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Schatten classes on compact manifolds: Kernel conditions ☆

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ABSTRACT

In this paper we give criteria on integral kernels ensuring that integral operators on compact manifolds belong to Schatten classes. A specific test for nuclearity is established as well as the corresponding trace formulae. In the special case of compact Lie groups, kernel criteria in terms of (locally and globally) hypoelliptic operators are also given.

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1. Introduction

Given a closed smooth manifold M (smooth manifold without boundary) endowed with a positive measure dx, in this paper we give sufficient conditions on Schwartz integral kernels in order to ensure that the corresponding integral operators belong to different Schatten classes. The problem of finding such criteria on different kinds of domains is classical and has been much studied, e.g. the paper [2] by Birman and Solomyak is a good introduction to the subject. In particular, it is well known that the smoothness of the kernel is related to the behaviour of the singular numbers.

In this paper we present criteria for Schatten classes and, in particular, for the trace class operators on compact smooth manifolds without boundary. Compact Lie groups will also be considered as a special case since then additional results can be obtained, also allowing criteria in terms of hypoelliptic operators such as the sub-Laplacian. The sufficient conditions on integral kernels $K(x,y)$ for Schatten classes will require regularity of a certain order in either x or y, or both.

We note that already some results of Birman and Solomyak [2] can be extended to compact manifolds but our approach allows one to be flexible about sets of variables in which one imposes the regularity of the kernel.

In order to obtain criteria for general Schatten classes we will use the well-known method of factorisation, particularly in the way applied by O'Brien in [20]. For applications to trace formulas of Schrödinger operators see also [21].

Schatten classes of pseudo-differential operators in the setting of the Weyl–Hörmander calculus have been considered in [39,40,6,33]. Schatten classes on compact Lie groups and s-nuclear operators on L^p spaces from the point of view of symbols have been respectively studied by the authors in [11] and [10]. In the subsequent part of the present paper we establish the characterisation of Schatten classes on closed manifolds in terms of symbols that we will introduce for this purpose.

In his classical book (cf. [37, Prop. 3.5, page 174]) Mitsuo Sugiura gives a trace class criterion for integral operators on $L^2(\mathbb{T}^1)$ with C^2-kernels. More precisely, the theorem asserts that every kernel in $C^2(\mathbb{T}^2)$ begets a trace class operator on $L^2(\mathbb{T}^1)$: if $K(\theta,\phi)$ is a C^2-function on $\mathbb{T}^2$, then the integral operator L on $L^2(\mathbb{T}^1)$ defined by

$$(Lf)(\theta) = \int_0^{2\pi} K(\theta,\phi) f(\phi)\, d\phi, \tag{1.1}$$

is trace class and has the trace

$$\operatorname{Tr}(L) = \frac{1}{2\pi}\int_0^{2\pi} K(\theta,\theta)\, d\theta. \tag{1.2}$$

The proof of this result relies on the connection between the absolute convergence of Fourier coefficients of the kernel and the trace class property (traceability) of the corresponding operator.

However, in this paper we show that such type of results can be significantly improved by using a different approach. Associating a discrete Fourier Analysis with an elliptic operator on a compact manifold, we will establish the aforementioned connection in the setting of general closed manifolds, also weakening the known assumptions on the kernel for the operator to be trace class and for the trace formula (1.2) to hold. Thus, in this respect, a direct extension of the method employed by Sugiura leads to weaker results than our approach, for closed manifolds of dimension higher than 2, and we discuss this at the end of Section 4.

To formulate the notions more precisely, let H be a complex Hilbert space endowed with an inner product denoted by $(\cdot,\cdot)$, and let $T : H \to H$ be a linear compact operator. If we denote by $T^* : H \to H$ the adjoint of T, then the linear operator $(T^*T)^{\frac{1}{2}} : H \to H$ is positive and compact. Let $(\psi_k)_k$ be an orthonormal basis for H consisting of eigenvectors of $|T| = (T^*T)^{\frac{1}{2}}$, and let $s_k(T)$ be the eigenvalue corresponding to the eigenvector ψ_k, $k = 1, 2, \ldots$. The non-negative numbers $s_k(T)$, $k = 1, 2, \ldots$, are called the singular values of $T : H \to H$. If

$$\sum_{k=1}^{\infty} s_k(T) < \infty,$$

then the linear operator $T : H \to H$ is said to be in the *trace class* S_1. It can be shown that $S_1(H)$ is a Banach space in which the norm $\|\cdot\|_{S_1}$ is given by

$$\|T\|_{S_1} = \sum_{k=1}^{\infty} s_k(T), \quad T \in S_1,$$

multiplicities counted. Let $T : H \to H$ be an operator in $S_1(H)$ and let $(\phi_k)_k$ be any orthonormal basis for H. Then, the series $\sum_{k=1}^{\infty} (T\phi_k, \phi_k)$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $(\phi_k)_k$. Thus, we can define the trace $\mathrm{Tr}(T)$ of any linear operator $T : H \to H$ in S_1 by

$$\mathrm{Tr}(T) := \sum_{k=1}^{\infty} (T\phi_k, \phi_k),$$

where $\{\phi_k : k = 1, 2, \ldots\}$ is any orthonormal basis for H. If the singular values are square-summable T is called a *Hilbert–Schmidt* operator. It is clear that every trace class operator is a Hilbert–Schmidt operator. More generally, if $0 < p < \infty$ and the sequence of singular values is p-summable, then T is said to belong to the Schatten class $S_p(H)$, and it is well known that each $S_p(H)$ is an ideal in $\mathcal{L}(H)$. If $1 \leqslant p < \infty$, a norm is associated with $S_p(H)$ by

$$\|T\|_{S_p} = \left(\sum_{k=1}^{\infty} \bigl(s_k(T)\bigr)^p\right)^{\frac{1}{p}}.$$

If $1 \leqslant p < \infty$ the class $S_p(H)$ becomes a Banach space endowed by the norm $\|T\|_{S_p}$. If $p = \infty$ we define $S_\infty(H)$ as the class of bounded linear operators on H, with $\|T\|_{S_\infty} := \|T\|_{op}$, the operator norm. In the case $0 < p < 1$ the quantity $\|T\|_{S_p}$ only defines a quasinorm, and $S_p(H)$ is also complete.

The Schatten classes are nested, with

$$S_p \subset S_q, \quad \text{if } 0 < p < q \leqslant \infty, \tag{1.3}$$

and satisfy the important multiplication property (cf. [17,32,14])

$$S_p S_q \subset S_r, \tag{1.4}$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 0 < p < q \leqslant \infty.$$

We will apply (1.4) for factorising our operators T in the form $T = AB$ with $A \in S_p$ and $B \in S_q$, and from this we deduce that $T \in S_r$.

A nice basic introduction to the study of the trace class is included in the book [19] by Peter Lax. For the basic theory of Schatten classes we refer the reader to [14,22,32,27].

In this paper we consider integral operators which is not restrictive in view of the Schwartz integral kernel theorem on closed manifolds. If $H = L^2(\Omega, \mathcal{M}, m)$, it is well known that T is a Hilbert–Schmidt operator if and only if T can be represented by an integral kernel $K = K(x, y) \in L^2(\Omega \times \Omega, m \otimes m)$. In this paper we are interested in the case when Ω is a closed manifold (which we denote by M). In particular, we note that in view of (1.3) the condition $K \in L^2(M \times M)$ implies that $T \in S_p$ for all $p \geqslant 2$.

