

Nonharmonic Analysis of Boundary Value Problems

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Dedicated to the memory of Professor Louis Boutet de Monvel (1941–2014).

In this paper, we develop the global symbolic calculus of pseudo-differential operators generated by a boundary value problem for a given (not necessarily self-adjoint or elliptic) differential operator. For this, we also establish elements of a non-self-adjoint distribution theory and the corresponding biorthogonal Fourier analysis. There are no assumptions on the regularity of the boundary which is allowed to have arbitrary singularities. We give applications of the developed analysis to obtain a priori estimates for solutions of boundary value problems that are elliptic within the constructed calculus.

1 Introduction

In this paper, we are interested in questions devoted to the global solvability and further properties of boundary value problems in \mathbb{R}^n . Given a problem for some pseudo-differential operator A with fixed boundary conditions in a domain $\Omega \subset \mathbb{R}^n$, the main idea for our analysis is to develop a suitable pseudo-differential calculus in which the given boundary value problem can be solved and its solution can be efficiently estimated. Such pseudo-differential calculus is developed in terms of a "model" operator L with the same boundary conditions in Ω for which we can introduce and work with

Received March 28, 2015; Revised July 12, 2015; Accepted July 23, 2015

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the global Fourier analysis expressed in terms of its eigenfunctions. In general, such a model operator L does not have to be self-adjoint, so we will be working with biorthogonal systems rather than with an orthonormal basis to take into account a possible non-self-adjointness. The operator L also does not have to be elliptic.

Different powerful approaches to boundary value problems for pseudo-differential operators have been already developed, see, for example, Boutet de Monvel [6] and many subsequent works by, among others, the Maz'ya school (see, e.g., [40]), Melrose school (see, e.g., [41]), or Schulze school (see, e.g., [31]), see also approaches in, for example, Eskin [23], Schrohe and Schulze [59], Melo et al. [44], Mitrea and Nistor [45], Plamenevskii [48], and references therein.

However, our approach is rather different from all these by being global in nature. An example of such an approach is the toroidal calculus of pseudo-differential operators on the torus \mathbb{T}^n or of the periodic pseudo-differential operators on \mathbb{R}^n . A global analysis of pseudo-differential operators on the torus based on the Fourier series representations of functions with further applications to the spectral theory was originated by Agranovich [1], with further developments of its different aspects by Agranovich [2], Amosov [3], Elschner [22], McLean [42], Melo [43], Prössdorf and Schneider [50], Saranen and Wendland [58], Turunen and Vainikko [70], Vainikko and Lifanov [71], and others. However, most of these papers deal with one-dimensional cases or with classes of operators rather than with classes of symbols. A consistent development of the application of the classical Fourier series techniques in the analysis of pseudo-differential operators on the torus was developed by the first author and Turunen in [52, 54] and can be also found in the monograph [53]. For further extensions of this periodic analysis to the almost periodic setting see, for example, Wahlberg [72, 73]. The classical Fourier series on a circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be viewed as a unitary transform in the Hilbert space $L^2(0, 1)$ generated by the operator of differentiation $(-i \frac{d}{dx})$ with periodic boundary conditions, because the system of exponents $\{\exp(2\pi i \lambda x), \lambda \in \mathbb{Z}\}$ is a system of its eigenfunctions.

The analysis of this paper is the development of such ideas to a more general setting without assuming that the problem has symmetries. Instead of the differential operator $(-i \frac{d}{dx})$ in the space $L^2(0, 1)$, we consider a differential operator L of order m with smooth coefficients, in the Hilbert space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open subset. We assume that L is equipped with some boundary conditions leading to a discrete spectrum with its family of eigenfunctions yielding a (biorthogonal) basis in $L^2(\Omega)$. Moreover, L does not have to be self-adjoint. General biorthogonal systems have been investigated by Bari [4] which is a setting convenient for our constructions; see also Gelfand [30]. Similar (slightly more general but essentially the same) systems are also

called “Hilbert systems” or “quasi-orthogonal systems” by Bari [4] and Kac et al. [32], respectively.

We then investigate the associated spaces of test functions, distributions, “convolutions”, Fourier transforms, Sobolev spaces $H_L^s(\Omega)$ and $l^p(L)$ spaces on the “dual”, associated to L , and their properties such as the Hausdorff–Young inequality, interpolation, and duality. A strong characteristic feature of this analysis is that it is build upon biorthogonal systems rather than more familiar orthonormal bases. Consequently, we introduce difference operators acting on Fourier coefficients, and the subsequent symbolic calculus of pseudo-differential operators generated by a differential operator L . A formula for compositions of pseudo-differential operators and other elements of the symbolic calculus are obtained. It is shown that pseudo-differential operators are bounded on L^2 under certain conditions on their symbols. We also analyze ellipticity and a priori estimates for operators within this calculus.

The exponential systems $\{e^{2\pi i\lambda x}\}_{\lambda \in \Lambda}$ on $L^2(0, 1)$ for a discrete set Λ possibly containing $\lambda \notin \mathbb{Z}$ have been considered by Paley and Wiener [47] who called such systems the *nonharmonic Fourier series* to emphasize the distinction with the usual (harmonic) Fourier series when $\Lambda = \mathbb{Z}$. For further explanations and developments of the nonharmonic analysis, we refer to survey papers by Sedletsii [62, 63] (see also an earlier survey [61]). The difference between the harmonic and nonharmonic Fourier series in our context is already exhibited by the operator $L = -i \frac{d}{dx}$ in the space $L^2(0, 1)$, but with boundary conditions $hy(0) = y(1)$ for a fixed $h > 0$. In this case, the series of eigenfunctions (a building block for our analysis) is “harmonic” for $h = 1$ and “nonharmonic” for $h \neq 1$. In Example 2.1, we explain this further and also complement it with a number of explicit formulae.

From this point of view, the analysis of pseudo-differential operators on the torus using the classical exponential bases as in [54], or further extensions using representation coefficients on compact Lie groups as in [53, 55], both fall within the realm of “harmonic” analysis. The latter approach has further, still “harmonic” extensions, for example, for the global analysis of pseudo-differential operators on the Heisenberg group [26], graded Lie groups [24, 25, 27], or general type I locally compact groups [39].

In the analysis of the present paper such symmetries are in general lost, nevertheless we attempt to still mimic the harmonic analysis constructions but in the new “nonharmonic” setting. Therefore, to also emphasize such a difference, we may call our analysis the “nonharmonic analysis of boundary value problems”. In spirit, this is similar to the global pseudo-differential analysis on closed manifolds as in [15, 16] partly based on the “nonharmonic” analysis on compact manifold by

Seeley [64, 65]. Such analysis becomes effective in a number of problems, for example, it was recently used in [17] to produce sharp kernel conditions for Schatten classes of operators on compact manifolds, and in [14] to give characterizations of Komatsu-type classes of functions and distributions, in particular for classes of Gevrey functions and ultradistributions, on a compact manifold, extending the characterization given for analytic functions by Seeley [65].

The analysis of [15] deals with general compact manifolds, but is simplified by the facts that there are no boundary conditions, the operator L is self-adjoint, elliptic and positive, and the considered calculus is that of invariant operators.

We keep the setting of this paper rather abstract, in particular not relying on a specific form of boundary conditions of the operator for our analysis. Certainly, if more information on the operator L and its properties are available, more conclusions can be drawn. In Section 2, we give several examples of operators and boundary conditions. In a somewhat related setting, the global pseudo-differential analysis based on an elliptic self-adjoint pseudo-differential operator on a closed manifold has been recently developed in [15].

Although in this paper we do not give explicit applications to partial differential equations, these will appear elsewhere. For example, the analysis developed here could allow one to treat classes of PDE problems in cylindrical domains of finite length without assuming periodic boundary conditions on the top and bottom edges of the cylindrical domain, see, for example, Denk and Nau [19] for this kind of problems. Also, in subsequent works we will apply the pseudo-differential analysis developed here to problems in punctured domains with δ -type potentials, for PDE problems of the type that appeared in [35, 37].

Let us formulate the main assumptions of this paper. We will consider a differential operator L of order m with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$ equipped with some boundary conditions. In order to describe the abstract scheme, we will denote the boundary conditions by (BC) without specifying them further in the general framework. Concerning the boundary conditions, we will assume that

(BC) *the boundary conditions (BC) are linear, that is, they are preserved under linear combinations or, in other words, the spaces of functions satisfying (BC) are linear.*

In this paper, we prefer to think of the operator in terms of its boundary conditions instead of domain, in view of the planned further applications. However, in the paper we may use both points of view.

Later on, once introducing topologies on spaces of functions in the domain of L , we will assume the condition (BC+) that the boundary conditions define a closed space. In Section 2, we give different examples of operators L and boundary conditions (BC).

The assumption (BC) may be reformulated by saying that the domain $\text{Dom}(L)$ of the operator L is linear, and the condition (BC+) by saying that $\text{Dom}(L)$ and $\text{Dom}(L^*)$ are closed in the topologies of $C_L^\infty(\bar{\Omega})$ and $C_{L^*}^\infty(\bar{\Omega})$, respectively, with the latter spaces and their topologies introduced in Definition 3.1.

Also, we will be working with discrete sets of eigenvalues and eigenfunctions indexed by a countable set \mathcal{I} . However, in different problems it may be more convenient to make different choices for this set, for example, $\mathcal{I} = \mathbb{N}$ or \mathbb{Z} or \mathbb{Z}^k , etc. In order to allow different applications, we will be denoting it by \mathcal{I} , and without loss of generality we will assume that

$$\mathcal{I} \text{ is a subset of } \mathbb{Z}^K \text{ for some } K \geq 1. \quad (1.1)$$

For simplicity, one can think of $\mathcal{I} = \mathbb{Z}$ or $\mathcal{I} = \mathbb{N} \cup \{0\}$ throughout this paper. Thus, throughout this paper we will be always working in the following setting.

Assumption 1.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. By L_Ω , we denote a differential operator L of order m with smooth coefficients in Ω , equipped with some linear boundary conditions (BC). Assume that L_Ω has a discrete spectrum $\{\lambda_\xi \in \mathbb{C} : \xi \in \mathcal{I}\}$ on $L^2(\Omega)$, where \mathcal{I} is a countable set as in (1.1), and we order the eigenvalues with the occurring multiplicities in the ascending order:

$$|\lambda_j| \leq |\lambda_k| \quad \text{for } |j| \leq |k|. \quad (1.2)$$

□

Let us denote by u_ξ the eigenfunction of L_Ω corresponding to the eigenvalue λ_ξ for each $\xi \in \mathcal{I}$, so that

$$L u_\xi = \lambda_\xi u_\xi \quad \text{in } \Omega, \quad \text{for all } \xi \in \mathcal{I}. \quad (1.3)$$

Here the eigenfunctions u_ξ satisfy the boundary conditions (BC) discussed earlier. The conjugate spectral problem is

$$L^* v_\xi = \bar{\lambda}_\xi v_\xi \quad \text{in } \Omega \quad \text{for all } \xi \in \mathcal{I}, \quad (1.4)$$

which we equip with the conjugate boundary conditions which we may denote by (BC)*. This adjoint problem will be denoted by L_Ω^* .

Let $\|u_\xi\|_{L^2} = 1$ and $\|v_\xi\|_{L^2} = 1$ for all $\xi \in \mathcal{I}$. Here, we can take biorthogonal systems $\{u_\xi\}_{\xi \in \mathcal{I}}$ and $\{v_\xi\}_{\xi \in \mathcal{I}}$, that is,

$$(u_\xi, v_\eta)_{L^2} = 0 \quad \text{for } \xi \neq \eta \quad \text{and} \quad (u_\xi, v_\eta)_{L^2} = 1 \quad \text{for } \xi = \eta, \tag{1.5}$$

where

$$(f, g)_{L^2} := \int_{\Omega} f(x)\overline{g(x)} \, dx$$

is the usual inner product of the Hilbert space $L^2(\Omega)$. From Bari’s work [4], it follows that the system $\{u_\xi : \xi \in \mathcal{I}\}$ is a basis in $L^2(\Omega)$ if and only if the system $\{v_\xi : \xi \in \mathcal{I}\}$ is a basis in $L^2(\Omega)$. So, from now on we will also assume the following.

Assumption 1.2. The system $\{u_\xi : \xi \in \mathcal{I}\}$ is a basis in $L^2(\Omega)$, that is, for every $f \in L^2(\Omega)$ there exists a unique series $\sum_{\xi \in \mathcal{I}} a_\xi u_\xi(x)$ that converges to f in $L^2(\Omega)$. □

Therefore, by Bari [4], the system $\{v_\xi : \xi \in \mathcal{I}\}$ is also a basis in $L^2(\Omega)$. Also, Assumption 1.2 will imply that the spaces $C_L^\infty(\bar{\Omega})$ and $C_{L^*}^\infty(\bar{\Omega})$ of test functions introduced in Section 3 are dense in $L^2(\Omega)$.

Let us define the weight

$$\langle \xi \rangle := (1 + |\lambda_\xi|^2)^{\frac{1}{2m}}, \tag{1.6}$$

which will be instrumental in measuring the growth/decay of Fourier coefficients and of symbols. To give its interpretation in terms of the operator analysis, we can define the operator L° by setting its values on the basis u_ξ by

$$L^\circ u_\xi := \bar{\lambda}_\xi u_\xi \quad \text{for all } \xi \in \mathcal{I}. \tag{1.7}$$

If L is self-adjoint, we have $L^\circ = L^* = L$. Consequently, we can informally think of $\langle \xi \rangle$ as of the eigenvalues of the positive (first order) operator $(I + L^\circ L)^{\frac{1}{2m}}$.

With a similar definition for $(L^*)^\circ$, we can observe that $(L^*)^\circ = (L^\circ)^*$.

The following technical definition will be useful to single out the case when the eigenfunctions of both L and L^* do not have zeros (WZ stands for “without zeros”).

Definition 1.3. The system $\{u_\xi : \xi \in \mathcal{I}\}$ is called a WZ-system if the functions $u_\xi(x)$, $v_\xi(x)$ do not have zeros on the domain $\bar{\Omega}$ for all $\xi \in \mathcal{I}$, and if there exist $C > 0$ and $N \geq 0$ such

that

$$\inf_{x \in \Omega} |u_\xi(x)| \geq C \langle \xi \rangle^{-N},$$

$$\inf_{x \in \Omega} |v_\xi(x)| \geq C \langle \xi \rangle^{-N},$$

as $|\xi| \rightarrow \infty$. □

We note that, in particular, a WZ-system cannot be all real-valued due to orthogonality relations (1.5). Several examples of WZ-systems will be given in Section 2, but a typical example is the system $\{e^{2\pi i \lambda x}\}_{\lambda \in \mathbb{Z}}$ for $L = -i \frac{d}{dx}$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

In the sequel, unless stated otherwise, whenever we will use inverses u_ξ^{-1} of the functions u_ξ , we will suppose that the system $\{u_\xi : \xi \in \mathcal{I}\}$ is a WZ-system. However, we will also try to mention explicitly when we make such an additional assumption.

The paper is organized as follows.

- Section 2: we give examples of operators L and of different boundary conditions yielding different types of biorthogonal systems.
- Section 3: we introduce elements of the global theory of distributions $\mathcal{D}'_L(\Omega)$ in Ω adapted to the boundary value problem L_Ω .
- Section 4: we develop the Fourier transform induced by L , which is the decomposition of elements of $\mathcal{D}'_L(\Omega)$ with respect to the eigenfunctions of L . Here is a point when both biorthogonal bases u_ξ and v_ξ came actively into play.
- Section 5: we introduce L -convolution \star_L , which is an operation resembling the usual convolution. Despite the lack of any symmetries in our problem, a number of useful properties of such L -convolution can still be obtained.
- Section 6: we introduce the space l^2_L for which the Plancherel identity for the L -Fourier transform holds. Consequently, we introduce Sobolev spaces $\mathcal{H}^s_L(\Omega)$ and describe their Fourier images.
- Section 7: we introduce the spaces $l^p(L)$ and $l^p(L^*)$ extending the spaces l^2_L and $l^2_{L^*}$ to the l^p -setting. We show that these spaces are interpolation spaces and satisfy the expected duality properties. Moreover, we obtain the Hausdorff–Young inequality for the L -Fourier transform in these spaces.
- Section 8: we prove the Schwartz kernel theorem in the distribution spaces $\mathcal{D}'_L(\Omega)$. This is necessary to set up the subsequent framework of the symbolic analysis and of the definition of the symbol as the L -Fourier transform of the L -convolution kernel.

- Section 9: we introduce difference operators acting on Fourier coefficients and on symbols. Keeping in mind ideas from the Calderón–Zygmund theory, these are defined as multiplications on the inverse Fourier transform side by functions vanishing at an anticipated singular support of the integral kernel. Due to the lack of symmetries (as compared, e.g., to the cases of the torus or of compact Lie groups) these difference operators also depend on the points x of the space.
- Section 10: the notion of difference operators is used to define Hörmander-type classes induced by the boundary value problem L_Ω and to develop elements of its symbolic calculus.
- Section 11: we derive some properties of the integral kernels of pseudo-differential operators.
- Section 12: we show that operators that are elliptic in the constructed symbol classes have both left and right parametrices and provide a formula for it.
- Section 13: we discuss possible Sobolev embedding theorems. In particular, it seems that in order to have a meaningful collection of embeddings further assumptions on the boundary value problem L_Ω may be needed.
- Section 14: we prove a criterion for the L^2 -boundedness of pseudo-differential operators in terms of their symbols, and extend it to Sobolev spaces as well. An application is given to obtain a priori estimates for solutions to boundary value problems to elliptic operators.

Most results, especially those starting from Section 5 appear to be new already in the case when the problem L_Ω is self-adjoint.

2 Examples of Operators L and Boundary Conditions

In this section, we give several examples of the operator L and of boundary conditions (BC). The following example shows that among other things, we can extend to the non-self-adjoint setting the toroidal calculus (with periodic boundary conditions) developed in [54].

Example 2.1. For $h > 0$, let the operator $O_h^{(1)}$ be given by the expression

$$O_h^{(1)} := -i \frac{\partial}{\partial x}$$

on the domain $\Omega = (0, 1)$ with the boundary condition

$$hy(0) = y(1).$$

In the case $h = 1$, we get the operator $O_1^{(1)}$ with periodic boundary conditions. In this case, the $O_1^{(1)}$ -pseudo-differential calculus developed in this paper coincides with the toroidal calculus some aspects of which were investigated in the works by Agranovich [1, 2], Turunen and Vainikko [70], and which was then consistently developed by Ruzhansky and Turunen in [54]. Thus, of main interest to us here will be the calculus generated by $O_h^{(1)}$ with $h \neq 1$.

It is easy to check that for $h \neq 1$ the operator $O_h^{(1)}$ is not self-adjoint. Spectral properties of the operator $O_h^{(1)}$ are well known (see Titchmarsh [69] and Cartwright [8]): with $\mathcal{I} = \mathbb{Z}$,

A. $O_h^{(1)}$ has a discrete spectrum and its eigenvalues satisfy

$$\lambda_j = -i \ln h + 2j\pi, \quad j \in \mathbb{Z}.$$

B. The system of eigenfunctions

$$\{u_j(x) = h^x e^{2\pi i x j}, \quad j \in \mathbb{Z}\}$$

of the operator $O_h^{(1)}$ is a minimal system in the space $L^2(\Omega)$, and the biorthogonal system to $\{u_j(x) = h^x e^{2\pi i x j}, \quad j \in \mathbb{Z}\}$ in $L^2(\Omega)$ is

$$\{v_j(x) = h^{-x} e^{2\pi i x j}, \quad j \in \mathbb{Z}\}.$$

C. The system of eigenfunctions of the operator $O_h^{(1)}$ is a Riesz basis in $L^2(\Omega)$. These families also form WZ-systems (WZ, as in Definition 1.3).

