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Kernel and symbol criteria for Schatten classes and r-nuclearity on compact manifolds

Critères portant sur des symboles et noyaux pour les classes de Schatten et r-nucléarité sur les variétés compactes

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ABSTRACT

In this Note, we present criteria on both symbols and integral kernels ensuring that the corresponding operators on compact manifolds belong to Schatten classes. A specific test for nuclearity is established as well as the corresponding trace formulae. In the special case of compact Lie groups, kernel criteria in terms of (locally and globally) hypoelliptic operators are also given. A notion of invariant operator and its full symbol associated with an elliptic operator are introduced. Some applications to the study of r-nuclearity on L^p spaces are also obtained.

RÉSUMÉ

Nous présentons dans cette Note des critères sur des symboles et noyaux pour s'assurer de ce que les opérateurs correspondants sur des variétés compactes appartiennent à une classe de Schatten. Les opérateurs à trace sont considérés comme un cas spécial. Nous introduisons aussi des notions d'opérateur invariant et de symbole global associés à un opérateur elliptique et les appliquons à l'etude de la nucléarité.

Version française abrégée

À un opérateur elliptique positif E sur une variété compacte M munie d'une mesure positive dx, nous avons d'abord, dans [10], associé une analyse de Fourier discrète et des espaces de Sobolev, puis nous avons utilisé ces espaces pour formuler des conditions suffisantes optimales sur des noyaux pour s'assurer de ce que les opérateurs intégraux correspondants appartenaient à une classe de Schatten. Ces conditions ont été obtenues en appliquant la méthode de factorisation bien connue ainsi que des estimations optimales pour l'appartenance des fractions négatives de $(I + E)$ aux idéaux de Schatten.

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En particulier, pour les opérateurs à trace, nous proposons des formulations pour la trace, en utilisant un processus de moyennisation du noyau sur la diagonale.

Dans [8], nous avons aussi introduit une notion d'opérateur invariant relatif à E et développé une notion de *symbol matriciel global* associé aux décompositions de l'espace $L^2(M, \mathrm{d}x)$ en espaces propres de l'opérateur E. Nous caractérisons les opérateurs invariants appartenant aux classes de Schatten en termes de leur symboles, donnons des conditions suffisantes pour la nucléarité au sens de Grothendieck sur les espaces de Lebesgue, et incluons quelques exemples. Nous prouvons aussi que la définition de l'invariance introduite ici prolonge celle connue sur les groupes de Lie compacts et que les résultats concernant les opérateurs invariants englobent ceux présentés dans [7] et [9].

1. Introduction

Let M be a closed smooth manifold (smooth manifold without boundary) endowed with a positive measure $\mathrm{d}x$. We denote by $\Psi^\nu(M)$ the usual Hörmander class of pseudo-differential operators of order $\nu \in \mathbb{R}$. In this paper, we will be using the class $\Psi^\nu_{\mathrm{cl}}(M)$ of classical operators (see, e.g., [11]). Furthermore, we denote by $\Psi^\nu_+(M)$ the class of positive definite operators in $\Psi^\nu_{\mathrm{cl}}(M)$, and by $\Psi^\nu_{\mathrm{e}}(M)$ the class of elliptic operators in $\Psi^\nu_{\mathrm{cl}}(M)$. Finally, $\Psi^\nu_{+\mathrm{e}}(M) := \Psi^\nu_+(M) \cap \Psi^\nu_{\mathrm{e}}(M)$ will denote the class of classical positive elliptic pseudo-differential operators of order ν.

We associate a discrete Fourier analysis with the operator $E \in \Psi^\nu_{+\mathrm{e}}(M)$ inspired by that considered by Seeley [20,21]—see also [13]. The eigenvalues of E form a sequence $\{\lambda_j\}_{0\le j<\infty}$, with $\lambda_0 := 0$ and with multiplicities d_j. The corresponding orthonormal basis of $L^2(M, \mathrm{d}x)$, consisting of eigenfunctions of E, will be denoted by $\{e_j^k\}_{j\ge 0}^{1\le k\le d_j}$. Relative to this basis, Fourier coefficients, a Plancherel identity and a Fourier inversion formula can be obtained.

By introducing suitable Sobolev spaces on $M \times M$ adapted to E, we give first sufficient Sobolev-type conditions on Schwartz integral kernels in order to ensure that the corresponding integral operators belong to different Schatten classes. Second, we introduce notions of invariant operators, Fourier multipliers and full matrix-symbols relative to E. We apply those notions to characterise Schatten classes and to find sufficient conditions for nuclearity and r-nuclearity in the sense of Grothendieck [12] for invariant operators in terms of the full matrix-symbol.

