H}}^2 \, \mathrm{d}\mu_{K/H}(kH) \\ &= \int_{K/H} \|f_\phi(xk) - f_\phi(yk)\|_{\mathcal{H}}^2 \, \mathrm{d}\mu_{K/H}(kH) \\ &\leq \varepsilon^2. \end{aligned}$$

Hence $f_\phi^K \in C_\psi(G, \mathcal{H}_\psi) \subset \mathrm{Ind}_\psi^G \mathcal{H}_\psi$, so that we indeed have a mapping $(f_\phi \mapsto f_\phi^K) : C_\phi(G, \mathcal{H}) \to C_\psi(G, \mathcal{H}_\psi)$.

Next, we claim that $f_\phi \mapsto f_\phi^K$ defines a surjective linear isometry $\mathrm{Ind}_\phi^G \mathcal{H} \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$. Isometricity follows by

$$\begin{aligned} \left\|f_\phi^K\right\|_{\mathrm{Ind}_\psi^G \mathcal{H}_\psi}^2 &= \int_{G/K} \left\|f_\phi^K(x)\right\|_{\mathcal{H}_\psi}^2 \, \mathrm{d}\mu_{G/K}(xK) \\ &= \int_{G/K} \int_{K/H} \left\|f_\phi^K(x)(k)\right\|_{\mathcal{H}}^2 \, \mathrm{d}\mu_{K/H}(kH) \, \mathrm{d}\mu_{G/K}(xK) \\ &= \int_{G/K} \int_{K/H} \|f_\phi(xk)\|_{\mathcal{H}}^2 \, \mathrm{d}\mu_{K/H}(kH) \, \mathrm{d}\mu_{G/K}(xK) \\ &= \int_{G/H} \|f_\phi(x)\|_{\mathcal{H}}^2 \, \mathrm{d}\mu_{G/H}(xH) \\ &= \|f_\phi\|_{\mathrm{Ind}_\phi^G \mathcal{H}}^2 . \end{aligned}$$

How about the surjectivity? The representation space $\mathrm{Ind}_\psi^G \mathcal{H}_\psi$ is the closure of $C_\psi(G, \mathcal{H}_\psi)$, and $\mathcal{H}_\psi$ is the closure of $C_\phi(K, \mathcal{H})$. Consequently, $\mathrm{Ind}_\psi^G \mathcal{H}_\psi$ is the closure of the vector space

$$\begin{aligned} C_\psi(G, C_\phi(K, \mathcal{H})) := \{ & g \in C(G, C(K, \mathcal{H})) \mid \forall x \in G \ \forall k \in K \ \forall h \in H : \\ & g(xk) = \psi(k)^* g(x), \ g(x)(kh) = \phi(h)^* g(x)(k) \}. \end{aligned}$$

Given $g \in C_\psi(G, C_\phi(K, \mathcal{H}))$, define $f_\phi \in C_\phi(G, \mathcal{H})$ by $f_\phi(x) := g(x)(e)$. Then $f_\phi^K = g$, because

$$f_\phi^K(x)(k) = f_\phi(xk) = g(xk)(e) = \psi(k)^* g(x)(e) = g(x)(k).$$

Thus $(f_\phi \mapsto f_\phi^K) : C_\phi(G, \mathcal{H}) \to C_\psi(G, C_\phi(K, \mathcal{H}))$ is a linear isometric bijection. Hence this mapping can be extended uniquely to a linear isometric bijection $A : \mathrm{Ind}_\phi^G \mathcal{H} \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$.

Finally, $A \in \mathrm{Hom}\left(\mathrm{Ind}_H^G \phi, \mathrm{Ind}_K^G \mathrm{Ind}_H^K \phi\right)$, since

$$\begin{aligned} A\left(\mathrm{Ind}_H^G \phi(y) f_\phi(x)\right) &= A f_\phi(y^{-1}x) \\ &= f_\phi^K(y^{-1}x) \\ &= \mathrm{Ind}_K^G \psi(y) f_\phi^K(x) \\ &= \mathrm{Ind}_K^G \psi(y) A f_\phi(x). \end{aligned}$$

This completes the proof. □

**Exercise 7.9.15.** Let $H$ be a closed subgroup of a compact group $G$. Let $\phi = (h \mapsto I) \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$, where $I = (u \mapsto u) : \mathcal{H} \to \mathcal{H}$.
a) Show that $\mathrm{Ind}_\phi^G \mathcal{H} \cong L^2(G/H, \mathcal{H})$, where the $L^2(G/H, \mathcal{H})$ inner product is given by

$$\langle f_{G/H}, g_{G/H} \rangle_{L^2(G/H,\mathcal{H})} := \int_{G/H} \langle f_{G/H}(xH), g_{G/H}(xH) \rangle_{\mathcal{H}} \, \mathrm{d}\mu_{G/H}(xH),$$

when $f_{G/H}, g_{G/H} \in C(G/H, \mathcal{H})$.
b) Let $K < G$ be closed. Let $\pi_K$ and $\pi_G$ be the left regular representations of $K$ and $G$, respectively. Prove that $\pi_G \sim \mathrm{Ind}_K^G \pi_K$.

*Remark* 7.9.16 (**Multiplicity of a representation**). A fundamental result for induced representations is the *Frobenius Reciprocity Theorem* 7.9.17, stated below without a proof. Let $G$ be a compact group and $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ be strongly continuous. Let $n([\xi], \phi) \in \mathbb{N}$ denote the *multiplicity of* $[\xi] \in \widehat{G}$ *in* $\phi$, defined as follows: if $\phi = \bigoplus_{j=1}^k \phi_j$, where each $\phi_j$ is a continuous irreducible unitary representation, then

$$n([\xi], \phi) := \left| \{ j \in \{1, \ldots, k\} : \ [\phi_j] = [\xi] \} \right|.$$

That is, $n([\xi], \phi)$ is the number of times $\xi$ may occur as an irreducible component in a direct sum decomposition of $\phi$.

**Theorem 7.9.17** (**Frobenius Reciprocity Theorem**). *Let $G$ be a compact group and $H < G$ be closed. Let $\xi, \eta$ be continuous such that $[\xi] \in \widehat{G}$ and $[\eta] \in \widehat{H}$. Then*

$$n\left([\xi], \mathrm{Ind}_H^G \eta\right) = n\left([\eta], \mathrm{Res}_H^G \xi\right),$$

*where $\mathrm{Res}_H^G \xi$ is the restriction*[1] *of $\xi$ to $H$.*

[1]see (7.4) for the definition of $\mathrm{Res}_H^G \xi$.

*Example.* Let $[\xi] \in \widehat{G}$, $H = \{e\}$ and $\eta = (e \mapsto I) \in \mathrm{Hom}(H, \mathcal{U}(\mathbb{C}))$. Then $\pi_L \sim \mathrm{Ind}_H^G \eta$ by Exercise 7.9.15, and $\widehat{H} = \{[\eta]\}$, so that

$$\begin{aligned} n\left([\xi], \mathrm{Ind}_H^G \eta\right) &= n\left([\xi], \pi_L\right) \\ &\overset{\text{Peter-Weyl}}{=} \dim(\xi) \\ &= \dim(\xi)\; n\left([\eta], \eta\right) \\ &= n\left([\eta], \bigoplus_{j=1}^{\dim(\xi)} \eta\right) \\ &= n\left([\eta], \mathrm{Res}_H^G \xi\right). \end{aligned}$$

As it should be, this is in accordance with the Frobenius Reciprocity Theorem 7.9.17.

*Example.* Let $[\xi], [\eta] \in \widehat{G}$. Then by the Frobenius Reciprocity Theorem 7.9.17,

$$\begin{aligned} n\left([\xi], \mathrm{Ind}_G^G \eta\right) &= n\left([\eta], \mathrm{Res}_G^G \xi\right) \\ &= n\left([\eta], \xi\right) \\ &= \begin{cases} 1, & \text{when } [\xi] = [\eta], \\ 0, & \text{when } [\xi] \neq [\eta]. \end{cases} \end{aligned}$$

Let $\phi$ be a finite-dimensional continuous unitary representation of $G$. Then $\phi = \bigoplus_{j=1}^{k} \xi_k$, where each $\xi_k$ is irreducible. Thereby

$$\mathrm{Ind}_G^G \phi \sim \bigoplus_{j=1}^{k} \mathrm{Ind}_G^G \xi_j \sim \bigoplus_{j=1}^{k} \xi_j \sim \phi;$$

in other words, induction practically does nothing in this case.

# Chapter 8

# Linear Lie groups

In this chapter we study linear Lie groups, i.e. Lie groups which are closed subgroups of $\mathrm{GL}(n, \mathbb{C})$. But first some words about the general Lie groups:

**Definition 8.0.18 (Lie groups).** A *Lie group* is a $C^\infty$-manifold which is also a group such that the group operations are $C^\infty$-smooth.

We will be mostly interested in the non-commutative Lie groups in view of the following:

*Remark* 8.0.19 (**Commutative Lie groups**). In the introduction to Part II we mentioned that in the case of commutative groups it is sufficient to study cases of $\mathbb{T}^n$ and $\mathbb{R}^n$. Indeed, we have the following two facts:

- Any compact commutative Lie group is isomorphic to the product of a torus with a finite commutative group.
- Any connected commutative Lie group is isomorphic to the product of a torus and the Euclidean space. In other words, if $G$ is a connected commutative Lie group then $G \cong \mathbb{T}^n \times \mathbb{R}^m$ for some $n, m$.

We will not prove these facts here but refer to e.g. [20, p. 25] for further details.

**Definition 8.0.20 (Linear Lie groups).** A *linear Lie group* is a Lie group which is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$.

There is a result stating that any compact Lie group is diffeomorphic to a linear Lie group, and thereby the matrix groups are especially interesting. In fact, we have:

**Corollary 8.0.21 (Universality of unitary groups).** *Let $G$ be a compact Lie group. Then there is some $n \in \mathbb{N}$ such that $G$ is isomorphic to a subgroup of $\mathrm{U}(n)$.*

## 8.1 Exponential map

The fundamental tool for studying linear Lie groups is the matrix exponential map, treated below.