D. The resolvent of the operator $O_h^{(1)}$ is

$$(O_h^{(1)} - \lambda I)^{-1} f(x) = \frac{i}{\Delta(\lambda)} e^{i\lambda(x+1)} \int_0^1 e^{-i\lambda t} f(t) dt + i e^{i\lambda x} \int_0^x e^{-i\lambda t} f(t) dt,$$

where

$$\Delta(\lambda) = h - e^{i\lambda}.$$

The above example fits into our framework once we index the family of eigenvalues and of the corresponding eigenfunctions by $\mathcal{I} = \mathbb{Z}$ which is a choice we made for the (discrete) index set. In Section 5, we will discuss convolutions generated by our operators L_Ω . In this example, the convolution generated by the operator $O_h^{(1)}$ has the

following explicit form:

$$(g \star_{O_h^{(1)}} f)(x) = \int_0^x g(x-t)f(t) dt + \frac{1}{h} \int_x^1 g(1+x-t)f(t) dt.$$

For more details on this particular convolution, see [36, 38]. □

Remark 2.2. The toroidal pseudo-differential calculus on all higher dimensional tori \mathbb{T}^n , $n \geq 1$, as outlined in [51] and then consistently developed in [54], cannot be covered by the first order differential operator $O_1^{(1)}$. But in this case we can take the second-order operator. Namely, identifying the torus \mathbb{T}^n with the cube $[0, 1]^n$ with periodic boundary conditions, we can take $L := \Delta$ to be the Laplacian with the periodic boundary conditions on the boundary of $\Omega = (0, 1)^n$. See Example 2.5 for further details. □

Example 2.3. The operator $L = i \frac{d}{dt}$ with $\Omega = (-a, a)$ and the boundary condition

$$\int_{-a}^a y(t) d\sigma(t) = 0, \quad \text{var } \sigma(t) < \infty, \quad 0 < a < \infty,$$

has the eigenfunctions in the form $\{\exp(i\lambda_k t)\}_{\lambda_k \in \Lambda}$, where $\Lambda \subset \mathbb{C}$ is the collection of zeros of the Fourier transform $\widehat{d\sigma}$ of the measure $d\sigma(t)$. It becomes a biorthogonal system or a Riesz basis under a number of properties of Λ , see Sedletskii [63] for a thorough review of this topic, see also [62]. □

Example 2.4. Combining Examples 2.1 and 2.3, we can consider operator $L = -i \frac{d}{dx}$ with $\Omega = (0, 1)$ with the domain

$$\text{Dom}(L) = \left\{ y \in W_2^1[0, 1] : ay(0) + by(1) + \int_0^1 y(x)q(x) dx = 0 \right\},$$

where $a \neq 0, b \neq 0$, and $q \in C^1[0, 1]$. We assume that

$$a + b + \int_0^1 q(x) dx = 1$$

so that the inverse L^{-1} exists and is bounded. Following [34], we have the following properties, with $\mathcal{I} = \mathbb{Z}$:

A. The operator L has a discrete spectrum and its eigenvalues can be enumerated so that

$$\lambda_j = -i \ln \left(-\frac{a}{b} \right) + 2j\pi + \alpha_j, \quad j \in \mathbb{Z},$$

and for any $\epsilon > 0$ we have

$$\sum_{j \in \mathbb{Z}} |\alpha_j|^{1+\epsilon} < \infty.$$

If m_j denotes the multiplicity of the eigenvalue λ_j , then $m_j = 1$ for sufficiently large j .

B. The system of extended eigenfunctions

$$\left\{ u_{jk}(x) = \frac{(ix)^k}{k!} e^{i\lambda_j x} : 0 \leq k \leq m_j - 1, j \in \mathbb{Z} \right\} \tag{2.1}$$

of the operator L is a minimal system in the space $L^2(0, 1)$, and its biorthogonal system is given by

$$v_{jk}(x) = \lim_{\lambda \rightarrow \lambda_j} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{(\lambda - \lambda_j)^{m_j}}{\Delta(\lambda)} (ib e^{i\lambda(1-x)} + i \int_x^1 e^{i\lambda(t-x)} q(t) dt) \right),$$

$0 \leq k \leq m_j - 1, j \in \mathbb{Z}$, where

$$\Delta(\lambda) = a + b e^{i\lambda} + \int_0^1 e^{i\lambda x} q(x) dx.$$

The eigenvalues of L are determined by the equation $\Delta(\lambda) = 0$.

C. The system $\{u_{jk}\}_{0 \leq k \leq m_j - 1, j \in \mathbb{Z}}$ of extended eigenfunctions (2.1) of the operator L is a Riesz basis in $L^2(0, 1)$. Any $f \in \text{Dom}(L)$ has a decomposition in a uniformly convergent series of functions in (2.1). Moreover, the eigenfunctions $e^{i\lambda_j x}$ satisfy

$$\sum_{j \in \mathbb{Z}} \|e^{i\lambda_j x} - e^{i2\pi j x}\|_{L^2(0,1)}^2 < \infty. \tag{2.2}$$

In particular, this implies that modulo finitely many elements, the system (2.1) is a WZ-system (WZ, as in Definition 1.3). □

Example 2.5. We now consider operator $L = O_h^{(n)}$, the analog of Example 2.1 in higher dimensions. Let

$$\Omega := (0, 1)^n \text{ and } h > 0, \text{ that is, } h = (h_1, \dots, h_n) \in \mathbb{R}^n : h_j > 0 \text{ for every } j = 1, \dots, n.$$

The operator $O_h^{(n)}$ on Ω is defined by the differential operator

$$O_h^{(n)} := \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \tag{2.3}$$

together with the boundary conditions (BC):

$$h_j f(x)|_{x_j=0} = f(x)|_{x_j=1}, \quad h_j \frac{\partial f}{\partial x_j}(x) \Big|_{x_j=0} = \frac{\partial f}{\partial x_j}(x) \Big|_{x_j=1}, \quad j = 1, \dots, n, \tag{2.4}$$

and the domain

$$\text{Dom}(O_h^{(n)}) = \{f \in L^2(\Omega) : \Delta f \in L^2(\Omega) : f \text{ satisfies (2.4)}\}.$$

In order to describe the corresponding biorthogonal system, we first note that since $b^0 = 1$ for all $b > 0$, we can define $0^0 = 1$. In particular, we write

$$h^x = h_1^{x_1} \dots h_n^{x_n} = \prod_{j=1}^n h_j^{x_j}$$

for $x \in [0, 1]^n$. Then, with $\mathcal{I} = \mathbb{Z}^n$, the system of eigenfunctions of the operator L_h is

$$\{u_\xi(x) = h^x e^{2\pi i x \cdot \xi}, \xi \in \mathbb{Z}^n\}$$

and the conjugate system is

$$\{v_\xi(x) = h^{-x} e^{2\pi i x \cdot \xi}, \xi \in \mathbb{Z}^n\},$$

where $x \cdot \xi = x_1 \xi_1 + \dots + x_n \xi_n$. Note that $u_\xi(x) = \prod_{j=1}^n u_{\xi_j}(x_j)$, where $u_{\xi_j}(x_j) = h_j^{x_j} e^{2\pi i x_j \xi_j}$. \square

Example 2.6. Various sine and cosine systems appear as biorthogonal systems as well. One interesting example is the collection of

$$\sin\left(k - \frac{1}{4}\right)t, \quad k \in \mathbb{N},$$

which appears as a system of eigenfunctions of the Sturm–Liouville problem after the separation of variables in the Lavrent’ev–Bicadze equation with special boundary conditions, see Ponomarev [49]. Shkalikov [67] showed that this system yields a Riesz basis in $L^2(0, \pi)$. See also Sedletskii [63, p. 146] for more perspective on this problem. \square

Example 2.7. Let $O^{(m)}$ be an ordinary differential operator in $L^2(0, 1)$ of order m generated by the differential expression

$$l(y) \equiv y^{(m)}(x) + \sum_{k=0}^{m-1} p_k(x) y^{(k)}(x), \quad 0 < x < 1, \tag{2.5}$$

with coefficients

$$p_k \in C^k[0, 1], \quad k = 0, 1, \dots, m - 1,$$

and boundary conditions

$$U_j(y) \equiv V_j(y) + \sum_{s=0}^{k_j} \int_0^1 y^{(s)}(t) \rho_{js}(t) dt = 0, \quad j = 1, \dots, m, \tag{2.6}$$

where

$$V_j(Y) \equiv \sum_{s=0}^j [\alpha_{js} Y^{(k_s)}(0) + \beta_{js} Y^{(k_s)}(1)],$$

with α_{js} and β_{js} some real numbers, and $\rho_{js} \in L^2(0, 1)$ for all j and s .

Furthermore, we suppose that the boundary conditions (2.6) are normed and strong regular in the sense considered by Shkalikov in [66]. Then it can be shown that the eigenvalues have the same algebraic and geometric multiplicities and, after a suitable adaption for our case, we have the following theorem.

Theorem 2.8 ([66]). The eigenfunctions of the operator $O^{(m)}$ with strong regular boundary conditions (2.6) form a Riesz basis in $L^2(0, 1)$. \square

In the monograph of Naimark [46], the spectral properties of differential operators generated by the differential expression (2.5) with the boundary conditions (2.6) without integral terms were considered. The statement as in Theorem 2.8 was established in this setting, with the asymptotic formula for the Weyl eigenvalue counting function $N(\lambda)$ in the form

$$N(\lambda) \sim C\lambda^{1/m} \quad \text{as } \lambda \rightarrow +\infty. \quad (2.7)$$

\square

Example 2.9. Let E_s be a realization in $L^2(\Omega)$ of a regular elliptic boundary value problem, that is, such that the underlying differential operator is uniformly elliptic and has smooth coefficients on an open bounded set $\Omega \subset \mathbb{R}^n$, and that the boundary conditions determining E_s are also regular in some sense. Suppose that E_s is a self-adjoint elliptic operator, so that E_s has a basis of eigenfunctions in $L^2(\Omega)$.

The earliest results on the asymptotic form of the eigenvalue counting function $N(\lambda)$ were obtained in 1911 by Weyl [74] for the case of the negative Laplacian $-\Delta$ in two dimensions. Using the theory of integral equations, Weyl derived the formula

$$N(\lambda) \sim \frac{\mu_2(\Omega)}{4\pi} \lambda \quad \text{as } \lambda \rightarrow +\infty, \quad (2.8)$$

where $\mu_2(\Omega)$ denotes the area of Ω . In three dimensions, this becomes

$$N(\lambda) \sim \frac{\mu_3(\Omega)}{6\pi^2} \lambda^{3/2} \quad \text{as } \lambda \rightarrow +\infty. \quad (2.9)$$

The problem was then developed by Courant [10–13], who extended the formulae of Weyl to further settings. In 1934, Carleman [7] introduced Tauberian methods reminiscent of

analytic number theory into the study of Weyl asymptotic formulae. Using Carleman’s results Clark [9] provided a rather general asymptotic formula

$$N(\lambda) \sim C\lambda^{n/m} \quad \text{as } \lambda \rightarrow +\infty, \tag{2.10}$$

for the operator E_s , where m is order of E_s . The second term of the spectral asymptotic was obtained by Duistermaat and Guillemin [20]. This has been extended further to elliptic systems, see the book [57] by Safarov and Vassiliev for a survey, as well as to systems with multiplicities, see Kamotski and Ruzhansky [33] and references therein. □

3 Global Distributions Generated by the Boundary Value Problem

In this section, we describe the spaces of distributions generated by the boundary value problem L_Ω and by its adjoint L_Ω^* and the related global Fourier analysis. The more far-reaching aim of this analysis is to establish a version of the Schwartz kernel theorem for the appearing spaces of distributions equipped with the corresponding boundary conditions. We first define the space $C_L^\infty(\bar{\Omega})$ of test functions.

Definition 3.1. The space $C_L^\infty(\bar{\Omega}) := \text{Dom}(L_\Omega^\infty)$ is called the space of test functions for L_Ω . Here we define

$$\text{Dom}(L_\Omega^\infty) := \bigcap_{k=1}^\infty \text{Dom}(L_\Omega^k),$$

where $\text{Dom}(L_\Omega^k)$, or just $\text{Dom}(L^k)$ for simplicity, is the domain of the operator L^k , in turn defined as

$$\text{Dom}(L^k) := \{f \in L^2(\Omega) : L^j f \in \text{Dom}(L), \quad j = 0, 1, 2, \dots, k - 1\}.$$

We note that in this way all operators $L^k, k \in \mathbb{N}$, are being equipped with the same boundary conditions (BC). The Fréchet topology of $C_L^\infty(\bar{\Omega})$ is given by the family of norms

$$\|\varphi\|_{C_L^k} := \max_{j \leq k} \|L^j \varphi\|_{L^2(\Omega)}, \quad k \in \mathbb{N}_0, \quad \varphi \in C_L^\infty(\bar{\Omega}). \tag{3.1}$$

Analogously to the L -case, we introduce the space $C_{L^*}^\infty(\bar{\Omega})$ corresponding to the adjoint operator L_Ω^* by

$$C_{L^*}^\infty(\bar{\Omega}) := \text{Dom}((L^*)^\infty) = \bigcap_{k=1}^\infty \text{Dom}((L^*)^k),$$

where $\text{Dom}((L^*)^k)$ is the domain of the operator $(L^*)^k$,

$$\text{Dom}((L^*)^k) := \{f \in L^2(\Omega) : (L^*)^j f \in \text{Dom}(L^*), j = 0, \dots, k - 1\},$$

which satisfy the adjoint boundary conditions corresponding to the operator L^* . The Fréchet topology of $C_{L^*}^\infty(\bar{\Omega})$ is given by the family of norms

$$\|\psi\|_{C_{L^*}^k} := \max_{j \leq k} \|(L^*)^j \psi\|_{L^2(\Omega)}, \quad k \in \mathbb{N}_0, \quad \psi \in C_{L^*}^\infty(\bar{\Omega}). \tag{3.2}$$

Since we have $u_\xi \in C_L^\infty(\bar{\Omega})$ and $v_\xi \in C_{L^*}^\infty(\bar{\Omega})$ for all $\xi \in \mathcal{I}$, we observe that Assumption 1.2 implies that the spaces $C_L^\infty(\bar{\Omega})$ and $C_{L^*}^\infty(\bar{\Omega})$ are dense in $L^2(\Omega)$. \square

We note that if L_Ω is self-adjoint, that is, if $L_\Omega^* = L_\Omega$ with the equality of domains, then $C_{L^*}^\infty(\bar{\Omega}) = C_L^\infty(\bar{\Omega})$.

In general, for functions $f \in C_L^\infty(\bar{\Omega})$ and $g \in C_{L^*}^\infty(\bar{\Omega})$, the L^2 -duality makes sense in view of the formula

$$(Lf, g)_{L^2(\Omega)} = (f, L^*g)_{L^2(\Omega)}. \tag{3.3}$$

Therefore, in view of the formula (3.3), it makes sense to define the distributions $\mathcal{D}'_L(\Omega)$ as the space which is dual to $C_{L^*}^\infty(\bar{\Omega})$. Note that the respective boundary conditions of L_Ω and L_Ω^* are satisfied by the choice of f and g in corresponding domains.

Definition 3.2. The space

$$\mathcal{D}'_L(\Omega) := \mathcal{L}(C_{L^*}^\infty(\bar{\Omega}), \mathbb{C})$$

of linear continuous functionals on $C_{L^*}^\infty(\bar{\Omega})$ is called the space of L -distributions. We can understand the continuity here either in terms of the topology (3.2) or in terms of sequences, see Proposition 3.3. For $w \in \mathcal{D}'_L(\Omega)$ and $\varphi \in C_{L^*}^\infty(\bar{\Omega})$, we shall write

$$w(\varphi) = \langle w, \varphi \rangle.$$

For any $\psi \in C_L^\infty(\bar{\Omega})$,

$$C_{L^*}^\infty(\bar{\Omega}) \ni \varphi \mapsto \int_\Omega \psi(x)\varphi(x) \, dx$$

is an L -distribution, which gives an embedding $\psi \in C_L^\infty(\bar{\Omega}) \hookrightarrow \mathcal{D}'_L(\Omega)$. We note that in the distributional notation formula (3.3) becomes

$$\langle L\psi, \varphi \rangle = \langle \psi, \overline{L^*\varphi} \rangle. \tag{3.4}$$

\square

With the topology on $C_L^\infty(\bar{\Omega})$ defined by (3.1), the space

$$\mathcal{D}'_{L^*}(\Omega) := \mathcal{L}(C_L^\infty(\bar{\Omega}), \mathbb{C})$$

of linear continuous functionals on $C_L^\infty(\bar{\Omega})$ is called the space of L^* -distributions.

Proposition 3.3. A linear functional w on $C_L^\infty(\bar{\Omega})$ belongs to $\mathcal{D}'_L(\Omega)$ if and only if there exist a constant $c > 0$ and a number $k \in \mathbb{N}_0$ with the property

$$|w(\varphi)| \leq c \|\varphi\|_{C_{L^*}^k} \quad \text{for all } \varphi \in C_L^\infty(\bar{\Omega}). \tag{3.5}$$

□

Proof. \Leftarrow . If w satisfies (3.5), it then follows that $w(\varphi_j) - w(\varphi) = w(\varphi_j - \varphi)$ converges to 0 as $j \rightarrow \infty$.

\Rightarrow . Now suppose that w does not satisfy condition (3.5). This means that for every $c > 0$ and every $k \in \mathbb{N}_0$, there is a $\varphi_{c,k} \in C_{L^*}^k(\bar{\Omega})$ for which

$$|w(\varphi_{c,k})| > c \|\varphi_{c,k}\|_{C_{L^*}^k}.$$

This implies

$$\|\psi_{c,k}\|_{C_{L^*}^k} < \frac{1}{c} \quad \text{and} \quad |w(\psi_{c,k})| = 1,$$

if we take $\psi_{c,k} = \lambda \varphi_{c,k}$ and $\lambda = \frac{1}{|w(\varphi_{c,k})|}$. The sequence $\{\psi_{k,k}\}_{k \in \mathbb{N}}$ converges to zero in $C_{L^*}^\infty(\bar{\Omega})$, while $w(\psi_{k,k})$ does not converge to zero. Therefore, w is not a distribution, which gives a contradiction. ■

The space $\mathcal{D}'_L(\Omega)$ has many similarities with the usual spaces of distributions. For example, suppose that for a linear continuous operator $D : C_L^\infty(\bar{\Omega}) \rightarrow C_L^\infty(\bar{\Omega})$ its adjoint D^* preserves the adjoint boundary conditions (domain) of L^*_Ω and is continuous on the space $C_{L^*}^\infty(\bar{\Omega})$, that is, that the operator $D^* : C_{L^*}^\infty(\bar{\Omega}) \rightarrow C_{L^*}^\infty(\bar{\Omega})$ is continuous. Then we can extend D to $\mathcal{D}'_L(\Omega)$ by

$$\langle Dw, \varphi \rangle := \langle w, \overline{D^* \varphi} \rangle \quad (w \in \mathcal{D}'_L(\Omega), \varphi \in C_{L^*}^\infty(\bar{\Omega})).$$

This extends (3.4) from L to other operators. The convergence in the linear space $\mathcal{D}'_L(\Omega)$ is the usual weak convergence with respect to the space $C_{L^*}^\infty(\bar{\Omega})$. The following principle of uniform boundedness is based on the Banach–Steinhaus Theorem applied to the Fréchet space $C_{L^*}^\infty(\bar{\Omega})$.

Lemma 3.4. Let $\{w_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathcal{D}'_L(\Omega)$ with the property that for every $\varphi \in C_{L^*}^\infty(\bar{\Omega})$, the sequence $\{w_j(\varphi)\}_{j \in \mathbb{N}}$ in \mathbb{C} is bounded. Then there exist constants $c > 0$ and $k \in \mathbb{N}_0$ such that

$$|w_j(\varphi)| \leq c \|\varphi\|_{C_{L^*}^k} \quad \text{for all } j \in \mathbb{N}, \varphi \in C_{L^*}^\infty(\bar{\Omega}). \quad (3.6)$$

□

The lemma above leads to the following property of completeness of the space of L -distributions.

Theorem 3.5. Let $\{w_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathcal{D}'_L(\Omega)$ with the property that for every $\varphi \in C_{L^*}^\infty(\bar{\Omega})$ the sequence $\{w_j(\varphi)\}_{j \in \mathbb{N}}$ converges in \mathbb{C} as $j \rightarrow \infty$. Denote the limit by $w(\varphi)$.

(i) Then $w : \varphi \mapsto w(\varphi)$ defines an L -distribution on Ω . Furthermore,

$$\lim_{j \rightarrow \infty} w_j = w \quad \text{in } \mathcal{D}'_L(\Omega).$$

(ii) If $\varphi_j \rightarrow \varphi$ in $C_{L^*}^\infty(\bar{\Omega})$, then

$$\lim_{j \rightarrow \infty} w_j(\varphi_j) = w(\varphi) \quad \text{in } \mathbb{C}. \quad \square$$

Proof. (i) Writing out the definitions, we find that w defines a linear functional on $C_{L^*}^\infty(\bar{\Omega})$. From the starting assumption, it follows that the sequence $\{w_j(\varphi)\}_{j \in \mathbb{N}}$ is bounded for every $\varphi \in C_{L^*}^\infty(\bar{\Omega})$, and thus we obtain an estimate of the form (3.6). Taking the limit in

$$|w(\varphi)| \leq |w(\varphi) - w_j(\varphi)| + |w_j(\varphi)| \leq |w(\varphi) - w_j(\varphi)| + c \|\varphi\|_{C_{L^*}^k}$$

as $j \rightarrow \infty$, we get

$$|w(\varphi)| \leq c \|\varphi\|_{C_{L^*}^k}$$

for all $\varphi \in C_{L^*}^\infty(\bar{\Omega})$. According to Proposition 3.3, this proves that $w \in \mathcal{D}'_L(\Omega)$, and $w_j \rightarrow w$ in $\mathcal{D}'_L(\Omega)$ now holds by definition.

(ii) Regarding the last assertion, we observe that if $\varphi_j \rightarrow \varphi$ in $C_{L^*}^\infty(\bar{\Omega})$, then by applying Lemma 3.4 once again, we obtain

$$|w_j(\varphi_j) - w(\varphi)| \leq |w_j(\varphi_j - \varphi)| + |w_j(\varphi) - w(\varphi)| \leq c \|\varphi_j - \varphi\|_{C_{L^*}^k} + |w_j(\varphi) - w(\varphi)|,$$

which converges to zero as $j \rightarrow \infty$. ■

The main tool in the proof of Theorem 3.5 was Lemma 3.4, which is based on the principle of uniform boundedness. It may be instructive to give another proof of Part (i) of Theorem 3.5 based on the method of the gliding hump.

