A SMOOTHING PROPERTY OF SCHRÖDINGER EQUATIONS
AND
A GLOBAL EXISTENCE RESULT TO DERIVATIVE NONLINEAR EQUATIONS *

Mitsuru Sugimoto
Department of Mathematics, Osaka University
sugimoto@math.wani.osaka-u.ac.jp

Michael Ruzhansky
Department of Mathematics, Imperial College
ruzh@ic.ac.uk

appeared in:

1. Derivative Nonlinear Schrödinger Equation

What is the condition of the initial data $\varphi$ for equation

$$
\begin{cases}
(i\partial_t + \Delta_x) u(t, x) = |\nabla u(t, x)|^N \\
u(0, x) = \varphi(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n
\end{cases}
$$

(1)

to have time global solution? Here are some answers:

- (Chihara 1996 $\cdots N \geq 3$). Smooth, rapidly decay, and sufficiently small.

*This work was completed with the aid of “UK-Japan Joint Project Grant” by “The Royal Society” and “Japan Society for the Promotion of Science”.

1
In the condition above, can we weaken the smoothness assumption for \( \varphi \)? The purpose of this article is to answer it affirmatively. We can replace the regularity index \( \left\lfloor \frac{n}{2} \right\rfloor + 5 \) by smaller one if the non-linear term has a structure! We consider equation

\[
\begin{cases}
(i\partial_t + \Delta_x) u(t, x) = \left| \left( \frac{x}{\langle x \rangle} \wedge \nabla_x \right) u(t, x) \right|^N \\
u(0, x) = \varphi(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n.
\end{cases}
\]

(2)

Here \( \langle x \rangle = (1 + |x|^2)^{1/2} \)

\[
\frac{x}{\langle x \rangle} \wedge \nabla_x = \left( \frac{x_i}{\langle x \rangle} \partial_{x_j} - \frac{x_j}{\langle x \rangle} \partial_{x_i} \right)_{i<j}.
\]

**Theorem 1.** Assume \( N \geq 4, \ s > \left( \frac{n+3}{2} \right) + \left( \frac{N-3}{2N-2} \right) \).

Suppose that \( \langle x \rangle \langle D_x \rangle^s \varphi \in L^2 \) and its \( L^2 \)-norm is sufficiently small. Then equation (2) has a time global solution.

But we will have questions:

- **Question 1 :** \( N = 2, 3 \)?
- **Question 2 :** The relation between the linear term and the structure of nonlinear term?

To answer them, we generalize the linear term \(-\Delta_x\) to \( L_p \), where

\[
L_p = p(D_x)^2; \quad p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0),
\]

\[
p(\xi) > 0, \quad p(\lambda\xi) = \lambda^p(\xi) \quad (\lambda > 0).
\]

By finding the structure \( \sigma(X, D_x) \) for \( L_p \) instead of \( \frac{x}{\langle x \rangle} \wedge \nabla_x \) for \(-\Delta_x\), we answer the questions. Assume that the Gaussian curvature of the hypersurface

\[
\Sigma_p = \{ \xi; p(\xi) = 1 \}
\]

never vanishes \( (p(\xi) = |\xi| \) is the usual case!), and consider the following generalized equation:

\[
\begin{cases}
(i\partial_t - L_p) u(t, x) = \sigma(X, D_x) u^N \\
u(0, x) = \varphi(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n.
\end{cases}
\]

(3)
A smoothing property of Schrödinger equations

(In the case $N = 2$, replace $\sigma(X, D_x)$ by $\sigma(X, D_x)^2$.)

We remark that the classical orbit $\{(x(t), \xi(t)); t \geq 0\}$ associated to (3) is the solution of

$$\begin{cases}
\dot{x}(t) = \nabla_{\xi} p^2(\xi(t)), & \dot{\xi}(t) = 0 \\
x(0) = 0, & \xi(0) = k.
\end{cases} \quad (4)$$

We define the set of the path of all classical orbits:

$$\Gamma_p = \{(x(t), \xi(t)); \text{sol. of (4)}, t \geq 0, k \in \mathbb{R}^n \setminus 0\} = \{(t \nabla p(\xi), \xi); \xi \in \mathbb{R}^n \setminus 0, t \geq 0\}.$$ By using these terminology, we assume the structure for the non-linear
term:

$$\sigma(x, \xi) \geq 0, \quad \sigma(x, \xi) = 0 \quad \text{on} \quad \Gamma_p,$$

$$\sigma(x, \xi) \sim \langle x \rangle^a |\xi|^b \quad \text{on} \quad \Gamma_p.$$ (5)