Let us endow $\mathbb{C}^n$ with the Euclidean inner product

$$(x, y) \mapsto \langle x, y\rangle_{\mathbb{C}^n} := \sum_{j=1}^{n} x_j \overline{y_j}.$$

The corresponding norm is $x \mapsto \|x\|_{\mathbb{C}^n} := \langle x, x\rangle_{\mathbb{C}^n}^{1/2}$. We identify the matrix algebra $\mathbb{C}^{n\times n}$ with $\mathcal{L}(\mathbb{C}^n)$, the algebra of linear operators $\mathbb{C}^n \to \mathbb{C}^n$. Let us endow $\mathbb{C}^{n\times n} \cong \mathcal{L}(\mathbb{C}^n)$ with the operator norm

$$Y \mapsto \|Y\|_{\mathcal{L}(\mathbb{C}^n)} := \sup_{x\in\mathbb{C}^n:\ \|x\|_{\mathbb{C}^n}\leq 1} \|Yx\|_{\mathbb{C}^n}.$$

Notice that $\|XY\|_{\mathcal{L}(\mathbb{C}^n)} \leq \|X\|_{\mathcal{L}(\mathbb{C}^n)}\|Y\|_{\mathcal{L}(\mathbb{C}^n)}$. For a matrix $X \in \mathbb{C}^{n\times n}$, the *exponential* $\exp(X) \in \mathbb{C}^{n\times n}$ is defined by the power series

$$\exp(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

where $X^0 := I$; this series converges in the Banach space $\mathbb{C}^{n\times n} \cong \mathcal{L}(\mathbb{C}^n)$, because

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left\|X^k\right\|_{\mathcal{L}(\mathbb{C}^n)} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|X\|_{\mathcal{L}(\mathbb{C}^n)}^k = \mathrm{e}^{\|X\|_{\mathcal{L}(\mathbb{C}^n)}} < \infty.$$

**Proposition 8.1.1.** *Let* $X, Y \in \mathbb{C}^{n\times n}$. *If* $XY = YX$ *then* $\exp(X+Y) = \exp(X)\exp(Y)$. *Therefore* $\exp : \mathbb{C}^{n\times n} \to \mathrm{GL}(n, \mathbb{C})$ *satisfies* $\exp(-X) = \exp(X)^{-1}$.

*Proof.* Now

$$\begin{aligned}
\exp(X+Y) &= \lim_{l\to\infty} \sum_{k=0}^{2l} \frac{1}{k!} (X+Y)^k \\
&\overset{XY=YX}{=} \lim_{l\to\infty} \sum_{k=0}^{2l} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} X^i Y^{k-i} \\
&= \lim_{l\to\infty} \left( \sum_{i=0}^{l} \frac{1}{i!} X^i \sum_{j=0}^{l} \frac{1}{j!} Y^j + \sum_{\substack{i,j:\ i+j\leq 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} X^i Y^j \right) \\
&= \lim_{l\to\infty} \left( \sum_{i=0}^{l} \frac{1}{i!} X^i \sum_{j=0}^{l} \frac{1}{j!} Y^j \right) \\
&= \exp(X)\exp(Y),
\end{aligned}$$

since the remainder term satisfies

$$\begin{aligned}\left\| \sum_{\substack{i,j:\ i+j\le 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} X^i Y^j \right\|_{\mathcal{L}(\mathbb{C}^n)} &\le \sum_{\substack{i,j:\ i+j\le 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} \|X\|_{\mathcal{L}(\mathbb{C}^n)}^i \|Y\|_{\mathcal{L}(\mathbb{C}^n)}^j \\ &\le l(l+1)\frac{1}{(l+1)!} c^{2l} \\ &\xrightarrow[l\to\infty]{} 0,\end{aligned}$$

where $c := \max\left(1, \|X\|_{\mathcal{L}(\mathbb{C}^n)}, \|Y\|_{\mathcal{L}(\mathbb{C}^n)}\right)$.

Consequently, $I = \exp(0) = \exp(X)\exp(-X) = \exp(-X)\exp(X)$, so that we get $\exp(-X) = \exp(X)^{-1}$. □

**Exercise 8.1.2.** Verify the estimates and the ranges of the summation indices in the proof of Proposition 8.1.1.

**Lemma 8.1.3.** *Let $X \in \mathbb{C}^{n\times n}$ and $P \in \mathrm{GL}(n,\mathbb{C})$. Then*

$$\begin{aligned}\exp\left(X^{\mathrm{T}}\right) &= \exp(X)^{\mathrm{T}},\\ \exp(X^*) &= \exp(X)^*,\\ \exp(PXP^{-1}) &= P\exp(X)P^{-1}.\end{aligned}$$

*Proof.* For the adjoint $X^*$,

$$\exp(X^*) = \sum_{k=0}^{\infty} \frac{1}{k!}(X^*)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(X^k)^* = \left(\sum_{k=0}^{\infty} \frac{1}{k!} X^k\right)^* = \exp(X)^*,$$

and similarly for the transpose $X^{\mathrm{T}}$. Finally,

$$\exp(PXP^{-1}) = \sum_{k=0}^{\infty} \frac{1}{k!}(PXP^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} PX^kP^{-1} = P\exp(X)P^{-1}.$$

□

**Proposition 8.1.4.** *If $\lambda \in \mathbb{C}$ is an eigenvalue of $X \in \mathbb{C}^{n\times n}$ then $\mathrm{e}^{\lambda}$ is an eigenvalue of $\exp(X)$. Consequently*

$$\det(\exp(X)) = \mathrm{e}^{\mathrm{Tr}(X)}.$$

*Proof.* Choose $P \in \mathrm{GL}(n,\mathbb{C})$ such that $Y := PXP^{-1} \in \mathbb{C}^{n\times n}$ is upper triangular; the eigenvalues of $X$ and $Y$ are the same, and for triangular matrices the eigenvalues are the diagonal elements. Since $Y^k$ is upper triangular for every $k \in \mathbb{N}$, $\exp(Y)$ is upper triangular. Moreover, $(Y^k)_{jj} = (Y_{jj})^k$, so that $(\exp(Y))_{jj} = \mathrm{e}^{Y_{jj}}$. The eigenvalues of $\exp(X)$ and $\exp(Y) = P\exp(X)P^{-1}$ are the same. The determinant of a matrix is the product of its eigenvalues; the trace of a matrix is the sum of its eigenvalues; this implies the last claim. □

*Remark* 8.1.5. Recall that $\mathrm{HOM}(G, H)$ is the set of continuous homomorphisms from $G$ to $H$, see Definition 7.1.12.

**Theorem 8.1.6** (**The form of** $\mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C}))$**).** *We have*

$$\mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C})) = \left\{ t \mapsto \exp(tX) \mid X \in \mathbb{C}^{n \times n} \right\}.$$

*Proof.* It is clear that $(t \mapsto \exp(tX)) \in \mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C}))$, since it is continuous and $\exp(sX)\exp(tX) = \exp((s+t)X)$.

Let $\phi \in \mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C}))$. Then $\phi(s+t) = \phi(s)\phi(t)$ implies that

$$\left( \int_0^h \phi(s)\ \mathrm{d}s \right) \phi(t) = \int_0^h \phi(s+t)\ \mathrm{d}s = \int_t^{t+h} \phi(u)\ \mathrm{d}u.$$

Recall that if $\|I - A\|_{\mathcal{L}(\mathbb{C}^n)} < 1$ then $A \in \mathbb{C}^{n \times n}$ is invertible; now

$$\begin{aligned} \left\| I - \frac{1}{h} \int_0^h \phi(s)\ \mathrm{d}s \right\|_{\mathcal{L}(\mathbb{C}^n)} &= \left\| \frac{1}{h} \int_0^h (I - \phi(s))\ \mathrm{d}s \right\|_{\mathcal{L}(\mathbb{C}^n)} \\ &\leq \sup_{s:\ |s| \leq |h|} \|I - \phi(s)\|_{\mathcal{L}(\mathbb{C}^n)} \\ &< 1 \end{aligned}$$

when $|h|$ is small enough, because $\phi(0) = I$ and $\phi$ is continuous. Therefore $\int_0^h \phi(s)\ \mathrm{d}s$ is invertible for small $|h|$, and we get

$$\phi(t) = \left( \int_0^h \phi(s)\ \mathrm{d}s \right)^{-1} \int_t^{t+h} \phi(u)\ \mathrm{d}u.$$

Since $\phi$ is continuous, this formula states that $\phi$ is differentiable. Now

$$\phi'(t) = \lim_{s \to 0} \frac{\phi(s+t) - \phi(t)}{s} = \lim_{s \to 0} \frac{\phi(s) - \phi(0)}{s}\ \phi(t) = X\ \phi(t),$$

where $X := \phi'(0)$. Hence the initial value problem

$$\begin{cases} \psi'(t) = X\ \psi(t), \quad \psi : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C}), \\ \psi(0) = I \end{cases}$$

has the solutions $\psi = \phi$ and $\psi = \phi_X := (t \mapsto \exp(tX))$. Define $\alpha : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ by $\alpha(t) := \phi(t)\ \phi_X(-t)$. Then $\alpha(0) = \phi(0)\ \phi_X(0) = I$ and

$$\begin{aligned} \alpha'(t) &= \phi'(t)\ \phi_X(-t) - \phi(t)\ \phi_X'(-t) \\ &= X\ \phi(t)\ \phi_X(-t) - \phi(t)\ X\ \phi_X(-t) \\ &= 0, \end{aligned}$$

since $X\ \phi(t) = \phi(t)\ X$. Thus $\alpha(t) = I$ for all $t \in \mathbb{R}$, so that $\phi = \phi_X$. □

**Proposition 8.1.7 (Logarithms).** *Let $A \in \mathbb{C}^{n\times n}$ be such that $\|I-A\|_{\mathcal{L}(\mathbb{C}^n)} < 1$. The* logarithm

$$\log(A) := -\sum_{k=1}^{\infty} \frac{1}{k}(I-A)^k$$

*is well-defined, and* $\exp(\log(A)) = A$. *Moreover, there exists $r > 0$ such that* $\log(\exp(X)) = X$ *if* $\|X\|_{\mathcal{L}(\mathbb{C}^n)} < r$.

