GLOBAL $L^2$-BOUNDEDNESS THEOREMS FOR A CLASS OF FOURIER INTEGRAL OPERATORS

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Abstract. The local $L^2$-mapping property of Fourier integral operators has been established in Hörmander [15] and in Eskin [13]. In this paper, we treat the global $L^2$-boundedness for a class of operators that appears naturally in many problems. As a consequence, we improve known global results for several classes of pseudo-differential and Fourier integral operators, as well as extend previous results of Asada and Fujiwara [1] or Kumano-go [18]. As an application, we show a global smoothing estimate for generalized Schrödinger equations which extends the results of Ben-Artzi and Devinatz [2], Walther [29], and [30].

1. Introduction

We consider (Fourier integral) operators, which can be globally written in the form

$$ Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) \, dy \, d\xi \quad (x \in \mathbb{R}^n), $$

where $a(x,y,\xi)$ is an amplitude function and $\phi(x,y,\xi)$ is a real phase function of the form

$$ \phi(x,y,\xi) = x \cdot \xi + \varphi(y,\xi). $$

Note that, by the equivalence of phase function theorem, Fourier integral operators with the local graph condition can always be written in this form microlocally. Although, due to the nontriviality of the Maslov cohomology class, general globally defined Fourier integral operators can not be written in this form with a single globally defined real valued phase $\phi$, the class (1.1) nevertheless appears in a number of important applications. It will be clear below how such operators naturally arise in global smoothing problems if we use an adaptation of the Egorov theorem.

Local $L^2$ mapping property of the operator (1.1) has been established in Hörmander [15] and in Eskin [13]. One of the aims of this paper is to establish global $L^2$-boundedness properties of operators (1.1). Analogous properties can be then easily obtained for adjoint operators as well.

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We will try to make as few assumptions as possible in the spirit of global $L^2$-estimates for pseudo-differential operators (see Calderón–Vaillancourt [6], Childs [8], Coifman-Meyer [9], Cordes [10]). In fact, due to the progress on $L^2$-bounds for pseudo-differential operators, Corollary 2.4 will not only extend these $L^2$-boundedness results to more general operators (1.1), but will also reduce the number of assumptions on the amplitude in the case of pseudo-differential operators, compared to the above mentioned papers (see also Sugimoto [27]).

Global $L^2$-boundedness of operators (1.1) has been previously studied by Asada-Fujiwara [1], Kumano-Go [18], Boulkhemair [5]. However, there one had to make a quite restrictive and not always natural assumption on the boundedness of $\partial_\xi \partial_\xi \phi$, which fails in many important cases. In Coriasco [12] and Boggia-Buzano-Rodino [4] such results are applied to obtain global estimates of solutions to some classes of hyperbolic equations. Here again they required quite strong decay properties of derivatives of both phase and amplitude. We will remove all these assumptions and will give more general $L^2$-estimates. In fact, in global estimates of Section 2 we will actually impose only a finite number of conditions on the phase and the amplitude, compared to infinitely many in the above mentioned papers, by keeping track of the number of derivatives.

As a consequence of our $L^2$-estimates, we can treat canonical transforms. Operators that appear there are of the form (1.1) with phase function

$$\phi(x, y, \xi) = x \cdot \xi - y \cdot p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|},$$

where $p(\xi)$ is a positively homogeneous function of degree one. If we take $p(\xi) = |\xi|$, then we have $\phi(x, y, \xi) = x \cdot \xi - y \cdot \xi$, and operator $T$ defined by (1.1) is a pseudo-differential operator. Furthermore, operator $T$ with general (1.2) is used to transform the Fourier multiplier $L_p = p(D_x)^2 = \mathcal{F}_\xi^{-1} p(\xi)^2 \mathcal{F}_x$ to the Laplacian $-\Delta$, where $\mathcal{F}_x$ ($\mathcal{F}_\xi^{-1}$ resp.) denotes the (inverse resp.) Fourier transform. In fact, we have a relation

$$T \cdot (-\Delta) \cdot T^{-1} = L_p$$

under a certain condition on $p(\xi)$ if we take 1 as the amplitude function $a(x, y, \xi)$ (see Section 4). The $L^2$-property of the Laplacian is well known in various situations. Our objective is to know the $L^2$-property of operator $T$, so that we can extract the $L^2$-property of operator $L_p$ from that of the Laplacian. This approach allows us to give a general treatment of several smoothing problems, including those treated by e.g. Ben-Artzi and Klainerman [3], Simon [24], Kato and Yajima [16], or Walther [29].