(In the case $N = 2$, assume $\sigma(x, \xi) \sim \langle x \rangle^a |\xi|^{1/2}$ instead.) Here we have used the notation

$$\sigma(x, \xi) \sim \langle x \rangle^a |\xi|^b$$

$$\iff$$

$$\begin{cases}
\sigma(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus 0)), \\
\sigma(\lambda x, \xi) = \lambda^a \sigma(x, \xi); (\lambda > 1, |x| \gg 1), \\
\sigma(x, \lambda \xi) = \lambda^b \sigma(x, \xi); (\lambda > 0).
\end{cases}$$ Under these assumptions, we have the following theorem which answers our questions:

**Theorem 2.** Suppose $n \geq 2$, $s > (n + 3)/2$. Assume that $\langle x \rangle (D_x)^s \varphi \in L^2$ and its $L^2$-norm is sufficiently small. Then equation (3) has a time global solution. (In the case $N = 2$, assume $\langle x \rangle^2 (D_x)^s \varphi \in L^2$ instead.)

Examples of nonlinear terms which satisfy (5) in the case $L_p = -\Delta_x$:

$$\sigma(x, \xi) = \left| \frac{x}{|x|} \wedge \xi \right|^2 |\xi|^{-1} \quad \text{for large } x \ (N \geq 3),$$

$$\sigma(x, \xi)^2 = \left| \frac{x}{|x|} \wedge \xi \right|^4 |\xi|^{-3} \quad \text{for large } x \ (N = 2).$$
2. Smoothing effect of Schrödinger equations

The proof of Theorem 2 is based on a global smoothing property of generalized Schrödinger equations. Let \( L_p = p(D_x)^2 \) be as before, and consider equation
\[
\begin{aligned}
(i \partial_t - L_p) u(t, x) &= 0 \\
u(0, x) &= \varphi(x) \in L^2(\mathbb{R}^n).
\end{aligned}
\]  

(6)

**Theorem 3.** Suppose \( n \geq 2 \). Assume
\[
\sigma(x, \xi) = 0 \text{ on } \Gamma_p, \quad \sigma(x, \xi) \sim \langle x \rangle^{-1/2} |\xi|^{1/2}.
\]

Then the solution \( u \) to equation (6) satisfies
\[
\sigma(X, D_x)u(t, x) \in L^2(\mathbb{R}_t \times \mathbb{R}_x^2).
\]  

(7)

There are some previous result without the structure assumption \( \sigma(x, \xi) = 0 \) on \( \Gamma_p \). In fact, (7) is true for the following:

- (Ben-Artzi & Klainerman 1992) \( \sigma(x, \xi) = \langle x \rangle^{-s}|\xi|^{1/2} \) (\( s > 1/2 \))
- (Kato & Yajima 1989) \( \sigma(x, \xi) = |x|^{\alpha-1}|\xi|^\alpha \) (\( 0 < \alpha < 1/2 \))

Theorem 3 means that we can take the critical index \( s = 1/2 \) or \( \alpha = 1/2 \) if we assume the structure condition.

We have a similar result for inhomogeneous equations
\[
\begin{aligned}
(i \partial_t - L_p) u(t, x) &= f(t, x) \\
u(0, x) &= 0.
\end{aligned}
\]  

(8)

**Theorem 4.** Suppose \( n \geq 2 \) and \( k \in \mathbb{N} \). Assume
\[
\begin{aligned}
\sigma(x, \xi) &\geq 0, \quad \sigma(x, \xi) = 0 \text{ on } \Gamma_p \\
\sigma(x, \xi) &\sim \langle x \rangle^{-1/(2k)+1}|\xi|^{1/k}.
\end{aligned}
\]

Then the solution \( u \) to (8) satisfies the estimate
\[
\|\sigma(X, D_x)^k u\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\langle x \rangle^{1/2+k} f\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)}.
\]

Combining Theorems 3 and 4, we have an estimate for equation
\[
\begin{aligned}
(i \partial_t - L_p) u(t, x) &= f(t, x) \\
u(0, x) &= \varphi(x).
\end{aligned}
\]  

(9)
Corollary 5. Suppose $n \geq 2$, $s, \tilde{s} \geq 0$, and $k \in \mathbb{N}$. Assume
\[
\sigma(x, \xi) \geq 0, \quad \sigma(x, \xi) = 0 \text{ on } \Gamma_p
\]
\[
\sigma(x, \xi) \sim \langle x \rangle^{0}\langle \xi \rangle^{1/k}.
\]
Then the solution $u$ to (9) satisfies the estimate
\[
\left\| \langle x \rangle^{-1/2+k}\sigma(X, D_x)^k u \right\|_{H^s_t(H^\tilde{s}_x)} \leq C \left\| \langle x \rangle^{1/2} \phi \right\|_{L^2(\mathbb{R}^n)} + C \left\| \langle x \rangle^{1/2+k} f \right\|_{H^s_t(H^\tilde{s}_x)}.
\]