*Proof.* Let $c := \|I-A\| < 1$ for a matrix $A \in \mathbb{C}^{n\times n}$. Then

$$\sum_{k=1}^{\infty} \frac{1}{k}\left\|(I-A)^k\right\|_{\mathcal{L}(\mathbb{C}^n)} \leq \sum_{k=1}^{\infty} \frac{1}{k}\|I-A\|_{\mathcal{L}(\mathbb{C}^n)}^k \leq \sum_{k=1}^{\infty} c^k = \frac{c}{1-c} < \infty,$$

so that $\log(A)$ is well-defined. Noticing that $I$ and $A$ commute, we have

$$\exp(\log(A)) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\sum_{l=1}^{\infty} \frac{1}{l}(I-A)^l\right)^k = A,$$

because if $|1-a| < 1$ for a *number* $a \in \mathbb{C}$, then

$$\mathrm{e}^{\ln a} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\sum_{l=1}^{\infty} \frac{1}{l}(1-a)^l\right)^k = a. \tag{8.1}$$

Due to the continuity of the exponential function, there exists $r > 0$ such that $|1-\mathrm{e}^x| < 1$ if $x \in \mathbb{C}$ satisfies $|x| < r$, and then

$$\ln(\mathrm{e}^x) = -\sum_{l=1}^{\infty} \frac{1}{l}(1-\mathrm{e}^x)^l = -\sum_{l=1}^{\infty} \frac{1}{l}\left(1-\sum_{k=0}^{\infty} \frac{1}{k!}x^k\right)^l = x, \tag{8.2}$$

so that if $X \in \mathbb{C}^{n\times n}$ satisfies $\|X\|_{\mathcal{L}(\mathbb{C})} < r$ then

$$\log(\exp(X)) = -\sum_{l=1}^{\infty} \frac{1}{l}(I-\exp(X))^l = -\sum_{l=1}^{\infty} \frac{1}{l}\left(I-\sum_{k=0}^{\infty} \frac{1}{k!}X^k\right)^l = X.$$

□

**Exercise 8.1.8.** Find an estimate for $r$ in Proposition 8.1.7.

**Exercise 8.1.9.** Justify formulae (8.1) and (8.2) and their matrix forms.

**Corollary 8.1.10.** *Let $r$ be as above and* $\mathbb{B} := \left\{X \in \mathbb{C}^{n\times n} : \|X\|_{\mathcal{L}(\mathbb{C}^n)} < r\right\}$. *Then* $(X \mapsto \exp(X)) : \mathbb{B} \to \exp(\mathbb{B})$ *is a diffeomorphism (i.e. a bijective $C^\infty$-smooth mapping).*

*Proof.* As exp and log are defined by power series, they are not just $C^\infty$-smooth but also analytic. □

**Lemma 8.1.11.** *Let* $X, Y \in \mathbb{C}^{n\times n}$. *Then*

$$\exp(X+Y) = \lim_{m\to\infty} \left(\exp(X/m)\ \exp(Y/m)\right)^m$$

*and*

$$\exp([X,Y]) = \lim_{m\to\infty} \{\exp(X/m), \exp(Y/m)\}^{m^2},$$

*where* $[X,Y] := XY - YX$ *and* $\{a,b\} := aba^{-1}b^{-1}$.

*Proof.* As $t \to 0$,

$$\begin{aligned}\exp(tX)\ \exp(tY) &= \left(I + tX + \frac{t^2}{2}X^2 + \mathcal{O}(t^3)\right) \\ &\quad\cdot \left(I + tY + \frac{t^2}{2}Y^2 + \mathcal{O}(t^3)\right) \\ &= I + t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3),\end{aligned}$$

so that

$$\begin{aligned}&\{\exp(tX), \exp(tY)\} \\ =\ & \left(I + t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3)\right) \\ &\cdot \left(I - t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3)\right) \\ =\ & I + t^2(XY - YX) + \mathcal{O}(t^3) \\ =\ & I + t^2[X,Y] + \mathcal{O}(t^3).\end{aligned}$$

Since exp is an injection in a neighbourhood of the origin $0 \in \mathbb{C}^{n\times n}$, we have

$$\exp(tX)\ \exp(tY) = \exp\left(t(X+Y) + \mathcal{O}(t^2)\right),$$

$$\{\exp(tX), \exp(tY)\} = \exp\left(t^2[X,Y] + \mathcal{O}(t^3)\right)$$

as $t \to 0$. Notice that $\exp(X)^m = \exp(mX)$ for all $m \in \mathbb{N}$. Therefore we get

$$\begin{aligned}\lim_{m\to\infty} \left(\exp(X/m)\ \exp(Y/m)\right)^m &= \lim_{m\to\infty} \exp\left(X + Y + \mathcal{O}(m^{-1})\right) \\ &= \exp(X+Y),\end{aligned}$$

$$\begin{aligned}\lim_{m\to\infty} \{\exp(X/m), \exp(Y/m)\}^{m^2} &= \lim_{m\to\infty} \exp\left([X,Y] + \mathcal{O}(m^{-1})\right) \\ &= \exp([X,Y]).\end{aligned}$$

□

## 8.2 No small subgroups for Lie, please

**Definition 8.2.1** (**"No small subgroups" property).** A topological group is said to have the "*no small subgroups*" property if there exists a neighbourhood of the neutral element containing no non-trivial subgroups.

We shall show that this property characterises Lie groups among compact groups.

*Example.* Let $\{G_j\}_{j\in J}$ be an infinite family of compact groups each having more than one element. Let us consider the compact product group $G := \prod_{j\in J} G_j$. Let

$$H_j := \{x \in G \mid \forall i \in J \setminus \{j\} : x_i = e_{G_i}\}.$$

Then $G_j \cong H_j < G$, and $H_j$ is a non-trivial subgroup of $G$. If $V \subset G$ is a neighbourhood of $e \in G$ then it contains all but perhaps finitely many $H_j$, due to the definition of the product topology. Hence in this case $G$ "has small subgroups" (i.e. has not the "no small subgroups" property).

**Theorem 8.2.2** (**Kernels of representations).** *Let $G$ be a compact group and $V \subset G$ open such that $e \in V$. Then there exists $\phi \in \mathrm{HOM}(G, \mathrm{U}(n))$ for some $n \in \mathbb{Z}^+$ such that* $\mathrm{Ker}(\phi) \subset V$.

*Proof.* First, $\{e\} \subset G$ and $G \setminus V \subset G$ are disjoint closed subsets of a compact Hausdorff space $G$. By Urysohn's Lemma (Theorem A.12.11), there exists $f \in C(G)$ such that $f(e) = 1$ and $f(G \setminus V) = \{0\}$. Since trigonometric polynomials are dense in $C(G)$ by Theorem 7.6.2, we may take $p \in \mathrm{TrigPol}(G)$ such that $\|p - f\|_{C(G)} < 1/2$. Then

$$\mathcal{H} := \mathrm{span}\{\pi_R(x)p \mid x \in G\} \subset L^2(\mu_G)$$

is a finite-dimensional vector space, and $\mathcal{H}$ inherits the inner product from $L^2(\mu_G)$. Let $A : \mathcal{H} \to \mathbb{C}^n$ be a linear isometry, where $n = \dim(\mathcal{H})$. Let us identify $\mathcal{U}(\mathbb{C}^n)$ with $\mathrm{U}(n)$. Define $\phi \in \mathrm{Hom}(G, \mathrm{U}(n))$ by

$$\phi(x) := A\ \pi_R(x)|_{\mathcal{H}}\ A^{-1}.$$

Then $\phi$ is clearly a continuous unitary representation. For every $x \in G \setminus V$,

$$|p(x) - 0| = |p(x) - f(x)| \leq \|p - f\|_{C(G)} < 1/2,$$

so that $p(x) \neq p(e)$, because

$$|p(e) - 1| = |p(e) - f(e)| \leq \|p - f\|_{C(G)} < 1/2;$$

consequently $\pi_R(x)p \neq p$. Thus $\mathrm{Ker}(\phi) \subset V$. □

**Corollary 8.2.3** (**Characterisation of linear Lie groups).** *Let $G$ be a compact group. Then $G$ has no small subgroups if and only if it is isomorphic to a linear Lie group.*

*Proof.* Let $G$ be a compact group without small subgroups. By Theorem 8.2.2, for some $n \in \mathbb{Z}^+$ there exists an injective $\phi \in \mathrm{HOM}(G, \mathrm{U}(n))$. Then $(x \mapsto \phi(x)) : G \to \phi(G)$ is an isomorphism, and a homeomorphism by Proposition A.12.7, because $\phi$ is continuous, $G$ is compact and $\mathrm{U}(n)$ is Hausdorff. Thus $\phi(G) < \mathrm{U}(n) < \mathrm{GL}(n, \mathbb{C})$ is a compact linear Lie group.

Conversely, suppose $G < \mathrm{GL}(n, \mathbb{C})$ is closed. Recall that the mapping $(X \mapsto \exp(X)) : \mathbb{B} \to \exp(\mathbb{B})$ is a homeomorphism, where

$$\mathbb{B} = \left\{ X \in \mathbb{C}^{n \times n} : \ \|X\|_{\mathcal{L}(\mathbb{C}^n)} < r \right\}$$

for some small $r > 0$. Hence $V := \exp(\mathbb{B}/2) \cap G$ is a neighbourhood of $I \in G$. In the search of a contradiction, suppose there exists a nontrivial subgroup $H < G$ such that $A \in H \subset V$ and $A \neq I$. Then $0 \neq \log(A) \in \mathbb{B}/2$, so that $m \log(A) \in \mathbb{B} \setminus (\mathbb{B}/2)$ for some $m \in \mathbb{Z}^+$. Thereby

$$\exp(m \log(A)) = \exp(\log(A))^m = A^m \in H \subset V \subset \exp(\mathbb{B}/2),$$

but also

$$\exp(m \log(A)) \in \exp(\mathbb{B} \setminus (\mathbb{B}/2)) = \exp(\mathbb{B}) \setminus \exp(\mathbb{B}/2);$$

this is a contradiction. □

*Remark* 8.2.4. Actually, it is shown above that Lie groups have no small subgroups; compactness played no role in this part of the proof.