For $p < 2$, the situation is much more subtle, and the Schatten classes $S_p(L^2)$ cannot be characterised as in the case $p = 2$ by a property analogous to the square integrability of integral kernels. This is a crucial fact that we now briefly describe. A classical result of Carleman [7] from 1916 gives the construction of a periodic *continuous* function $\varkappa(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$ for which the Fourier coefficients c_k satisfy

$$\sum_{k=-\infty}^{\infty} |c_k|^p = \infty \quad \text{for any } p < 2. \tag{1.5}$$

Now, using this and considering the normal operator

$$Tf = f * \varkappa \tag{1.6}$$

acting on $L^2(\mathbb{T}^1)$ one obtains that the sequence $(c_k)_k$ forms a complete system of eigenvalues of this operator corresponding to the complete orthonormal system

$$\phi_k(x) = e^{2\pi ikx}, \qquad T\phi_k = c_k\phi_k.$$

The system ϕ_k is also complete for T^*, $T^*\phi_k = \overline{c_k}\phi_k$, so that the singular values of T are given by $s_k(T) = |c_k|$, and hence by (1.5) we have

$$\sum_{k=-\infty}^{\infty} s_k(T)^p = \infty \quad \text{for any } p < 2.$$

In other words, in contrast to the case of the class S_2 of Hilbert–Schmidt operators which is characterised by the square integrability of the kernel, Carleman's result shows that below the index $p = 2$ the class of kernels generating operators in the Schatten class S_p cannot be characterised by criteria of the type

$$\iint \left|K(x,y)\right|^\alpha dx\,dy < \infty,$$

since the kernel $K(x,y) = \varkappa(x-y)$ of the operator T in (1.6) satisfies any kind of integral condition of such form due to the boundedness of $\varkappa$.

This example demonstrates the relevance of obtaining criteria for operators to belong to Schatten classes for $p < 2$ and, in particular, motivates the results in this paper. Among other things, we may also note that the continuity of the kernel (as in the above example) also does not guarantee that the operator would belong to any of the Schatten classes S_p with $p < 2$. Therefore, it is natural to ask what regularity imposed on the kernel would guarantee such inclusions (for example, the C^2 condition in Sugiura's result mentioned earlier does imply the traceability on $\mathbb{T}^1$). Thus, these questions will be the main interest of the present paper.

As for criteria for operators to belong to Schatten classes S_p for $0 < p < 2$, a simplified version of our result in given in Theorem 3.6, for kernels in Sobolev spaces can be stated as follows:

Theorem 1.1. *Let M be a closed manifold of dimension n. Let $K \in H^\mu(M \times M)$ for some $\mu > 0$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x) = \int_M K(x,y)f(y)\,dy,$$

is in the Schatten classes $S_p(L^2(M))$ for

$$p > \frac{2n}{n + 2\mu}.$$

In particular, if $\mu > \frac{n}{2}$, then T is trace class.

This result improves, for example, Sugiura's result for the operator (1.1). Theorem 1.1 follows from the main result Theorem 3.6 giving criteria in terms of the mixed Sobolev spaces, Proposition 4.3, and Corollary 4.2. In particular, the use of mixed Sobolev spaces in Theorem 3.6 allows us to formulate criteria requiring different (smaller) regularities of $K(x,y)$ in x and y, or only in one of these variables.

We note that the situation for Schatten classes S_p for $p>2$ is simpler and, in fact, similar to that of $p=2$. For example, for left-invariant operators on compact Lie groups G, i.e. for convolution operators of the form $Tf = f * \varkappa$, it was shown in [11] that

$$\varkappa \in L^{p'}(G), \quad 1 \leqslant p' \leqslant 2 \quad \Longrightarrow \quad T \in S_p\big(L^2(G)\big), \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

The converse of this is also true but for interchanged indices, i.e.

$$T \in S_p\big(L^2(G)\big), \quad 1 \leqslant p \leqslant 2 \quad \Longrightarrow \quad \varkappa \in L^{p'}(G).$$

We refer to [11] for this as well as for the symbolic characterisation of Schatten classes in the setting of compact Lie groups.

In this work we allow singularities in the kernel so that the formula (1.2) would need to be modified in order for the integral over the diagonal to make sense. In such case, in order to calculate the trace of an integral operator using a non-continuous kernel along the diagonal, one idea is to average it to obtain an integrable function. Such an averaging process has been introduced by Weidmann [42] in the Euclidean setting, and applied by Brislawn in [3,4] for integral operators on $L^2(\mathbb{R}^n)$ and on $L^2(\Omega, \mathcal{M}, \mu)$, respectively, where Ω is a second countable topological space endowed with a σ-finite Borel measure. The corresponding extensions to the L^p setting have been established in [8] and [9]. The L^2 regularity of such an averaging process is a consequence of the L^2-boundedness of the martingale maximal function. Denoting by $\widetilde{K}(x,x)$ the pointwise values of this averaging process, Brislawn [4] proved the following formula for a trace class operator T on $L^2(\mu)$, for the extension to L^p see [8]:

$$\operatorname{Tr}(T) = \int_{\Omega} \widetilde{K}(x,x)\, d\mu(x). \tag{1.7}$$

In Section 2 we describe the discrete Fourier analysis involved in our problem and establish several relations between eigenvalues and their multiplicities for elliptic positive pseudo-differential operators. Then in Section 3 we establish our criteria for Schatten classes on compact manifolds and, in particular, for the trace class in Section 4. For this, we briefly recall the definition of the averaging process involved in the formula (1.7). We also explain another method relating the convergence of the Fourier coefficients of the kernel with the traceability. In Section 5 the special case of compact Lie groups is considered where we show that the criteria can be also given using hypoelliptic operators.

2. Fourier analysis associated with an elliptic operator

In this section we start by recording some basic elements of Fourier analysis on compact manifolds which will be useful for our analysis.

Let M be a compact smooth manifold of dimension n without boundary, endowed with a fixed volume dx. We denote by $\Psi^\nu(M)$ the Hörmander class of pseudo-differential operators of order $\nu \in \mathbb{R}$, i.e. operators which, in every coordinate chart, are operators in Hörmander classes on $\mathbb{R}^n$ with symbols in $S^\nu_{1,0}$, see e.g. [31] or [23]. In this paper we will be using the class $\Psi^\nu_{cl}(M)$ of classical operators, i.e. operators with symbols having (in all local coordinates) an asymptotic expansion of the symbol in positively homogeneous components (see e.g. [13]). Furthermore, we denote by $\Psi^\nu_+(M)$ the class of positive definite operators in $\Psi^\nu_{cl}(M)$, and by $\Psi^\nu_e(M)$ the class of elliptic operators in $\Psi^\nu_{cl}(M)$. Finally,

$$\Psi^\nu_{+e}(M) := \Psi^\nu_+(M) \cap \Psi^\nu_e(M)$$

will denote the class of classical positive elliptic pseudo-differential operators of order ν. We note that complex powers of such operators are well-defined, see e.g. Seeley [29]. In fact, all pseudo-differential operators considered in this paper will be classical, so we may omit explicitly mentioning it every time, but we note that we could equally work with general operators in $\Psi^\nu(M)$ since their powers have similar properties, see e.g. [35].

We now associate a discrete Fourier analysis with the operator $E \in \Psi^\nu_{+e}(M)$ inspired by those considered by Seeley [28,30], see also [15]. However, we adapt it to our purposes and prove several auxiliary statements concerning the eigenvalues of E and their multiplicities, useful to us in the sequel.

The eigenvalues of E form a sequence $\{\lambda_j\}$, with multiplicities taken into account. For each eigenvalue λ_j, there is the corresponding finite dimensional eigenspace F_j of functions on M, which are smooth due to the ellipticity of E. We set

$$d_j := \dim F_j, \quad \text{and} \quad F_0 := \ker E, \qquad \lambda_0 := 0.$$

We also set $d_0 := \dim F_0$. Since the operator E is elliptic, it is Fredholm, hence also $d_0 < \infty$ (we can refer to [1] for various properties of F_0 and d_0).

We fix an orthonormal basis of $L^2(M, dx)$ consisting of eigenfunctions of E:

$$\{e^k_j\}^{1 \leqslant k \leqslant d_j}_{j \geqslant 0}, \tag{2.1}$$

where $\{e^k_j\}^{1 \leqslant k \leqslant d_j}$ is an orthonormal basis of F_j. We denote by $(\cdot,\cdot)$ the standard inner product on $L^2(M)$ associated with its volume element. Let $P_j : L^2(M) \to F_j$ be the corresponding orthogonal projections on F_j. We observe that

$$P_j f = \sum_{k=1}^{d_j} (f, e^k_j) e^k_j,$$

for $f \in L^2(M)$. The Fourier inversion formula takes the form

$$f = \sum_{j=0}^{\infty} P_j f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (f, e_j^k) e_j^k, \tag{2.2}$$

for each $f \in L^2(M)$, and where the convergence is understood with respect to the $L^2(M)$-norm.