Proof. Suppose that w does not belong to $\mathcal{D}'_L(\Omega)$. Then there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}}$ in $C^\infty_{L^*}(\bar{\Omega})$ that converges to zero in $C^\infty_{L^*}(\bar{\Omega})$, while $\{w(\varphi_j)\}_{j \in \mathbb{N}}$ does not converge to zero as $j \rightarrow \infty$. Hence, by passing to a subsequence if necessary, we can arrange that there exists $c > 0$ such that $|w(\varphi_j)| \geq c$. We can assume that $\|\varphi_j\|_{C^j_{L^*}} \leq \frac{1}{4^j}$ if we replace $\{\varphi_j\}_{j \in \mathbb{N}}$ by a suitable subsequence if necessary. Accordingly, upon writing φ_j for $2^j \varphi_j$, we obtain that $\varphi_j \rightarrow 0$ in $C^\infty_{L^*}(\bar{\Omega})$, while $|w(\varphi_j)| \rightarrow \infty$ as $j \rightarrow \infty$.

Next, we define a subsequence of $\{\varphi_j\}_{j \in \mathbb{N}}$, say $\{\psi_j\}_{j \in \mathbb{N}}$ in $C^\infty_{L^*}(\bar{\Omega})$, and a subsequence of $\{w_j\}_{j \in \mathbb{N}}$, say $\{v_j\}_{j \in \mathbb{N}}$ in $\mathcal{D}'_L(\Omega)$, as follows. Select ψ_1 such that $|w(\psi_1)| > 2$. As $w_j(\psi_1) \rightarrow w(\psi_1)$, we may choose v_1 such that $|v_1(\psi_1)| > 2$. Now proceed by induction on j . Thus, assume that ψ_k and v_k have been chosen, for $1 \leq k < j$. Then select ψ_j from the sequence $\{\varphi_j\}_{j \in \mathbb{N}}$ such that

$$\begin{aligned} \text{(a)} \quad & \|\psi_j\|_{C^j_{L^*}} < \frac{1}{2^j}, \\ \text{(b)} \quad & |v_k(\psi_j)| < \frac{1}{2^{j-k}} \quad (1 \leq k < j), \\ \text{(c)} \quad & |w(\psi_j)| > \sum_{1 \leq k < j} |w(\psi_k)| + j + 1. \end{aligned} \tag{3.7}$$

Condition (a) can be satisfied because of the properties of the φ_i ; and (b) because of $\varphi_j \rightarrow 0$ in $C^\infty_{L^*}(\bar{\Omega})$ and all v_k belong to $\mathcal{D}'_L(\Omega)$, for $1 \leq k < j$; whereas (c) holds because $|w(\varphi_j)| \rightarrow \infty$ as $j \rightarrow \infty$. In addition, since $\lim_{j \rightarrow \infty} w_j(\psi) = w(\psi)$, for all $\psi \in C^\infty_{L^*}(\bar{\Omega})$, condition (c) implies that we may select v_j from the sequence $\{w_j\}_{j \in \mathbb{N}}$ such that

$$|v_j(\psi_j)| > \sum_{1 \leq k < j} |v_j(\psi_k)| + j + 1. \tag{3.8}$$

Now, set $\psi := \sum_{k \in \mathbb{N}} \psi_k$. According to (a), the series on the right-hand side converges in $C^\infty_{L^*}(\bar{\Omega})$, which leads to $\psi \in C^\infty_{L^*}(\bar{\Omega})$. Obviously, for any j ,

$$v_j(\psi) = \sum_{1 \leq k < j} v_j(\psi_k) + v_j(\psi_j) + \sum_{j < k} v_j(\psi_k),$$

hence

$$|v_j(\psi)| \geq |v_j(\psi_j)| - \sum_{1 \leq k < j} |v_j(\psi_k)| - \sum_{j < k} |v_j(\psi_k)| > j + 1 - 1 = j,$$

on account of (3.8) and (b). On the other hand, $\{v_j\}_{j \in \mathbb{N}}$ being a subsequence of $\{w_j\}_{j \in \mathbb{N}}$ implies $\lim_{j \rightarrow \infty} v_j(\psi) = w(\psi)$. Summarizing these properties, we have arrived at a contradiction. ■

Similarly to the previous case, we have analogs of Proposition 3.3 and Theorem 3.5 for L^* -distributions.

4 L -Fourier Transform

In this section, we define the L -Fourier transform generated by our boundary value problem L_Ω and its main properties. The main difference between the self-adjoint and non-self-adjoint problems L_Ω is that in the latter case we have to make sure that we use the right functions from the available biorthogonal families of u_ξ and v_ξ . We start by defining the spaces that we will obtain on the Fourier transform side.

From now on, we will assume that the boundary conditions are closed under taking limits in the strong uniform topology to ensure that the strongly convergent series preserve the boundary conditions. More precisely, from now on:

(BC+) *assume that, with L_0 denoting L or L^* , if $f_j \in C_{L_0}^\infty(\bar{\Omega})$ satisfies $f_j \rightarrow f$ in $C_{L_0}^\infty(\bar{\Omega})$, then $f \in C_{L_0}^\infty(\bar{\Omega})$.*

Let $\mathcal{S}(\mathcal{I})$ denote the space of rapidly decaying functions $\varphi: \mathcal{I} \rightarrow \mathbb{C}$. That is, $\varphi \in \mathcal{S}(\mathcal{I})$ if for any $M < \infty$, there exists a constant $C_{\varphi, M}$ such that

$$|\varphi(\xi)| \leq C_{\varphi, M} \langle \xi \rangle^{-M}$$

holds for all $\xi \in \mathcal{I}$. Here $\langle \xi \rangle$ is already adapted to our boundary value problem since it is defined by (1.6).

The topology on $\mathcal{S}(\mathcal{I})$ is given by the seminorms p_k , where $k \in \mathbb{N}_0$ and

$$p_k(\varphi) := \sup_{\xi \in \mathcal{I}} \langle \xi \rangle^k |\varphi(\xi)|.$$

Continuous linear functionals on $\mathcal{S}(\mathcal{I})$ are of the form

$$\varphi \mapsto \langle u, \varphi \rangle := \sum_{\xi \in \mathcal{I}} u(\xi) \varphi(\xi),$$

where functions $u: \mathcal{I} \rightarrow \mathbb{C}$ grow at most polynomially at infinity, that is, there exist constants $M < \infty$ and $C_{u, M}$ such that

$$|u(\xi)| \leq C_{u, M} \langle \xi \rangle^M$$

holds for all $\xi \in \mathcal{I}$. Such distributions $u: \mathcal{I} \rightarrow \mathbb{C}$ form the space of distributions which we denote by $\mathcal{S}'(\mathcal{I})$. We now define the L -Fourier transform on $C_L^\infty(\bar{\Omega})$.

Definition 4.1. We define the L -Fourier transform

$$(\mathcal{F}_L f)(\xi) = (f \mapsto \hat{f}) : C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{S}(\mathcal{I})$$

by

$$\hat{f}(\xi) := (\mathcal{F}_L f)(\xi) = \int_{\Omega} f(x) \overline{v_{\xi}(x)} \, dx. \tag{4.1}$$

Analogously, we define the L^* -Fourier transform

$$(\mathcal{F}_{L^*} f)(\xi) = (f \mapsto \hat{f}_*) : C_{L^*}^\infty(\bar{\Omega}) \rightarrow \mathcal{S}(\mathcal{I})$$

by

$$\hat{f}_*(\xi) := (\mathcal{F}_{L^*} f)(\xi) = \int_{\Omega} f(x) \overline{u_{\xi}(x)} \, dx. \tag{4.2}$$

□

Expressions (4.1) and (4.2) are well defined by the Cauchy–Schwarz inequality, for example,

$$|\hat{f}(\xi)| = \left| \int_{\Omega} f(x) \overline{v_{\xi}(x)} \, dx \right| \leq \|f\|_{L^2} \|v_{\xi}\|_{L^2} = \|f\|_{L^2} < \infty. \tag{4.3}$$

Moreover, we have the following proposition.

Proposition 4.2. The L -Fourier transform \mathcal{F}_L is a bijective homeomorphism from $C_L^\infty(\bar{\Omega})$ to $\mathcal{S}(\mathcal{I})$. Its inverse

$$\mathcal{F}_L^{-1} : \mathcal{S}(\mathcal{I}) \rightarrow C_L^\infty(\bar{\Omega})$$

is given by

$$(\mathcal{F}_L^{-1} h)(x) = \sum_{\xi \in \mathcal{I}} h(\xi) u_{\xi}(x), \quad h \in \mathcal{S}(\mathcal{I}), \tag{4.4}$$

so that the Fourier inversion formula becomes

$$f(x) = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) u_{\xi}(x) \quad \text{for all } f \in C_L^\infty(\bar{\Omega}). \tag{4.5}$$

Similarly, $\mathcal{F}_{L^*} : C_{L^*}^\infty(\bar{\Omega}) \rightarrow \mathcal{S}(\mathcal{I})$ is a bijective homeomorphism and its inverse

$$\mathcal{F}_{L^*}^{-1} : \mathcal{S}(\mathcal{I}) \rightarrow C_{L^*}^\infty(\bar{\Omega})$$

is given by

$$(\mathcal{F}_{L^*}^{-1} h)(x) := \sum_{\xi \in \mathcal{I}} h(\xi) v_{\xi}(x), \quad h \in \mathcal{S}(\mathcal{I}), \tag{4.6}$$

so that the conjugate Fourier inversion formula becomes

$$f(x) = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) v_\xi(x) \quad \text{for all } f \in C_L^\infty(\bar{\Omega}). \tag{4.7}$$

□

Proof. The proof is largely similar to the standard case, so we only indicate a few key points due to biorthogonality. We show first that for any $f \in C_L^\infty(\bar{\Omega})$ we have $\hat{f} \in \mathcal{S}(\mathcal{I})$, that is, that for any $M < \infty$ there exists a constant C such that

$$|\hat{f}(\xi)| \leq C \langle \xi \rangle^{-M}$$

holds for all $\xi \in \mathcal{I}$. Indeed, for any $M \in \mathbb{N}$ and $\lambda_\xi \neq 0$ we get

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\Omega} f(x) \overline{v_\xi(x)} \, dx \right| = \left| \int_{\Omega} f(x) \frac{(L^*)^M v_\xi(x)}{\lambda_\xi^M} \, dx \right| \\ &= \left| \frac{1}{\lambda_\xi^M} \int_{\Omega} L^M f(x) \overline{v_\xi(x)} \, dx \right| \leq C \|L^M f\|_{L^2(\Omega)} \langle \xi \rangle^{-mM} \end{aligned}$$

by the Cauchy–Schwarz inequality. In view of (3.1), this also shows that \mathcal{F}_L is continuous from $C_L^\infty(\bar{\Omega})$ to $\mathcal{S}(\mathcal{I})$.

Now, in view of (BC+), for any $h \in \mathcal{S}(\mathcal{I})$ the formula (4.4) defines a function $\mathcal{F}_L^{-1} h \in C_L^\infty(\bar{\Omega})$ with Fourier coefficients $h(\xi)$ due to biorthogonality relations (1.5). If two function $f_1, f_2 \in C_L^\infty(\bar{\Omega})$ have the same Fourier coefficients $\hat{f}_1(\xi) = \hat{f}_2(\xi)$ for all $\xi \in \mathcal{I}$, since the linear span $\{u_\xi\}_{\xi \in \mathcal{I}}$ is dense in $C_L^\infty(\bar{\Omega})$, we have

$$f_1(x) = \sum_{\xi \in \mathcal{I}} \hat{f}_1(\xi) u_\xi(x) = \sum_{\xi \in \mathcal{I}} \hat{f}_2(\xi) u_\xi(x) = f_2(x).$$

The continuity of $\mathcal{F}_L^{-1} : \mathcal{S}(\mathcal{I}) \rightarrow C_L^\infty(\bar{\Omega})$ readily follows as well. The properties of the conjugate Fourier transform \mathcal{F}_{L^*} can be seen in an analogous way. ■

By dualizing the inverse L -Fourier transform $\mathcal{F}_L^{-1} : \mathcal{S}(\mathcal{I}) \rightarrow C_L^\infty(\bar{\Omega})$, the L -Fourier transform extends uniquely to the mapping

$$\mathcal{F}_L : \mathcal{D}'_L(\Omega) \rightarrow \mathcal{S}'(\mathcal{I})$$

by the formula

$$\langle \mathcal{F}_L w, \varphi \rangle := \left\langle w, \overline{\mathcal{F}_{L^*}^{-1} \varphi} \right\rangle \quad \text{with } w \in \mathcal{D}'_L(\Omega), \varphi \in \mathcal{S}(\mathcal{I}). \tag{4.8}$$

It can be readily seen that if $w \in \mathcal{D}'_L(\Omega)$, then $\hat{w} \in \mathcal{S}'(\mathcal{I})$. The reason for taking complex conjugates in (4.9) is that, if $w \in C^\infty_L(\bar{\Omega})$, we have the equality

$$\begin{aligned} \langle \hat{w}, \varphi \rangle &= \sum_{\xi \in \mathcal{I}} \hat{w}(\xi) \varphi(\xi) = \sum_{\xi \in \mathcal{I}} \left(\int_{\Omega} w(x) \overline{v_\xi(x)} \, dx \right) \varphi(\xi) \\ &= \int_{\Omega} w(x) \overline{\left(\sum_{\xi \in \mathcal{I}} \varphi(\xi) v_\xi(x) \right)} \, dx = \int_{\Omega} w(x) \overline{(\mathcal{F}_{L^*}^{-1} \bar{\varphi})} \, dx = \langle w, \overline{\mathcal{F}_{L^*}^{-1} \bar{\varphi}} \rangle. \end{aligned}$$

Analogously, we have the mapping

$$\mathcal{F}_{L^*} : \mathcal{D}'_{L^*}(\Omega) \rightarrow \mathcal{S}'(\mathcal{I})$$

defined by the formula

$$\langle \mathcal{F}_{L^*} w, \varphi \rangle := \langle w, \overline{\mathcal{F}_{L^*}^{-1} \bar{\varphi}} \rangle \quad \text{with } w \in \mathcal{D}'_{L^*}(\Omega), \varphi \in \mathcal{S}(\mathcal{I}). \tag{4.9}$$

It can be also seen that if $w \in \mathcal{D}'_{L^*}(\Omega)$, then $\hat{w} \in \mathcal{S}'(\mathcal{I})$.

We note that since systems of u_ξ and of v_ξ are Riesz bases, we can also compare L^2 -norms of functions with sums of squares of Fourier coefficients. The following statement follows from the work of Bari [4, Theorem 9].

Lemma 4.3. There exist constants $k, K, m, M > 0$ such that for every $f \in L^2(\Omega)$ we have

$$m^2 \|f\|_{L^2}^2 \leq \sum_{\xi \in \mathcal{I}} |\hat{f}(\xi)|^2 \leq M^2 \|f\|_{L^2}^2$$

and

$$k^2 \|f\|_{L^2}^2 \leq \sum_{\xi \in \mathcal{I}} |\hat{f}_*(\xi)|^2 \leq K^2 \|f\|_{L^2}^2. \quad \square$$

However, we note that the Plancherel identity can be also achieved in suitably defined l^2 -spaces of Fourier coefficients, see Proposition 6.1.

5 L-Convolution

Let us introduce a notion of the L -convolution, an analog of the convolution adapted to the boundary problem L_Ω .

Definition 5.1 (*L*-Convolution). For $f, g \in C_L^\infty(\bar{\Omega})$, define their *L*-convolution by

$$(f \star_L g)(x) := \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \hat{g}(\xi) u_\xi(x). \quad (5.1)$$

By Proposition 4.2, it is well defined and we have $f \star_L g \in C_L^\infty(\bar{\Omega})$.

Moreover, due to the rapid decay of *L*-Fourier coefficients of functions in $C_L^\infty(\bar{\Omega})$ compared to a fixed polynomial growth of elements of $\mathcal{S}'(\mathcal{I})$, the definition (5.1) still makes sense if $f \in \mathcal{D}'_L(\Omega)$ and $g \in C_L^\infty(\bar{\Omega})$, with $f \star_L g \in C_L^\infty(\bar{\Omega})$. \square

Analogously to the *L*-convolution, we can introduce the *L**-convolution. Thus, for $f, g \in C_{L^*}^\infty(\bar{\Omega})$, we define the *L**-convolution using the *L**-Fourier transform by

$$(f \star_{L^*} g)(x) := \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) \hat{g}_*(\xi) v_\xi(x). \quad (5.2)$$

Its properties are similar to those of the *L*-convolution, so we may formulate and prove only the latter.

Remark 5.2. Informally, expanding the definitions of the Fourier transforms in (5.1), we can also write

$$(f \star_L g)(x) := \int_\Omega \int_\Omega F(x, y, z) f(y) g(z) dy dz, \quad (5.3)$$

where

$$F(x, y, z) = \sum_{\xi \in \mathcal{I}} u_\xi(x) \overline{v_\xi(y)} \overline{v_\xi(z)}.$$

The latter series should be understood in the sense of distributions.

In the case of operator $L = O_1^{(1)}$ generated by the operator of differentiation with periodic boundary condition on the interval $(0, 1)$, see the case $h = 1$ in Example 2.1 as in [54], we have

$$F(x, y, z) = \delta(x - y - z).$$

For any $h > 0$, it can be shown that the convolution generated by the operator $O_h^{(1)}$ from Example 2.1 has also the following integral form:

$$(f \star_{O_h^{(1)}} g)(x) = \int_0^x f(x-t)g(t) dt + \frac{1}{h} \int_x^1 f(1+x-t)g(t) dt,$$

see [36, 38]. \square

Proposition 5.3. For any $f, g \in C_L^\infty(\bar{\Omega})$, we have

$$\widehat{f \star_L g} = \hat{f} \hat{g}.$$

The convolution is commutative and associative. If $g \in C_L^\infty(\bar{\Omega})$, then for all $f \in \mathcal{D}'_L(\Omega)$ we have

$$f \star_L g \in C_L^\infty(\bar{\Omega}). \tag{5.4}$$

If $f, g \in L^2(\Omega)$, then $f \star_L g \in L^1(\Omega)$ with

$$\|f \star_L g\|_{L^1} \leq C |\Omega|^{1/2} \|f\|_{L^2} \|g\|_{L^2},$$

where $|\Omega|$ is the volume of Ω , with C independent of f, g, Ω . □

Proof. By direct calculation, we get

$$\begin{aligned} \mathcal{F}_L(f \star_L g)(\xi) &= \int_{\Omega} \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \hat{g}(\eta) u_{\eta}(x) \overline{v_{\xi}(x)} \, dx \\ &= \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \hat{g}(\eta) \int_{\Omega} u_{\eta}(x) \overline{v_{\xi}(x)} \, dx \\ &= \hat{f}(\xi) \hat{g}(\xi). \end{aligned}$$

This also implies the commutativity of the convolution in view of the bijectivity of the Fourier transform. For the associativity, let $f, g, h \in C_L^\infty(\bar{\Omega})$. We can argue similarly using the Fourier transform or, by the definition and direct calculations, we have

$$\begin{aligned} ((f \star_L g) \star_L h)(x) &= \sum_{\xi \in \mathcal{I}} \left[\int_{\Omega} \left(\sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \hat{g}(\eta) u_{\eta}(y) \right) \overline{v_{\xi}(y)} \, dy \right] \hat{h}(\xi) u_{\xi}(x) \\ &= \sum_{\xi \in \mathcal{I}} \left[\sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \hat{g}(\eta) \int_{\Omega} u_{\eta}(y) \overline{v_{\xi}(y)} \, dy \right] \hat{h}(\xi) u_{\xi}(x) \\ &= \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \hat{g}(\xi) \hat{h}(\xi) u_{\xi}(x) \\ &= \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \left[\sum_{\eta \in \mathcal{I}} \hat{g}(\eta) \hat{h}(\eta) \int_{\Omega} u_{\eta}(y) \overline{v_{\xi}(y)} \, dy \right] u_{\xi}(x) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \left[\int_{\Omega} \left(\sum_{\eta \in \mathcal{I}} \hat{g}(\eta) \hat{h}(\eta) u_{\eta}(y) \right) \overline{v_{\xi}(y)} \, dy \right] u_{\xi}(x) \\
 &= (f \star_L (g \star_L h))(x).
 \end{aligned}$$

The associativity is proved. For (5.4), we note that

$$L^k(f \star_L g)(x) = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \hat{g}(\xi) \lambda_{\xi}^k u_{\xi}(x),$$

and the series converges absolutely since $\hat{g} \in S(\mathcal{I})$. By (BC+), the boundary conditions are also satisfied since they are satisfied by u_{ξ} . This shows that $f \star_L g \in C_L^{\infty}(\bar{\Omega})$.