We should mention here that the global $L^2$-boundedness for the mentioned example (1.2) is not covered by previous results, for example, Asada and Fujiwara [1], Kumano-go [18], or Boulkhemair [5]. The basic reason is that in
the case of (1.2) the second order derivative \(\partial_\xi \partial_\xi \phi(x, y, \xi)\) is a product of \(y\) with a homogeneous of order \(-1\) function of \(\xi\), so it is not globally bounded on \(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\). This makes the above mentioned results immediately non applicable.

For example, the result of [1] is motivated by the construction of fundamental solution of Schrödinger equation in the way of Feynman’s path integral, and it requires the boundedness of all the derivatives of entries of the matrix

\[
\begin{pmatrix}
\partial_x \partial_y \phi & \partial_x \partial_\xi \phi \\
\partial_\xi \partial_y \phi & \partial_\xi \partial_\xi \phi
\end{pmatrix}.
\]

For the details, see [1] and references cited there. With our example (1.2), this assumption fails (since the boundedness of the entries of \(\partial_\xi \partial_\xi \phi\) fails).

In this paper, we develop a different global \(L^2\)-theory which does not require these decay or boundedness assumptions. In particular, it includes the case of example (1.2).

For \(m \in \mathbb{R}\), let \(L^2_m(\mathbb{R}^n)\) be the set of measurable functions \(f\) such that the norm

\[
\|f\|_{L^2_m(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \langle x \rangle^m |f(x)|^2 \, dx \right)^{1/2}; \quad \langle x \rangle^m = (1 + |x|^2)^{m/2}
\]

is finite. The following is a simplified version of our main result (Theorem 3.1):

**Theorem 1.1.** Let operator \(T\) be defined by (1.1), i.e. let

\[
Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(y, \xi))} a(x, y, \xi) u(y) \, dy \, d\xi,
\]

where \(\phi(y, \xi) \in C^\infty(\mathbb{R}_y^n \times \mathbb{R}_\xi^n)\) is a real-valued function, and amplitude function \(a(x, y, \xi) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n)\). Assume that

\[
|\det \partial_y \partial_\xi \phi(y, \xi)| \geq C > 0,
\]

and all the derivatives of entries of \(\partial_y \partial_\xi \phi\) are bounded. Also assume that

\[
\begin{align*}
|\partial_\xi^\alpha \phi(y, \xi)| & \leq C_\alpha \langle y \rangle & \text{for all } |\alpha| \geq 1, \\
|\partial_\xi^\alpha \partial_\xi^\beta \partial_\xi^\gamma a(x, y, \xi)| & \leq C_{\alpha \beta \gamma} \langle x \rangle^{-|\alpha|} & \text{for all } \alpha, \beta, \text{ and } \gamma.
\end{align*}
\]

Then \(T\) is bounded on \(L^2_m(\mathbb{R}^n)\) for any \(m \in \mathbb{R}\).
This theorem says that, if amplitude functions $a(x, y, \xi)$ have some decaying properties with respect to $x$, we do not need the boundedness of $\partial_y \partial_\xi \phi$ for the $L^2$-boundedness, as required in the above mentioned papers, and we can have weighted estimates, as well. (The same is true when both phase and amplitude functions have some decaying properties with respect to $y$. See Theorem 3.1.) In particular, assumption (1.3) does allow for growth in $y$ and does not require any decay of derivatives with respect to $\xi$. Note also that assumption (1.4) is automatically satisfied if the amplitude $a$ is independent of $x$ while being bounded in the other variables (together with its derivatives). In particular, it does imply the weighted $L^2$-boundedness for adjoint operators, which are of the form

$$Su(y) = \int_{\mathbb{R}^n} e^{i\varphi(y, \xi)} a(y, \xi) \hat{u}(\xi) d\xi,$$

and which appear, for example, as solutions to first order hyperbolic equations. The $L^2$-boundedness of adjoint operators $S^*$ is also in Theorem 2.5 (and hence also of $S$; this is covered by Theorem 2.1 as well).

We also note that it is enough to reduce assumptions of Theorem 1.1 to a finite number of derivatives. We keep track of this number and give a more general version of this result in Theorem 3.1 and Remark 3.1.