Definition 2.1. Let $u \in \mathcal{D}'(M)$ and let $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be a nonnegative integer. We define $\widehat{u}(j) \in F_j^*$ by $\widehat{u}(j)(\varphi) := u(\varphi)$, for $\varphi \in F_j$.

For the distributional valuations we use the notation $u(\varphi)$ or $\langle u, \varphi \rangle$. If $\varphi \in \mathcal{D}(M)$ and $u \in \mathcal{D}'(M)$ we have

$$\varphi = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (\varphi, e_j^k) e_j^k,$$

and

$$u(\varphi) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (\varphi, e_j^k) u(e_j^k). \tag{2.3}$$

We note that the same type of formula holds for operators from $\mathcal{C}^\infty(M)$ to $\mathcal{D}'(M)$, namely, if $T : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ is a linear continuous operator, we have

$$T(\varphi) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (\varphi, e_j^k) T(e_j^k), \tag{2.4}$$

with an appropriate distributional understanding of convergence. The Fourier coefficients of $u \in \mathcal{D}'(M)$ associated with the basis (2.1) can be obtained from (2.3):

$$\widehat{u}(j)(\varphi) = \sum_{k=1}^{d_j} (\varphi, e_j^k) u(e_j^k).$$

In particular, if $u \in L^2(M)$ we obtain

$$\widehat{u}(j)(\varphi) = \sum_{k=1}^{d_j} (\varphi, e_j^k) (u, \overline{e_j^k}).$$

From the paragraph above we can deduce:

Proposition 2.2. *If $u \in \mathcal{D}'(M)$ then $\widehat{u}(j) \circ P_j$ is in $\mathcal{D}'(M)$, and*

$$u = \sum_{j=0}^{\infty} \widehat{u}(j) \circ P_j \quad \text{in } \mathcal{D}'(M). \tag{2.5}$$

Proof. If $\varphi \in \mathcal{D}(M)$ is a test function we have

$$\begin{aligned}\sum_{j=0}^{\infty} \big(\widehat{u}(j) \circ P_j\big)(\varphi) &= \sum_{j \geqslant 0} \widehat{u}(j)(P_j\varphi) = \sum_{j=0}^{\infty} \widehat{u}(j) \sum_{k=1}^{d_j} \big(\varphi, e_j^k\big) e_j^k \\ &= \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \big(\varphi, e_j^k\big) \widehat{u}(j)\big(e_j^k\big) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \big(\varphi, e_j^k\big) u\big(e_j^k\big) = u(\varphi),\end{aligned}$$

in view of (2.3), completing the proof. □

Comparing the Fourier inversion formula (2.2) in $L^2(M)$ with the formula (2.5) in $\mathcal{D}'(M)$, for a function $f \in L^2(M)$, we can identify its distributional Fourier coefficients $\widehat{f}(j)$ with their action on the basis of F_j given by

$$\widehat{f}(j,k) := \big(f, e_j^k\big).$$

We will also denote sometimes by $\mathcal{F}$ the Fourier transform associating with $f \in L^2(M)$ its Fourier coefficients. Since $\{e_j^k\}_{j \geqslant 0}^{1 \leqslant k \leqslant d_j}$ forms a complete orthonormal system in $L^2(M)$, for all $f \in L^2(M)$ we have the Plancherel formula

$$\|f\|_{L^2(M)}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \big|\big(f, e_j^k\big)_{L^2}\big|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \big|\widehat{f}(j,k)\big|^2. \tag{2.6}$$

Since the criteria that we will obtain depend (a priori) on the choice of the orthonormal basis $\{e_j^k\}$, the asymptotics of the corresponding eigenvalues play an essential role. We now establish several simple but useful relations between the eigenvalues λ_j and their multiplicities d_j.

Proposition 2.3. *Let M be a closed manifold of dimension n, and let $E \in \Psi_{+e}^{\nu}(M)$, with $\nu > 0$. Then there exists a constant $C > 0$ such that*

$$d_j \leqslant C(1 + \lambda_j)^{\frac{n}{\nu}} \tag{2.7}$$

for all $j \geqslant 1$. Moreover, we have

$$\sum_{j=1}^{\infty} d_j (1 + \lambda_j)^{-q} < \infty \quad \text{if and only if} \quad q > \frac{n}{\nu}. \tag{2.8}$$

Proof. We observe that $(1+\lambda_j)^{1/\nu}$ is an eigenvalue of the first-order elliptic positive operator $(I+E)^{1/\nu}$ of multiplicity d_j. The Weyl formula for the eigenvalue counting function for the operator $(I+E)^{1/\nu}$ yields

$$\sum_{j:\ (1+\lambda_j)^{1/\nu}\leqslant\lambda} d_j = C_0\lambda^n + O\big(\lambda^{n-1}\big)$$

as $\lambda\to\infty$. This implies $d_j \leqslant C(1+\lambda_j)^{n/\nu}$ for sufficiently large λ_j, implying (2.7).

We now prove (2.8). Let us denote $T := (I+E)^{-q/2}$. The eigenvalues of T are $(1+\lambda_j)^{-q/2}$ with multiplicities d_j, therefore, we obtain

$$\sum_{j=0}^{\infty} d_j(1+\lambda_j)^{-q} = \|T\|_{S_2}^2 \asymp \|K\|_{L^2(M\times M)}^2. \tag{2.9}$$

At the same time, by the functional calculus of pseudo-differential operators, we know that $T\in\Psi^{-\nu q/2}(M)$, so that its kernel $K(x,y)$ is smooth for $x\neq y$, and near the diagonal $x=y$, identifying points with their local coordinates, it satisfies the estimate

$$\big|K(x,y)\big| \leqslant C_\alpha|x-y|^{-\alpha},$$

for any $\alpha > n-\nu q/2$, see e.g. [13] or [23, Theorem 2.3.1], and the order is sharp with respect to the order of the operator. Thus, we get that $K\in L^2(M\times M)$ if and only if we can choose α such that $n > 2\alpha > 2n-\nu q$, which together with (2.9) implies (2.8). □

3. Schatten classes on compact manifolds

Before stating our first result, we point out that a look at the proof of (1.2) (cf. [37, Prop. 3.5]) shows that statement can be already improved in the following way:

Proposition 3.1. *Let $\Delta = \frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial\phi^2}$ be the Laplacian on $\mathbb{T}^2$. Let $K(\theta,\phi)$ be a measurable function on $\mathbb{T}^2$ and suppose that there exists an integer $q>1$ such that $\Delta^{\frac{q}{2}}K \in L^2(\mathbb{T}^1\times\mathbb{T}^1)$. Then the integral operator L on $L^2(\mathbb{T}^1)$, defined by*

$$(Lf)(\theta) = \int_0^{2\pi} K(\theta,\phi)f(\phi)\,d\phi,$$

is trace class and has the trace

$$\mathrm{Tr}(L) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{K}(\theta,\theta)\,d\theta,$$

where $\widetilde{K}$ stands for the averaging process described in Section 4.

Our criteria for Schatten classes will also depend on a test of square integrability operating on the kernels through an elliptic operator, and the result of Proposition 3.1 will be improved in Theorem 3.6 (see specifically Corollary 4.2) by using a different approach to the problem. In the auxiliary next lemma we show that such condition is independent of the choice of an elliptic operator.