For the last statement, by simple calculations we get

$$\begin{aligned}
 \int_{\Omega} |(f \star_L g)(x)| \, dx &\leq \int_{\Omega} \sum_{\xi \in \mathcal{I}} |\hat{f}(\xi) \hat{g}(\xi)| |u_{\xi}(x)| \, dx \\
 &\leq \sum_{\xi \in \mathcal{I}} |\hat{f}(\xi)| |\hat{g}(\xi)| \|u_{\xi}\|_{L^1} \\
 &\leq C \|f\|_{L^2} \|g\|_{L^2} \sup_{\xi \in \mathcal{I}} \|u_{\xi}\|_{L^1},
 \end{aligned}$$

the latter estimate by Lemma 4.3. Since Ω is a bounded set, by the Cauchy–Schwarz inequality we have

$$\|u_{\xi}\|_{L^1} \leq |\Omega|^{1/2} \|u_{\xi}\|_{L^2} = |\Omega|^{1/2}$$

for all $\xi \in \mathcal{I}$, where $|\Omega|$ is the volume of Ω . This inequality implies the statement. ■

6 Plancherel Formula, Sobolev Spaces $\mathcal{H}_L^s(\Omega)$, and their Fourier Images

In this section, we discuss Sobolev spaces adapted to L_{Ω} and their images under the L -Fourier transform. We start with the L^2 -setting, where we can recall inequalities between L^2 -norms of functions and sums of squares of their Fourier coefficients, see Lemma 4.3. However, below we show that we actually have the Plancherel identity in a suitably defined space l_L^2 and its conjugate $l_{L^*}^2$.

Let us denote by

$$l_L^2 = l^2(L)$$

the linear space of complex-valued functions a on \mathcal{I} such that $\mathcal{F}_L^{-1} a \in L^2(\Omega)$, that is, if there exists $f \in L^2(\Omega)$ such that $\mathcal{F}_L f = a$. Then the space of sequences l_L^2 is a Hilbert

space with the inner product

$$(a, b)_{l_L^2} := \sum_{\xi \in \mathcal{I}} a(\xi) \overline{(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} b)(\xi)} \tag{6.1}$$

for arbitrary $a, b \in l_L^2$. The reason for this choice of the definition is the following formal calculation:

$$\begin{aligned} (a, b)_{l_L^2} &= \sum_{\xi \in \mathcal{I}} a(\xi) \overline{(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} b)(\xi)} \\ &= \sum_{\xi \in \mathcal{I}} a(\xi) \int_{\Omega} \overline{(\mathcal{F}_L^{-1} b)(x)} u_{\xi}(x) \, dx \\ &= \int_{\Omega} \left[\sum_{\xi \in \mathcal{I}} a(\xi) u_{\xi}(x) \right] \overline{(\mathcal{F}_L^{-1} b)(x)} \, dx \\ &= \int_{\Omega} (\mathcal{F}_L^{-1} a)(x) \overline{(\mathcal{F}_L^{-1} b)(x)} \, dx \\ &= (\mathcal{F}_L^{-1} a, \mathcal{F}_L^{-1} b)_{L^2}, \end{aligned} \tag{6.2}$$

which implies the Hilbert space properties of the space of sequences l_L^2 . The norm of l_L^2 is then given by the formula

$$\|a\|_{l_L^2} = \left(\sum_{\xi \in \mathcal{I}} a(\xi) \overline{(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} a)(\xi)} \right)^{1/2} \quad \text{for all } a \in l_L^2. \tag{6.3}$$

We note that individual terms in this sum may be complex-valued but the whole sum is real and non-negative due to formula (6.2).

Analogously, we introduce the Hilbert space

$$l_{L^*}^2 = l^2(L^*)$$

as the space of functions a on \mathcal{I} such that $\mathcal{F}_{L^*}^{-1} a \in L^2(\Omega)$, with the inner product

$$(a, b)_{l_{L^*}^2} := \sum_{\xi \in \mathcal{I}} a(\xi) \overline{(\mathcal{F}_L \circ \mathcal{F}_{L^*}^{-1} b)(\xi)} \tag{6.4}$$

for arbitrary $a, b \in l_{L^*}^2$. The norm of $l_{L^*}^2$ is given by the formula

$$\|a\|_{l_{L^*}^2} = \left(\sum_{\xi \in \mathcal{I}} a(\xi) \overline{(\mathcal{F}_L \circ \mathcal{F}_{L^*}^{-1} a)(\xi)} \right)^{1/2}$$

for all $a \in l_{L^*}^2$. The spaces of sequences l_L^2 and $l_{L^*}^2$ are thus generated by biorthogonal systems $\{u_\xi\}_{\xi \in \mathcal{I}}$ and $\{v_\xi\}_{\xi \in \mathcal{I}}$. The reason for their definition in the above forms becomes clear again in view of the following Plancherel identity.

Proposition 6.1 (Plancherel’s identity). If $f, g \in L^2(\Omega)$, then $\hat{f}, \hat{g} \in l_L^2$, $\hat{f}_*, \hat{g}_* \in l_{L^*}^2$, and the inner products (6.1), (6.4) take the form

$$(\hat{f}, \hat{g})_{l_L^2} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{g}_*(\xi)}$$

and

$$(\hat{f}_*, \hat{g}_*)_{l_{L^*}^2} = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) \overline{\hat{g}(\xi)}.$$

In particular, we have

$$\overline{(\hat{f}, \hat{g})_{l_L^2}} = (\hat{g}_*, \hat{f}_*)_{l_{L^*}^2}.$$

The Parseval identity takes the form

$$(f, g)_{L^2} = (\hat{f}, \hat{g})_{l_L^2} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{g}_*(\xi)}. \tag{6.5}$$

Furthermore, for any $f \in L^2(\Omega)$, we have $\hat{f} \in l_L^2$, $\hat{f}_* \in l_{L^*}^2$, and

$$\|f\|_{L^2} = \|\hat{f}\|_{l_L^2} = \|\hat{f}_*\|_{l_{L^*}^2}. \tag{6.6}$$

□

Proof. By the definition, we get

$$(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} \hat{g})(\xi) = (\mathcal{F}_{L^*} g)(\xi) = \hat{g}_*(\xi)$$

and

$$(\mathcal{F}_L \circ \mathcal{F}_{L^*}^{-1} \hat{g}_*)(\xi) = (\mathcal{F}_L g)(\xi) = \hat{g}(\xi).$$

Hence, it follows that

$$(\hat{f}, \hat{g})_{l_L^2} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} \hat{g})(\xi)} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{g}_*(\xi)}$$

and

$$(\hat{f}_*, \hat{g}_*)_{l_{L^*}^2} = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) \overline{(\mathcal{F}_L \circ \mathcal{F}_{L^*}^{-1} \hat{g}_*)(\xi)} = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) \overline{\hat{g}(\xi)}.$$

To show Parseval’s identity (6.5), using these properties and the biorthogonality of u_ξ ’s to v_η ’s, we can write

$$\begin{aligned} (f, g)_{L^2} &= \left(\sum_{\xi \in \mathcal{I}} \hat{f}(\xi) u_\xi, \sum_{\eta \in \mathcal{I}} \hat{g}_*(\eta) v_\eta \right) \\ &= \sum_{\xi \in \mathcal{I}} \sum_{\eta \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{g}_*(\eta)} (u_\xi, v_\eta)_{L^2} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{g}_*(\xi)} = \left(\hat{f}, \hat{g} \right)_{l^2_\xi}, \end{aligned}$$

proving (6.5). Taking $f = g$, we get

$$\|f\|_{L^2}^2 = (f, f)_{L^2} = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) \overline{\hat{f}_*(\xi)} = \left(\hat{f}, \hat{f} \right)_{l^2_\xi} = \left\| \hat{f} \right\|_{l^2_\xi}^2,$$

proving the first equality in (6.6). Then, by checking that

$$(f, f)_{L^2} = \overline{(f, f)_{L^2}} = \sum_{\xi \in \mathcal{I}} \overline{\hat{f}(\xi)} \hat{f}_*(\xi) = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) \overline{\hat{f}(\xi)} = \left(\hat{f}_*, \hat{f}_* \right)_{l^2_{\xi^*}} = \left\| \hat{f}_* \right\|_{l^2_{\xi^*}}^2,$$

the proofs of (6.6) and of Proposition 6.1 are complete. ■

Now we introduce Sobolev spaces generated by the operator L_Ω .

Definition 6.2 (Sobolev spaces $\mathcal{H}_L^s(\Omega)$). For $f \in \mathcal{D}'_L(\Omega) \cap \mathcal{D}'_{L^*}(\Omega)$ and $s \in \mathbb{R}$, we say that

$$f \in \mathcal{H}_L^s(\Omega) \text{ if and only if } \langle \xi \rangle^s \hat{f}(\xi) \in l^2_\xi.$$

We define the norm on $\mathcal{H}_L^s(\Omega)$ by

$$\|f\|_{\mathcal{H}_L^s(\Omega)} := \left(\sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}(\xi) \overline{\hat{f}_*(\xi)} \right)^{1/2}. \tag{6.7}$$

The Sobolev space $\mathcal{H}_L^s(\Omega)$ is then the space of L -distributions f for which we have $\|f\|_{\mathcal{H}_L^s(\Omega)} < \infty$. Similarly, we can define the space $\mathcal{H}_{L^*}^s(\Omega)$ by the condition

$$\|f\|_{\mathcal{H}_{L^*}^s(\Omega)} := \left(\sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}_*(\xi) \overline{\hat{f}(\xi)} \right)^{1/2} < \infty. \tag{6.8}$$

□

We note that the expressions in (6.7) and (6.8) are well defined since the sum

$$\sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}(\xi) \overline{\hat{f}_*(\xi)} = \left(\langle \xi \rangle^s \hat{f}(\xi), \langle \xi \rangle^s \hat{f}(\xi) \right)_{l^2_{\mathcal{I}}} \geq 0$$

is real and non-negative. Consequently, since we can write the sum in (6.8) as the complex conjugate of that in (6.7), and with both being real, we see that the spaces $\mathcal{H}_L^s(\Omega)$ and $\mathcal{H}_{L^*}^s(\Omega)$ coincide as sets. Moreover, we have the following proposition.

Proposition 6.3. For every $s \in \mathbb{R}$, the Sobolev space $\mathcal{H}_L^s(\Omega)$ is a Hilbert space with the inner product

$$(f, g)_{\mathcal{H}_L^s(\Omega)} := \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}(\xi) \overline{\hat{g}(\xi)}.$$

Similarly, the Sobolev space $\mathcal{H}_{L^*}^s(\Omega)$ is a Hilbert space with the inner product

$$(f, g)_{\mathcal{H}_{L^*}^s(\Omega)} := \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}_*(\xi) \overline{\hat{g}_*(\xi)}.$$

For every $s \in \mathbb{R}$, the Sobolev spaces $\mathcal{H}^s(\Omega)$, $\mathcal{H}_L^s(\Omega)$, and $\mathcal{H}_{L^*}^s(\Omega)$ are isometrically isomorphic. \square

Proof. The spaces $\mathcal{H}_L^0(\Omega)$ and $\mathcal{H}_L^s(\Omega)$ are isometrically isomorphic by the canonical isomorphism

$$\varphi_s : \mathcal{H}_L^0(\Omega) \rightarrow \mathcal{H}_L^s(\Omega),$$

defined by

$$\varphi_s f(x) := \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{-s} \hat{f}(\xi) u_{\xi}(x).$$

Indeed, φ_s is a linear isometry between $\mathcal{H}_L^t(\Omega)$ and $\mathcal{H}_L^{t+s}(\Omega)$ for every $s \in \mathbb{R}$, and it is true that

$$\varphi_{s_1} \varphi_{s_2} = \varphi_{s_1+s_2} \quad \text{and} \quad \varphi_s^{-1} = \varphi_{-s}.$$

Then the completeness of $L^2(\Omega) = \mathcal{H}_L^0(\Omega)$ is transferred to that of $\mathcal{H}_L^s(\Omega)$ for every $s \in \mathbb{R}$.

As $L^2(\Omega) = \mathcal{H}_L^0(\Omega)$, the spaces $L^2(\Omega)$ and $\mathcal{H}_L^s(\Omega)$ are isometrically isomorphic for every $s \in \mathbb{R}$. Hence, the Sobolev spaces $\mathcal{H}^s(\Omega)$ and $\mathcal{H}_L^s(\Omega)$ are also isometrically isomorphic for every $s \in \mathbb{R}$. The arguments for the space $\mathcal{H}_{L^*}^s(\Omega)$ are all similar. \blacksquare

7 Spaces $l^p(L)$ and $l^p(L^*)$

In this section, we describe the p -Lebesgue versions of the spaces of Fourier coefficients. These spaces can be considered as the extension of the usual l^p spaces on the discrete set \mathcal{I} adapted to the fact that we are dealing with biorthogonal systems.

Definition 7.1. Thus, we introduce the spaces $l_L^p = l^p(L)$ as the spaces of all $a \in \mathcal{S}'(\mathcal{I})$ such that

$$\|a\|_{l^p(L)} := \left(\sum_{\xi \in \mathcal{I}} |a(\xi)|^p \|u_\xi\|_{L^\infty(\Omega)}^{2-p} \right)^{1/p} < \infty \quad \text{for } 1 \leq p \leq 2 \tag{7.1}$$

and

$$\|a\|_{l^p(L)} := \left(\sum_{\xi \in \mathcal{I}} |a(\xi)|^p \|v_\xi\|_{L^\infty(\Omega)}^{2-p} \right)^{1/p} < \infty \quad \text{for } 2 \leq p < \infty, \tag{7.2}$$

and, for $p = \infty$,

$$\|a\|_{l^\infty(L)} := \sup_{\xi \in \mathcal{I}} \left(|a(\xi)| \cdot \|v_\xi\|_{L^\infty(\Omega)}^{-1} \right) < \infty. \quad \square$$

Remark 7.2. We note that in the case of $p = 2$, we have already defined the space $l^2(L)$ by the norm (6.3). There is no problem with this since the norms (7.1) and (7.2) with $p = 2$ are equivalent to that in (6.3). Indeed, by Lemma 4.3 the first one gives a homeomorphism between $l^p(L)$ with $p = 2$ just defined and $L^2(\Omega)$ while the space $l^2(L)$ defined by (6.3) is isometrically isomorphic to $L^2(\Omega)$ by the Plancherel identity in Proposition 6.1. Therefore, both norms lead to the same space which we denote by $l^2(L)$. The norms (7.1) and (7.2) with $p = 2$ and the one in (6.3) are equivalent, but there are advantages in using both of them. Thus, the norms (7.1) and (7.2) allow us to view $l^2(L)$ as a member of the scale of spaces $l^p(L)$ for $1 \leq p \leq \infty$ with subsequent functional analytic properties, while the norm (6.3) is the one for which the Plancherel identity (6.6) holds. \square

Analogously, we also introduce spaces $l_{L^*}^p = l^p(L^*)$ as the spaces of all $b \in \mathcal{S}'(\mathcal{I})$ such that the following norms are finite:

$$\|b\|_{l^p(L^*)} = \left(\sum_{\xi \in \mathcal{I}} |b(\xi)|^p \|v_\xi\|_{L^\infty(\Omega)}^{2-p} \right)^{1/p} \quad \text{for } 1 \leq p \leq 2,$$

$$\|b\|_{l^p(L^*)} = \left(\sum_{\xi \in \mathcal{I}} |b(\xi)|^p \|u_\xi\|_{L^\infty(\Omega)}^{2-p} \right)^{1/p} \quad \text{for } 2 \leq p < \infty,$$

$$\|b\|_{l^\infty(L^*)} = \sup_{\xi \in \mathcal{I}} \left(|b(\xi)| \cdot \|u_\xi\|_{L^\infty(\Omega)}^{-1} \right).$$

Before we discuss several basic properties of the spaces $l^p(L)$, we recall a useful fact on the interpolation of weighted spaces from Bergh and Löfström [5, Theorem 5.5.1].

Theorem 7.3 (Interpolation of weighted spaces). Let us write $d\mu_0(x) = \omega_0(x)d\mu(x)$, $d\mu_1(x) = \omega_1(x)d\mu(x)$, and write $L^p(\omega) = L^p(\omega d\mu)$ for the weight ω . Suppose that $0 < p_0, p_1 < \infty$. Then

$$\left(L^{p_0}(\omega_0), L^{p_1}(\omega_1) \right)_{\theta, p} = L^p(\omega),$$

where $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\omega = \omega_0^{\frac{p(1-\theta)}{p_0}} \omega_1^{\frac{p\theta}{p_1}}$. □

From this, it is easy to check that we obtain the following corollary.

Corollary 7.4 (Interpolation of $l^p(L)$ and $l^p(L^*)$ spaces). For $1 \leq p \leq 2$, we have

$$\begin{aligned} \left(l^1(L), l^2(L) \right)_{\theta, p} &= l^p(L), \\ \left(l^1(L^*), l^2(L^*) \right)_{\theta, p} &= l^p(L^*), \end{aligned}$$

where $0 < \theta < 1$ and $p = \frac{2}{2-\theta}$. □

Remark 7.5. The reason that the interpolation above is restricted to $1 \leq p \leq 2$ is that the definition of l^p -spaces changes when we pass $p = 2$, in the sense that we use different families of biorthogonal systems u_ξ and v_ξ for $p < 2$ and for $p > 2$. We note that if the boundary value problem $L_\Omega = L_\Omega^*$ is self-adjoint, so that we can take $u_\xi = v_\xi$ for all $\xi \in \mathcal{I}$, then the scales $l^p(L)$ and $l^p(L^*)$ coincide and satisfy interpolation properties for all $1 \leq p < \infty$. □

Using these interpolation properties, we can establish further properties of the Fourier transform and its inverse.

Theorem 7.6 (Hausdorff–Young inequality). Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There is a constant $C_p \geq 1$ such that for all $f \in L^p(\Omega)$ and $a \in l^p(L)$ we have

$$\left\| \hat{f} \right\|_{l^{p'}(L)} \leq C_p \|f\|_{L^p(\Omega)} \quad \text{and} \quad \left\| \mathcal{F}_L^{-1} a \right\|_{L^{p'}(\Omega)} \leq C_p \|a\|_{l^p(L)}. \tag{7.3}$$

Similarly, we also have

$$\left\| \hat{f}_* \right\|_{l^{p'}(L^*)} \leq C_p \|f\|_{L^p(\Omega)} \quad \text{and} \quad \left\| \mathcal{F}_{L^*}^{-1} b \right\|_{L^{p'}(\Omega)} \leq C_p \|b\|_{l^p(L^*)}, \tag{7.4}$$

for all $b \in l^p(L^*)$. □

It follows from the proof that if L_Ω is self-adjoint, then the l^2_L -norms discussed in Remark 7.2 coincide, and so we can put $C_p = 1$ in inequalities (7.3) and (7.4). If L_Ω is not self-adjoint, C_p may in principle depend on L and its domain through constants from inequalities in Lemma 4.3.

Proof of Theorem 7.6. First we note that the proofs of (7.3) and (7.4) are similar, so it suffices to prove only (7.3). Then we observe that (7.3) would follow from the $L^1(\Omega) \rightarrow l^\infty(L)$ and $l^1(L) \rightarrow L^\infty(\Omega)$ boundedness in view of the Plancherel identity in Proposition 6.1 by interpolation, see, for example, Bergh and Löfström [5, Corollary 5.5.4]. We note that in view of the discussion in Remark 7.2 we write constants C_p in inequalities (7.3) and (7.4). In particular, if L_Ω is self-adjoint, we can put $C_p = 1$.

Thus, we can assume that $p = 1$. Using the definition $\hat{f}(\xi) = \int_\Omega f(x) \overline{v_\xi(x)} dx$, we get

$$|\hat{f}(\xi)| \leq \int_\Omega |f(x)| |\overline{v_\xi(x)}| dx \leq \|\overline{v_\xi}\|_{L^\infty} \|f\|_{L^1}.$$

Therefore,

$$\|\hat{f}\|_{l^\infty(L)} = \sup_{\xi \in \mathcal{I}} |\hat{f}(\xi)| \|v_\xi\|_{L^\infty}^{-1} \leq \|f\|_{L^1},$$

which gives the first inequality in (7.3) for $p = 1$. For the second one, using

$$(\mathcal{F}_L^{-1}a)(x) = \sum_{\xi \in \mathcal{I}} a(\xi) u_\xi(x)$$

we have

$$|(\mathcal{F}_L^{-1}a)(x)| \leq \sum_{\xi \in \mathcal{I}} |a(\xi)| |u_\xi(x)| \leq \sum_{\xi \in \mathcal{I}} |a(\xi)| \|u_\xi\|_{L^\infty} = \|a\|_{l^1(L)},$$

from which we get

$$\|\mathcal{F}_L^{-1}a\|_{L^\infty} \leq \|a\|_{l^1(L)},$$

completing the proof. ■

We now turn to the duality between spaces $l^p(L)$ and $l^q(L^*)$.