**Exercise 8.2.5.** Use the Peter–Weyl Theorem 7.5.14 to provide an alternative proof for Theorem 8.2.2. Hint: For each $x \in G \setminus V$ there exists $\phi_x \in \mathrm{HOM}(G, \mathrm{U}(n_x))$ such that $x \notin \mathrm{Ker}(\phi_x)$, because...

## 8.3 Lie groups and Lie algebras

Next we deal with representation theory of Lie groups. We introduce Lie algebras, which sometimes still bear the archaic label "*infinitesimal groups*", quite adequately describing their essence: a Lie algebra is a sort of locally linearised version of a Lie group.

**Definition 8.3.1** (**Lie algebras**)**.** A $\mathbb{K}$-*Lie algebra* is a $\mathbb{K}$-vector space $V$ endowed with a bilinear mapping $((a, b) \mapsto [a, b]_V = [a, b]) : V \times V \to V$ satisfying

$$[a, a] = 0 \quad \text{and} \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in V$; the second identity is called the *Jacobi identity*. Notice that here $[a, b] = -[b, a]$ for all $a, b \in V$. A vector subspace $W \subset V$ of a Lie algebra $V$ is called a *Lie subalgebra* if $[a, b] \in W$ for all $a, b \in W$ (and thus $W$ is a Lie algebra in its own right). A linear mapping $A : V_1 \to V_2$ between Lie algebras $V_1, V_2$ is called a *Lie algebra homomorphism* if $[Aa, Ab]_{V_2} = A[a, b]_{V_1}$ for all $a, b \in V_1$.

*Example.* 1. For a $\mathbb{K}$-vector space $V$, the trivial Lie product $[a,b] := 0$ gives a trivial Lie algebra.

2. A $\mathbb{K}$-algebra $\mathcal{A}$ can be endowed with the canonical Lie product

$$(a,b) \mapsto [a,b] := ab - ba;$$

this Lie algebra is denoted by $\mathrm{Lie}_{\mathbb{K}}(\mathcal{A})$. Important special cases of such Lie algebras are

$$\mathrm{Lie}_{\mathbb{K}}(\mathbb{C}^{n\times n}) \cong \mathrm{Lie}_{\mathbb{K}}(\mathrm{End}(\mathbb{C}^n)), \quad \mathrm{Lie}_{\mathbb{K}}(\mathrm{End}(V)), \quad \mathrm{Lie}_{\mathbb{K}}(\mathcal{L}(X)),$$

where $X$ is a normed space and $\mathrm{End}(V)$ is the algebra of linear operators $V \to V$ on a vector space $V$. For short, let

$$\mathfrak{gl}(V) := \mathrm{Lie}_{\mathbb{R}}(\mathrm{End}(V)).$$

3. **(Derivations of algebras).** Let $\mathcal{D}(\mathcal{A})$ be the $\mathbb{K}$-vector space of *derivations* of a $\mathbb{K}$-algebra $\mathcal{A}$; that is, $D \in \mathcal{D}(\mathcal{A})$ if it is a linear mapping $\mathcal{A} \to \mathcal{A}$ satisfying the *Leibniz property*

$$D(ab) = D(a)\ b + a\ D(b)$$

for all $a, b \in \mathcal{A}$. Then $\mathcal{D}(\mathcal{A})$ has a Lie algebra structure given by $[D,E] := DE - ED$. An important special case is $\mathcal{A} = C^\infty(M)$, where $M$ is a $C^\infty$-manifold; if $C^\infty(M)$ is endowed with the local uniform convergence for all derivatives topology, then $D \in \mathcal{D}(C^\infty(M))$ is continuous if and only if it is a linear first-order partial differential operator with smooth coefficients (alternatively, a smooth vector field on $M$).

**Definition 8.3.2.** The *Lie algebra* $\mathfrak{Lie}(G) = \mathfrak{g}$ *of a linear Lie group* $G$ is introduced in the following Theorem 8.3.3:

**Theorem 8.3.3 (Lie algebras of linear Lie groups).** *Let $G < \mathrm{GL}(n,\mathbb{C})$ be closed. The $\mathbb{R}$-vector space*

$$\mathfrak{Lie}(G) = \mathfrak{g} := \left\{ X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R} :\ \exp(tX) \in G \right\}$$

*is a Lie subalgebra of the $\mathbb{R}$-Lie algebra* $\mathrm{Lie}_{\mathbb{R}}(\mathbb{C}^{n\times n}) \cong \mathfrak{gl}(\mathbb{C}^n)$.

*Proof.* Let $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Trivially, $\exp(t\lambda X) \in G$ for all $t \in \mathbb{R}$, yielding $\lambda X \in \mathfrak{g}$. Since $G$ is closed and exp is continuous,

$$\begin{aligned} G \ni (\exp(tX/m)\ \exp(tY/m))^m &\xrightarrow[m\to\infty]{} \exp\left(t(X+Y)\right) \in G \\ G \ni \{\exp(tX/m), \exp(Y/m)\}^{m^2} &\xrightarrow[m\to\infty]{} \exp\left(t[X,Y]\right) \in G \end{aligned}$$

by Lemma 8.1.11. Thereby $X+Y, [X,Y] \in \mathfrak{g}$. □

**Exercise 8.3.4.** Let $X \in \mathbb{C}^{n\times n}$ be such that $\exp(tX) = I$ for all $t \in \mathbb{R}$. Show that $X = 0$.

**Exercise 8.3.5.** Let $\mathfrak{g} \subset \mathbb{C}^{n\times n}$ be the Lie algebra of a linear Lie group $G < \mathrm{GL}(n,\mathbb{R})$. Show that $\mathfrak{g} \subset \mathbb{R}^{n\times n}$.

**Definition 8.3.6** (**Dimension of a linear Lie group).** Let $G$ be a linear Lie group and $\mathfrak{g} = \mathfrak{Lie}(G)$. The *dimension of* $G$ is $\dim(G) := \dim(\mathfrak{g}) = k$, hence $\mathfrak{g} \cong \mathbb{R}^k$ as a vector space.

*Remark* 8.3.7 (**Exponential coordinates**). From Theorem 8.1.6 it follows that

$$\mathrm{HOM}(\mathbb{R}, G) = \{t \mapsto \exp(tX) \mid X \in \mathfrak{g}\}.$$

The mapping $(X \mapsto \exp(X)) : \mathfrak{g} \to G$ is a diffeomorphism in a small neighbourhood of $0 \in \mathfrak{g}$. Hence, given a vector space basis for $\mathfrak{g} \cong \mathbb{R}^k$, a small neighbourhood of $\exp(0) = I \in G$ is endowed with the so-called *exponential coordinates*. If $G$ is compact and connected then $\exp(\mathfrak{g}) = G$, so that the exponential map may "wrap $\mathfrak{g}$ around $G$"; we shall not prove this.

*Remark* 8.3.8. Informally speaking, if $X, Y \in \mathfrak{g}$ are near $0 \in \mathfrak{g}$, $x := \exp(X)$ and $y := \exp(Y)$ then $x, y \in G$ are near $I \in G$ and

$$\exp(X + Y) \approx xy, \quad \exp([X, Y]) \approx \{x, y\} = xyx^{-1}y^{-1}.$$

In a sense, the Lie algebra $\mathfrak{g}$ is the infinitesimally linearised $G$ near $I \in G$.

*Remark* 8.3.9 (**Lie algebra as invariant vector fields**). The Lie algebra $\mathfrak{g}$ can be identified with the tangent space of $G$ at the identity $I \in G$. Using left-translations (resp. right-translations), $\mathfrak{g}$ can be identified with the set of left-invariant (resp. right-invariant) vector fields on $G$, and vector fields have a natural interpretation as first-order partial differential operators on $G$: For $x \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define

$$\begin{aligned} L_X f(x) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t} f\left(x \ \exp(tX)\right)\right|_{t=0}, \\ R_X f(x) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t} f\left(\exp(tX) \ x\right)\right|_{t=0}. \end{aligned}$$

Then $\pi_L(y) L_X f = L_X \pi_L(y) f$ and $\pi_R(y) R_X f = R_X \pi_R(y) f$ for all $y \in G$, where $\pi_L, \pi_R$ are the left and right regular representations of $G$, respectively.

**Definition 8.3.10** (**Abbreviations for Lie algebras).** Some usual abbreviations are

$$\begin{aligned} \mathfrak{gl}(n,\mathbb{K}) &= \mathfrak{Lie}(\mathrm{GL}(n,\mathbb{K})), \\ \mathfrak{sl}(n,\mathbb{K}) &= \mathfrak{Lie}(\mathrm{SL}(n,\mathbb{K})), \\ \mathfrak{o}(n) &= \mathfrak{Lie}(\mathrm{O}(n)), \\ \mathfrak{so}(n) &= \mathfrak{Lie}(\mathrm{SO}(n)), \\ \mathfrak{u}(n) &= \mathfrak{Lie}(\mathrm{U}(n)), \\ \mathfrak{su}(n) &= \mathfrak{Lie}(\mathrm{SU}(n)), \end{aligned}$$

and so on.