Lemma 3.2. *Let M be a closed manifold. Let $E_1, E_2 \in \Psi_e^\nu(M)$ with $\nu \in \mathbb{R}$ and let $h \in \mathcal{D}'(M)$. Then $E_1 h \in L^2(M)$ if and only if $E_2 h \in L^2(M)$.*

Proof. Let us suppose that $E_1 h \in L^2(M)$ and consider a parametrix $L_1 \in \Psi^{-\nu}(M)$ of E_1. In particular, there exists $R \in \Psi^{-\infty}(M)$ such that

$$L_1 E_1 - R = I.$$

Then

$$\begin{aligned} E_2 h &= E_2(L_1 E_1 - R)h \\ &= (E_2 L_1)(E_1 h) - (E_2 R)h, \end{aligned}$$

with $E_2 L_1 \in \Psi^0(M), E_2 R \in \Psi^{-\infty}(M)$. Since $E_1 h \in L^2(M)$ we obtain $(E_2 L_1)(E_1 h) \in L^2(M)$; the fact that $E_2 R$ is smoothing gives us $(E_2 R)h \in L^2(M)$, implying that $E_2 h \in L^2(M)$. $\square$

We first establish a simple observation for powers of positive elliptic operators to belong to Schatten classes S_p on $L^2(M)$.

Proposition 3.3. *Let M be a closed manifold of dimension n, and let $E \in \Psi_{+e}^\nu(M)$ be a positive elliptic pseudo-differential operator of order $\nu > 0$. Let $0 < p < \infty$. Then*

$$(I + E)^{-\alpha} \in S_p\big(L^2(M)\big) \quad \textit{if and only if} \quad \alpha > \frac{n}{p\nu}. \tag{3.1}$$

Proof. Let λ_j denote the eigenvalues of E, each λ_j having the multiplicity d_j. Then the operator $(I + E)^{-\alpha}$ is positive definite, its singular values are $(1 + \lambda_j)^{-\alpha}$ with multiplicities d_j. Therefore,

$$\big\|(I + E)^{-\alpha}\big\|_{S_p}^p = \sum_{j=0}^{\infty} d_j (1 + \lambda_j)^{-\alpha p},$$

which is finite if and only if $\alpha p > \frac{n}{\nu}$ by (2.8), implying the statement. $\square$

If P is a pseudo-differential operator on M, for a function (or distribution) on $M \times M$, we will use the notation $P_y K(x, y)$ to indicate that the operator P is acting on the

y-variable, the second factor of the product $M \times M$. For a positive elliptic operator $P \in \Psi^{\nu}_{+e}(M)$, by the elliptic regularity, the Sobolev space $H^{\mu}(M)$ can be characterised as the space of all distributions $f \in \mathcal{D}'(M)$ such that $(I+P)^{\frac{\mu}{\nu}} f \in L^2(M)$, and this characterisation is independent of the choice of operator P (see also Lemma 3.2).

We now define Sobolev spaces $H^{\mu_1,\mu_2}_{x,y}(M \times M)$ of mixed regularity $\mu_1, \mu_2 \geqslant 0$. We observe that for $K \in L^2(M \times M)$, we have

$$\|K\|^2_{L^2(M\times M)} = \int\limits_{M\times M} \bigl|K(x,y)\bigr|^2\,dx\,dy = \int\limits_M \biggl(\int\limits_M \bigl|K(x,y)\bigr|^2\,dy\biggr)\,dx,$$

or we can also write this as

$$K \in L^2(M \times M) \quad \Longleftrightarrow \quad K \in L^2_x\bigl(M, L^2_y(M)\bigr). \tag{3.2}$$

In particular, this means that K_x defined by $K_x(y) = K(x,y)$ is well-defined for almost every $x \in M$ as a function in $L^2_y(M)$.

Definition 3.4. Let $K \in L^2(M\times M)$ and let $\mu_1, \mu_2 \geqslant 0$. We say that $K \in H^{\mu_1,\mu_2}_{x,y}(M\times M)$ if $K_x \in H^{\mu_2}(M)$ for almost all $x \in M$, and if $(I+P_x)^{\frac{\mu_1}{\nu}} K_x \in L^2_x(M, H^{\mu_2}_y(M))$ for some $P \in \Psi^{\nu}_{+e}(M)$, $\nu > 0$. We set

$$\|K\|_{H^{\mu_1,\mu_2}_{x,y}(M\times M)} := \biggl(\int\limits_M \bigl\|(I+P_x)^{\frac{\mu_1}{\nu}} K_x\bigr\|^2_{H^{\mu_2}(M)}\,dx\biggr)^{1/2}.$$

By the elliptic regularity it follows that different choices of operators $P \in \Psi^{\nu}_{+e}(M)$, $\nu > 0$, give equivalent norms on the space $H^{\mu_1,\mu_2}_{x,y}(M \times M)$. Thus, for operators $E_j \in \Psi^{\nu_j}_{+e}(M)$ $(j = 1,2)$ with $\nu_j > 0$, we can formulate Definition 3.4 in an alternative (and perhaps more practical) way:

Definition 3.5. For operators $E_j \in \Psi^{\nu_j}_{+e}(M)$ $(j = 1,2)$ with $\nu_j > 0$, we define

$$K \in H^{\mu_1,\mu_2}_{x,y}(M \times M) \quad \Longleftrightarrow \quad (I+E_1)^{\frac{\mu_1}{\nu_1}}_x (I+E_2)^{\frac{\mu_2}{\nu_2}}_y K \in L^2(M \times M), \tag{3.3}$$

where the expression on the right hand side means that we are applying pseudo-differential operators on M separately in x and y. We note that these operators commute since they are acting on different sets of variables of K.

As we have noted above, the definition does not depend on a particular choice of operators $E_j \in \Psi^{\nu_j}_{+e}(M)$, with the norms of K induced by (3.3) being all equivalent to each other and to that in Definition 3.4. In Proposition 4.3 we establish some properties of the spaces $H^{\mu_1,\mu_2}_{x,y}$, namely, we will show the inclusions between the mixed and the standard Sobolev spaces on the compact (closed) manifold $M \times M$ as

$$H^{\mu_1+\mu_2}(M \times M) \subset H^{\mu_1,\mu_2}_{x,y}(M \times M) \subset H^{\min(\mu_1,\mu_2)}(M \times M),$$

for any $\mu_1, \mu_2 \geqslant 0$.

We will now give our main criteria for Schatten classes.

Theorem 3.6. *Let M be a closed manifold of dimension n and let $\mu_1, \mu_2 \geqslant 0$. Let $K \in L^2(M \times M)$ be such that $K \in H^{\mu_1,\mu_2}_{x,y}(M \times M)$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x) = \int_M K(x, y)f(y)\, dy,$$

is in the Schatten classes $S_r(L^2(M))$ for

$$r > \frac{2n}{n + 2(\mu_1 + \mu_2)}.$$

Remark 3.7. The value for r comes from the relation

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2},$$

for some $0 < p_1, p_2 < \infty$, where the condition $r > \frac{2n}{n+2(\mu_1+\mu_2)}$ comes from $\mu_j > \frac{n}{p_j \nu_j}$ by a suitable application of (3.1). Also, since then $r = \frac{2p_1p_2}{p_1p_2+2(p_1+p_2)}$, the range for r is the interval $(0, 2)$ since, in general, $0 < p_j < \infty$. Therefore, Theorem 3.6 provides a sufficient condition for Schatten classes S_r for $0 < r < 2$. For $\mu_1, \mu_2 = 0$ the conclusion is trivial and can be sharpened to include $r = 2$.

Remark 3.8. We note that for $\mu_1 = 0$, Theorem 3.6 says that for $K \in L^2(M, H^\mu(M))$, we have that the corresponding operator T satisfies $T \in S_r$ for

$$r > \frac{2n}{n + 2\mu}.$$

In this case no regularity in the x-variable is imposed on the kernel.

We also note that the 'dual' result with $\mu_2 = 0$ imposing no regularity of K with respect to y also follows directly from it by considering the adjoint operator T^* and using the equality $\|T^*\|_{S_r} = \|T\|_{S_r}$.

Proof of Theorem 3.6. Let, for example, $E = (I+\Delta_M)^{\frac{1}{2}}$, where Δ_M is a positive definite elliptic differential operator of order 2, and $E = \overline{E}$. The existence of such Δ_M follows, for example, from the Whitney embedding theorem.