Theorem 7.7 (Duality of $l^p(L)$ and $l^{p'}(L^*)$). Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$(l^p(L))' = l^{p'}(L^*) \quad \text{and} \quad (l^p(L^*))' = l^p(L). \quad \square$$

Proof. The proof is rather standard but we give some details for clarity. The duality can be given by the form

$$(\sigma_1, \sigma_2) = \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \sigma_2(\xi)$$

for $\sigma_1 \in l^p(L)$ and $\sigma_2 \in l^{p'}(L^*)$. Assume that $1 < p \leq 2$. Then, if $\sigma_1 \in l^p(L)$ and $\sigma_2 \in l^{p'}(L^*)$, we have

$$\begin{aligned} |(\sigma_1, \sigma_2)| &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \sigma_2(\xi) \right| \\ &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \|u_\xi\|_{L^\infty}^{\frac{2}{p}-1} \|u_\xi\|_{L^\infty}^{-\left(\frac{2}{p}-1\right)} \sigma_2(\xi) \right| \\ &\leq \left(\sum_{\xi \in \mathcal{I}} |\sigma_1(\xi)|^p \|u_\xi\|_{L^\infty}^{p\left(\frac{2}{p}-1\right)} \right)^{\frac{1}{p}} \left(\sum_{\xi \in \mathcal{I}} |\sigma_2(\xi)|^{p'} \|u_\xi\|_{L^\infty}^{-p'\left(\frac{2}{p}-1\right)} \right)^{\frac{1}{p'}} \\ &= \|\sigma_1\|_{l^p(L)} \|\sigma_2\|_{l^{p'}(L^*)}, \end{aligned}$$

where we used that $2 \leq p' < \infty$ and that $\frac{2}{p} - 1 = 1 - \frac{2}{p}$ in the last line. Assume now that $2 < p < \infty$. Then, if $\sigma_1 \in l^p(L)$ and $\sigma_2 \in l^{p'}(L^*)$, we have

$$\begin{aligned} |(\sigma_1, \sigma_2)| &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \sigma_2(\xi) \right| \\ &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \|v_\xi\|_{L^\infty}^{\frac{2}{p}-1} \|v_\xi\|_{L^\infty}^{-\left(\frac{2}{p}-1\right)} \sigma_2(\xi) \right| \\ &\leq \left(\sum_{\xi \in \mathcal{I}} |\sigma_1(\xi)|^p \|v_\xi\|_{L^\infty}^{p\left(\frac{2}{p}-1\right)} \right)^{\frac{1}{p}} \left(\sum_{\xi \in \mathcal{I}} |\sigma_2(\xi)|^{p'} \|v_\xi\|_{L^\infty}^{-p'\left(\frac{2}{p}-1\right)} \right)^{\frac{1}{p'}} \\ &= \|\sigma_1\|_{l^p(L)} \|\sigma_2\|_{l^{p'}(L^*)}. \end{aligned}$$

Let now $p = 1$. In this case, we get

$$\begin{aligned} |(\sigma_1, \sigma_2)| &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \sigma_2(\xi) \right| \\ &= \left| \sum_{\xi \in \mathcal{I}} \sigma_1(\xi) \|u_\xi\|_{L^\infty} \|u_\xi\|_{L^\infty}^{-1} \sigma_2(\xi) \right| \end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{\xi \in \mathcal{I}} |\sigma_1(\xi)| \|u_\xi\|_{L^\infty} \right) \sup_{\xi \in \mathcal{I}} |\sigma_2(\xi)| \|u_\xi\|_{L^\infty}^{-1} \\ &= \|\sigma_1\|_{l^1(L)} \|\sigma_2\|_{l^\infty(L^*)}. \end{aligned}$$

The proofs for the adjoint spaces are similar. ■

8 Schwartz' Kernel Theorem

This section is devoted to establishing the Schwartz kernel theorem in the spaces of distributions $\mathcal{D}'_L(\Omega)$. In this analysis, as well as in establishing further estimates for the integral kernels in Section 11, we will need the following assumption which may be also regarded as the definition of the number s_0 . So, from now on we will make the following assumption.

Assumption 8.1. Assume that the number $s_0 \in \mathbb{R}$ is such that we have

$$\sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{-s_0} < \infty. \quad \square$$

Recalling the operator L° in (1.7) the assumption (8.1) is equivalent to assuming that the operator $(I + L^\circ L)^{-\frac{s_0}{4m}}$ is Hilbert–Schmidt on $L^2(\Omega)$. Indeed, recalling the definition of $\langle \xi \rangle$ in (1.6), namely that $\langle \xi \rangle$ are the eigenvalues of $(I + L^\circ L)^{-\frac{s_0}{2m}}$, the condition that the operator $(I + L^\circ L)^{-\frac{s_0}{4m}}$ is Hilbert–Schmidt is equivalent to the condition that

$$\|(I + L^\circ L)^{-\frac{s_0}{4m}}\|_{\text{HS}}^2 \cong \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{-s_0} < \infty. \tag{8.1}$$

If L is elliptic, we may expect that we can take any $s_0 > n$ but this depends on the boundary conditions in general. The order s_0 will enter the regularity properties of the Schwartz kernels.

We will use the notation

$$C_L^\infty(\bar{\Omega} \times \bar{\Omega}) := C_L^\infty(\bar{\Omega}) \otimes C_L^\infty(\bar{\Omega}),$$

and for the corresponding dual space we write

$$\mathcal{D}'_L(\Omega \times \Omega) := (C_L^\infty(\bar{\Omega} \times \bar{\Omega}))'.$$

The purpose of the subsequent discussion is to show that for a continuous linear operator

$$A: C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_L(\Omega)$$

there exists the kernel $K \in \mathcal{D}'_L(\Omega \times \Omega)$ such that

$$\langle Af, g \rangle = \int_{\Omega} \int_{\Omega} K(x, y) f(x)g(y) \, dx \, dy,$$

and, using the notion of the L -convolution in Section 5, the convolution kernel $k_A(x) \in \mathcal{D}'_L(\Omega)$, such that

$$Af(x) = (k_A(x) \star_L f)(x).$$

Here as usual, we identify an integrable function w in, for example, $C_L^\infty(\bar{\Omega})$, with the distribution

$$C_L^\infty(\bar{\Omega}) \ni \varphi \mapsto \langle w, \varphi \rangle = \int_{\Omega} w(x)\varphi(x) \, dx,$$

and we shall use the integral as a notation for the value $\langle w, \varphi \rangle$ of w at φ also when w is an arbitrary distribution in $\mathcal{D}'_L(\Omega)$.

Consider the space \mathcal{A} of all separately continuous bilinear functionals A on $C_L^\infty(\bar{\Omega}) \times C_L^\infty(\bar{\Omega})$ with the topology of uniform convergence on products of bounded sets in $C_L^\infty(\bar{\Omega})$. Any distribution w in $\mathcal{D}'_L(\Omega \times \Omega)$ gives rise to such a functional $A \in \mathcal{A}$ by specialization to products of functions

$$(\Lambda w)(f, g) := \langle w, f(x)g(y) \rangle =: A(f, g). \quad (8.2)$$

The kernel theorem says that the mapping

$$\Lambda: w \mapsto A$$

is a linear homeomorphism between $\mathcal{D}'_L(\Omega \times \Omega)$ and \mathcal{A} . In particular, for every $A \in \mathcal{A}$ there is precisely one “kernel” $K \in \mathcal{D}'_L(\Omega \times \Omega)$ such that

$$A(f, g) = \int_{\Omega} \int_{\Omega} K(x, y) f(x)g(y) \, dx \, dy.$$

Such theorem was proved by Schwartz [60] for standard distributions, but then much simplified proofs have been given, for instance, by Ehrenpreis [21] and by Gask [29].

Any function h in $C_L^\infty(\bar{\Omega} \times \bar{\Omega})$ can be expanded in an L -Fourier series

$$h(x, y) = \sum_{\xi, \eta \in \mathcal{I}} a_{\xi\eta} u_\xi(x) u_\eta(y). \tag{8.3}$$

The coefficients in (8.3) are given by

$$a_{\xi\eta} = \int_\Omega \int_\Omega h(x, y) \overline{v_\xi(x)} \overline{v_\eta(y)} \, dx \, dy.$$

Integration by parts in these formulae and the Cauchy–Schwarz inequality yield the estimates

$$|a_{\xi\eta}| \leq C_{j_1, j_2} |h|_{j_1+j_2} (1 + \langle \xi \rangle)^{-mj_1} (1 + \langle \eta \rangle)^{-mj_2}, \tag{8.4}$$

where C_{j_1, j_2} is a constant independent of h , $j_1, j_2 \in \mathbb{N}_0$, and

$$|h|_j := \sum_{k_1+k_2 \leq j} \left\| L_x^{k_1} L_y^{k_2} h(x, y) \right\|_{L^2(\Omega \times \Omega)}.$$

Rescaling the coefficients $a_{\xi\eta}$, we can write (8.3) in the form

$$h(x, y) = \sum_{\xi, \eta \in \mathcal{I}} b_{\xi\eta} f_\xi(x) g_\eta(y) \tag{8.5}$$

with $f_\xi(x)$ and $g_\eta(y)$ proportional to $u_\xi(x)$ and $u_\eta(y)$, with new coefficients $b_{\xi\eta}$. The proportionality factors shall be chosen in a suitable way, expressed in the following discussion.

Lemma 8.2. Let h be a function in $C_L^\infty(\bar{\Omega} \times \bar{\Omega})$ and k and l be given positive integers. Then f_ξ and g_η in (8.5) can be chosen such that

$$|f_\xi|_k \leq 1 \quad \text{and} \quad |g_\eta|_l \leq 1$$

for all ξ and η , and

$$\sum_{\xi, \eta \in \mathcal{I}} |b_{\xi\eta}| \leq C |h|_{k+l+2s_0},$$

with a constant C independent of h , and the number s_0 is the one from Assumption (8.1). □

Proof. We write (8.3) as

$$h(x, y) = \sum_{\xi, \eta \in \mathcal{I}} a_{\xi\eta} (1 + \langle \xi \rangle)^{mk} (1 + \langle \eta \rangle)^{ml} [(1 + \langle \xi \rangle)^{-mk} u_\xi(x)] [(1 + \langle \eta \rangle)^{-ml} u_\eta(y)]$$

and choose the functions in square brackets for f_ξ and g_η . The estimates (8.4) and some straightforward calculations then give the lemma. ■

From Lemma 8.2, we readily obtain the following corollary that expresses the fact that if h is in some bounded set in $C_L^\infty(\bar{\Omega} \times \bar{\Omega})$, the expansion (8.5) can be made such that (8.6) holds with f_ξ and g_η in fixed bounded sets in $C_L^\infty(\bar{\Omega})$.

Corollary 8.3. Let $\{r_\nu\}_{\nu=1}^\infty$ be a sequence of positive real numbers. Then there exists another sequence $\{s_\nu\}_1^\infty$ of positive real numbers such that for every $h \in C_L^\infty(\bar{\Omega} \times \bar{\Omega})$ satisfying

$$|h|_\nu \leq r_\nu, \quad \nu = 1, 2, \dots,$$

we can choose f_ξ and g_η in (8.5) so that we have

$$|f_\xi|_\nu \leq s_\nu, \quad |g_\eta|_\nu \leq s_\nu, \quad \nu = 1, 2, \dots,$$

for all ξ and η , and also

$$\sum_{\xi, \eta \in \mathcal{I}} |b_{\xi\eta}| \leq 1. \tag{8.6}$$

□

Since $A \in \mathcal{A}$ is continuous, there exist a constant C and integers k and l (depending on A) for which

$$|A(f, g)| \leq C |f|_k |g|_l, \quad f, g \in C_L^\infty(\bar{\Omega}). \tag{8.7}$$

As stated above, there is a mapping Λ of $\mathcal{D}'_L(\bar{\Omega} \times \bar{\Omega})$ into \mathcal{A} , defined by (8.2). We shall now first prove that the range of this mapping is the whole of \mathcal{A} and that it is one-to-one.

Theorem 8.4. For any separately continuous functional A on $C_L^\infty(\bar{\Omega}) \times C_L^\infty(\bar{\Omega})$, there exists precisely one distribution u in $\mathcal{D}'_L(\Omega \times \Omega)$ such that

$$(\Lambda u)(f, g) := \langle u, f(x)g(y) \rangle = A(f, g) \tag{8.8}$$

holds for all (f, g) in $C_L^\infty(\bar{\Omega}) \times C_L^\infty(\bar{\Omega})$. The mapping Λ defined by (8.8) is a linear homeomorphism. □

Proof. Let us write an arbitrary h in $C_L^\infty(\bar{\Omega} \times \bar{\Omega})$ in the form given by Lemma 8.2. If k and l are integers such that (8.7) holds for our given A , we find

$$\sum_{\xi, \eta \in \mathcal{I}} |b_{\xi\eta}| |A(f_\xi, g_\eta)| \leq C |h|_{k+l+2s_0}, \tag{8.9}$$

with C independent of h . We define u by

$$\langle u, h \rangle := \sum_{\xi, \eta \in \mathcal{I}} b_{\xi\eta} A(f_\xi, g_\eta) \tag{8.10}$$

and conclude from (8.9) that u is an L -distribution on $\Omega \times \Omega$ of order $k + l + 2s_0$. It is clear that (8.8) holds for this u and also that u is uniquely determined by A : indeed, if A vanishes it follows from (8.10) that $u(h) = 0$ on all finite sums $h = \sum_{\xi, \eta \in \mathcal{I}} b_{\xi\eta} f_{\xi} g_{\eta}$, and as the set of such sums is dense in $C_L^{\infty}(\bar{\Omega} \times \bar{\Omega})$, the L -distribution u must vanish.

Let us now show that the mapping Λ defined by (8.8) is a linear homeomorphism. In view of Proposition 3.3 and Lemma 3.4, the topologies on \mathcal{A} and $\mathcal{D}'_L(\bar{\Omega} \times \bar{\Omega})$ can be defined by the seminorms, which can be also expressed as

$$\rho_{B_x B_y}(A) = \sup |A(f, g)|, \quad f \in B_x, g \in B_y,$$

$$\varrho_{B_{xy}}(u) = \sup |\langle u, h \rangle|, \quad h \in B_{xy},$$

where B_x, B_y are bounded sets in $C_L^{\infty}(\bar{\Omega})$ and B_{xy} is a bounded set in $C_L^{\infty}(\bar{\Omega} \times \bar{\Omega})$.

It is clear that Λ is linear. Let us show that Λ and Λ^{-1} are both continuous.

Let $\rho_{B_x B_y}$ be an arbitrary seminorm on \mathcal{A} . Then

$$\rho_{B_x B_y}(\Lambda u) = \rho_{B_x B_y}(A) = \sup_{f \in B_x, g \in B_y} |A(f, g)| = \sup_{f \in B_x, g \in B_y} |\langle u, fg \rangle|.$$

It is easy to see that for any bounded sets $B_x \subset C_L^{\infty}(\bar{\Omega})$ and $B_y \subset C_L^{\infty}(\bar{\Omega})$ there exists a bounded set $B_{xy} \subset C_L^{\infty}(\bar{\Omega} \times \bar{\Omega})$ such that all products fg are in B_{xy} whenever f is in B_x and g is in B_y . Hence,

$$\sup_{f \in B_x, g \in B_y} |\langle u, fg \rangle| \leq \sup_{h \in B_{xy}} |\langle u, h \rangle| = \varrho_{B_{xy}}(u),$$

and so Λ is continuous. Conversely, if $\varrho_{B_{xy}}$ is a seminorm on $C_L^{\infty}(\bar{\Omega} \times \bar{\Omega})$, we find

$$\varrho_{B_{xy}}(\Lambda^{-1} A) = \varrho_{B_{xy}}(u) = \sup_{h \in B_{xy}} |\langle u, h \rangle| = \sup_{h \in B_{xy}} \left| \sum_{\xi, \eta \in \mathcal{I}} b_{\xi\eta} A(f_{\xi}, g_{\eta}) \right|,$$

where h has been expanded as in Corollary 8.3, and thus

$$\sup_{h \in B_{xy}} |A(f_{\xi}, g_{\eta})| \leq \sup_{f \in B_x, g \in B_y} |A(f, g)| = \rho_{B_x B_y}(A),$$

if B_x and B_y are those bounded sets in $C_L^{\infty}(\bar{\Omega})$ which contain all f_{ξ} and g_{η} according to Lemma 8.3. From (8.6), we now conclude that

$$\varrho_{B_{xy}}(\Lambda^{-1} A) \leq \sup_{h \in B_{xy}} |A(f_{\xi}, g_{\eta})| \sum_{\xi, \eta \in \mathcal{I}} |b_{\xi\eta}| \leq \rho_{B_x B_y}(A),$$

and thus Λ^{-1} is also continuous. This completes the proof of the theorem. ■

Summarizing what we have proved, for any linear continuous operator

$$A: C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_L(\Omega)$$

there exists a kernel $K_A \in \mathcal{D}'_L(\Omega \times \Omega)$ such that for all $f \in C_L^\infty(\bar{\Omega})$, we can write, in the sense of distributions,

$$Af(x) = \int_{\Omega} K_A(x, y) f(y) dy. \quad (8.11)$$

As usual, K_A is called the Schwartz kernel of A . For $f \in C_L^\infty(\bar{\Omega})$, using the Fourier series formula

$$f(y) = \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) u_\eta(y),$$

we can also write

$$Af(x) = \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \int_{\Omega} K_A(x, y) u_\eta(y) dy. \quad (8.12)$$

Suppose now that $\{u_\xi : \xi \in \mathcal{I}\}$ is a WZ-system in the sense of Definition 1.3. Let us introduce the L -distribution $k_A \in \mathcal{D}'_L(\Omega \times \Omega)$ by the formula

$$k_A(x, z) := k_A(x)(z) := \sum_{\eta \in \mathcal{I}} u_\eta^{-1}(x) \int_{\Omega} K_A(x, y) u_\eta(y) dy u_\eta(z). \quad (8.13)$$

Since for some $C > 0$ and $N \geq 0$, we have by Definition 1.3

$$\inf_{x \in \Omega} |u_\eta(x)| \geq C \langle \eta \rangle^{-N},$$

the series in (8.13) is converges in the sense of L -distributions. Formula (8.13) means that the Fourier transform of k_A in the second variable satisfies

$$\hat{k}_A(x, \eta) u_\eta(x) = \int_{\Omega} K_A(x, y) u_\eta(y) dy. \quad (8.14)$$

Combining this and (8.12), we get

$$Af(x) = \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \int_{\Omega} K_A(x, y) u_\eta(y) dy = \sum_{\eta \in \mathcal{I}} \hat{f}(\eta) \hat{k}_A(x, \eta) u_\eta(x) = (f \star_L k_A(x))(x),$$

where in the last equality we used the notion of the L -convolution in Definition 5.1. Summarizing this calculation as well as an analogous argument for the adjoint operator L^* , we record the following proposition.

Proposition 8.5. Suppose that $\{u_\xi : \xi \in \mathcal{I}\}$ is a WZ-system in the sense of Definition 1.3. Then for a linear continuous operator

$$A : C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_L(\Omega)$$

there exists the convolution kernel $k_A \in \mathcal{D}'_L(\Omega \times \Omega)$ such that

$$Af(x) = (f \star_L k_A)(x), \quad f \in C_L^\infty(\bar{\Omega}),$$

where we write

$$k_A(x)(y) := k_A(x, y)$$

in the sense of distributions. The convolution kernel k_A and the Schwartz kernel K_A of an operator A are related by formulae (8.11)–(8.14).

Also, for any linear continuous operator

$$A : C_{L^*}^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_{L^*}(\Omega)$$

there exists a kernel $\tilde{K}_A \in \mathcal{D}'_{L^*}(\Omega \times \Omega)$ such that for all $f \in C_{L^*}^\infty(\bar{\Omega})$ we have

$$Af(x) = \int_{\Omega} \tilde{K}_A(x, y) f(y) \, dy.$$

If, in addition, $\{v_\xi : \xi \in \mathcal{I}\}$ is a WZ-system, then for a linear continuous operator $A : C_{L^*}^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_{L^*}(\Omega)$ there exists the convolution kernel $\tilde{k}_A \in \mathcal{D}'_{L^*}(\Omega \times \Omega)$, such that

$$Af(x) = (f \star_{L^*} \tilde{k}_A)(x), \quad f \in C_{L^*}^\infty(\bar{\Omega}),$$

where we write

$$\tilde{k}_A(x)(y) := \tilde{k}_A(x, y)$$

in the sense of distributions. □

In the last formula, we refer to (5.2) for the definition of the L^* -convolution \star_{L^*} .

9 L -Quantization and Full Symbols

In this section, we describe the L -quantization induced by the boundary value problem L_Ω . From now on, we will assume that the system of functions $\{u_\xi : \xi \in \mathcal{I}\}$ is a WZ-system in the sense of Definition 1.3. Later, we will make some remarks on what happens when this assumption is not satisfied.

Definition 9.1 (*L*-Symbols of operators on Ω). The *L*-symbol of a linear continuous operator

$$A: C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_L(\Omega)$$

at $x \in \Omega$ and $\xi \in \mathcal{I}$ is defined by

$$\sigma_A(x, \xi) := \widehat{k_A(x)}(\xi) = \mathcal{F}_L(k_A(x))(\xi).$$

Hence, we can also write

$$\sigma_A(x, \xi) = \int_{\Omega} k_A(x, y) \overline{v_\xi(y)} \, dy = \langle k_A(x), \bar{v}_\xi \rangle. \quad \square$$

By the *L*-Fourier inversion formula, the convolution kernel can be regained from the symbol:

$$k_A(x, y) = \sum_{\xi \in \mathcal{I}} \sigma_A(x, \xi) u_\xi(y), \tag{9.1}$$

all in the sense of *L*-distributions. We now show that an operator *A* can be represented by its symbol.