**Exercise 8.3.11.** Calculate the dimensions of the linear Lie groups mentioned in Definition 8.3.10.

**Proposition 8.3.12.** *Let $G, H$ be linear Lie groups having the respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $\psi \in \mathrm{HOM}(G,H)$. Then for every $X \in \mathfrak{g}$ there exists a unique $Y \in \mathfrak{h}$ such that $\psi(\exp(tX)) = \exp(tY)$ for all $t \in \mathbb{R}$.*

*Proof.* Let $X \in \mathfrak{g}$. Then $\phi := (t \mapsto \psi(\exp(tX))) : \mathbb{R} \to H$ is a continuous homomorphism, so that $\phi = (t \mapsto \exp(tY))$, where $Y = \phi'(0) \in \mathfrak{h}$. □

**Proposition 8.3.13.** *Let $F, G, H$ be closed subgroups of $\mathrm{GL}(n,\mathbb{C})$, with their respective Lie algebras $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$. Then*
(a) *$H < G \Rightarrow \mathfrak{h} \subset \mathfrak{g}$,*
(b) *the Lie algebra of $F \cap G$ is $\mathfrak{f} \cap \mathfrak{g}$,*
(c) *the Lie algebra $\mathfrak{c}_I$ of the component $C_I < G$ of the neutral element $I$ is $\mathfrak{g}$.*

*Proof.*
(a): If $H < G$ and $X \in \mathfrak{h}$ then $\exp(tX) \in H \subset G$ for all $t \in \mathbb{R}$, so that $X \in \mathfrak{g}$.
(b): Let $\mathfrak{e}$ be the Lie algebra of $F \cap G$. By (a), $\mathfrak{e} \subset \mathfrak{f} \cap \mathfrak{g}$. If $X \in \mathfrak{f} \cap \mathfrak{g}$ then $\exp(tX) \in F \cap G$ for all $t \in \mathbb{R}$, so that $X \in \mathfrak{e}$. Hence $\mathfrak{e} = \mathfrak{f} \cap \mathfrak{g}$.
(c): By (a), $\mathfrak{c}_I \subset \mathfrak{g}$. Let $X \in \mathfrak{g}$. Now the connectedness of $\mathbb{R}$ (Theorem A.16.9) and the continuity of $t \mapsto \exp(tX)$ by Proposition A.16.3 imply the connectedness of

$$\{\exp(tX) : \ t \in \mathbb{R}\} \ni \exp(0) = I.$$

Thereby $\{\exp(tX) : \ t \in \mathbb{R}\} \subset C_I$, so that $X \in \mathfrak{c}_I$. □

*Example* (**Lie algebra of SL**$(n,\mathbb{K})$). Let us compute the Lie algebra $\mathfrak{sl}(n,\mathbb{K})$ of the linear Lie group

$$\mathrm{SL}(n,\mathbb{K}) = \{A \in \mathrm{GL}(n,\mathbb{K}) \mid \det(A) = 1\}.$$

Notice that $\mathfrak{sl}(n,\mathbb{K}) \subset \mathbb{K}^{n\times n}$ by Exercise 8.3.5. Hence

$$\begin{aligned} & \mathfrak{sl}(n,\mathbb{K}) := \\ := \ & \left\{X \in \mathbb{K}^{n\times n} \mid \forall t \in \mathbb{R} : \ \exp(tX) \in \mathrm{SL}(n,\mathbb{K})\right\} \\ = \ & \left\{X \in \mathbb{K}^{n\times n} \mid \forall t \in \mathbb{R} : \ \exp(tX) \in \mathbb{K}^{n\times n}, \ \det(\exp(tX)) = 1\right\}. \end{aligned}$$

Let $\{\lambda_j\}_{j=1}^n \subset \mathbb{C}$ be the set of the eigenvalues of $X \in \mathbb{K}^{n\times n}$. The characteristic polynomial $(z \mapsto \det(zI - X)) : \mathbb{C} \to \mathbb{C}$ of $X$ satisfies

$$\begin{aligned} \det(zI - X) &= \prod_{j=1}^n (z - \lambda_j) \\ &= z^n - z^{n-1}\sum_{j=1}^n \lambda_j + \ldots + (-1)^n \prod_{j=1}^n \lambda_j \\ &= z^n - z^{n-1}\mathrm{Tr}(X) + \ldots + (-1)^n \det(X), \end{aligned}$$

We know that $X$ is similar to an upper triangular matrix $Y = PXP^{-1}$ for some $P \in \mathrm{GL}(n, \mathbb{K})$. Since

$$\begin{aligned}\det(zI - PXP^{-1}) &= \det(P(zI - X)P^{-1}) \\ &= \det(P)\,\det(zI - X)\,\det(P^{-1}) \\ &= \det(zI - X),\end{aligned}$$

the eigenvalues of $X$ and $Y$ are the same, and they are on the diagonal of $Y$. Evidently, $\{\mathrm{e}^{\lambda_j}\}_{j=1}^n \subset \mathbb{C}$ is the set of the eigenvalues of both $\exp(Y)$ and $\exp(X) = P^{-1}\exp(Y)P$. Since the determinant is the product of the eigenvalues and the trace is the sum of the eigenvalues, we have

$$\det(\exp(X)) = \prod_{j=1}^{n} \mathrm{e}^{\lambda_j} = \mathrm{e}^{\sum_{j=1}^n \lambda_j} = \mathrm{e}^{\mathrm{Tr}(X)}$$

(see also Proposition 8.1.4). Therefore $X \in \mathfrak{sl}(n, \mathbb{K})$ if and only if $\mathrm{Tr}(X) = 0$ and $\exp(tX) \in \mathbb{K}^{n\times n}$ for all $t \in \mathbb{R}$. Thus

$$\mathfrak{sl}(n, \mathbb{K}) = \left\{X \in \mathbb{K}^{n\times n} \mid \mathrm{Tr}(X) = 0\right\}$$

as the reader may check.

Next we ponder the relationship between Lie group and Lie algebra homomorphisms.

**Definition 8.3.14** (**Differential homomorphisms).** Let $G, H$ be linear Lie groups with respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. The *differential homomorphism of* $\psi \in \mathrm{HOM}(G, H)$ is the mapping $\psi' = \mathfrak{Lie}(\psi) : \mathfrak{g} \to \mathfrak{h}$ defined by

$$\psi'(X) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\psi(\exp(tX))\right|_{t=0}.$$

*Remark* 8.3.15. Above, $\psi'$ is well-defined since $f := (t \mapsto \psi(\exp(tX))) \in \mathrm{HOM}(\mathbb{R}, H)$ is of the form $t \mapsto \exp(tY)$ for some $Y \in \mathfrak{h}$, as a consequence of Theorem 8.1.6. Moreover, $Y = f'(0) = \psi'(X)$ holds, so that

$$\psi(\exp(tX)) = \exp(t\psi'(X)).$$

**Theorem 8.3.16.** *Let $F, G, H$ be linear Lie groups with respective Lie algebras $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$. Let $\phi \in \mathrm{HOM}(F, G)$ and $\psi \in \mathrm{HOM}(G, H)$. The mapping $\psi' : \mathfrak{g} \to \mathfrak{h}$ defined in Definition 8.3.14 is a Lie algebra homomorphism. Moreover,*

$$(\psi \circ \phi)' = \psi'\phi' \quad \text{and} \quad \mathrm{Id}_G' = \mathrm{Id}_{\mathfrak{g}},$$

*where* $\mathrm{Id}_G = (x \mapsto x) : G \to G$ *and* $\mathrm{Id}_{\mathfrak{g}} = (X \mapsto X) : \mathfrak{g} \to \mathfrak{g}$.

*Proof.* Let $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \psi'(\lambda X) &= \frac{\mathrm{d}}{\mathrm{d}t}\psi(\exp(t\lambda X))|_{t=0} \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}t}\psi(\exp(tX))|_{t=0} \\ &= \lambda\psi'(X). \end{aligned}$$

If $t \in \mathbb{R}$ then

$$\begin{aligned} \exp\left(t\psi'(X+Y)\right) &= \psi\left(\exp(tX+tY)\right) \\ &= \psi\left(\lim_{m\to\infty}\left(\exp(tX/m)\ \exp(tY/m)\right)^m\right) \\ &= \lim_{m\to\infty}\left(\psi(\exp(tX/m))\ \psi(\exp(tY/m))\right)^m \\ &= \lim_{m\to\infty}\left(\exp(t\psi'(X)/m)\ \exp(t\psi'(Y)/m)\right)^m \\ &= \exp(t(\psi'(X)+\psi'(Y))), \end{aligned}$$

so that $t\psi'(X+Y) = t\left(\psi'(X)+\psi'(Y)\right)$ for small enough $|t|$, as we recall that exp is injective in a small neighbourhood of $0 \in \mathfrak{g}$. Consequently, $\psi' : \mathfrak{g} \to \mathfrak{h}$ is linear. Next,

$$\begin{aligned} \exp\left(t\psi'([X,Y])\right) &= \psi\left(\exp(t[X,Y])\right) \\ &= \psi\left(\lim_{m\to\infty}\{\exp(tX/m), \exp(tY/m)\}^{m^2}\right) \\ &= \lim_{m\to\infty}\{\exp(t\psi'(X)/m), \exp(t\psi'(Y)/m)\}^{m^2} \\ &= \exp\left(t[\psi'(X), \psi'(Y)]\right), \end{aligned}$$

so that we get $\psi'([X,Y]) = [\psi'(X), \psi'(Y)]$. Thus $\psi' : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

If $Z \in \mathfrak{f}$ then

$$\begin{aligned} (\psi\circ\phi)'(Z) &= \frac{\mathrm{d}}{\mathrm{d}t}\psi\left(\phi(\exp(tZ))\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\psi\left(\exp(t\phi'(Z))\right)|_{t=0} \\ &= \psi'(\phi'(Z)). \end{aligned}$$

Finally, $\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)|_{t=0} = X$, yielding $\mathrm{Id}'_G = \mathrm{Id}_{\mathfrak{g}}$. □

*Remark* 8.3.17. Notice that isomorphic linear Lie groups must have isomorphic Lie algebras. Now we know that a continuous Lie group homomorphism $\psi$ can naturally be linearised to get a Lie algebra homomorphism $\psi'$, so that we have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H, \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow{\psi'} & \mathfrak{h}. \end{array}$$

What if we are given a Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$, does there exists $\phi \in \mathrm{HOM}(G, H)$ such that $\phi' = f$? This problem is studied in the following two exercises.

**Definition 8.3.18** (**Simply connected spaces).** A topological space $X$ is said to be *simply connected* if $X$ is path-connected and if every closed curve in $X$ can be shrunken to a point continuously in the set $X$.

**Exercise 8.3.19.** Show that the groups $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$ are both connected and simply connected.