We now explain the plan of this paper. In Section 2, we show the global $L^2$-boundedness of a class of oscillatory integral operators, which generalizes a standard local result explained in Stein [25]. By using it, we prove various type of the $L^2$-boundedness of Fourier integral operators. Some of them are extension of previous results on the $L^2$-boundedness of pseudo-differential operators with non-regular symbols. It is worth mentioning that, in general, we do not necessarily need the standard homogeneity assumption for the phase function in the frequency variable. In addition, we impose the boundedness condition on only a finite number of the derivatives of phase functions, instead of infinitely many as in [1] and [18].

In Section 3, we state and prove our main result Theorem 3.1. We remark that it (together with Theorem 2.5) substantially weakens the assumptions for the $L^2$-boundedness of SG pseudo-differential (as in Cordes [11]) and SG Fourier integral operators (as in Coriasco [12]). These operators are used to handle the SG hyperbolic partial differential equations (roughly speaking, certain equations with coefficients of polynomial growth). The class of symbols $SG^{m_1, m_2}$ is defined as a space of smooth functions $a = a(y, \xi) \in C^\infty(R^n_y \times R^n_\xi)$ satisfying the estimate

$$|\partial_y^\beta \partial_\xi^\gamma a(y, \xi)| \leq C_{\beta, \gamma} (\langle y \rangle^{m_1 - |\beta|} \langle \xi \rangle^{m_2 - |\gamma|})$$

for all $\beta$ and $\gamma$. SG Fourier integral operators are operators of the form

$$Tu(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(y, \xi) u(y) d\xi dy$$
(or its adjoint), where \( a \in SG^{m_1,m_2} \) and \( \varphi \in SG^{1,1} \), which also satisfies
\[
C_1 \langle y \rangle \leq \langle \partial_\xi \varphi(y,\xi) \rangle \leq C_2 \langle y \rangle, \quad C_1 \langle \xi \rangle \leq \langle \partial_y \varphi(y,\xi) \rangle \leq C_2 \langle \xi \rangle,
\]
for some \( C_1, C_2 > 0 \). A result in [11] for SG pseudo-differential and its extension in [12] for SG Fourier integral operators states that under these assumptions on the phase \( \phi \), and for \( a \in SG^{0,0} \), the corresponding operator \( T \) is bounded on \( L^2(\mathbb{R}^n) \). Without going much into detail, let us mention here that statements of our results replace the strong decay assumptions \( \phi \in SG^{1,1}, a \in SG^{0,0} \), by (a finite number of) boundedness conditions, for \( T \) to be still bounded in \( L^2(\mathbb{R}^n) \).

In Section 4, we exhibit an example of how to use our main result. We mainly focus on the problem of global smoothing property of generalized Schrödinger equations
\[
(1.5) \quad \begin{cases}
(i\partial_t + Q(D))u(t,x) = 0, \\
u(0,x) = f(x).
\end{cases}
\]
Ben-Artzi and Devinatz [2] showed a global smoothing estimate to equation (1.5), where the symbol \( Q(\xi) \) of \( Q(D) \) is a real polynomial of principal type. Walther [30] considered the case of radially symmetric \( Q(\xi) \). By using our result Theorem 3.1, we can treat more general case (see Theorems 4.2 and 4.3). More refined applications to this subject will be shown in our forthcoming papers [22] and [23].

In subsequent work [21], we will establish properties of operators (1.1) in weighted Sobolev spaces, which will have several further applications of these results to hyperbolic equations as well as global canonical transforms.

### 2. Global \( L^2 \)-estimates

First of all, we confirm a basic result on the \( L^2 \)-boundedness of a class of oscillatory integral operators, based on the argument of Fujiwara [14], which is a global version of a proposition in Stein [25, p.377].

Here and hereafter, the capital \( C \) (sometimes with some suffixes) always denotes a positive constant which may differ on each occasion. We also note that we will use the standard notation \( \partial^\alpha_x = \partial^\alpha_{x_1} \cdots \partial^\alpha_{x_n} \). By \( \partial_x \) or by \( \nabla_x \), we will denote the gradient.