(i) We first suppose that $\mu_1, \mu_2 > 0$. By Proposition 3.3 with $\alpha = \mu_2$ and $\nu = 1$ we get, in particular, that $E_y^{-\mu_2} \in S_{p_2}(L^2(M))$ for

$$\mu_2 > \frac{n}{p_2},$$

the first ingredient in the proof.

Now let us consider the kernel $B(x,y)$ of the operator $E_y^{-\mu_2} \in S_{p_2}(L^2(M))$ for $\mu_2 > \frac{n}{p_2}$. If $f \in L^2(M)$, let

$$g(y) := \int_M B(y,z) f(z)\, dz,$$

so that

$$E_y^{\mu_2} g(y) = f(y).$$

First we note that by (3.3) the condition $K \in H_{x,y}^{\mu_1;\mu_2}(M \times M)$ can be written as $E_x^{\mu_1} E_y^{\mu_2} K \in L^2(M \times M)$, with $\nu_1 = \nu_2 = 1$. Since $E_x^{\mu_1} E_y^{\mu_2} K \in L^2(M \times M)$, we have $E_x^{\mu_1} E_y^{\mu_2} K(x,\cdot) \in L_y^2(M)$ for almost every x, and this fact will justify the use of scalar products in the next argument.

We observe that

$$\begin{aligned}
E_x^{\mu_1} T f(x) &= \int_M E_x^{\mu_1} K(x,y) f(y)\, dy \\
&= \int_M E_x^{\mu_1} K(x,y) E_y^{\mu_2} g(y)\, dy \\
&= \left(E_x^{\mu_1} K(x,\cdot), E_y^{\mu_2} \bar{g}\right)_{L^2(M)} \\
&= \left(E_y^{\mu_2} E_x^{\mu_1} K(x,\cdot), \bar{g}\right)_{L^2(M)} \\
&= \left(E_x^{\mu_1} E_y^{\mu_2} K(x,\cdot), \bar{g}\right)_{L^2(M)} \\
&= \int_M E_x^{\mu_1} E_y^{\mu_2} K(x,y) g(y)\, dy \\
&= \int_M E_x^{\mu_1} E_y^{\mu_2} K(x,y) \left(\int_M B(y,z) f(z)\, dz \right) dy.
\end{aligned}$$

Now, setting

$$A(x,y) := E_x^{\mu_1} E_y^{\mu_2} K(x,y),$$

we have shown that

$$E_x^{\mu_1} Tf(x) = \int_M E_x^{\mu_1} K(x,y) f(y)\, dy = \int_M A(x,y) \left(\int_M B(y,z) f(z)\, dz \right) dy,$$

thus we have factorised the operator $E_x^{\mu_1} T$ with $A \in S_2$ and $B \in S_{p_2}$. By (1.4) we get that $E_x^{\mu_1} T \in S_t$ with $\frac{1}{t} = \frac{1}{2} + \frac{1}{p_2}$.

On the other hand, since $E_x^{-\mu_1} \in S_{p_1}$ for $\mu_1 > \frac{n}{p_1}$, we obtain

$$T = E_x^{-\mu_1} E_x^{\mu_1} T \in S_r,$$

with

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{t} = \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2}.$$

Using inequalities

$$p_1 > \frac{n}{\mu_1} \quad \text{and} \quad p_2 > \frac{n}{\mu_2},$$

this is equivalent to

$$r > \frac{2n}{n + 2(\mu_1 + \mu_2)}.$$

(*ii*) If $\mu_1 = 0$ and $\mu_2 > 0$, just by removing the operator $E_x^{\mu_1}$ in the argument above we get the desired result.

(*iii*) If $\mu_1 > 0$ and $\mu_2 = 0$. This is a consequence of case (*ii*) proceeding by duality, considering the adjoint of the operator T and applying the fact that $\|T\|_{S_r} = \|T^*\|_{S_r}$. □

4. Trace class operators and their traces

We shall now briefly recall the averaging process which is required for the study of trace formulae for kernels with discontinuities along the diagonal. We start by defining the martingale maximal function. Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space and let $\{\mathcal{M}_j\}_j$ be a sequence of sub-σ-algebras such that

$$\mathcal{M}_j \subset \mathcal{M}_{j+1} \quad \text{and} \quad \mathcal{M} = \bigcup_j \mathcal{M}_j.$$

In order to define conditional expectations we assume that μ is σ-finite on each $\mathcal{M}_j$. In that case, if $f \in L^p(\mu)$, then $E(f|\mathcal{M}_n)$ exists. We say that a sequence $\{f_j\}_j$ of functions on Ω is a *martingale* if each f_j is $\mathcal{M}_j$-measurable and

$$E(f_j|\mathcal{M}_k) = f_k \quad \text{for } k < j. \tag{4.1}$$

In order to obtain a generalisation of the Hardy–Littlewood maximal function we consider the particular case of martingales generated by a single $\mathcal{M}$-measurable function f. The *martingale maximal function* is defined by

$$Mf(x) := \sup_j E\big(|f|\big|\mathcal{M}_j\big)(x). \tag{4.2}$$

This martingale can be defined, in particular, on a second countable topological space endowed with a σ-finite Borel measure. For our purposes in the study of the kernel the sequence of σ-algebras is constructed from a corresponding increasing sequence of partitions $\mathcal{P}_j \times \mathcal{P}_j$ of $\Omega \times \Omega$ with $\Omega = M$, the closed manifold.

Now, for each $(x, y) \in M \times M$ there is a unique $C_j(x) \times C_j(y) \in \mathcal{P}_j \times \mathcal{P}_j$ containing (x, y). Those sets $C_j(x)$ replace the cubes in $\mathbb{R}^n$ in the definition of the classical Hardy–Littlewood maximal function. We refer to Doob [12] for more details on the martingale maximal function and its properties.

We denote by $A_j^{(2)}$ the averaging operators on $\Omega \times \Omega$: Let $K \in L^1_{loc}(\mu \otimes \mu)$, then the averaging $A_j^{(2)}$ is defined $\mu \otimes \mu$-almost everywhere (cf. [4]) by

$$A_j^{(2)} K(x, y) := \frac{1}{\mu(C_j(x))\mu(C_j(y))} \int\limits_{C_j(x)} \int\limits_{C_j(y)} K(s, t)\, d\mu(t)\, d\mu(s). \tag{4.3}$$

The averaging process will be applied to the kernels $K(x, y)$ of our operators. As a consequence of the fundamental properties of the martingale maximal function it can be deduced that

$$\widetilde{K}(x, y) := \lim_{j \to \infty} A_j^{(2)} K(x, y),$$

is defined almost everywhere and that it agrees with $K(x, y)$ in the points of continuity. We observe that if $K(x, y)$ is the integral kernel of a trace class operators, then $K(x, y)$ is, in particular, square integrable, and hence by the Hölder inequality it is integrable on the compact manifold $M \times M$.

In the sequel in this section, we can always assume that $K \in L^2(M \times M)$ since it is not restrictive because the trace class is included in the Hilbert–Schmidt class, and the square integrability of the kernel is then a necessary condition.

As a corollary of Theorem 3.6, for the trace class operators we have:

Corollary 4.1. *Let M be a closed manifold of dimension n and let $K \in L^2(M \times M)$, $\mu_1, \mu_2 \geqslant 0$, be such that*

$$\mu_1 + \mu_2 > \frac{n}{2}$$

and $K \in H^{\mu_1,\mu_2}_{x,y}(M \times M)$. Then the integral operator T on $L^2(M)$, defined by

$$(Tf)(x) = \int\limits_M K(x,y)f(y)\,dy$$

is trace class and its trace is given by

$$\operatorname{Tr}(T) = \int\limits_M \widetilde{K}(x,x)\,dx. \tag{4.4}$$

Proof. We observe that to get $r = 1$ from Theorem 3.6, we require the following inequality to hold:

$$1 > \frac{2n}{n + 2(\mu_1 + \mu_2)}.$$

But this is equivalent to $\mu_1 + \mu_2 > \frac{n}{2}$. The trace formula is a consequence of (1.7). □

Corollary 4.1 improves, in particular, Proposition 3.1:

Corollary 4.2. *Let M be a smooth closed manifold of dimension n. Let $K \in L^2(M \times M)$ be such that $K \in H^\mu(M \times M)$ for $\mu > \frac{n}{2}$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x) = \int\limits_M K(x,y)f(y)\,dy,$$

is trace class on $L^2(M)$ and its trace is given by (4.4).