**Exercise 8.3.20.** Show that the groups $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$ are connected but not simply connected.

**Exercise 8.3.21.** Let $G, H$ be linear Lie groups such that $G$ is simply connected. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Show that there exists $\phi \in \mathrm{HOM}(G, H)$ such that $\phi' = f$. (This is a rather demanding task unless one knows that $\exp : \mathfrak{g} \to G$ is surjective and uses Lemma 8.1.11. A proof can be found e.g. in [37].)

**Exercise 8.3.22.** Related to Exercise 8.3.21, give an example of a non-simply-connected $G$ and a homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ which is not of the form $f = \phi'$.

**Lemma 8.3.23.** *Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group $G$, and*

$$S := \left\{ \exp(X_1) \cdots \exp(X_m) \mid m \in \mathbb{Z}^+,\ \{X_j\}_{j=1}^m \subset \mathfrak{g} \right\}.$$

*Then $S = C_I$, the component of $I \in G$.*

*Proof.* Now $S < G$ is path-connected, since

$$(t \mapsto \exp(tX_1) \cdots \exp(tX_m)) : [0, 1] \to S$$

is continuous, connecting $I \in S$ to the point $\exp(X_1) \cdots \exp(X_m) \in S$. For a small enough neighbourhood $U \subset \mathfrak{g}$ of $0 \in \mathfrak{g}$, we have a homeomorphism $(X \mapsto \exp(X)) : U \to \exp(U)$. Because of

$$\exp(X_1) \cdots \exp(X_m) \ \in\ \exp(X_1) \cdots \exp(X_m) \exp(U) \ \subset\ S,$$

it follows that $S < G$ is open. But open subgroups are always closed, as the reader can easily verify. Thus $S \ni I$ is connected, closed and open, so that $S = C_I$. $\square$

**Corollary 8.3.24.** *Let $G, H$ be linear Lie groups and $\phi, \psi \in \mathrm{HOM}(G, H)$. Then:*
(a) $\mathfrak{Lie}(\mathrm{Ker}(\psi)) = \mathrm{Ker}(\psi')$.
(b) *If $G$ is connected and $\phi' = \psi'$ then $\phi = \psi$.*
(c) *Let $H$ be connected; then $\psi'$ is surjective if and only if $\psi$ is surjective.*

*Proof.*
(a) $\mathrm{Ker}(\psi) < G < \mathrm{GL}(n, \mathbb{C})$ is a closed subgroup, since $\psi$ is a continuous homomorphism. Thereby

$$\begin{aligned}
\mathfrak{Lie}(\mathrm{Ker}(\psi)) &= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R}:\ \exp(tX) \in \mathrm{Ker}(\psi)\right\} \\
&= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R}:\ \exp(t\psi'(X)) = \psi(\exp(tX)) = I\right\} \\
&= \left\{X \in \mathbb{C}^{n\times n} \mid \psi'(X) = 0\right\} \\
&= \mathrm{Ker}(\psi').
\end{aligned}$$

(b) Take $A \in G$. Then $A = \exp(X_1)\cdots\exp(X_m)$ for some $\{X_j\}_{j=1}^m \subset \mathfrak{g}$ by Lemma 8.3.23, so that

$$\begin{aligned}
\phi(A) &= \exp\left(\phi'(X_1)\right)\cdots\exp\left(\phi'(X_m)\right) \\
&= \exp\left(\psi'(X_1)\right)\cdots\exp\left(\psi'(X_m)\right) \\
&= \psi(A).
\end{aligned}$$

(c) Suppose $\psi' : \mathfrak{g} \to \mathfrak{h}$ is surjective. Let $B \in H$. Now $H$ is connected, so that Lemma 8.3.23 says that $B = \exp(Y_1)\cdots\exp(Y_m)$ for some $\{Y_j\}_{j=1}^m \subset \mathfrak{h}$. Exploit the surjectivity of $\psi'$ to obtain $X_j \in \mathfrak{g}$ such that $\psi'(X_j) = Y_j$. Then

$$\begin{aligned}
\psi\left(\exp(X_1)\cdots\exp(X_m)\right) &= \psi\left(\exp(X_1)\right)\cdots\psi\left(\exp(X_m)\right) \\
&= \exp(Y_1)\cdots\exp(Y_m) \\
&= B.
\end{aligned}$$

Conversely, suppose $\psi : G \to H$ is surjective. Trivially, $\psi'(0) = 0 \in \mathfrak{h}$; let $0 \neq Y \in \mathfrak{h}$. Let $r_0 := r/\|Y\|$, where $r$ is as in Proposition 8.1.7; notice that if $|t| < r_0$ then $\log(\exp(tY)) = tY$. The surjectivity of $\psi$ guarantees that for every $t \in \mathbb{R}$ there exists $A_t \in G$ such that $\psi(A_t) = \exp(tY)$. The set $R := \{A_t :\ 0 < t < r_0\}$ is uncountable, so that it has an accumulation point $x \in \mathbb{C}^{n\times n}$; and $x \in G$, because $R \subset G$ and $G \subset \mathbb{C}^{n\times n}$ is closed. Let $\varepsilon > 0$. Then there exist $s, t \in ]0, r_0[$ such that $s \neq t$ and

$$\|A_s - x\| < \varepsilon, \quad \|A_t - x\| < \varepsilon, \quad \left\|A_s^{-1} - x^{-1}\right\| < \varepsilon.$$

Thereby

$$\begin{aligned}
\left\|A_s^{-1}A_t - I\right\| &= \left\|A_s^{-1}(A_t - A_s)\right\| \\
&\leq \left\|A_s^{-1}\right\|\left(\|A_t - x\| + \|x - A_s\|\right) \\
&\leq \left(\|x^{-1}\| + \varepsilon\right) 2\varepsilon.
\end{aligned}$$

Hence we demand $\|A_s^{-1}A_t - I\| < 1$ and $\|\psi(A_s^{-1}A_t) - I\| < 1$, yielding

$$\psi(A_s^{-1}A_t) = \psi(A_s)^{-1}\psi(A_t) = \exp((t-s)Y).$$

Consequently

$$\psi'\left(\log(A_s^{-1}A_t)\right) = (t-s)Y.$$

Therefore $\psi'\left(\frac{1}{t-s}\log(A_s^{-1}A_t)\right) = Y$. □

**Definition 8.3.25** (**Adjoint representation of Lie groups).** The *adjoint representation of a linear Lie group* $G$ is the mapping $\mathrm{Ad} \in \mathrm{HOM}(G, \mathrm{Aut}(\mathfrak{g}))$ defined by

$$\mathrm{Ad}(A)X := AXA^{-1},$$

where $A \in G$ and $G \in \mathfrak{g}$.

*Remark* 8.3.26. Indeed, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, because

$$\exp\left(t\mathrm{Ad}(A)X\right) = \exp\left(tAXA^{-1}\right) = A\exp\left(tX\right)A^{-1}$$

belongs to $G$ if $A \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. It is a homomorphism, since

$$\mathrm{Ad}(AB)X = ABXB^{-1}A^{-1} = \mathrm{Ad}(A)(BXB^{-1}) = \mathrm{Ad}(A)\ \mathrm{Ad}(B)\ X,$$

and Ad is trivially continuous.

**Exercise 8.3.27.** Let $\mathfrak{g}$ be a Lie algebra. Consider $\mathrm{Aut}(\mathfrak{g})$ as a linear Lie group. Show that $\mathfrak{Lie}(\mathrm{Aut}(\mathfrak{g}))$ and $\mathfrak{gl}(\mathfrak{g})$ are isomorphic as Lie algebras.

**Definition 8.3.28** (**Adjoint representation of Lie algebras).** The *adjoint representation of the Lie algebra* $\mathfrak{g}$ *of a linear Lie group* $G$ is the differential representation

$$\mathrm{ad} = \mathrm{Ad}' : \mathfrak{g} \to \mathfrak{Lie}(\mathrm{Aut}(\mathfrak{g})) \cong \mathfrak{gl}(\mathfrak{g}),$$

that is $\mathrm{ad}(X) := \mathrm{Ad}'(X)$, so that

$$\begin{aligned}
\mathrm{ad}(X)Y &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tX)Y\exp(-tX)\right)|_{t=0} \\
&= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\right)Y\exp(-tX) + \exp(tX)Y\frac{\mathrm{d}}{\mathrm{d}t}\exp(-tX)\right)|_{t=0} \\
&= XY - YX \\
&= [X,Y].
\end{aligned}$$

*Remark* 8.3.29. Notice that the diagram commutes:

$$\begin{array}{ccc}
G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(G) \\
{\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\
\mathfrak{g} & \xrightarrow{\mathrm{Ad}'=\mathrm{ad}} & \mathfrak{Lie}(\mathrm{Aut}(\mathfrak{g})).
\end{array}$$

### 8.3.1 Universal enveloping algebra

Here we discuss the universal enveloping algebra.

*Remark* 8.3.30 (**Universal enveloping algebra informally**). We are going to study higher order partial differential operators on $G$. Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group $G$. Next we construct a natural associative algebra $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}$ modulo an ideal, enabling embedding $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. Recall that $\mathfrak{g}$ can be interpreted as the vector space of first-order left (or right) -translation invariant partial differential operators on $G$. Consequently, $\mathcal{U}(\mathfrak{g})$ can be interpreted as the vector space of finite-order left (or right) -translation invariant partial differential operators on $G$.