Theorem 9.2 (*L*-quantization). Let

$$A: C_L^\infty(\bar{\Omega}) \rightarrow C_L^\infty(\bar{\Omega})$$

be a continuous linear operator with *L*-symbol σ_A . Then

$$Af(x) = \sum_{\xi \in \mathcal{I}} u_\xi(x) \sigma_A(x, \xi) \hat{f}(\xi) \tag{9.2}$$

for every $f \in C_L^\infty(\bar{\Omega})$ and $x \in \Omega$. The *L*-symbol σ_A satisfies

$$\sigma_A(x, \xi) = u_\xi(x)^{-1} (Au_\xi)(x) \tag{9.3}$$

for all $x \in \Omega$ and $\xi \in \mathcal{I}$. □

Proof. Let us define a convolution operator $A_{x_0} \in \mathcal{L}(C_L^\infty(\bar{\Omega}))$ by the kernel

$$k_{x_0}(x) := k_A(x_0, x),$$

that is, by

$$A_{x_0} f(x) := (f \star_L k_{x_0})(x),$$

with the usual distributional interpretation of the appearing quantities. Thus,

$$\sigma_{A_{x_0}}(x, \xi) = \widehat{k_{x_0}}(\xi) = \sigma_A(x_0, \xi),$$

so that we have

$$A_{x_0} f(x) = \sum_{\xi \in \mathcal{I}} \widehat{A_{x_0} f}(\xi) u_\xi(x) = \sum_{\xi \in \mathcal{I}} \widehat{f}(\xi) \sigma_A(x_0, \xi) u_\xi(x),$$

where we used that $\widehat{f \star_L k_{x_0}} = \widehat{f} \widehat{k_{x_0}}$ by the same calculations as in Lemma 5.3. This implies (9.2) because

$$Af(x) = A_x f(x).$$

For (9.3), we can then calculate

$$u_\xi(x)^{-1} (Au_\xi)(x) = u_\xi(x)^{-1} \sum_{\eta \in \mathcal{I}} u_\eta(x) \sigma_A(x, \eta) \hat{u}_\xi(\eta) = \sigma_A(x, \xi),$$

completing the proof. ■

As a consequence of the proof and of various formulae for kernels and convolutions, we can collect several formulae for the symbol under the assumption that the biorthogonal system u_ξ is a WZ-system.

Corollary 9.3. We have the following equivalent formulae for L -symbols:

- (i) $\sigma_A(x, \xi) = \int_{\Omega} k_A(x, y) \overline{v_\xi(y)} \, dy;$
- (ii) $\sigma_A(x, \xi) = u_\xi^{-1}(x) (Au_\xi)(x);$
- (iii) $\sigma_A(x, \xi) = u_\xi^{-1}(x) \int_{\Omega} K_A(x, y) u_\xi(y) \, dy;$
- (iv) $\sigma_A(x, \xi) = u_\xi^{-1}(x) \int_{\Omega} \int_{\Omega} F(x, y, z) k_A(x, y) u_\xi(z) \, dy \, dz.$

Here and in the sequel, we write $u_\xi^{-1}(x) = u_\xi(x)^{-1}$. Formula (iii) also implies

$$(v) \quad K_A(x, y) = \sum_{\xi \in \mathcal{I}} u_\xi(x) \sigma_A(x, \xi) \overline{v_\xi(y)}.$$

□

In the case when $\{u_\xi : \xi \in \mathcal{I}\}$ is not a WZ-system, we can still understand the L -symbol σ_A of the operator A as a function on $\bar{\Omega} \times \mathcal{I}$, for which the equality

$$u_\xi(x)\sigma_A(x, \xi) = \int_{\Omega} K_A(x, y)u_\xi(y) dy$$

holds for all ξ in \mathcal{I} and for $x \in \bar{\Omega}$. Of course, this implies certain restrictions on the zeros of the Schwartz kernel K_A . Such restrictions may be considered natural from the point of view of the scope of problems that can be treated by our approach in the case when the eigenfunctions $u_\xi(x)$ may vanish at some points x .

Similarly, we can introduce an analogous notion of the L^* -quantization.

Definition 9.4 (L^* -Symbols of operators on Ω). The L^* -symbol of a linear continuous operator

$$A : C_{L^*}^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_{L^*}(\Omega)$$

at $x \in \Omega$ and $\xi \in \mathcal{I}$ is defined by

$$\tau_A(x, \xi) := \widehat{\tilde{k}_A(x)_*}(\xi) = \mathcal{F}_{L^*}(\tilde{k}_A(x))(\xi).$$

We can also write

$$\tau_A(x, \xi) = \int_{\Omega} \tilde{k}_A(x, y)\overline{u_\xi(y)} dy = \langle \tilde{k}_A(x), \overline{u_\xi} \rangle. \quad \square$$

By the L^* -Fourier inversion formula, the convolution kernel can be regained from the symbol:

$$\tilde{k}_A(x, y) = \sum_{\xi \in \mathcal{I}} \tau_A(x, \xi)v_\xi(y) \tag{9.4}$$

in the sense of L^* -distributions. Analogously to the L -quantization, we have the following corollary.

Corollary 9.5 (L^* -quantization). Let τ_A be the L^* -symbol of a continuous linear operator

$$A : C_{L^*}^\infty(\bar{\Omega}) \rightarrow C_{L^*}^\infty(\bar{\Omega}).$$

Then

$$Af(x) = \sum_{\xi \in \mathcal{I}} v_\xi(x)\tau_A(x, \xi)\hat{f}_*(\xi) \tag{9.5}$$

for every $f \in C_{L^*}^\infty(\bar{\Omega})$ and $x \in \Omega$. For all $x \in \Omega$ and $\xi \in \mathcal{I}$, we have

$$\tau_A(x, \xi) = v_\xi(x)^{-1}(Av_\xi)(x). \tag{9.6}$$

We also have the following equivalent formulae for the L^* -symbol:

$$(i) \quad \tau_A(x, \xi) = \int_{\Omega} \tilde{k}_A(x, y) \overline{u_{\xi}(y)} \, dy,$$

$$(ii) \quad \tau_A(x, \xi) = v_{\xi}^{-1}(x) \int_{\Omega} \tilde{K}_A(x, y) v_{\xi}(y) \, dy. \quad \square$$

We now briefly describe the notion of Fourier multipliers which is a natural name for operators with symbols independent of x . In [18], the analysis of this paper is applied to investigate the spectral properties of such operators, so we can be brief here.

Definition 9.6. Let $A : C_L^{\infty}(\bar{\Omega}) \rightarrow C_L^{\infty}(\bar{\Omega})$ be a continuous linear operator. We will say that A is an L -Fourier multiplier if it satisfies

$$\mathcal{F}_L(Af)(\xi) = \sigma(\xi)\mathcal{F}_L(f)(\xi), \quad f \in C_L^{\infty}(\bar{\Omega}),$$

for some $\sigma : \mathcal{I} \rightarrow \mathbb{C}$. Analogously, we define L^* -Fourier multipliers: Let $B : C_{L^*}^{\infty}(\bar{\Omega}) \rightarrow C_{L^*}^{\infty}(\bar{\Omega})$ be a continuous linear operator. We will say that B is an L^* -Fourier multiplier if it satisfies

$$\mathcal{F}_{L^*}(Bf)(\xi) = \tau(\xi)\mathcal{F}_{L^*}(f)(\xi), \quad f \in C_{L^*}^{\infty}(\bar{\Omega}),$$

for some $\tau : \mathcal{I} \rightarrow \mathbb{C}$. □

As used in [18], we have the following simple relation between the symbols of an operator and its adjoint.

Proposition 9.7. The operator A is an L -Fourier multiplier by $\sigma(\xi)$ if and only if A^* is an L^* -Fourier multiplier by $\overline{\sigma(\xi)}$. □

Proof. It is enough to prove the “only if” implication. First, by the Parseval identity

$$(Af, g)_{L^2} = \sum_{\xi \in \mathcal{I}} \widehat{Af}(\xi) \overline{\widehat{g}(\xi)} = \sum_{\xi \in \mathcal{I}} \sigma(\xi) \widehat{f}(\xi) \overline{\widehat{g}(\xi)} = \sum_{\xi \in \mathcal{I}} \widehat{f}(\xi) \overline{\sigma(\xi) \widehat{g}(\xi)}.$$

At the same time

$$(Af, g)_{L^2} = (f, A^*g)_{L^2} = \sum_{\xi \in \mathcal{I}} \widehat{f}(\xi) \overline{\widehat{A^*g}(\xi)}.$$

Consequently,

$$\widehat{A^*g}(\xi) = \overline{\sigma(\xi)} \widehat{g}(\xi),$$

that is, A^* is an L^* -multiplier by $\overline{\sigma(\xi)}$. ■

10 Difference Operators and Symbolic Calculus

In this section, we discuss difference operators that will be instrumental in defining symbol classes for the symbolic calculus of operators. An interesting new feature of these operators compared with previous settings is that they will be also dependent on a point $x \in \Omega$.

Let $q_j \in C^\infty(\Omega \times \Omega)$, $j = 1, \dots, l$, be a given family of smooth functions. We will call the collection of q_j 's *L-strongly admissible* if the following properties hold:

- for every $x \in \Omega$, the multiplication by $q_j(x, \cdot)$ is a continuous linear mapping on $C_L^\infty(\bar{\Omega})$, for all $j = 1, \dots, l$;
- $q_j(x, x) = 0$ for all $j = 1, \dots, l$;
- $\text{rank}(\nabla_y q_1(x, y), \dots, \nabla_y q_l(x, y))|_{y=x} = n$;
- the diagonal in $\Omega \times \Omega$ is the only set when all of q_j 's vanish:

$$\bigcap_{j=1}^l \{(x, y) \in \Omega \times \Omega : q_j(x, y) = 0\} = \{(x, x) : x \in \Omega\}.$$

We note that the first property above implies that for every $x \in \Omega$, the multiplication by $q_j(x, \cdot)$ is also well defined and extends to a continuous linear mapping on $\mathcal{D}'_L(\Omega)$. Also, the last property above contains the second one but we chose to still give it explicitly for the clarity of the exposition.

The collection of q_j 's with the above properties generalizes the notion of a strongly admissible collection of functions for difference operators introduced in [56] in the context of compact Lie groups. We will use the multi-index notation

$$q^\alpha(x, y) := q_1^{\alpha_1}(x, y) \cdots q_l^{\alpha_l}(x, y).$$

Analogously, the notion of an L^* -strongly admissible collection suitable for the conjugate problem is that of a family $\tilde{q}_j \in C^\infty(\Omega \times \Omega)$, $j = 1, \dots, l$, satisfying the properties:

- for every $x \in \Omega$, the multiplication by $\tilde{q}_j(x, \cdot)$ is a continuous linear mapping on $C_{L^*}^\infty(\bar{\Omega})$, for all $j = 1, \dots, l$;
- $\tilde{q}_j(x, x) = 0$ for all $j = 1, \dots, l$;
- $\text{rank}(\nabla_y \tilde{q}_1(x, y), \dots, \nabla_y \tilde{q}_l(x, y))|_{y=x} = n$;
- the diagonal in $\Omega \times \Omega$ is the only set when all of \tilde{q}_j 's vanish:

$$\bigcap_{j=1}^l \{(x, y) \in \Omega \times \Omega : \tilde{q}_j(x, y) = 0\} = \{(x, x) : x \in \Omega\}.$$

We also write

$$\tilde{q}^\alpha(x, y) := \tilde{q}_1^{\alpha_1}(x, y) \cdots \tilde{q}_l^{\alpha_l}(x, y).$$

We now record the Taylor expansion formula with respect to a family of q_j 's, which follows from expansions of functions g and $q^\alpha(e, \cdot)$ by the common Taylor series.

Proposition 10.1. Any smooth function $g \in C^\infty(\Omega)$ can be approximated by Taylor polynomial-type expansions, that is, for $e \in \Omega$, we have

$$\begin{aligned} g(x) &= \sum_{|\alpha| < N} \frac{1}{\alpha!} D_x^{(\alpha)} g(x) \Big|_{x=e} q^\alpha(e, x) + \sum_{|\alpha|=N} \frac{1}{\alpha!} q^\alpha(e, x) g_N(x) \\ &\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} D_x^{(\alpha)} g(x) \Big|_{x=e} q^\alpha(e, x) \end{aligned} \tag{10.1}$$

in a neighborhood of $e \in \Omega$, where $g_N \in C^\infty(\Omega)$ and $D_x^{(\alpha)} g(x)|_{x=e}$ can be found from the recurrent formulae: $D_x^{(0, \dots, 0)} := I$ and for $\alpha \in \mathbb{N}_0^l$,

$$\partial_x^\beta g(x)|_{x=e} = \sum_{|\alpha| \leq |\beta|} \frac{1}{\alpha!} [\partial_x^\beta q^\alpha(e, x)] \Big|_{x=e} D_x^{(\alpha)} g(x)|_{x=e},$$

where $\beta = (\beta_1, \dots, \beta_n)$ and $\partial_x^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$. □

Analogously, any function $g \in C^\infty(\Omega)$ can be approximated by Taylor polynomial-type expansions corresponding to the adjoint problem, that is, we have

$$\begin{aligned} g(x) &= \sum_{|\alpha| < N} \frac{1}{\alpha!} \tilde{D}_x^{(\alpha)} g(x) \Big|_{x=e} \tilde{q}^\alpha(e, x) + \sum_{|\alpha|=N} \frac{1}{\alpha!} \tilde{q}^\alpha(e, x) g_N(x) \\ &\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \tilde{D}_x^{(\alpha)} g(x) \Big|_{x=e} \tilde{q}^\alpha(e, x) \end{aligned} \tag{10.2}$$

in a neighborhood of $e \in \Omega$, where $g_N \in C^\infty(\Omega)$ and $\tilde{D}_x^{(\alpha)} g(x)|_{x=e}$ are found from the recurrent formula: $\tilde{D}^{(0, \dots, 0)} := I$ and for $\alpha \in \mathbb{N}_0^l$,

$$\partial_x^\beta g(x)|_{x=e} = \sum_{|\alpha| \leq |\beta|} \frac{1}{\alpha!} [\partial_x^k \tilde{q}^\alpha(e, x)] \Big|_{x=e} \tilde{D}_x^{(\alpha)} g(x)|_{x=e},$$

where $\beta = (\beta_1, \dots, \beta_n)$, and ∂^β is defined as in Proposition 10.1.

It can be seen that operators $D^{(\alpha)}$ and $\tilde{D}^{(\alpha)}$ are differential operators of order $|\alpha|$. We now define difference operators acting on Fourier coefficients. Since the problem

in general may lack any invariance or symmetry structure, the introduced difference operators will depend on a point x where they will be taken when applied to symbols.

Definition 10.2. For WZ-systems, we define difference operator $\Delta_{q,(x)}^\alpha$ acting on Fourier coefficients by any of the following equal expressions:

$$\begin{aligned} \Delta_{q,(x)}^\alpha \hat{f}(\xi) &= u_\xi^{-1}(x) \int_\Omega \left[\int_\Omega q^\alpha(x, y) F(x, y, z) f(z) dz \right] u_\xi(y) dy \\ &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) \hat{f}(\eta) u_\eta(x) \\ &= u_\xi^{-1}(x) ([q^\alpha(x, \cdot) u_\xi(\cdot)] \star_L f)(x). \end{aligned}$$

Analogously, we define the difference operator $\tilde{\Delta}_{q,(x)}^\alpha$ acting on adjoint Fourier coefficients by

$$\tilde{\Delta}_{q,(x)}^\alpha \hat{f}_*(\xi) := v_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_{L^*}(\tilde{q}^\alpha(x, \cdot) v_\xi(\cdot))(\eta) \hat{f}_*(\eta) v_\eta(x). \quad \square$$

For simplicity, if there is no confusion, for a fixed collection of q_j 's, instead of $\Delta_{q,(x)}$ and $\tilde{\Delta}_{q,(x)}$ we will often simply write $\Delta_{(x)}$ and $\tilde{\Delta}_{(x)}$.

Recalling that the general philosophy behind the symbolic constructions and the definition of the classes of symbols is that since the symbol is the Fourier transform of the (convolution) kernel of the operator, the difference conditions correspond to the multiplication of the kernel by functions vanishing on its singular support and, therefore, lead to the improved behavior reducing the strength of the singularity. Indeed, applying difference operators to a symbol and using formulae from Section 9, we obtain

$$\begin{aligned} \Delta_{(x)}^\alpha a(x, \xi) &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) a(x, \eta) u_\eta(x) \\ &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) \int_\Omega K(x, y) u_\eta(y) dy \\ &= u_\xi^{-1}(x) \int_\Omega K(x, y) \left[\sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) u_\eta(y) \right] dy \\ &= u_\xi^{-1}(x) \int_\Omega q^\alpha(x, y) K(x, y) u_\xi(y) dy. \end{aligned} \tag{10.3}$$

In view of the first property of the strongly admissible collections, for each $x \in \Omega$, the multiplication by $q^\alpha(x, \cdot)$ is well defined on $\mathcal{D}'_L(\Omega)$. Therefore, we can write (10.3) also in

the distributional form

$$\Delta_{(x)}^\alpha a(x, \xi) = u_\xi^{-1}(x) \langle q^\alpha(x, \cdot)K(x, \cdot), u_\xi \rangle,$$

providing more light on the nature of the difference operators applied to symbols. In view of the preceding discussion this and the latter formula (10.3) yield indeed the justification of the definition of difference operators as in Definition 10.2.

Plugging the expression (v) from Corollary 9.3 for the kernel in terms of the symbol into (10.3), namely, using

$$K(x, y) = \sum_{\eta \in \mathcal{I}} u_\eta(x) a(x, \eta) \overline{v_\eta(y)},$$

we record another useful form of (10.3) to be used later as

$$\begin{aligned} \Delta_{(x)}^\alpha a(x, \xi) &= u_\xi^{-1}(x) \int_\Omega q^\alpha(x, y) \left[\sum_{\eta \in \mathcal{I}} u_\eta(x) a(x, \eta) \overline{v_\eta(y)} \right] u_\xi(y) \, dy \\ &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} u_\eta(x) a(x, \eta) \left[\int_\Omega q^\alpha(x, y) \overline{v_\eta(y)} u_\xi(y) \, dy \right], \end{aligned} \tag{10.4}$$

with the usual distributional interpretation of all the steps. In the sequel, we will also require the L^* -version of this formula, which we record now as

$$\tilde{\Delta}_{(x)}^\alpha a(x, \xi) = v_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} v_\eta(x) a(x, \eta) \left[\int_\Omega \tilde{q}^\alpha(x, y) \overline{u_\eta(y)} v_\xi(y) \, dy \right]. \tag{10.5}$$

Using such difference operators and derivatives $D^{(\alpha)}$ from Proposition 10.1, we can now define classes of symbols.

Definition 10.3 (Symbol class $S_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$). Let $m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$. The L -symbol class $S_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$ consists of those functions $a(x, \xi)$ which are smooth in x for all $\xi \in \mathcal{I}$, and which satisfy

$$|\Delta_{(x)}^\alpha D_x^{(\beta)} a(x, \xi)| \leq C_{\alpha\beta m} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \tag{10.6}$$

for all $x \in \bar{\Omega}$, for all $\alpha, \beta \geq 0$, and for all $\xi \in \mathcal{I}$. Here the operators $D_x^{(\beta)}$ are defined in Proposition 10.1. We will often denote them simply by $D^{(\beta)}$.

The class $S_{1,0}^m(\bar{\Omega} \times \mathcal{I})$ will be often denoted by writing simply $S^m(\bar{\Omega} \times \mathcal{I})$. In (10.6), we assume that the inequality is satisfied for $x \in \Omega$ and it extends to the closure

$\bar{\Omega}$. Furthermore, we define

$$S_{\rho,\delta}^\infty(\bar{\Omega} \times \mathcal{I}) := \bigcup_{m \in \mathbb{R}} S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$$

and

$$S^{-\infty}(\bar{\Omega} \times \mathcal{I}) := \bigcap_{m \in \mathbb{R}} S^m(\bar{\Omega} \times \mathcal{I}).$$

When we have two L -strongly admissible collections, expressing one in terms of the other similarly to Proposition 10.1 and arguing similarly to [56], we can convince ourselves that for $\rho > \delta$ the definition of the symbol class does not depend on the choice of an L -strongly admissible collection.

Analogously, we define the L^* -symbol class $\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ as the space of those functions $a(x, \xi)$ which are smooth in x for all $\xi \in \mathcal{I}$, and which satisfy

$$\left| \tilde{\Delta}_{(x)}^\alpha \tilde{D}^{(\beta)} a(x, \xi) \right| \leq C_{\alpha\beta m} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}$$

for all $x \in \bar{\Omega}$, for all $\alpha, \beta \geq 0$, and for all $\xi \in \mathcal{I}$. Similarly, we can define classes $\tilde{S}_{\rho,\delta}^\infty(\bar{\Omega} \times \mathcal{I})$ and $\tilde{S}^{-\infty}(\bar{\Omega} \times \mathcal{I})$. \square

If $a \in S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$, it is convenient to denote by $a(X, D) = \text{Op}_L(a)$ the corresponding L -pseudo-differential operator defined by

$$\text{Op}_L(a) f(x) = a(X, D) f(x) := \sum_{\xi \in \mathcal{I}} u_\xi(x) a(x, \xi) \hat{f}(\xi). \tag{10.7}$$

The set of operators $\text{Op}_L(a)$ of the form (10.7) with $a \in S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ will be denoted by $\text{Op}_L(S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$, or by $\Psi_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$. If an operator A satisfies $A \in \text{Op}_L(S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$, we denote its L -symbol by $\sigma_A = \sigma_A(x, \xi)$, $x \in \bar{\Omega}$, $\xi \in \mathcal{I}$. Naturally, $\sigma_{a(X,D)}(x, \xi) = a(x, \xi)$.