The problem of finding such criteria on different kinds of domain is classical and has been much studied (cf. [3,7,9,23,4] and references therein).

2. Kernel conditions for Schatten classes on compact manifolds

We first define Sobolev spaces $H^{\mu_1,\mu_2}_{x,y}(M \times M)$ of mixed regularity $\mu_1, \mu_2 \ge 0$.

Definition 2.1. Let $K \in L^2(M \times M)$ and let $\mu_1, \mu_2 \ge 0$. For operators $E_j \in \Psi^{\nu_j}_{+\mathrm{e}}(M)$ $(j = 1, 2)$ with $\nu_j > 0$, we define:

$$K \in H^{\mu_1,\mu_2}_{x,y}(M \times M) \quad \Longleftrightarrow \quad (I + E_1)_x^{\frac{\mu_1}{\nu_1}} (I + E_2)_y^{\frac{\mu_2}{\nu_2}} K \in L^2(M \times M), \tag{1}$$

where the expression on the right-hand side means that we are applying pseudo-differential operators on M separately in x and y. We note that these operators commute since they are acting on different sets of variables of K.

By the elliptic regularity the spaces $H^{\mu_1,\mu_2}_{x,y}(M \times M)$ do not depend on a particular choice of E_1, E_2 as above. We will now give our main kernel criterion obtained in [10] for Schatten classes.

Theorem 2.1. *Let M be a closed manifold of dimension n and let $\mu_1, \mu_2 \ge 0$. Let $K \in L^2(M \times M)$ be such that $K \in H^{\mu_1,\mu_2}_{x,y}(M \times M)$. Then the corresponding integral operator T_K on $L^2(M)$ given by $T_K f(x) = \int_M K(x, y) f(y)\,\mathrm{d}y$ is in the Schatten classes $S_r(L^2(M))$ for $r > \frac{2n}{n+2(\mu_1+\mu_2)}$.*

We formulate below the results on the trace class. Due to possible singularities of the kernel $K(x, y)$ along the diagonal, we will require an averaging process on K, as described in [10] (see also [2,5]). We denote by $\widetilde{K}(x, y)$ the pointwise values of such process defined a.e. As a corollary of Theorem 2.1, for the trace class we have:

Corollary 2.2. *Let M be a closed manifold of dimension n and let $K \in L^2(M \times M)$, $\mu_1, \mu_2 \ge 0$, be such that $\mu_1 + \mu_2 > \frac{n}{2}$ and $K \in H^{\mu_1,\mu_2}_{x,y}(M \times M)$. Then the corresponding integral operator T_K on $L^2(M)$ is trace class and its trace is given by*

$$\mathrm{Tr}(T_K) = \int_M \widetilde{K}(x, x)\,\mathrm{d}x. \tag{2}$$

We also obtain several corollaries in terms of the derivatives of the kernel. We denote by $C_x^\alpha C_y^\beta(M \times M)$ the space of functions of class C^β with respect to y and C^α with respect to x.

Corollary 2.3. *Let M be a closed manifold of dimension n. Let $K \in C_x^{\ell_1} C_y^{\ell_2}(M \times M)$ for some even integers $\ell_1, \ell_2 \in 2\mathbb{N}_0$ such that $\ell_1 + \ell_2 > \frac{n}{2}$. Then T_K is in $S_1(L^2(M))$, and its trace is given by*

$$\operatorname{Tr}(T_K) = \int_M K(x,x)\,dx. \tag{3}$$

The corollary above is sharp (cf. Remark 4.5 of [10]) as a consequence of classical results for the convergence of Fourier series on the torus (cf. [22], Ch. VII; [25]).