**Definition 8.3.31** (**Universal enveloping algebra).** Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Let

$$\mathcal{T} := \bigoplus_{m=0}^{\infty} \otimes^m \mathfrak{g}$$

be the tensor product algebra of $\mathfrak{g}$, where $\otimes^m \mathfrak{g}$ denotes the $m$-fold tensor product $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$; that is, $\mathcal{T}$ is the linear span of the elements of the form

$$\lambda_{00}\mathbf{1} + \sum_{m=1}^{M} \sum_{k=1}^{K_m} \lambda_{mk}\ X_{mk1} \otimes \cdots \otimes X_{mkm},$$

where $\mathbf{1}$ is the formal unit element of $\mathcal{T}$, $\lambda_{mk} \in \mathbb{K}$, $X_{mkj} \in \mathfrak{g}$ and $M, K_m \in \mathbb{Z}^+$; the product of $\mathcal{T}$ is begotten by the tensor product, i.e.

$$(X_1 \otimes \cdots \otimes X_p)(Y_1 \otimes \cdots \otimes Y_q) := X_1 \otimes \cdots \otimes X_p \otimes Y_1 \otimes \cdots \otimes Y_q$$

is extended to a unique bilinear mapping $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$. Let $\mathcal{J}$ be the (two-sided) ideal in $\mathcal{T}$ spanned by the set

$$\mathcal{O} := \{X \otimes Y - Y \otimes X - [X, Y] :\ X, Y \in \mathfrak{g}\};$$

i.e. $\mathcal{J} \subset \mathcal{T}$ is the smallest vector subspace such that $\mathcal{O} \subset \mathcal{J}$ and $DE, ED \in \mathcal{J}$ for every $D \in \mathcal{J}$ and $E \in \mathcal{T}$ (in a sense, $\mathcal{J}$ is a "huge zero" in $\mathcal{T}$). The quotient algebra

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}/\mathcal{J}$$

is called the *universal enveloping algebra of* $\mathfrak{g}$.

**Definition 8.3.32** (**Canonical mapping of a Lie algebra).** Let $\iota : \mathcal{T} \to \mathcal{U}(\mathfrak{g}) = \mathcal{T}/\mathcal{J}$ be the quotient mapping $t \mapsto t + \mathcal{J}$. A natural interpretation is that $\mathfrak{g} \subset \mathcal{T}$. The restricted mapping $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is called the *canonical mapping of* $\mathfrak{g}$.

*Remark* 8.3.33. Notice that $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{Lie}_{\mathbb{K}}(\mathcal{U}(\mathfrak{g}))$ is a Lie algebra homomorphism: it is linear and

$$\begin{aligned} \iota|_{\mathfrak{g}}([X,Y]) &= \iota([X,Y]) \\ &= \iota(X \otimes Y - Y \otimes X) \\ &= \iota(X)\iota(Y) - \iota(Y)\iota(X) \\ &= \iota|_{\mathfrak{g}}(X)\iota|_{\mathfrak{g}}(Y) - \iota|_{\mathfrak{g}}(Y)\iota|_{\mathfrak{g}}(X) \\ &= [\iota|_{\mathfrak{g}}(X), \iota|_{\mathfrak{g}}(Y)]. \end{aligned}$$

**Theorem 8.3.34 (Universality of the enveloping algebra).** *Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra, $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ its canonical mapping, $\mathcal{A}$ an associative $\mathbb{K}$-algebra, and*

$$\sigma : \mathfrak{g} \to \mathrm{Lie}_{\mathbb{K}}(\mathcal{A})$$

*a Lie algebra homomorphism. Then there exists a unique algebra homomorphism*

$$\tilde{\sigma} : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$$

*satisfying $\tilde{\sigma}\,(\iota|_{\mathfrak{g}}(X)) = \sigma(X)$ for all $X \in \mathfrak{g}$, i.e.*

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\tilde{\sigma}} & \mathcal{A} \\ {\scriptstyle \iota|_{\mathfrak{g}}}\uparrow & & \| \\ \mathfrak{g} & \xrightarrow{\sigma} & \mathrm{Lie}_{\mathbb{K}}(\mathcal{A}). \end{array}$$

*Proof.* Let us define a linear mapping $\sigma_0 : \mathcal{T} \to \mathcal{A}$ by

$$\sigma_0(X_1 \otimes \cdots \otimes X_m) := \sigma(X_1) \cdots \sigma(X_m). \tag{8.3}$$

Then $\sigma_0(\mathcal{J}) = \{0\}$, since

$$\begin{aligned} \sigma_0(X \otimes Y - Y \otimes X - [X,Y]) &= \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) - \sigma([X,Y]) \\ &= \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) - [\sigma(X), \sigma(Y)] \\ &= 0. \end{aligned}$$

Hence if $t, u \in \mathcal{T}$ and $t - u \in \mathcal{J}$ then $\sigma_0(t) = \sigma_0(u)$. Thereby we may define $\tilde{\sigma} := (t + \mathcal{J} \mapsto \sigma_0(t)) : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$. Finally, it is clear that $\tilde{\sigma}$ is an algebra homomorphism making the diagram above commute. The uniqueness is clear by construction since (8.3) must hold. □

**Corollary 8.3.35 (Ado–Iwasawa Theorem).** *Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group $G$. Then the canonical mapping $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

*Proof.* Let $\sigma = (X \mapsto X) : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$. Due to the universality of $\mathcal{U}(\mathfrak{g})$ there exists an $\mathbb{R}$-algebra homomorphism $\tilde{\sigma} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}^{n\times n}$ such that $\sigma(X) = \tilde{\sigma}\left(\iota|_{\mathfrak{g}}(X)\right)$ for all $X \in G$, i.e.

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\tilde{\sigma}} & \mathbb{C}^{n\times n} \\ {\scriptstyle \iota|_{\mathfrak{g}}}\uparrow & & \Vert \\ \mathfrak{g} & \xrightarrow{\sigma} & \mathfrak{gl}(n, \mathbb{C}). \end{array}$$

Then $\iota|_{\mathfrak{g}}$ is injective because $\sigma$ is injective. □

*Remark* 8.3.36. By the Ado–Iwasawa Theorem (Corollary 8.3.35), the Lie algebra $\mathfrak{g}$ of a linear Lie group can be considered as a Lie subalgebra of $\mathrm{Lie}_{\mathbb{R}}\left(\mathcal{U}(\mathfrak{g})\right)$.

**Definition 8.3.37 (ad).** Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Let us define the linear mapping $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $\mathrm{ad}(X)Z := [X, Z]$.

*Remark* 8.3.38. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and $X, Z \in G$. Since

$$\begin{aligned} 0 &= [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \\ &= [[X,Y],Z] - ([X,[Y,Z]] - [Y,[X,Z]]) \\ &= \mathrm{ad}([X,Y])Z - [\mathrm{ad}(X), \mathrm{ad}(Y)]Z, \end{aligned}$$

we notice that

$$\mathrm{ad}([X,Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)],$$

i.e. ad is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

**Definition 8.3.39 (Killing form and semisimple Lie groups).** The *Killing form* of the Lie algebra $\mathfrak{g}$ is the bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, defined by

$$B(X,Y) := \mathrm{Tr}\left(\mathrm{ad}(X)\ \mathrm{ad}(Y)\right)$$

(recall that by Exercise B.5.40, on a finite-dimensional vector space the trace can be defined independent of any inner product). A ($\mathbb{R}$- or $\mathbb{C}$-)Lie algebra $\mathfrak{g}$ is called *semisimple* if its Killing form is *non-degenerate*, i.e. if

$$\forall X \in \mathfrak{g} \setminus \{0\}\ \exists Y \in \mathfrak{g} :\ B(X,Y) \neq 0;$$

equivalently, $B$ is non-degenerate if the matrix $\left(B(X_i, X_j)\right)_{i,j=1}^n$ is invertible, where $\{X_j\}_{j=1}^n \subset \mathfrak{g}$ is a vector space basis. A connected linear Lie group is called *semisimple* if its Lie algebra is semisimple.

*Example.* Linear Lie groups $\mathrm{SL}(n, \mathbb{K})$ and $\mathrm{SO}(n)$ are semisimple, but $\mathrm{GL}(n)$ is not semisimple.

*Remark* 8.3.40. Since $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$, we have

$$B(X,Y) = B(Y,X).$$

We also have

$$B(X, [Y, Z]) = B([X, Y], Z),$$

because

$$\mathrm{Tr}(a(bc - cb)) = \mathrm{Tr}(abc) - \mathrm{Tr}(acb) = \mathrm{Tr}(abc) - \mathrm{Tr}(bac) = \mathrm{Tr}((ab - ba)c)$$

yields

$$\begin{aligned} B(X, [Y, Z]) &= \mathrm{Tr}\,(\mathrm{ad}(X)\ \mathrm{ad}([Y, Z])) \\ &= \mathrm{Tr}\,(\mathrm{ad}(X)\ [\mathrm{ad}(Y), \mathrm{ad}(Z)]) \\ &= \mathrm{Tr}\,([\mathrm{ad}(X), \mathrm{ad}(Y)]\ \mathrm{ad}(Z)) \\ &= \mathrm{Tr}\,(\mathrm{ad}([X, Y])\ \mathrm{ad}(Z)) \\ &= B([X, Y], Z). \end{aligned}$$

It can be proven that the Killing form of the Lie algebra of a compact linear Lie group is negative semi-definite, i.e. $B(X, X) \leq 0$. On the other hand, if the Killing form of a Lie group is negative definite, i.e. $B(X, X) < 0$ whenever $X \neq 0$, then the group is compact.

### 8.3.2 Casimir element and Laplace operator

Here we discuss some properties of the Casimir element and the corresponding Laplace operator.

**Definition 8.3.41** (**Casimir element**). Let $\mathfrak{g}$ be a semisimple $\mathbb{K}$-Lie algebra with a vector space basis $\{X_j\}_{j=1}^n \subset \mathfrak{g}$. Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be the Killing form of $\mathfrak{g}$, and define the matrix $R \in \mathbb{K}^{n \times n}$ by $R_{ij} := B(X_i, X_j)$. Let

$$X^i := \sum_{j=1}^n \left(R^{-1}\right)_{ij} X_j,$$

so that $\{X^i\}_{i=1}^n$ is another vector space basis for $\mathfrak{g}$. Then the *Casimir element* $\Omega \in \mathcal{U}(\mathfrak{g})$ *of* $\mathfrak{g}$ is defined by

$$\Omega := \sum_{i=1}^n X_i X^i.$$

*Remark* 8.3.42. The Casimir element $\Omega \in \mathcal{U}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ of a compact semisimple linear Lie group $G$ can be considered as an elliptic linear second-order (left and right) translation invariant partial differential operator. In a sense, the Casimir operator is an analogy of the Euclidean Laplace operator

$$\mathcal{L} = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

Such a Laplace operator can be constructed for any compact Lie group $G$, and with it we may define Sobolev spaces on $G$ nicely, etc.