Analogously, if $a \in \tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$, we denote by $a(X, D) = \text{Op}_{L^*}(a)$ the corresponding L^* -pseudo-differential operator defined by

$$\text{Op}_{L^*}(a) f(x) = a(X, D) f(x) := \sum_{\xi \in \mathcal{I}} v_\xi(x) a(x, \xi) \hat{f}_*(\xi). \tag{10.8}$$

The set of operators $\text{Op}_{L^*}(a)$ of the form (10.8) with $a \in \tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ will be denoted by $\text{Op}_{L^*}(\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$, or by $\tilde{\Psi}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$.

Remark 10.4 (Topology on $S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ ($\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$)). The set $S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ ($\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$) of symbols has a natural topology. Let us consider the functions $P_{\alpha\beta}^j : S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}) \rightarrow \mathbb{R}$

$(\tilde{p}_{\alpha\beta}^j : \tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}) \rightarrow \mathbb{R})$ defined by

$$p_{\alpha\beta}^j(\sigma) := \sup \left[\frac{|\Delta_{(x)}^\alpha D^{(\beta)} \sigma(x, \xi)|}{\langle \xi \rangle^{l-\rho|\alpha|+\delta|\beta|}} : (x, \xi) \in \bar{\Omega} \times \mathcal{I} \right]$$

$$\left(\tilde{p}_{\alpha\beta}^j(\sigma) := \sup \left[\frac{|\tilde{\Delta}_{(x)}^\alpha \tilde{D}^{(\beta)} \sigma(x, \xi)|}{\langle \xi \rangle^{l-\rho|\alpha|+\delta|\beta|}} : (x, \xi) \in \bar{\Omega} \times \mathcal{I} \right] \right).$$

Now $\{p_{\alpha\beta}^j\}$ ($\{\tilde{p}_{\alpha\beta}^j\}$) is a countable family of seminorms, and they define a Fréchet topology on $S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ ($\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{Z})$). Due to the bijective correspondence of $\text{Op}_L(S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$ and $S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I})$ ($\text{Op}_{L^*}(\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$ and $\tilde{S}_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{Z})$), this directly topologizes also the set of operators. These spaces are not normable, and the topologies have but a marginal role. □

The notion of a symbol can be naturally extended to that of an amplitude.

Definition 10.5 (*L*-amplitudes). The class $\mathcal{A}_{\rho,\delta}^m(\bar{\Omega})$ of *L*-amplitudes consists of the functions $a(x, y, \xi)$ which are smooth in x and y for all $\xi \in \mathcal{I}$ and which satisfy

$$\left| \Delta_{(x)}^\alpha \Delta_{(y)}^{\alpha'} D_x^{(\beta)} D_y^{(\gamma)} a(x, y, \xi) \right| \leq C_{\alpha\alpha'\beta\gamma m} \langle \xi \rangle^{m-\rho(|\alpha|+|\alpha'|)+\delta(|\beta|+|\gamma|)} \tag{10.9}$$

for all $x, y \in \bar{\Omega}$, for all $\alpha, \alpha', \beta, \gamma \geq 0$, and for all $\xi \in \mathcal{I}$. Such a function a will be also called an *L*-amplitude of order $m \in \mathbb{R}$ of type (ρ, δ) . Formally, we may also define

$$(\text{Op}_L(a) f)(x) := \sum_{\xi \in \mathcal{I}} \int_{\Omega} u_\xi(x) \overline{v_\xi(y)} a(x, y, \xi) f(y) dy$$

for $f \in C_L^\infty(\bar{\Omega})$. Sometimes we may denote $\text{Op}_L(a)$ by $a(X, Y, D)$. We also write $\mathcal{A}^m(\bar{\Omega}) := \mathcal{A}_{1,0}^m(\bar{\Omega})$ as well as

$$\mathcal{A}^{-\infty}(\bar{\Omega}) := \bigcap_{m \in \mathbb{R}} \mathcal{A}^m(\bar{\Omega}) \quad \text{and} \quad \mathcal{A}_{\rho,\delta}^\infty(\bar{\Omega}) := \bigcup_{m \in \mathbb{R}} \mathcal{A}_{\rho,\delta}^m(\bar{\Omega}). \tag{10.10}$$

Clearly, we can regard the *L*-symbols as a special class of *L*-amplitudes, namely the ones independent of the middle argument. Analogously, the class $\tilde{\mathcal{A}}_{\rho,\delta}^m(\bar{\Omega})$ of *L*^{*}-amplitudes consists of the functions $a(x, y, \xi)$ which are smooth in x and y for all $\xi \in \mathcal{I}$ and which satisfy

$$\left| \tilde{\Delta}_{(x)}^\alpha \tilde{\Delta}_{(y)}^{\alpha'} \tilde{D}_x^{(\beta)} \tilde{D}_y^{(\gamma)} a(x, y, \xi) \right| \leq C_{\alpha\alpha'\beta\gamma m} \langle \xi \rangle^{m-\rho(|\alpha|+|\alpha'|)+\delta(|\beta|+|\gamma|)} \tag{10.10}$$

for all $x, y \in \bar{\Omega}$, for all $\alpha, \alpha', \beta, \gamma \geq 0$, and for all $\xi \in \mathcal{I}$. Formally, we may also write

$$(\text{Op}_{L^*}(a)f)(x) := \sum_{\xi \in \mathcal{I}} \int_{\Omega} v_{\xi}(x) \overline{u_{\xi}(y)} a(x, y, \xi) f(y) \, dy$$

for $f \in C_{L^*}^{\infty}(\bar{\Omega})$. We also write $\tilde{\mathcal{A}}^m(\bar{\Omega}) := \tilde{\mathcal{A}}_{1,0}^m(\bar{\Omega})$ as well as $\tilde{\mathcal{A}}^{-\infty}(\bar{\Omega}) := \bigcap_{m \in \mathbb{R}} \tilde{\mathcal{A}}^m(\bar{\Omega})$ and $\tilde{\mathcal{A}}_{\rho,\delta}^{\infty}(\bar{\Omega}) := \bigcup_{m \in \mathbb{R}} \tilde{\mathcal{A}}_{\rho,\delta}^m(\bar{\Omega})$.

Definition 10.6 (Equivalence of amplitudes). We say that amplitudes a, a' are $m(\rho, \delta)$ -equivalent ($m \in \mathbb{R}$), $a \stackrel{m,\rho,\delta}{\sim} a'$, if $a - a' \in \mathcal{A}_{\rho,\delta}^m(\bar{\Omega})$; they are asymptotically equivalent, $a \sim a'$ (or $a \stackrel{-\infty}{\sim} a'$ if we need additional clarity), if $a - a' \in \mathcal{A}^{-\infty}(\bar{\Omega})$. For the corresponding operators, we also write $\text{Op}(a) \stackrel{m,\rho,\delta}{\sim} \text{Op}(a')$ and $\text{Op}(a) \sim \text{Op}(a')$ (or $\text{Op}(a) \stackrel{-\infty}{\sim} \text{Op}(a')$ if we need additional clarity), respectively. It is obvious that $\stackrel{m,\rho,\delta}{\sim}$ and \sim are equivalence relations. □

From the algebraic point of view, we could handle the amplitudes, symbols, and operators modulo the equivalence relation \sim , because the L -pseudo-differential operators form a $*$ -algebra with $\text{Op}(S^{-\infty}(\bar{\Omega} \times \mathcal{I}))$ as a subalgebra.

The next theorem is a prelude to asymptotic expansions, which are the main tool in the symbolic analysis of L -pseudo-differential operators.

Theorem 10.7 (Asymptotic sums of symbols). Let $(m_j)_{j=0}^{\infty} \subset \mathbb{R}$ be a sequence such that $m_j > m_{j+1}$, and $m_j \rightarrow -\infty$ as $j \rightarrow \infty$, and $\sigma_j \in S_{\rho,\delta}^{m_j}(\bar{\Omega} \times \mathcal{I})$ for all $j \in \mathbb{N}$. Then there exists an L -symbol $\sigma \in S_{\rho,\delta}^{m_0}(\bar{\Omega} \times \mathcal{I})$ such that for all $N \in \mathbb{N}$,

$$\sigma \stackrel{m_N,\rho,\delta}{\sim} \sum_{j=0}^{N-1} \sigma_j. \quad \square$$

Proof. The proof is rather standard. Choose a function $\chi \in C^{\infty}(\mathbb{R})$ satisfying $|\xi| \geq 1 \Rightarrow \chi(\xi) = 1$ and $|\xi| \leq \frac{1}{2} \Rightarrow \chi(\xi) = 0$; otherwise $0 \leq \chi(\xi) \leq 1$. Take a sequence $(\varepsilon_j)_{j=0}^{\infty}$ of positive real numbers such that $\varepsilon_j > \varepsilon_{j+1}$, and $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and define $\chi_j \in C^{\infty}(\mathbb{R})$ by $\chi_j(\xi) := \chi(\varepsilon_j \xi)$. Since $\chi_j(\xi) = 1$ for sufficiently large ξ , we get $\chi_j \sigma_j \in S_{\rho,\delta}^{m_j}(\bar{\Omega} \times \mathcal{I})$ for each j . For any fixed $\xi \in \mathcal{I}$, the function $\chi_j(\xi) \sigma_j(x, \xi)$ vanishes, when j is large enough. This justifies the definition

$$\sigma(x, \xi) := \sum_{j=0}^{\infty} \chi_j(\xi) \sigma_j(x, \xi),$$

and clearly $\sigma \in S_{\rho,\delta}^{m_0}(\bar{\Omega} \times \mathcal{I})$. Furthermore,

$$\sigma(x, \xi) - \sum_{j=0}^{N-1} \sigma_j(x, \xi) = \sum_{j=0}^{N-1} [\chi_j(\xi) - 1] \sigma_j(x, \xi) + \sum_{j=N}^{\infty} \chi_j(\xi) \sigma_j(x, \xi).$$

Recall that $\varepsilon_j > \varepsilon_{j+1}$, and $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, so that the $\sum_{j=0}^{N-1}$ part of the sum vanishes, whenever ξ is large. This shows that

$$\sigma(x, \xi) - \sum_{j=0}^{N-1} \chi_j(\xi) \sigma_j(x, \xi) \in S_{\rho,\delta}^{m_N}(\bar{\Omega} \times \mathcal{I})$$

finishing the proof. ■

We will now look at the formula for the symbol of the adjoint operator. Let $A \in \text{Op}_L(S_{\rho,\delta}^m(\bar{\Omega} \times \mathcal{I}))$. By the definition of the adjoint operator, we have

$$(Au_\xi, v_\eta)_{L^2} = (u_\xi, A^*v_\eta)_{L^2}$$

or

$$\int_{\Omega} Au_\xi(x) \overline{v_\eta(x)} \, dx = \int_{\Omega} u_\xi(x) \overline{A^*v_\eta(x)} \, dx$$

for $\xi, \eta \in \mathcal{I}$. Plugging in the integral expressions, we get

$$\begin{aligned} \int_{\Omega} \left[\int_{\Omega} K_A(x, y) u_\xi(y) \, dy \right] \overline{v_\eta(x)} \, dx &= \int_{\Omega} u_\xi(x) \overline{\left[\int_{\Omega} K_{A^*}(x, y) v_\eta(y) \, dy \right]} \, dx \\ &= \int_{\Omega} u_\xi(y) \overline{\left[\int_{\Omega} \overline{K_{A^*}(y, x)} v_\eta(x) \, dx \right]} \, dy \end{aligned}$$

for $\xi, \eta \in \mathcal{I}$, where we swapped x and y in the last formula. Consequently, we get the familiar property

$$K_{A^*}(x, y) = \overline{K_A(y, x)}.$$

Now, using this and formula (ii) in Corollary 9.5, and then formula (v) in Corollary 9.3 and the Taylor expansion in Proposition 10.1, we can write for the L^* -symbol τ_{A^*} of A^* that

$$\begin{aligned} v_\xi(x) \tau_{A^*}(x, \xi) &= \int_{\Omega} K_{A^*}(x, y) v_\xi(y) \, dy \\ &= \int_{\Omega} \overline{K_A(y, x)} v_\xi(y) \, dy \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega} \sum_{\eta \in \mathcal{I}} \overline{u_{\eta}(y) \sigma_A(y, \eta)} v_{\eta}(x) v_{\xi}(y) \, dy \\
 &\sim \int_{\Omega} \sum_{\eta \in \mathcal{I}} \overline{u_{\eta}(y)} \sum_{\alpha} \frac{1}{\alpha!} \overline{D_x^{(\alpha)} \sigma_A(x, \eta)} q^{\alpha}(x, y) v_{\eta}(x) v_{\xi}(y) \, dy
 \end{aligned}$$

as an asymptotic sum. Formally, regrouping terms for each α , we obtain

$$\tau_{A^*}(x, \xi) \sim v_{\xi}(x)^{-1} \sum_{\alpha} \frac{1}{\alpha!} \sum_{\eta \in \mathcal{I}} v_{\eta}(x) \overline{D_x^{(\alpha)} \sigma_A(x, \eta)} \int_{\Omega} \overline{q^{\alpha}(x, y) u_{\eta}(y)} v_{\xi}(y) \, dy.$$

Using the L^* -version of the difference formula (10.5), taking

$$\tilde{q}(x, y) := \overline{q(x, y)}$$

we can write this as

$$\tau_{A^*}(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \overline{\tilde{\Delta}_x^{\alpha} D_x^{(\alpha)} \sigma_A(x, \xi)}.$$

Making rigorous estimates for the remainder in a routine way, and assuming in the following theorem that for every $x \in \Omega$, the multiplication by $q_j(x, \cdot)$ preserves both spaces $C_L^{\infty}(\bar{\Omega})$ and $C_{L^*}^{\infty}(\bar{\Omega})$, we have proved the following theorem.

Theorem 10.8 (Adjoint operators). Let $0 \leq \delta < \rho \leq 1$. Let $A \in \text{Op}_L(S_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I}))$. Assume that the conjugate symbol class $\tilde{S}_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$ is defined with strongly admissible functions $\tilde{q}_j(x, y) := \overline{q_j(x, y)}$ which are L^* -strongly admissible. Then the adjoint of A satisfies $A^* \in \text{Op}_{L^*}(\tilde{S}_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I}))$, with its L^* -symbol $\tau_{A^*} \in \tilde{S}_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$ having the asymptotic expansion

$$\tau_{A^*}(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \overline{\tilde{\Delta}_x^{\alpha} D_x^{(\alpha)} \sigma_A(x, \xi)}. \quad \square$$

We now treat symbols of the amplitude operators.

Theorem 10.9 (Amplitude symbols). Let $0 \leq \delta < \rho \leq 1$. For every amplitude $a \in \mathcal{A}_{\rho, \delta}^m(\bar{\Omega})$, there exists a unique L -symbol $\sigma \in S_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$ satisfying $\text{Op}_L(a) = \text{Op}_L(\sigma)$, where

$$\sigma(x, \xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_{(x)}^{\alpha} D_y^{(\alpha)} a(x, y, \xi) \Big|_{y=x}. \quad (10.11)$$

□

Proof. As a linear operator on $C_L^{\infty}(\bar{\Omega})$, the operator $\text{Op}_L(a)$ possesses the unique L -symbol $\sigma = \sigma_{\text{Op}_L(a)}$, but at the moment we do not yet know whether $\sigma \in S_{\rho, \delta}^m(\bar{\Omega} \times \mathcal{I})$.

By Theorem 9.2, the L -symbol is computed from

$$\sigma(x, \xi) = u_\xi^{-1}(x)(\text{Op}_L(a)u_\xi)(x) = u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \int_{\Omega} u_\eta(x) \overline{v_\eta(y)} a(x, y, \eta) u_\xi(y) \, dy.$$

Let us denote

$$k_a(x, y, z) := (\mathcal{F}_L^{-1} a(x, y, \cdot))(z)$$

that is, we get

$$a(x, y, \xi) = (\mathcal{F}_L k_a(x, y, \cdot))(\xi).$$

Then we have

$$\begin{aligned} \sigma(x, \xi) &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \int_{\Omega} \int_{\Omega} u_\eta(x) \overline{v_\eta(y)} \overline{v_\eta(z)} k_a(x, y, z) u_\xi(y) \, dy dz \\ &= u_\xi^{-1}(x) \int_{\Omega} \int_{\Omega} F(x, y, z) k_a(x, y, z) u_\xi(y) \, dy dz. \end{aligned}$$

Now we approximate the function $k_a(x, \cdot, z) \in C_L^\infty(\bar{\Omega})$ by Taylor polynomial-type expansions, by using Proposition 10.1, we have

$$\begin{aligned} \sigma(x, \xi) &\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \int_{\Omega} \int_{\Omega} F(x, y, z) q^\alpha(x, y) [D_y^{(\alpha)} k_a(x, y, z)]_{y=x} u_\xi(y) u_\xi^{-1}(x) \, dz dy \\ &\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_{(x)}^\alpha D_y^{(\alpha)} a(x, y, \xi) \Big|_{y=x}. \end{aligned}$$

Omitting a routine verification of the properties of the remainder, this yields the statement. ■

We now formulate the composition formula.

Theorem 10.10. Let $m_1, m_2 \in \mathbb{R}$ and $\rho > \delta \geq 0$. Let $A, B : C_L^\infty(\bar{\Omega}) \rightarrow C_L^\infty(\bar{\Omega})$ be continuous and linear, and assume that their L -symbols satisfy

$$\begin{aligned} |\Delta_{(x)}^\alpha \sigma_A(x, \xi)| &\leq C_\alpha \langle \xi \rangle^{m_1 - \rho|\alpha|}, \\ |D^{(\beta)} \sigma_B(x, \xi)| &\leq C_\beta \langle \xi \rangle^{m_2 + \delta|\beta|}, \end{aligned}$$

for all $\alpha, \beta \geq 0$, uniformly in $x \in \bar{\Omega}$ and $\xi \in \mathcal{I}$. Then

$$\sigma_{AB}(x, \xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta_{(x)}^\alpha \sigma_A(x, \xi)) D^{(\alpha)} \sigma_B(x, \xi), \tag{10.12}$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\left| \sigma_{AB}(x, \xi) - \sum_{|\alpha| < N} \frac{1}{\alpha!} (\Delta_{(x)}^\alpha \sigma_A(x, \xi)) D^{(\alpha)} \sigma_B(x, \xi) \right| \leq C_N \langle \xi \rangle^{m_1 + m_2 - (\rho - \delta)N}. \quad \square$$

Proof. First, by the Schwartz kernel theorem from Section 8, we have

$$\begin{aligned} ABf(x) &= (k_A(x) \star_L Bf)(x) \\ &= \int_{\Omega} \left[\int_{\Omega} F(x, y, z) k_A(x, z) dz \right] (Bf)(y) dy \\ &= \int_{\Omega} \left(\left[\int_{\Omega} F(x, y, z) k_A(x, z) dz \right] \right. \\ &\quad \left. \times \int_{\Omega} \left[\int_{\Omega} F(y, s, t) k_B(y, t) dt \right] f(s) ds \right) dy. \end{aligned}$$

Hence,

$$\begin{aligned} \sigma_{AB}(x, \xi) &= u_{\xi}^{-1}(x) (A(Bu_{\xi}))(x) \\ &= u_{\xi}^{-1}(x) \int_{\Omega} \left(\left[\int_{\Omega} F(x, y, z) k_A(x, z) dz \right] \right. \\ &\quad \left. \times \int_{\Omega} \left[\int_{\Omega} F(y, s, t) k_B(y, t) dt \right] u_{\xi}(s) ds \right) dy. \end{aligned}$$

Now we approximate the function $k_B(\cdot, t) \in C_L^\infty(\bar{\Omega})$ by Taylor polynomial-type expansions. By using Proposition 10.1, we get

$$\begin{aligned} \sigma_{AB}(x, \xi) &\sim u_{\xi}^{-1}(x) \int_{\Omega} \left(\left[\int_{\Omega} F(x, y, z) k_A(x, z) dz \right] \right. \\ &\quad \left. \times \int_{\Omega} \left[\int_{\Omega} F(y, s, t) \sum_{\alpha \geq 0} \frac{1}{\alpha!} q^\alpha(x, y) D_x^{(\alpha)} k_B(x, t) dt \right] u_{\xi}(s) ds \right) dy \\ &= \sum_{\alpha \geq 0} \frac{1}{\alpha!} u_{\xi}^{-1}(x) \int_{\Omega} \left(\left[\int_{\Omega} F(x, y, z) q^\alpha(x, y) k_A(x, z) dz \right] \right. \\ &\quad \left. \times \int_{\Omega} \left[\int_{\Omega} F(y, s, t) D_x^{(\alpha)} k_B(x, t) dt \right] u_{\xi}(s) ds \right) dy. \end{aligned}$$

Since

$$\begin{aligned}
 & u_\xi^{-1}(y) \int_{\Omega} \left[\int_{\Omega} F(y, s, t) D_x^{(\alpha)} k_B(x, t) dt \right] u_\xi(s) ds \\
 &= u_\xi^{-1}(y) \int_{\Omega} \left[\int_{\Omega} \sum_{\eta \in \mathcal{I}} u_\eta(y) \overline{v_\eta(s) v_\eta(t)} D_x^{(\alpha)} k_B(x, t) dt \right] u_\xi(s) ds \\
 &= \sum_{\eta \in \mathcal{I}} u_\xi^{-1}(y) u_\eta(y) \int_{\Omega} \left[\int_{\Omega} D_x^{(\alpha)} k_B(x, t) \overline{v_\eta(t)} dt \right] u_\xi(s) \overline{v_\eta(s)} ds \\
 &= \sum_{\eta \in \mathcal{I}} u_\xi^{-1}(y) u_\eta(y) \left[\int_{\Omega} u_\xi(s) \overline{v_\eta(s)} ds \right] \times \left[\int_{\Omega} D_x^{(\alpha)} k_B(x, t) \overline{v_\eta(t)} dt \right] \\
 &= u_\xi^{-1}(y) u_\xi(y) D_x^{(\alpha)} \hat{k}_B(x, \xi) \\
 &= D_x^{(\alpha)} \hat{k}_B(x, \xi) \\
 &= D_x^{(\alpha)} \sigma_B(x, \xi),
 \end{aligned}$$

using Definition 10.2, we have

$$\sigma_{AB}(x, \xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta_{(x)}^\alpha \sigma_A(x, \xi)) D_x^{(\alpha)} \sigma_B(x, \xi).$$

Omitting a routine treatment of the remainder, this completes the proof. ■

11 Properties of Integral Kernels

We now establish some properties of Schwartz kernels of pseudo-differential operators with symbols in the introduced Hörmander-type classes. In the following Theorem 11.1, let us make the assumption on the growth of L^∞ -norms of the eigenfunctions u_ξ . Finding estimates for the norms $\|u_\xi\|_{L^\infty}$ in terms of the corresponding eigenvalues of L is a challenging problem even for self-adjoint operators L , see, for example, Sogge and Zelditch [68] and references therein. Thus, on tori or, more generally, on compact Lie groups, the eigenfunctions of the Laplacian can be chosen to be uniformly bounded. However, even for the Laplacian, on more general manifolds, such growth depends on the geometry of the manifold. We refer the reader to [15, Remark 8.9] for a more thorough discussion of this topic as well as for a list of relevant references.