Indeed, Corollary 4.2 follows from Corollary 4.1 and the inclusion

$$H^\mu(M \times M) \subset H^{0,\mu}_{x,y}(M \times M),$$

the latter being a special case of the following inclusions between usual and mixed Sobolev spaces:

Proposition 4.3. *Let M be a smooth closed manifold. Then we have the inclusions*

$$H^{\mu_1+\mu_2}(M \times M) \subset H^{\mu_1,\mu_2}_{x,y}(M \times M) \subset H^{\min(\mu_1,\mu_2)}(M \times M), \tag{4.5}$$

for any $\mu_1, \mu_2 \geqslant 0$.

Proof. Let Δ be a second order positive elliptic differential operator on M (such an operator exists e.g. by the Whitney embedding theorem), and let e_j^k denote the orthonormal basis and λ_j the corresponding eigenvalues, leading to the discrete Fourier analysis associated with Δ as described in Section 2. Then the products $e_j^k(x)e_l^m(y)$ give rise to an orthonormal basis in $L^2(M\times M)$. Consequently, using (3.3) with $E_1=E_2=\Delta$, we see that $K\in H^{\mu_1,\mu_2}_{x,y}(M\times M)$ is equivalent to the condition

$$\sum_{j=0}^{\infty}\sum_{k=1}^{d_j}\sum_{l=0}^{\infty}\sum_{m=1}^{d_l}(1+\lambda_j)^{\mu_1}(1+\lambda_l)^{\mu_2}\left|\widehat{K}(j,k,l,m)\right|^2<\infty, \tag{4.6}$$

where the Fourier coefficients $\widehat{K}(j,k,l,m)$ are determined by the Fourier series

$$K(x,y)=\sum_{j=0}^{\infty}\sum_{k=1}^{d_j}\sum_{l=0}^{\infty}\sum_{m=1}^{d_l}\widehat{K}(j,k,l,m)e_j^k(x)e_l^m(y).$$

On the other hand, we note that

$$(1+\Delta_x+\Delta_y)e_j^k(x)e_l^m(y)=(1+\lambda_j+\lambda_l)e_j^k(x)e_l^m(y),$$

which implies that $K\in H^{\mu}(M\times M)$ is equivalent to

$$\sum_{j=0}^{\infty}\sum_{k=1}^{d_j}\sum_{l=0}^{\infty}\sum_{m=1}^{d_l}(1+\lambda_j+\lambda_l)^{\mu}\left|\widehat{K}(j,k,l,m)\right|^2<\infty. \tag{4.7}$$

Comparing the expressions in (4.6) and (4.6), and using the inequalities

$$(1+\lambda_j+\lambda_l)^{\min(\mu_1,\mu_2)}\leqslant C(1+\lambda_j)^{\mu_1}(1+\lambda_l)^{\mu_2}\leqslant C(1+\lambda_j+\lambda_l)^{\mu_1+\mu_2},$$

we obtain the inclusions (4.5). □

We also obtain some corollaries in terms of the derivatives of the kernel. We denote by $C_x^\alpha C_y^\beta(M\times M)$ the space of functions of class C^β with respect to y and C^α with respect to x.

Corollary 4.4. *Let M be a closed manifold of dimension n. Let $K\in C_x^{\ell_1}C_y^{\ell_2}(M\times M)$ some even integers $\ell_1,\ell_2\in 2\mathbb{N}_0$ such that $\ell_1+\ell_2>\frac{n}{2}$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x)=\int_M K(x,y)f(y)\,dy,$$

is in $S_1(L^2(M))$, and its trace is given by

$$\operatorname{Tr}(T)=\int_M K(x,x)\,dx. \tag{4.8}$$

Proof. Let Δ_M be an elliptic positive definite second order differential operator on M, then $(I+\Delta_M)_x^{\frac{\ell_1}{2}}(I+\Delta_M)_y^{\frac{\ell_2}{2}}K \in C(M\times M) \subset L^2(M\times M)$. Now, by observing that $\ell_1+\ell_2 > \frac{n}{2}$ the result follows from Corollary 4.1. □

Remark 4.5. The index $\frac{n}{2}$ in Corollary 4.4 is sharp. Indeed, for the torus $\mathbb{T}^n$ with n even, there exist a function χ of class $C^{\frac{n}{2}}$ such that the series of Fourier coefficients diverges (cf. [34, Ch. VII], [41]). By considering the convolution kernel $K(x,y)=\chi(x-y)$, the singular values of the operator T given by $Tf = f*\chi$ agree with the absolute values of the Fourier coefficients. Hence, $T \notin S_1(L^2(\mathbb{T}^n))$ but $K \in C^{\frac{n}{2}}(M\times M)$ (we can think of e.g. $\ell_1 = 0$ and $\ell_2 = \frac{n}{2}$ in Corollary 4.4). On the other hand, concerning necessary conditions on the kernel, writing the convolution operator T in the pseudo-differential form

$$Tf(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\cdot\xi}\sigma(\xi)\widehat{f}(\xi)$$

with $\sigma(\xi) = \widehat{\chi}(\xi)$, it can be shown that $\sum_{\xi\in\mathbb{Z}^n}|\sigma(\xi)| < \infty$ if and only if the corresponding pseudo-differential operator T_σ is trace class on $L^2(\mathbb{T}^n)$ (cf. [11]). Hence, when dealing with a multiplier we can deduce that if T_σ is trace class then its kernel is continuous. This can be obtained from the formula for the convolution kernel

$$K(x,y) = \sum_{\xi\in\mathbb{Z}^n} e^{i(x-y)\xi}\sigma(\xi)$$

and the summability of σ. Therefore, the continuity of kernels is a necessary condition for traceability of convolution operators on $\mathbb{T}^n$. However, as we note from the example (1.6) on $M = \mathbb{T}^1$ the convolution kernel $K(x,y) = \varkappa(x-y)$ is continuous but the corresponding operator is not trace class.

We now make some remarks about the relation between the trace class property and the Fourier coefficients of the kernel, in the sense of Section 2.

The main idea in the proof by Sugiura of (1.2) (and then also of Proposition 3.1) consists in exploiting the underlying relation between the convergence of the series of Fourier coefficients of the kernel $K \in C^2(\mathbb{T}^1\times\mathbb{T}^1)$ and the traceability. This link is basically reduced to the application of the following classical result, see e.g. [37]:

Lemma 4.6. *Let H be a separable Hilbert space. If a bounded linear operator T on H satisfies*

$$\sum_{m,j\in\mathbb{N}} \left|(T\phi_j,\phi_m)\right| < \infty$$

for a fixed orthonormal basis $(\phi_j)_j$, then T is trace class.

By choosing an orthonormal basis $\{\phi_j\}$ consisting of eigenfunctions of the Laplacian on $\mathbb{T}^1 \times \mathbb{T}^1$ one can prove that the Fourier coefficients of K agree with the values $(T\phi_j, \phi_m)$. On the other hand, it can be shown that the series of Fourier coefficients converges absolutely and hence the traceability follows. The proof can be extended to smooth closed manifolds by using elliptic positive pseudo-differential operators instead of Laplacians on $M \times M$, and associating a discrete Fourier analysis with the cross product $M \times M$. However, this method leads to a weaker result, by furnishing the condition $K = K(x, y) \in C^\nu(M \times M)$ with $\nu > n$. We have improved that kind of result by obtaining a sharp condition on the regularity of K with respect to y as given in Corollary 4.4 (with $\ell_1 = 0$ and $\ell_2 > n/2$).