We now formulate some consequences on compact Lie groups combining results from [7]. For example, on the compact Lie group SU(2), let X, Y, Z be three left-invariant vector fields X, Y, Z such that $[X, Y] = Z$ (for example, these would be derivatives with respect to Euler angles at a point extended to the whole of SU(2) by the left-invariance). Let $\mathcal{L}_{\text{sub}} = X^2 + Y^2$ be the sub-Laplacian. Then we have:

$$0 < r < \infty \quad \text{and} \quad \alpha r > 4 \quad \Longrightarrow \quad (I - \mathcal{L}_{\text{sub}})^{-\alpha/2} \in S_r\big(L^2\big(\mathrm{SU}(2)\big)\big). \tag{4}$$

The same is true for $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ considered as the compact Lie group with the quaternionic product. Using this instead of elliptic operators, we can show:

Corollary 2.4. *Let $K \in L^2(\mathbb{S}^3 \times \mathbb{S}^3)$ be such that we have $(1 - \mathcal{L}_{\text{sub}})_x^{\mu_1/2}(1 - \mathcal{L}_{\text{sub}})_y^{\mu_2/2} K \in L^2(\mathbb{S}^3 \times \mathbb{S}^3)$ for some $\mu_1, \mu_2 \geq 0$. Then T_K is in $S_r(L^2(\mathbb{S}^3))$ for $r > \frac{4}{2+\mu_1+\mu_2}$. The same result holds on the compact Lie groups* SU(2) *and* SO(3).

We now argue that instead of the sub-Laplacian other globally hypoelliptic operators can be used, also those that are not necessarily covered by Hörmander's sum of the squares theorem. We will formulate this for the group SO(3), noting that, however, the same conclusion holds also on $\mathrm{SU}(2) \simeq \mathbb{S}^3$. We fix three left-invariant vector fields X, Y, Z on SO(3) associated with the derivatives with respect to the Euler angles, so that we also have $[X, Y] = Z$, see [17] or [18] for the detailed expressions. We consider the following family of 'Schrödinger' differential operators

$$\mathcal{H}_\gamma = iZ - \gamma\big(X^2 + Y^2\big),$$

for a parameter $0 < \gamma < \infty$. For $\gamma = 1$, it was shown in [19] that $\mathcal{H}_1 + cI$ is globally hypoelliptic if and only if $0 \notin \{c + \ell(\ell+1) - m(m+1) : \ell \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq \ell\}$. It has been also shown in [7, Section 4] that, if $\gamma > 1$, then $I + \mathcal{H}_\gamma$ is globally hypoelliptic, and

$$(I + \mathcal{H}_\gamma)^{-\alpha/2} \in S_p \quad \text{if and only if } \alpha p > 4.$$

As a consequence of this and following the argument in [10] for the proof of Theorem 2.1 with $I + \mathcal{H}_\gamma$ instead of $E = \Delta_M$ for the manifold $M = \mathrm{SO}(3)$, as well as Corollary 2.4, we obtain:

Corollary 2.5. *Let $K \in L^2(\mathrm{SO}(3) \times \mathrm{SO}(3))$ be such that $(I + \mathcal{H}_\gamma)_x^{\mu_1/2}(I + \mathcal{H}_\gamma)_y^{\mu_2/2} K \in L^2(\mathrm{SO}(3) \times \mathrm{SO}(3))$ for some $\mu_1, \mu_2 \geq 0$. Then the integral operator T_K on $L^2(\mathrm{SO}(3))$ is in S_r for $r > \frac{4}{2+\mu_1+\mu_2}$ and $\gamma > 1$.*

3. Symbols, Fourier multipliers and nuclearity

Let us now consider the concepts of invariant operator and corresponding full symbols introduced in [8]. The eigenvalues of $E \in \Psi_{+e}^{\nu}(M)$ (counted without multiplicities) form a sequence $\{\lambda_j\}$ which we order so that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots. \tag{5}$$

For each eigenvalue λ_j, there is the corresponding finite dimensional eigenspace H_j of functions on M, which are smooth due to the ellipticity of E. We set:

$$d_j := \dim H_j \quad \text{and} \quad H_0 := \ker E.$$

We also set $d_0 := \dim H_0$. Since the operator E is elliptic, it is Fredholm, hence also $d_0 < \infty$ (we can refer to [1,15] for various properties of H_0 and d_0). We fix an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of E:

$$\{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}, \tag{6}$$

where $\{e_j^k\}^{1\leq k\leq d_j}$ is an orthonormal basis of H_j. Let $P_j : L^2(M) \to H_j$ be the corresponding projection.