**Theorem 2.1.** Let \( \Gamma_x, \Gamma_y \subset \mathbb{R}^n \) be open cones. Let operator \( I_\varphi \) be defined by
\[
(2.1) \quad I_\varphi u(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,y)} a(x,y) u(y) \, dy,
\]
where \( a(x,y) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_y) \), \( \text{supp}\, a \subset \Gamma_x \times \Gamma_y \), and \( \varphi(x,y) \in C^\infty(\Gamma_x \times \Gamma_y) \) is a real-valued function. Assume that
\[
|\partial^\alpha_x \partial^\beta_y a(x,y)| \leq C_{\alpha\beta},
\]
for $|\alpha|, |\beta| \leq 2n + 1$. Also assume that
\begin{align*}
|\partial_y \varphi(x, y) - \partial_y \varphi(v, y)| \geq C|x - v| & \quad \text{for } x, v \in \Gamma_x, y \in \Gamma_y, \\
|\partial_x \varphi(x, y) - \partial_x \varphi(x, w)| \geq C|y - w| & \quad \text{for } x \in \Gamma_x, y, w \in \Gamma_y,
\end{align*}
and that
\begin{align*}
|\partial_x \partial_y \varphi(x, y)| \leq C_\alpha, \quad |\partial_x \partial_y \varphi(x, y)| \leq C_\beta & \quad \text{on } \text{supp } a
\end{align*}
for $1 \leq |\alpha|, |\beta| \leq 2n + 2$. Then operator $I_\varphi$ is $L^2(\mathbb{R}^n)$-bounded, and satisfies
\begin{align*}
\|I_\varphi\|_{L^2 \to L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n + 1} \|\partial_x \partial_y a(x, y)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.
\end{align*}

We note that the principal difference with previous $L^2$ results here is that we assume (2.3) only for mixed derivatives of the phase, i.e. we may exclude $\alpha = 0$ or $\beta = 0$ in this condition. Since we do not assume any homogeneity of phase $\varphi$, cones do not have any special meaning in this setting. However, we formulate it in this way for the convenience of future applications in Section 4, where we will need a microlocal version of this theorem and the phase will be homogeneous.

Remark 2.1. If $\Gamma_x = \Gamma_y = \mathbb{R}^n$, we may assume
\begin{align*}
|\det \partial_x \partial_y \varphi(x, y)| \geq C > 0, \quad \text{for all } x, y \in \mathbb{R}^n,
\end{align*}
instead of (2.2) since it follows from (2.3) and (2.4) by applying the mean value theorem to the global inverses of $\partial_y \varphi(\cdot, y)$ and $\partial_x \varphi(x, \cdot)$, which we get by the global inverse function theorem. In fact, this is the main case of Theorem 2.1. Condition (2.4) is the global version of the local graph condition, which is necessary even for the local $L^2$-boundedness of Fourier integral operators.

Proof of Theorem 2.1. Let $g \in C_0^\infty(\mathbb{R}^n)$ be a real-valued positive function such that $\{g_k(x)\}_{k \in \mathbb{Z}^n}$, where $g_k(x) = g(x - k)$, forms a partition of unity. We decompose the operator $I_\varphi$ as
\begin{align*}
I_\varphi = \sum_{(j,k) \in \mathbb{Z}^n \times \mathbb{Z}^n} I_{(j,k)},
\end{align*}
where $I_{(j,k)} = g_j I_\varphi g_k$, that is,
\begin{align*}
I_{(j,k)} u(x) = g_j(x) \int e^{i\varphi(x,z)} a(x, z) g_k(z) u(z) \, dz.
\end{align*}
We denote the adjoint of $I_{(j,k)}$ by $I^*_{(j,k)}$, that is,
\begin{align*}
I^*_{(j,k)} u(z) = g_k(z) \int e^{-i\varphi(y,z)} a(y, z) g_j(y) u(y) \, dy.
\end{align*}
Then we have
\begin{align*}
I_{(j,k)} I^*_{(l,m)} u(x) = \int K_{(j,k), (l,m)}(x, y) u(y) \, dy,
\end{align*}
where
\[ K_{(j,k),(l,m)}(x, y) = g_j(x)g_l(y) \int e^{i(\varphi(x,z) - \varphi(y,z))} a(x, z) \overline{a(y, z)} g_k(z)g_m(z) \, dz. \]