However, for the sake of completeness we establish below a result concerning the convergence of series of Fourier coefficients. The related problem concerning the convergence of Fourier series on compact manifolds has been studied by Taylor [38]. Taylor's paper also included a special version for compact Lie groups. Similar results for compact connected Lie groups from a different approach were obtained by Sugiura in [36]. In order to study such kind of convergence for a kernel K on $M \times M$ we will first apply the convergence criterion (2.8) to the manifold $M \times M$. In the following lemma we will associate with an elliptic positive pseudo-differential operator E of order ν on $M \times M$ an orthonormal basis of $L^2(M \times M)$ which we denote by $\{e_\ell^m\}_{\ell \geqslant 0}^{1 \leqslant m \leqslant d_\ell}$.

Lemma 4.7. *Let M be a compact smooth manifold of dimension n, and let $E \in \Psi_{+e}^\nu(M \times M)$ with $\nu > n$. Let $K \in L^2(M \times M)$ be such that $EK \in L^2(M \times M)$. Let λ_ℓ be the eigenvalues of E and let d_ℓ be the corresponding multiplicities. With respect to the orthonormal basis $\{e_\ell^m\}_{\ell \geqslant 0}^{1 \leqslant m \leqslant d_\ell}$ of $L^2(M \times M)$, $K(x, y)$ can be written as*

$$K(x, y) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \widehat{K}(\ell, m) e_\ell^m(x, y), \tag{4.9}$$

with $\sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} |\widehat{K}(\ell, m)| < \infty$, and the condition $EK \in L^2(M \times M)$ is independent of the choice of the operator E in $\Psi_{+e}^\nu(M \times M)$, and can be expressed as the Sobolev space condition $K \in H^\nu(M \times M)$.

Proof. We define

$$h(x, y) := EK(x, y),$$

which is in $L^2(M \times M)$. With respect to the orthonormal basis $\{e_\ell^m\}_{\ell \geqslant 0}^{1 \leqslant m \leqslant d_\ell}$ of $L^2(M \times M)$ the kernel $K(x, y)$ can be written as (4.9). At the same time, the Fourier coefficients of h satisfy, by the Plancherel formula (2.6),

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left|\widehat{h}(\ell, m)\right|^2 = \|h\|_{L^2(M \times M)}^2, \tag{4.10}$$

and

$$\widehat{h}(\ell, m) = \widehat{EK}(\ell, m) = \lambda_\ell \widehat{K}(\ell, m),$$

for $0 \leqslant \ell < \infty$ and $1 \leqslant m \leqslant d_\ell$.

For the Fourier coefficients of K, by the Cauchy–Schwartz inequality, (4.10) and the inequality (2.8) applied to E on $M \times M$ we obtain

$$\sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell} \left|\widehat{K}(\ell, m)\right| \leqslant C \sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell} \lambda_\ell \frac{|\widehat{K}(\ell, m)|}{1+\lambda_\ell} = C \sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell} \frac{|\widehat{h}(\ell, m)|}{1+\lambda_\ell}$$
$$\leqslant C\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell} \left|\widehat{h}(\ell, m)\right|^2\right)^{\frac{1}{2}} \left(\sum_{\ell} d_\ell (1+\lambda_\ell)^{-2}\right)^{\frac{1}{2}} \leqslant C\|h\|_{L^2},$$

where for the application of (2.8), we note that the convergence of the series $\sum_\ell d_\ell (1+\lambda_\ell)^{-2}$ on $M \times M$ with $q = 2$ is equivalent to the condition $\nu > n$. The independence of the choice of an elliptic positive pseudo-differential operator of order ν is an immediate consequence of Lemma 3.2 for the manifold $M \times M$. $\square$

5. Schatten classes on compact Lie groups

In this section we consider the conditions for Schatten classes for operators on compact Lie groups. We show that the conditions on the kernel can be also formulated in terms of hypoelliptic operators. This is done by combining the factorisation method used in the previous sections with recent results [11] by the authors on characterisation of invariant operators in Schatten classes on compact Lie groups. We start by describing the basic concepts we will require for this setting.

Let G be a compact Lie group of dimension n with the normalised Haar measure dx. Let $\widehat{G}$ denote the set of equivalence classes of continuous irreducible unitary representations of G. Since G is compact, the set $\widehat{G}$ is discrete. For $[\xi] \in \widehat{G}$, by choosing a basis in the representation space of ξ, we can view ξ as a matrix-valued function $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space of ξ. By the Peter–Weyl theorem the collection

$$\left\{\sqrt{d_\xi}\, \xi_{ij} :\ 1 \leqslant i, j \leqslant d_\xi,\ [\xi] \in \widehat{G}\right\}$$

is the orthonormal basis of $L^2(G)$. If $f \in L^1(G)$ we define its global Fourier transform at ξ by

$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx.$$

If ξ is a matrix representation, we have $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier inversion formula is a consequence of the Peter–Weyl theorem, and we have

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\bigl(\xi(x)\widehat{f}(\xi)\bigr). \tag{5.1}$$

For each $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplacian $\mathcal{L}_G$ (or the Casimir element of the universal enveloping algebra), with the same eigenvalues which we denote by $-\lambda^2_{[\xi]}$, so that we have

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda^2_{[\xi]} \xi_{ij}(x) \quad \text{for all } 1 \leqslant i, j \leqslant d_\xi. \tag{5.2}$$

The weight for measuring the decay or growth of Fourier coefficients in this setting is

$$\langle \xi \rangle := \bigl(1 + \lambda^2_{[\xi]}\bigr)^{\frac{1}{2}},$$

the eigenvalues of the (positive) elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$. The Parseval identity takes the form

$$\|f\|_{L^2(G)} = \biggl(\sum_{[\xi] \in \widehat{G}} d_\xi \bigl\|\widehat{f}(\xi)\bigr\|^2_{\mathtt{HS}}\biggr)^{\frac{1}{2}}, \quad \text{where } \bigl\|\widehat{f}(\xi)\bigr\|^2_{\mathtt{HS}} = \operatorname{Tr}\bigl(\widehat{f}(\xi)\widehat{f}(\xi)^*\bigr),$$

which gives the norm on $\ell^2(\widehat{G})$.

For a linear continuous operator A from $C^\infty(G)$ to $\mathcal{D}'(G)$ we define its *matrix-valued symbol* $\sigma_A(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ by

$$\sigma_A(x, \xi) := \xi(x)^* (A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}. \tag{5.3}$$

Then one has [23,24] the global quantization

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\bigl(\xi(x)\sigma_A(x, \xi)\widehat{f}(\xi)\bigr) \tag{5.4}$$

in the sense of distributions, and the sum is independent of the choice of a representation ξ from each equivalence class $[\xi] \in \widehat{G}$. If A is a linear continuous operator from $C^\infty(G)$ to $C^\infty(G)$, the series (5.4) is absolutely convergent and can be interpreted in the pointwise sense. We will also write $A = \operatorname{Op}(\sigma_A)$ for the operator A given by the formula (5.4). We refer to [23,24] for the consistent development of this quantization and the corresponding symbolic calculus.

In the subsequent Part II of this paper, we will relate the Fourier and symbolic analysis on general closed manifolds to those on compact Lie groups. So, now we will only concentrate on showing that the conditions for Schatten classes can be also formulated by examining the regularity of the kernel under the action of non-elliptic but hypoelliptic operators.