**Theorem 8.3.43** (**Properties of Casimir element**). *The Casimir element of a finite-dimensional semisimple $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is independent of the choice of the vector space basis $\{X_j\}_{j=1}^n \subset \mathfrak{g}$. Moreover,*

$$D\Omega = \Omega D.$$

*for all $D \in \mathcal{U}(\mathfrak{g})$.*

*Proof.* Let $\{X_j\}_{j=1}^n \subset \mathfrak{g}$, $R_{ij} = B(X_i, X_j)$ and $\Omega$ be as in Definition 8.3.41. To simplify notation, we consider only the case $\mathbb{K} = \mathbb{R}$. Let $\{Y_i\}_{i=1}^n \subset \mathfrak{g}$ be a vector space basis of $\mathfrak{g}$. Then there exists $A = (A_{ij})_{i,j=1}^n \in \mathrm{GL}(n, \mathbb{R})$ such that

$$\left\{ Y_i := \sum_{j=1}^n A_{ij} X_j \right\}_{i=1}^n .$$

Then

$$\begin{aligned} S &:= (B(Y_i, Y_j))_{i,j=1}^n \\ &= \left( B\Big( \sum_{k=1}^n A_{ik} X_k, \sum_{l=1}^n A_{jl} X_l \Big) \right)_{i,j=1}^n \\ &= \left( \sum_{k,l=1}^n A_{ik}\ B(X_k, X_l)\ A_{jl} \right)_{i,j=1}^n \\ &= A R A^{\mathrm{T}}; \end{aligned}$$

hence

$$S^{-1} = ((S^{-1})_{ij})_{i,j=1}^n = (A^{\mathrm{T}})^{-1} R^{-1} A^{-1}.$$

Let us now compute the Casimir element of $\mathfrak{g}$ with respect to the basis $\{Y_j\}_{j=1}^n$:

$$\begin{aligned} \sum_{i,j=1}^n (S^{-1})_{ij} Y_i Y_j &= \sum_{i,j=1}^n (S^{-1})_{ij} \sum_{k=1}^n A_{ik} X_k \sum_{l=1}^n A_{jl} X_l \\ &= \sum_{k,l=1}^n X_k X_l \sum_{i,j=1}^n A_{ik} (S^{-1})_{ij} A_{jl} \\ &= \sum_{k,l=1}^n X_k X_l \sum_{i,j=1}^n (A^{\mathrm{T}})_{ki} ((A^{\mathrm{T}})^{-1} R^{-1} A^{-1})_{ij} A_{jl} \\ &= \sum_{k,l=1}^n X_k X_l (R^{-1})_{kl}. \end{aligned}$$

Thus the definition of the Casimir element does not depend on the choice of a vector space basis.

We still have to prove that $\Omega$ commutes with every $D \in \mathcal{U}(\mathfrak{g})$. Since

$$B(X^i, X_j) = \sum_{k=1}^{n} (R^{-1})_{ik} B(X_k, X_j) = \sum_{k=1}^{n} (R^{-1})_{ik} R_{kj} = \delta_{ij},$$

we can extend $(X_i, X_j) \mapsto \langle X_i, X_j \rangle_{\mathfrak{g}} := B(X^i, X_j)$ uniquely to an inner product

$$((X, Y) \mapsto \langle X, Y \rangle_{\mathfrak{g}}) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

with respect to which the collection $\{X_i\}_{i=1}^n$ is an orthonormal basis. For the Lie product $(x, y) \mapsto [x, y] := xy - yx$ of $\mathrm{Lie}_{\mathbb{R}}(\mathcal{U}(\mathfrak{g}))$ we have

$$[x, yz] = [x, y]z + y[x, z],$$

so that for $D \in \mathfrak{g}$ we get

$$[D, \Omega] = [D, \sum_{i=1}^{n} X_i X^i] = \sum_{i=1}^{n} \left( [D, X_i] X^i + X_i [D, X^i] \right).$$

Let $c_{ij}, d_{ij} \in \mathbb{R}$ be defined by

$$[D, X_i] = \sum_{j=1}^{n} c_{ij} X_j, \quad [D, X^i] = \sum_{j=1}^{n} d_{ij} X^j.$$

Then

$$\begin{aligned} c_{ij} &= \langle X_j, [D, X_i] \rangle_{\mathfrak{g}} \\ &= B(X^j, [D, X_i]) \\ &= B([X^j, D], X_i) \\ &= B(-[D, X^j], X_i) \\ &= B(-\sum_{k=1}^{n} d_{jk} X^k, X_i) \\ &= -\sum_{k=1}^{n} d_{jk} B(X^k, X_i) \\ &= -\sum_{k=1}^{n} d_{jk} \langle X_k, X_i \rangle_{\mathfrak{g}} \\ &= -d_{ji}, \end{aligned}$$

so that

$$\begin{aligned} [D, \Omega] &= \sum_{i,j=1}^{n} (c_{ij} X_j X^i + d_{ij} X_i X^j) \\ &= \sum_{i,j=1}^{n} (c_{ij} + d_{ji}) X_j X^i \\ &= 0, \end{aligned}$$

i.e. $D\Omega = \Omega D$ for all $D \in \mathfrak{g}$. By induction, we may prove that

$$[D_1 D_2 \cdots D_m, \Omega] \; = \; D_1[D_2 \cdots D_m, \Omega] + [D_1, \Omega] D_2 \cdots D_m \; = \; 0$$

for every $\{D_j\}_{j=1}^m \subset \mathfrak{g}$, so that $D\Omega = \Omega D$ for all $D \in \mathcal{U}(\mathfrak{g})$. □

**Exercise 8.3.44.** How should the proof of Theorem 8.3.43 be modified if $\mathbb{K} = \mathbb{C}$ instead of $\mathbb{K} = \mathbb{R}$?

**Definition 8.3.45** (**Laplace operator on** $G$)**.** The Casimir element from Definition 8.3.41, also denoted by

$$\mathcal{L}_G := \Omega \in \mathcal{U}(\mathfrak{g}),$$

and viewed as a second order partial differential operator on $G$ is also called the *Laplace* operator on $G$. Here a vector field $Y \in \mathfrak{g}$ is viewed as a differential operator $Y \equiv D_Y : C^\infty(G) \to C^\infty(G)$, defined by

$$Y f(x) \equiv D_Y f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x \;\exp(tY))\Big|_{t=0}.$$

*Remark* 8.3.46. The Laplace operator $\mathcal{L}_G$ is a negative definite bi-invariant operator on $G$, by Theorem 8.3.43. If $G$ is equipped with the unique (up to a constant) bi-invariant Riemannian metric, $\mathcal{L}_G$ is its Laplace–Beltrami operator.

In the notation of right and left Peter–Weyl theorem in Theorem 7.5.14 and Remark 7.5.16, we denote

$$\mathcal{H}^\phi := \bigoplus_{i=1}^{\dim\phi} \mathcal{H}^\phi_{i,\cdot} = \bigoplus_{j=1}^{\dim\phi} \mathcal{H}^\phi_{\cdot,j}.$$

**Theorem 8.3.47** (**Eigenvalues of the Laplacian on** $G$)**.** *For every* $\phi \in \widehat{G}$ *the space* $\mathcal{H}^\phi$ *is an eigenspace of* $\mathcal{L}_G$ *and* $-\mathcal{L}_G|_{\mathcal{H}^\phi} = \lambda_\phi I$*, for some* $\lambda_\phi \geq 0$.

*Proof.* We will use the notation of Theorem 7.5.14. Note that by Theorem 8.3.43 the Laplace operator $\mathcal{L}_G$ is bi-invariant, so that it commutes with both $\pi_R(x)$ and $\pi_L(x)$, for all $x \in G$. Therefore, by the Peter–Weyl theorem it commutes with all $\phi \in \widehat{G}$. Thus $\mathcal{L}_G(\mathcal{H}^\phi_{\cdot,j}) \subset \mathcal{H}^\phi_{\cdot,j}$ and $\mathcal{L}_G(\mathcal{H}^\phi_{i,\cdot}) \subset \mathcal{H}^\phi_{i,\cdot}$, for all $1 \leq i,j \leq \dim(\phi)$. It follows that $\mathcal{L}_G \phi_{ij} \in \mathcal{H}^\phi_{i,\cdot} \bigcap \mathcal{H}^\phi_{\cdot,j} = \mathrm{span}(\phi_{ij})$, so that $\mathcal{L}_G \phi_{ij} = c_{ij}\phi_{ij}$ for some constants $c_{ij}$. Let us now determine these constants. We have

$$\begin{aligned}
(\mathcal{L}_G \pi_R(y)\phi_{ij})(x) &= \mathcal{L}_G(\phi_{ij}(xy)) \\
&= \mathcal{L}_G\left(\sum_{k=1}^{\dim(\phi)} \phi_{ik}(x)\phi_{kj}(y)\right) \\
&= \sum_{k=1}^{\dim(\phi)} c_{ik}\phi_{ik}(x)\phi_{kj}(y).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}(\pi_R(y)\mathcal{L}_G\phi_{ij})(x) &= c_{ij}\phi_{ij}(xy)\\ &= \sum_{k=1}^{\dim(\phi)} c_{ij}\phi_{ik}(x)\phi_{kj}(y).\end{aligned}$$

It follows now from the orthogonality Lemma 7.5.12 that $c_{ik}\phi_{kj}(y) = c_{ij}\phi_{kj}(y)$, or that $c_{ik} = c_{ij}$ for all $1 \leq i, j, k \leq \dim(\phi)$. A similar calculation with the left regular action $\pi_L(y)$ shows that $c_{kj} = c_{ij}$ for all $1 \leq i, j, k \leq \dim(\phi)$. Hence $\mathcal{L}_G\phi_{ij} = c\phi_{ij}$ for all $1 \leq i, j \leq \dim(\phi)$, and since $\mathcal{L}_G$ is negative definite, we obtain the statement with $\lambda_\phi := -c \geq 0$. □