Theorem 11.1 (Kernel of a pseudo-differential operator). Let μ_0 be a constant such that there is $C > 0$ such that for all $\xi \in \mathcal{I}$ we have

$$\|u_\xi\|_{L^\infty} \leq C \langle \xi \rangle^{\mu_0}.$$

Let $a \in S_{\rho,\delta}^\mu(\bar{\Omega} \times \mathcal{I})$, $\rho > 0$. Then the kernel $K(x, y)$ of the pseudo-differential operator $\text{Op}_L a$ satisfies

$$|(L_y^*)^k K(x, y)| \leq C_{Nk} |x - y|^{-N} \quad (11.1)$$

for any $N > (\mu + mk + 2\mu_0 + s_0)/\rho$ and $x \neq y$, where m is the order of the differential operator L and s_0 is the constant from Assumption 8.1. \square

In particular, if L is, for example, locally elliptic, (11.1) implies that for $x \neq y$, the kernel $K(x, y)$ is a smooth function. And, if $a \in S^{-\infty}(\bar{\Omega} \times \mathcal{I})$, then the integral kernel $K(x, y)$ of $\text{Op}_L a$ is smooth in x and y .

Proof. By Corollary 9.3, we have

$$a(x, \xi) = u_\xi^{-1}(x) \int_{\Omega} K(x, y) u_\xi(y) \, dy.$$

By using Definition 10.2 and by direct calculations, recalling (10.3) we have

$$\begin{aligned} \Delta_{(x)}^\alpha a(x, \xi) &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) a(x, \eta) u_\eta(x) \\ &= u_\xi^{-1}(x) \sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) \int_{\Omega} K(x, y) u_\eta(y) \, dy \\ &= u_\xi^{-1}(x) \int_{\Omega} K(x, y) \left[\sum_{\eta \in \mathcal{I}} \mathcal{F}_L(q^\alpha(x, \cdot) u_\xi(\cdot))(\eta) u_\eta(y) \right] \, dy \\ &= u_\xi^{-1}(x) \int_{\Omega} q^\alpha(x, y) K(x, y) u_\xi(y) \, dy, \end{aligned}$$

and also

$$\begin{aligned} u_\xi(x) \lambda_\xi^k \Delta_{(x)}^\alpha a(x, \xi) &= \int_{\Omega} q^\alpha(x, y) K(x, y) \lambda_\xi^k u_\xi(y) \, dy \\ &= \int_{\Omega} q^\alpha(x, y) K(x, y) L_y^k u_\xi(y) \, dy = \int_{\Omega} (L_y^*)^k (q^\alpha(x, y) K(x, y)) u_\xi(y) \, dy. \end{aligned}$$

This means that

$$(L_y^*)^k (q^\alpha(x, y) K(x, y)) = \mathcal{F}_L^{-1}(u_\xi(x) \lambda_\xi^k \Delta_{(x)}^\alpha a(x, \xi))(y).$$

Since it follows from assumptions that

$$\lambda_{\xi}^k \Delta_{(x)}^{\alpha} a(x, \xi) \in S^{\mu+m k-\rho|\alpha|}(\bar{\Omega} \times \mathcal{I}),$$

we have

$$\lambda_{\xi}^k |\Delta_{(x)}^{\alpha} a(x, \xi)| \leq C \langle \xi \rangle^{\mu+m k-\rho|\alpha|}.$$

We recall now the norm

$$\|a(x, \cdot)\|_{l^1(L)} = \sum_{\xi \in \mathcal{I}} |a(x, \xi)| \|u_{\xi}\|_{L^{\infty}(\Omega)}$$

from Section 7. It follows that

$$\|u_{\xi}(x) \lambda_{\xi}^k \Delta_{(x)}^{\alpha} a(x, \xi)\|_{l^1(L)} \leq C \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{\mu+m k-\rho|\alpha|} \|u_{\xi}\|_{L^{\infty}(\Omega)}^2 \leq C \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{\mu+m k-\rho|\alpha|+2\mu_0}.$$

Consequently, if

$$|\alpha| > (\mu + m k + 2\mu_0 + s_0)/\rho,$$

where s_0 is the constant from Assumption 8.1, we have that $u_{\xi}(x) \lambda_{\xi}^k \Delta_{(x)}^{\alpha} a(x, \xi)$ is in $l^1(L)$ with respect to ξ , and hence $(L_y^*)^k (q^{\alpha}(x, y) K(x, y))$ is in L^{∞} by the Hausdorff–Young inequality in Theorem 7.6. Since L_y^* is a differential operator, we also have

$$q^{\alpha}(x, y) (L_y^*)^k K(x, y) \in L^{\infty}(\Omega \times \Omega)$$

for such α . By the properties of q^{α} , it implies the statement of the theorem. ■

The singular support of $w \in \mathcal{D}'_L(\Omega)$ is defined as the complement of the set where w is smooth. Namely, $x \notin \text{sing supp } w$ if there is an open neighborhood U of x and a smooth function $f \in C^{\infty}_L(\bar{\Omega})$ such that $w(\varphi) = f(\varphi)$ for all $\varphi \in C^{\infty}_L(\bar{\Omega})$ with $\text{supp } \varphi \subset U$. As an immediate consequence of Theorem 11.1, we obtain the information on how the singular support is mapped by a pseudo-differential operator.

Corollary 11.2. Let $\sigma_A \in S^{\mu}_{\rho, \delta}(\bar{\Omega} \times \mathcal{I})$, $1 \geq \rho > \delta \geq 0$. Then for every $w \in \mathcal{D}'_L(\Omega)$, we have

$$\text{sing supp } Aw \subset \text{sing supp } w. \quad \square$$

For elliptic operators, in Corollary 12.2 we state also the inverse inclusion.

12 L -Elliptic Pseudo-Differential Operators

In this section, we discuss operators that are elliptic in the symbol classes generated by L . For such operators, we can obtain parametrix and then also a priori estimates by the properties of pseudo-differential operators in, for example, Sobolev spaces, once they are established in Section 14, see Theorem 14.3. Thus, from the asymptotic expansion for the composition of pseudo-differential operators, we get an expansion for a parametrix of an elliptic operator.

Theorem 12.1 (L -ellipticity). Let $1 \geq \rho > \delta \geq 0$. Let $\sigma_A \in S_{\rho,\delta}^{\mu}(\bar{\Omega} \times \mathcal{I})$ be elliptic in the sense that there exist constants $C_0 > 0$ and $N_0 \in \mathbb{N}$ such that

$$|\sigma_A(x, \xi)| \geq C_0 \langle \xi \rangle^{\mu} \quad (12.1)$$

for all $(x, \xi) \in \bar{\Omega} \times \mathcal{I}$ for which $\xi \geq N_0$; this is equivalent to assuming that there exists $\sigma_B \in S_{\rho,\delta}^{-\mu}(\bar{\Omega} \times \mathcal{I})$ such that $I - BA, I - AB$ are in $\text{Op}_L S^{-\infty}$. Let

$$A \sim \sum_{j=0}^{\infty} A_j,$$

$\sigma_{A_j} \in S_{\rho,\delta}^{\mu - (\rho - \delta)j}(\bar{\Omega} \times \mathcal{I})$. Then

$$B \sim \sum_{k=0}^{\infty} B_k,$$

where $B_k \in S_{\rho,\delta}^{-\mu - (\rho - \delta)k}(\bar{\Omega} \times \mathcal{I})$ is such that

$$\sigma_{B_0}(x, \xi) = 1/\sigma_{A_0}(x, \xi)$$

for large enough ξ , and recursively

$$\sigma_{B_N}(x, \xi) = \frac{-1}{\sigma_{A_0}(x, \xi)} \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{|\alpha|=N-j-k} \frac{1}{\alpha!} [\Delta_{(x)}^{\alpha} \sigma_{A_j}(x, \xi)] D_x^{(\alpha)} \sigma_{B_k}(x, \xi). \quad \square$$

Proof. Now $I \sim BA$, so that by the composition Theorem 10.10 we have

$$\begin{aligned} 1 &\sim \sigma_{BA}(x, \xi) \\ &\sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\alpha!} (\Delta_{(x)}^{\alpha} \sigma_B(x, \xi)) D_x^{(\alpha)} \sigma_A(x, \xi) \\ &\sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\alpha!} \left(\Delta_{(x)}^{\alpha} \sum_{k=0}^{\infty} \sigma_{B_k}(x, \xi) \right) D_x^{(\alpha)} \sum_{j=0}^{\infty} \sigma_{A_j}(x, \xi), \end{aligned}$$

where we want to solve it for σ_{B_k} . Note that A_0 is elliptic if and only if A is elliptic. Moreover, without a loss of generality we may assume that σ_{A_0} does not vanish anywhere. Obviously, we can demand that $1 = \sigma_{B_0}(x, \xi)\sigma_{A_0}(x, \xi)$, and that

$$0 = \sum_{j+k+|\alpha|=N} \frac{1}{\alpha!} [\Delta_{(x)}^\alpha \sigma_{B_k}(x, \xi)] D_x^{(\alpha)} \sigma_{A_j}(x, \xi).$$

Then the trivial solution of these equations is the recursion of the theorem. It is easy to check that $\sigma_{B_N} \in S_{\rho, \delta}^{-\mu - (\rho - \delta)N}(\bar{\Omega} \times \mathcal{I})$. Thus, $B \sim \sum_{k=0}^\infty B_k$. ■

Theorem 11.1 applied to the parametrix from in Theorem 12.1, implies the inverse inclusion to the singular supports from Corollary 11.2 for elliptic operators.

Corollary 12.2. Let $1 \geq \rho > \delta \geq 0$ and assume that $\sigma_A \in S_{\rho, \delta}^\mu(\bar{\Omega} \times \mathcal{I})$ is L -elliptic. Then for every $w \in \mathcal{D}'_L(\Omega)$, we have

$$\text{sing supp } Aw = \text{sing supp } w. \quad \square$$

13 Sobolev Embedding Theorem

In this section, we prove an example of a Sobolev embedding theorem for Sobolev spaces \mathcal{H}_L^s associated to L , considered in Section 6. However, only limited conclusions are possible in the abstract setting when no further specifics about L are available. Now, let $C(\Omega)$ be the Banach space under the norm

$$\|f\|_{C(\Omega)} := \sup_{x \in \Omega} |f(x)|.$$

We recall that we have a differential operator L of order m with smooth coefficients in the open set $\Omega \subset \mathbb{R}^n$, and also the operator L° from (1.7).

The following theorem is conditional to the local regularity estimate (13.1). It is satisfied with $\varkappa = 1$ if, for example, L is locally elliptic, that is, elliptic in the classical sense of \mathbb{R}^n . However, if L is, for example, a sum of squares satisfying Hörmander’s commutator condition, the number $\varkappa \geq 1$ may depend on the order to which the Hörmander condition is satisfied, see, for example, [28] in the context of compact Lie groups.

Theorem 13.1. Let k be an integer such that $k > n/2$. Let \varkappa be such that the operators L and L° satisfy the inequality

$$\left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{L^2(\Omega)} \leq C \|(I + L^\circ L)^{\frac{\varkappa k}{2m}} f\|_{L^2(\Omega)} \tag{13.1}$$

for all $f \in C^\infty(\Omega)$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Then we have the continuous embedding

$$\mathcal{H}_L^{\alpha,k}(\Omega) \hookrightarrow C(\Omega). \quad \square$$

Proof. By the local Sobolev embedding theorem at $x \in \Omega$, for $|\alpha| \leq k$, we have

$$|f(x)| \leq C \left(\sum_{|\alpha| \leq k} \int_{\Omega} \left| \frac{\partial^\alpha}{\partial Y^\alpha} f(y) \right|^2 dy \right)^{1/2}.$$

Thus, considering $f \in C^\infty(\Omega)$ without loss of generality due to the density of $C^\infty(\Omega)$, and for all $x \in \Omega$, in view of the assumption (13.1) we have

$$\begin{aligned} |f(x)| &\leq C \left(\sum_{|\alpha| \leq k} \int_{\Omega} \left| \frac{\partial^\alpha}{\partial Y^\alpha} f(y) \right|^2 dy \right)^{1/2} \\ &\leq C \left(\int_{\Omega} |(I + L \circ L)^{\frac{\alpha k}{2m}} f(y)|^2 dy \right)^{1/2} \\ &\leq C \left(\sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2\alpha k} \widehat{f}(\xi) \overline{\widehat{f}^*(\xi)} \right)^{1/2} = C \|f\|_{\mathcal{H}_L^{\alpha,k}(\Omega)}. \end{aligned}$$

is true. Hence, we obtain the statement of the theorem. ■

14 Conditions for L^2 -Boundedness

In this section, we will discuss what conditions on the L -symbol a guarantee the L^2 -boundedness of the corresponding pseudo-differential operator $\text{Op}_L(a) : C_L^\infty(\bar{\Omega}) \rightarrow \mathcal{D}'_L(\Omega)$.

Theorem 14.1. Let k be an integer $> n/2$. Let $a : \bar{\Omega} \times \mathcal{I} \rightarrow \mathbb{C}$ be such that

$$|\partial_x^\alpha a(x, \xi)| \leq C \quad \text{for all } (x, \xi) \in \bar{\Omega} \times \mathcal{I}, \tag{14.1}$$

and all $|\alpha| \leq k$, all $x \in \Omega$ and $\xi \in \mathcal{I}$. Then the operator $\text{Op}_L(a)$ extends to a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$. □

Proof. Let us define an operator A_Y by

$$A_Y f(x) := \sum_{\xi \in \mathcal{I}} \int_{\Omega} u_\xi(x) \overline{v_\xi(z)} a(y, \xi) f(z) dz,$$

so that $A_x f(x) = \text{Op}_L(a) f(x)$. Then

$$\|\text{Op}_L(a) f\|_{L^2(\Omega)}^2 = \int_{\Omega} |A_x f(x)|^2 dx \leq \int_{\Omega} \sup_{y \in \Omega} |A_y f(x)|^2 dx,$$

and by an application of the local Sobolev embedding theorem we get

$$\sup_{y \in \Omega} |A_y f(x)|^2 \leq C \sum_{|\alpha| \leq k} \int_{\Omega} |\partial_y^\alpha A_y f(x)|^2 dy.$$

Therefore, using the Fubini theorem to change the order of integration, we obtain

$$\begin{aligned} \|\text{Op}_L(a) f\|_{L^2(\Omega)}^2 &\leq C \sum_{|\alpha| \leq k} \int_{\Omega} \int_{\Omega} |\partial_y^\alpha A_y f(x)|^2 dx dy \\ &\leq C \sum_{|\alpha| \leq k} \sup_{y \in \Omega} \int_{\Omega} |\partial_y^\alpha A_y f(x)|^2 dx \\ &= C \sum_{|\alpha| \leq k} \sup_{y \in \Omega} \|\partial_y^\alpha A_y f\|_{L^2(\Omega)}^2 \\ &\leq C \sum_{|\alpha| \leq k} \sup_{y \in \Omega} \sup_{\xi \in \mathcal{I}} \|\partial_y^\alpha a(y, \xi)\|_{L^2(\Omega)}^2 \|f\|_{L^2(\Omega)}^2, \end{aligned}$$

using the L^2 -boundedness of multipliers with bounded symbols following from Lemma 4.3, completing the proof. ■

From a suitable adaption of the composition Theorem 10.10, using that by Proposition 10.1 the operators ∂_x^α and $D_x^{(\alpha)}$ can be expressed in terms of each other as linear combinations with smooth coefficients, we immediately obtain the result in Sobolev spaces.

Corollary 14.2. Let k be an integer $> n/2$. Let $\mu \in \mathbb{R}$ and let $a: \bar{\Omega} \times \mathbb{Z} \rightarrow \mathbb{C}$ be such that

$$|\partial_x^\alpha a(x, \xi)| \leq C \langle \xi \rangle^\mu \quad \text{for all } (x, \xi) \in \bar{\Omega} \times \mathcal{I}, \tag{14.2}$$

and for all α . Then operator $\text{Op}_L(a)$ extends to a bounded operator from $\mathcal{H}_L^s(\Omega)$ to $\mathcal{H}_L^{s-\mu}(\Omega)$, for any $s \in \mathbb{R}$. □

By using Theorem 12.1 and Corollary 14.2, we get the following theorem.

Theorem 14.3. Let A be an elliptic pseudo-differential operator with L -symbol $\sigma_A \in S^\mu(\bar{\Omega} \times \mathcal{I})$, $\mu \in \mathbb{R}$, and let $Au = f$ in Ω , $u \in \mathcal{H}_L^\infty(\Omega)$. Then we have the estimate

$$\|u\|_{\mathcal{H}_L^{s+\mu}(\Omega)} \leq C_{sN}(\|f\|_{\mathcal{H}_L^s(\Omega)} + \|u\|_{\mathcal{H}_L^{-N}(\Omega)})$$

for any $s, N \in \mathbb{R}$. □

Proof. Since A is an elliptic pseudo-differential operator with L -symbol $\sigma_A \in S^\mu(\bar{\Omega} \times \mathcal{I})$, by Theorem 12.1 there exists a pseudo-differential operator A^\sharp with symbol $\sigma_{A^\sharp} \in S^{-\mu}(\bar{\Omega} \times \mathcal{I})$, such that

$$AA^\sharp = A^\sharp A = I + R,$$

where $R \in \text{Op}_L(S^{-\infty})$. Thus,

$$A^\sharp f = A^\sharp Au = (I + R)u$$

and

$$u = A^\sharp f - Ru.$$

Now, for $f \in \mathcal{H}_L^s(\Omega)$ we have $A^\sharp f \in \mathcal{H}_L^{s+\mu}(\Omega)$. As $u \in \mathcal{H}_L^\infty(\Omega)$, there exists $s_0 \in \mathbb{R}$ such that $u \in \mathcal{H}_L^{s_0}(\Omega)$. Then $Ru \in C_L^\infty(\bar{\Omega})$. Hence, we have

$$u = (A^\sharp f - Ru) \in \mathcal{H}_L^{s+\mu}(\Omega).$$

By using Corollary 14.2, we complete the proof. ■

Funding

The first author was supported in parts by the EPSRC grant EP/K039407/1 and by the Leverhulme Grant RPG-2014-02. No new data was collected or generated during the course of the research. The work was also supported by the MESRK Grant 0773/GF4 of the Committee of Science, Ministry of Education, and Science of the Republic of Kazakhstan. Funding to pay the Open Access publication charges for this article was provided by EPSRC.

Acknowledgements

The authors would like to thank Julio Delgado for discussions. Applications of the approach of this paper to Schatten classes and nuclearity properties as well as estimates on eigenvalue asymptotics for boundary value problems will appear in [18].

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