The Fourier coefficients of $f \in L^2(M)$ with respect to the orthonormal basis $\{e_j^k\}$ will be denoted by

$$(\mathcal{F}f)(j,k) := \hat{f}(j,k) := \big(f, e_j^k\big). \tag{7}$$

We will call the collection of $\hat{f}(j,k)$ the *Fourier coefficients of f relative to E*, or simply the *Fourier coefficients of f*. If $f \in L^2(M)$, we also write

$$\hat{f}(j) = \begin{pmatrix} \hat{f}(j,1) \\ \vdots \\ \hat{f}(j,d_j) \end{pmatrix} \in \mathbb{C}^{d_j},$$

thus thinking of the Fourier transform always as a column vector. The following theorem proved in [8] is the base to introduce the concepts of invariant operators and full symbols relative to E.

Theorem 3.1. *Let M be a closed manifold and let $T : C^\infty(M) \to L^2(M)$ be a linear operator. Then the following conditions are equivalent:*

(i) *For each $j \in \mathbb{N}_0$, we have $T(H_j) \subset H_j$.*
(ii) *For each $j \in \mathbb{N}_0$ and $1 \le k \le j$, we have $TEe_j^k = ETe_j^k$.*
(iii) *For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that for all e_j^k*

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk}\delta_{j\ell}. \tag{8}$$

(iv) *For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that*

$$\widehat{Tf}(\ell) = \sigma(\ell)\hat{f}(\ell)$$

for all $f \in C^\infty(M)$.
The matrices $\sigma(\ell)$ in (iii) *and* (iv) *coincide. If T extends to a linear continuous operator $T : \mathcal{D}'(M) \to \mathcal{D}'(M)$, then the above properties are also equivalent to the following ones:*
(v) *for each $j \in \mathbb{N}_0$, we have $TP_j = P_jT$ on $C^\infty(M)$,*
(vi) *$TE = ET$ on $L^2(M)$.*

If any of the equivalent conditions (i)–(iv) of Theorem 3.1 are satisfied, we say that the operator $T : C^\infty(M) \to L^2(M)$ is *invariant (or is a Fourier multiplier) relative to E*. We can also say that T is E-invariant or is an E-multiplier. When there is no risk of confusion, we will just refer to such kind of operators as invariant operators or as multipliers. If T extends to a linear continuous operator $T : \mathcal{D}'(M) \to \mathcal{D}'(M)$, then we will say that T is *strongly invariant relative to E*.

The proposition below shows how invariant operators can be expressed in terms of their symbols.

Proposition 3.1. *An invariant operator T_σ associated with the symbol σ can be written in the following way:*

$$T_\sigma f(x) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell}\big(\sigma(\ell)\hat{f}(\ell)\big)_m e_\ell^m(x) = \sum_{\ell=0}^{\infty}\big[\sigma(\ell)\hat{f}(\ell)\big]^\top e_\ell(x),$$

where $[\sigma(\ell)\hat{f}(\ell)]$ denotes the column vector, and $[\sigma(\ell)\hat{f}(\ell)]^\top e_\ell(x)$ denotes the multiplication (the scalar product) of the column vector $[\sigma(\ell)\hat{f}(\ell)]$ with the column vector $e_\ell(x) = (e_\ell^1(x), \cdots, e_\ell^m(x))^\top$. In particular, we also have:

$$\big(T_\sigma e_j^k\big)(x) = \sum_{m=1}^{d_j}\sigma(j)_{mk}e_j^m(x). \tag{9}$$

If $\|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$ grows polynomially in ℓ and $f \in C^\infty(M)$, the convergence in (9) is uniform.

We can now formulate our characterisation of the membership of invariant operators in Schatten classes:

Theorem 3.2. *Let $0 < r < \infty$. An invariant operator $T : L^2(M) \to L^2(M)$ is in $S_r(L^2(M))$ if and only if its symbol σ_T satisfies $\sum_{\ell=0}^{\infty}\|\sigma_T(\ell)\|_{S_r}^r < \infty$. Moreover*

$$\|T\|_{S_r(L^2(M))}^r = \sum_{\ell=0}^{\infty}\big\|\sigma_T(\ell)\big\|_{S_r}^r.$$

If an invariant operator $T : L^2(M) \to L^2(M)$ is in the trace class $S_1(L^2(M))$, then

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty}\mathrm{Tr}\big(\sigma_T(\ell)\big).$$

We now turn to some applications to the nuclearity on $L^p(M)$ spaces. Let F_1 and F_2 be two Banach spaces and $0 < r \le 1$, a linear operator T from F_1 into F_2 is called *r-nuclear* if there exist sequences (x'_n) in F'_1 and (y_n) in F_2 so that

$$Ax = \sum_n \langle x, x'_n \rangle y_n \quad \text{and} \quad \sum_n \|x'_n\|^r_{F'_1} \|y_n\|^r_{F_2} < \infty. \tag{10}$$

This notion, developed by Grothendieck [12], extends the notion of Schatten classes to the setting of Banach spaces.