Instead of Proposition 3.3 for elliptic operators as the starting point, here we will use its analogue for hypoelliptic operators established in [11], for example its analogue for sub-Laplacians on compact Lie groups G. Let us write

$$\begin{aligned} \mathcal{L}_G &= X_1^2 + \cdots + X_{n-1}^2 + X_n^2, \\ \mathcal{L}_{sub} &= X_1^2 + \cdots + X_{n-1}^2, \end{aligned} \tag{5.5}$$

for a basis $X_1, \ldots, X_n$ of left-invariant vector fields on the Lie algebra $\mathfrak{g}$ of G, assuming that the span of the first commutators of $X_1, \ldots, X_{n-1}$ contains X_n. Then it was shown in [11] that

$$0 < r < \infty \quad \text{and} \quad \alpha r > 2n \quad \Longrightarrow \quad (I - \mathcal{L}_{sub})^{-\alpha/2} \in S_r\big(L^2(G)\big). \tag{5.6}$$

Here we can note that the powers of the hypoelliptic positive pseudo-differential operator $I - \mathcal{L}_{sub}$ are well-defined. There is a general theory, see e.g. [18], [16], or more recent results and references in [5]. However, if we observe that the matrix symbol of the sub-Laplacian (as well as the symbol of the operator $\mathcal{H}_\gamma$ is the sequel) are diagonal, a complex power of such operator may be defined by the quantization formula (5.4) using the matrix symbol being the corresponding complex power of the diagonal symbol of the operator. For left-invariant operators with diagonal matrix symbols (such as $\mathcal{L}_{sub}$ or $\mathcal{H}_\gamma$) all such approaches yield the same operators (see e.g. [26] for more details).

Consequently, arguing in the same way as in the proof of Theorem 3.6 we obtain:

Corollary 5.1. *Let G be a compact Lie group of dimension n, and let $\mathcal{L}_{sub}$ be a sub-Laplacian as in (5.5). Let $K \in L^2(M \times M)$ be such that*

$$(I - \mathcal{L}_{sub})_x^{\mu_1/2} (I - \mathcal{L}_{sub})_y^{\mu_2/2} K \in L^2(G \times G)$$

for some $\mu_1, \mu_2 \geqslant 0$. Then the integral operator T on $L^2(G)$, defined by

$$(Tf)(x) = \int_G K(x, y) f(y)\, dy,$$

is in the Schatten classes $S_r(L^2(G))$ for

$$r > \frac{2n}{n + (\mu_1 + \mu_2)}.$$

Proof. We argue similar to the proof of Theorem 3.6. In particular, we know from (5.6) that $(I - \mathcal{L}_{sub})^{-\mu_j/2} \in S_{p_j}$ for $\mu_j p_j > 2n$ $(j = 1, 2)$. From the relation

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2}$$

and $p_j > \frac{2n}{\mu_j}$, we get that under the condition $r > \frac{2n}{n+(\mu_1+\mu_2)}$ the operator T belongs to the Schatten class S_r on $L^2(G)$. $\square$

As it was noted in [11], the implication (5.6) can be improved for particular groups using their particular structure. For example, for the compact Lie group SU(2) we have, for three left-invariant vector fields X, Y, Z that $[X, Y] = Z$, and so with

$$\mathcal{L}_{sub} = X^2 + Y^2$$

we have

$$0 < r < \infty \quad \text{and} \quad \alpha r > 4 \quad \Longrightarrow \quad (I - \mathcal{L}_{sub})^{-\alpha/2} \in S_r\big(L^2\big(\mathrm{SU}(2)\big)\big). \tag{5.7}$$

The same is true for $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ considered as the compact Lie group with the quaternionic product. Using this instead of (5.6), we get a refinement of Corollary 5.1 in the setting of $\mathbb{S}^3 \simeq \mathrm{SU}(2)$:

Corollary 5.2. *Let $K \in L^2(\mathbb{S}^3 \times \mathbb{S}^3)$ be such that we have*

$$(1 - \mathcal{L}_{sub})_x^{\mu_1/2}(1 - \mathcal{L}_{sub})_y^{\mu_2/2} K \in L^2\big(\mathbb{S}^3 \times \mathbb{S}^3\big)$$

for some $\mu_1, \mu_2 \geqslant 0$. Then the integral operator T on $L^2(\mathbb{S}^3)$, defined by

$$(Tf)(x) = \int_{\mathbb{S}^3} K(x,y) f(y)\, dy,$$

is in the Schatten classes $S_r(L^2(\mathbb{S}^3))$ for

$$r > \frac{4}{2 + \mu_1 + \mu_2}.$$

The same result holds on compact Lie groups SU(2) *and* SO(3).

Proof. Using (5.7) with $\alpha = \mu_j$, the assumption $\mu_j p_j > 4$ implies that $(I - \mathcal{L}_{sub})^{-\mu_j/2} \in S_{p_j}$ for $p_j > \frac{4}{\mu_j} (j = 1, 2)$. From $\frac{1}{r} = \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2}$ and $p_j > \frac{4}{\mu_j}$ we obtain the condition $r > \frac{4}{2+\mu_1+\mu_2}$. $\square$

We now show that instead of the sub-Laplacian other globally hypoelliptic operators can be used, also those that are not necessarily covered by Hörmander's sum of the squares theorem. Instead of SU(2), for a change, we will formulate this for the group SO(3) noting that, however, the same conclusion holds also on $\mathrm{SU}(2) \simeq \mathbb{S}^3$. To formulate and motivate the result, we first briefly introduce some more notation concerning the group $G = \mathrm{SO}(3)$ of the 3×3 real orthogonal matrices of determinant one. For the details of the representation theory and the global quantization of operators on SO(3) we refer the reader to [23, Chapter 12]. The unitary dual in this case of $G = \mathrm{SO}(3)$ can be identified as $\widehat{G} \simeq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, so that

$$\widehat{\mathrm{SO}(3)} = \big\{ \big[t^\ell\big] : \ t^\ell \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)}, \ \ell \in \mathbb{N}_0 \big\}.$$

The dimension of each t^ℓ is $d_{t^\ell} = 2\ell + 1$.

As in the case of $\mathrm{SU}(2)$, let us fix three left-invariant vector fields X, Y, Z on $\mathrm{SO}(3)$ associated with the derivatives with respect to the Euler angles, so that we also have $[X, Y] = Z$, see [23] or [24] for the detailed expressions.

We will consider an *example* of an operator (on $\mathrm{SO}(3)$) which is not covered by Hörmander's sum of squares theorem. Namely, we consider the following family of 'Schrödinger' differential operators

$$\mathcal{H}_\gamma = iZ - \gamma\big(X^2 + Y^2\big),$$

for a parameter $0 < \gamma < \infty$. For $\gamma = 1$ it was shown in [25] that $\mathcal{H}_1 + cI$ is globally hypoelliptic if and only if

$$0 \notin \big\{ c + \ell(\ell+1) - m(m+1) : \ell \in \mathbb{N}, \ m \in \mathbb{Z}, \ |m| \leqslant \ell \big\}.$$

It has been also shown in [11, Section 4] that, if $\gamma > 1$, then $I + \mathcal{H}_\gamma$ is globally hypoelliptic, and

$$(I + \mathcal{H}_\gamma)^{-\alpha/2} \in S_p \quad \text{if and only if} \quad \alpha p > 4.$$

As a consequence of this and following the argument in the proof of Theorem 3.6 with $I + \mathcal{H}_\gamma$ instead of $E = \Delta_M$ for the manifold $M = \mathrm{SO}(3)$, as well as Corollary 5.2, we obtain:

Corollary 5.3. *Let $\gamma > 1$. Let $K \in L^2(\mathrm{SO}(3) \times \mathrm{SO}(3))$ be such that*

$$(I + \mathcal{H}_\gamma)_x^{\mu_1/2} (I + \mathcal{H}_\gamma)_y^{\mu_2/2} K \in L^2\big(\mathrm{SO}(3) \times \mathrm{SO}(3)\big)$$

for some $\mu_1, \mu_2 \geqslant 0$. Then the integral operator T on $L^2(\mathrm{SO}(3))$ defined by

$$(Tf)(x) = \int\limits_{\mathrm{SO}(3)} K(x, y) f(y)\, dy,$$

is in S_r on $L^2(\mathrm{SO}(3))$ for

$$r > \frac{4}{2 + \mu_1 + \mu_2}.$$

The same conclusion holds on $\mathrm{SU}(2) \simeq \mathbb{S}^3$. Again, if $\mu_1 = \mu_2 = 0$, the results have a trivial strengthening to include the case $r = 2$.

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