To prove Theorem 2, use Corollary 5 with $f = \left| \sigma(X, D_x)^k u \right|^N$. Then, the key point is that the space $H^s_t(H^\tilde{s}_x)$ is an algebra if $s > 1/2$ and $\tilde{s} > n/2$. In fact, we have
\[
\left\| \langle x \rangle^{1/2+k}\sigma(X, D_x)^k u \right\|_{H^s_t(H^\tilde{s}_x)} \leq C \left\| \langle x \rangle^{1/2} \phi \right\|_{L^2(\mathbb{R}^n)} + C \left\| \langle x \rangle^{1/2+k} f \right\|_{H^s_t(H^\tilde{s}_x)}.
\]

if $k \geq (N + 1)/(2N - 2)$. We can take $k = 1$ if $N \geq 3$ and $k = 2$ if $N = 2$. Hence we have a priori estimate
\[
\left\| \langle x \rangle^{-1/2+k}\sigma(X, D_x)^k u \right\|_{H^s_t(H^\tilde{s}_x)} \leq C \left\| \langle x \rangle^{1/2} \phi \right\|_{L^2(\mathbb{R}^n)} + C \left\| \langle x \rangle^{-1/2+k} \sigma(X, D_x)^k u \right\|_{H^s_t(H^\tilde{s}_x)}.
\]

3. Mapping property of Fourier integral operators

The proofs of theorems in previous sections will appear in our forthcoming papers. In this section, we mention the main tool for proving results in Section 2. Let
\[
Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} a(x, y, \xi) u(y) dyd\xi
\]
be a Fourier integral operator with an amplitude function $a(x, y, \xi)$ and a real phase function $\phi(x, y, \xi)$. Especially, we need the case
\[
\phi(x, y, \xi) = x \cdot \xi - y \cdot \frac{\nabla p(\xi)}{|\nabla p(\xi)|} \nabla p(\xi).
\]
In fact, we have
\[ L_p \cdot T = T \cdot (\triangle_x) \]
if \( a(x, y, \xi) = 1 \). By this relation, \( L^2 \)-property of \( \triangle_x \) can be interpreted as that of \( L_p \), so that we can prove Theorems 3 and 4.

To justify the argument above, we need the (weighted) global \( L^2 \)-boundedness of the operator \( T \). Then, what is the condition for \( T \) to be \( L^2(\mathbb{R}^n) \)-bounded? About this question, we knew only the following sufficient condition:

- (Asada-Fujiwara 1978) All the derivatives of \( a(x, y, \xi) \) and all the derivatives of each element of \( D(\phi) \) is bounded and

\[ |\det D(\phi)| \geq C > 0, \]

where
\[ D(\phi) = \begin{pmatrix} \phi_{xx} & \phi_{x\xi} \\ \phi_{y\xi} & \phi_{\xi\xi} \end{pmatrix}. \]

But we cannot use the result of Asada-Fujiwara since \( \phi_{\xi\xi} \) is not bounded for \( \phi \) given by (10). Hence we need the following extended result which can be seen in our preprint.

For \( m \in \mathbb{R} \), let \( L^2_m(\mathbb{R}^n) \) be the set of functions \( f \) such that the norm
\[ \|f\|_{L^2_m(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2} \]
is finite.

**Theorem 6.** Let \( \phi(x, y, \xi) = x \cdot \xi - \varphi(y, \xi) \). Assume that
\[ |\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0, \]
and all the derivatives of entries of \( \partial_y \partial_\xi \varphi \) are bounded. Also assume that
\[ |\partial^\alpha_\xi \varphi(y, \xi)| \leq C_\alpha(y) \quad \text{for all } |\alpha| \geq 1, \]
\[ |\partial^\alpha_\xi \partial^\beta_y \partial^\gamma_\xi a(x, y, \xi)| \leq C_{\alpha\beta\gamma}(x)^{-|\alpha|} \]
for all \( \alpha, \beta, \) and \( \gamma \), or that
\[ |\partial^\alpha_\xi \partial^\beta_y \varphi(y, \xi)| \leq C_\alpha(y)^{1-|\alpha|} \quad \text{for all } \alpha, |\beta| \geq 1, \]
\[ |\partial^\alpha_\xi \partial^\beta_y \partial^\gamma_\xi a(x, y, \xi)| \leq C_{\alpha\beta\gamma}(y)^{-|\beta|} \]
for all \( \alpha, \beta, \) and \( \gamma \). Then \( T \) is bounded on \( L^2_m(\mathbb{R}^n) \) for any \( m \in \mathbb{R} \).
A smoothing property of Schrödinger equations

References