In order to study nuclearity on $L^p(M)$ for a given compact manifold M of dimension n, we introduce a function $\Lambda(j,k;n,p)$ that controls the L^p-norms of the family of eigenfunctions $\{e^k_j\}$ of the operator E, i.e. we will suppose that $\Lambda(j,k;n,p)$ is such that we have the estimates:

$$\|e^k_j\|_{L^p(M)} \le \Lambda(j,k;n,p). \tag{11}$$

There are many things that can be said about the behaviour of $\Lambda(j,k;n,p)$ in different settings, see, e.g., results and discussions in [6,24,8].

We will use the following function $\tilde{p}$ for $1 \le p \le \infty$:

$$\tilde{p} := \begin{cases} 0, & \text{if } 1 \le p \le 2, \\ \frac{p-2}{p}, & \text{if } 2 < p < \infty, \\ 1, & \text{if } p = \infty. \end{cases} \tag{12}$$

For $1 \le p_1, p_2 \le \infty$ we denote their dual indices by $q_1 := p'_1$, $q_2 := p'_2$. The criterion for r-nuclearity now is:

Theorem 3.3. *Let $1 \le p_1, p_2 < \infty$ and $0 < r \le 1$. Let $T : L^{p_1}(M) \to L^{p_2}(M)$ be a strongly invariant linear continuous operator. Assume that its matrix-valued symbol $\sigma(\ell)$ satisfies:*

$$\sum_{\ell=0}^{\infty} \sum_{m,k=1}^{d_\ell} \left|\sigma(\ell)_{mk}\right|^r \Lambda(\ell,m;n,\infty)^{\tilde{p_2} r} \Lambda(\ell,k;n,\infty)^{\tilde{q_1} r} < \infty.$$

Then the operator $T : L^{p_1}(M) \to L^{p_2}(M)$ is r-nuclear.

In some cases it is possible to simplify the sufficient condition above when the control function $\Lambda(\ell,m;n,\infty)$ is independent of m. For instance a classical result (local Weyl law) due to Hörmander ([14, Theorem 5.1], [16, Chapter XXIX]) implies the following estimate:

Lemma 3.4. *Let M be a closed manifold of dimension n. Let $E \in \Psi^\nu_{+e}(M)$, then*

$$\|e^m_\ell\|_{L^\infty} \le C\lambda_\ell^{\frac{n-1}{2\nu}}. \tag{13}$$

Thus $\Lambda(\ell;n,\infty) = C\lambda_\ell^{\frac{n-1}{2\nu}}$ furnishes an example of Λ independent of m. For controls of type $\Lambda(\ell;n,\infty)$ we have a basis-independent condition:

Corollary 3.5. *Let $1 \le p_1, p_2 < \infty$ and $0 < r \le 1$. Let $T : L^{p_1}(M) \to L^{p_2}(M)$ be a strongly invariant formally self-adjoint continuous operator. Assume that its matrix-valued symbol $\sigma(\ell)$ satisfies:*

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|^r_{S_r} \Lambda(\ell;n,\infty)^{(\tilde{p_2}+\tilde{q_1})r} < \infty.$$

Then the operator $T : L^{p_1}(M) \to L^{p_2}(M)$ is r-nuclear. In particular, if its matrix-valued symbol $\sigma(\ell)$ satisfies:

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|^r_{S_r} \lambda_\ell^{\frac{(n-1)}{2\nu}(\tilde{p_2}+\tilde{q_1})r} < \infty, \tag{14}$$

then the operator $T : L^{p_1}(M) \to L^{p_2}(M)$ is r-nuclear.

We now give an example of the application of such results in the case of the sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$. We consider the Laplacian (the Casimir element) $E = -\mathcal{L}_{\mathbb{S}^3}$.

Corollary 3.6. *If $\alpha > \frac{3}{r} + \frac{1}{2}(\tilde{p_2} + \tilde{q_1})$, $0 < r \le 1$, $1 \le p_1, p_2 < \infty$, the operator $(I - \mathcal{L}_{\mathbb{S}^3})^{-\frac{\alpha}{2}}$ is r-nuclear from $L^{p_1}(\mathbb{S}^3)$ into $L^{p_2}(\mathbb{S}^3)$.*

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