By integration by parts, we have
\[ \int e^{i(\varphi(x,z) - \varphi(y,z))} a(x, z) \overline{a(y, z)} g_k(z)g_m(z) \, dz = \int e^{i(\varphi(x,z) - \varphi(y,z))} L^{2n+1} \left( a(x, z) \overline{a(y, z)} g_k(z)g_m(z) \right) \, dz, \]

where \( L \) is the transpose of the operator
\[ tL = \frac{1}{i} \frac{\partial_z \varphi(x, z) - \partial_z \varphi(y, z)}{|\partial_z \varphi(x, z) - \partial_z \varphi(y, z)|^2} \partial_z. \]

From the assumptions we obtain
\[ |\partial_z \varphi(x, z) - \partial_z \varphi(y, z)| \geq C |x - y| \]
and
\[ |\partial^\beta_z \varphi(x, z) - \partial^\beta_z \varphi(y, z)| \leq C_\beta |x - y| \]
for \( 1 \leq |\beta| \leq 2n + 2 \) since we may assume that \( \Gamma_{x} \) and \( \Gamma_{y} \) are proper cones.

Hence, we have
\[ |K_{(j,k),(l,m)}(x, y)| \leq CA^2 \frac{g_j(x)g_l(y)}{1 + |x - y|^{2n+1}} h(k - m), \]

where \( h \in C_0^\infty(\mathbb{R}^n) \) is a positive function \( (h(x) = \int g(z - x)g(z) \, dz) \), and
\[ A = \sup_{|\alpha|,|\beta| \leq 2n+1} \|\partial^\alpha_x \partial^\beta_y a\|_{L^\infty(\mathbb{R}^n_2 \times \mathbb{R}^n_2)}. \]

Then we have
\[ \sup_x \int |K_{(j,k),(l,m)}(x, y)| \, dy \leq CA^2 \frac{h(k - m)}{1 + |j - l|^{2n+1}}, \]
\[ \sup_y \int |K_{(j,k),(l,m)}(x, y)| \, dx \leq CA^2 \frac{h(k - m)}{1 + |j - l|^{2n+1}}, \]
which implies
\[ \|I_{(j,k)} I_{(l,m)}\|_{L^2 \rightarrow L^2} \leq CA^2 \frac{h(k - m)}{1 + |j - l|^{2n+1}}. \]

Here we have used the following lemma (see Stein [25, p.284]):

**Lemma 2.1.** Suppose \( S \) is given by
\[ (Sf)(x) = \int s(x, y)f(y) \, dy, \]
where the kernel \( s(x, y) \) satisfies
\[ \sup_x \int |s(x, y)| \, dy \leq 1, \quad \sup_y \int |s(x, y)| \, dx \leq 1. \]
Then $\|S\|_{L^2 \to L^2} \leq 1$.

By the same discussion, we have

$$\|I_{(j,k)}^* I_{(l,m)}\|_{L^2 \to L^2} \leq CA^2 \frac{h(j-l)}{1 + |k-m|^{2n+1}}.$$  

Then we have

$$\|I_{(j,k)} I_{(l,m)}\|_{L^2 \to L^2}, \|I_{(j,k)}^* I_{(l,m)}\|_{L^2 \to L^2} \leq CA^2 \{\gamma(j-l, k-m)\}^2,$$

where

$$\gamma(j_1, j_2) = \sqrt{\frac{h(j_2)}{1 + |j_1|^{2n+1}} + \frac{h(j_1)}{1 + |j_2|^{2n+1}}}$$

and it satisfies the estimate

$$\sum_{(j_1, j_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \gamma(j_1, j_2) < \infty.$$  

We have the desired result, by the following Cotlar’s lemma (see Calderón and Vaillancourt [6], Stein [25, Chapter VII, Section 2]):

**Lemma 2.2.** Assume a family of $L^2$-bounded operators $\{T_j\}_{j \in \mathbb{Z}^r}$ and positive constants $\{\gamma(j)\}_{j \in \mathbb{Z}^r}$ satisfy

$$\|T_i^* T_j\|_{L^2 \to L^2} \leq \{\gamma(i-j)\}^2, \quad \|T_j T_j^*\|_{L^2 \to L^2} \leq \{\gamma(i-j)\}^2,$$

and

$$M = \sum_{j \in \mathbb{Z}^r} \gamma(j) < \infty.$$  

Then the operator

$$T = \sum_{j \in \mathbb{Z}^r} T_j$$

satisfies

$$\|T\|_{L^2 \to L^2} \leq M.$$  

□

By using Theorem 2.1 on oscillatory integral operators (2.1), we can easily show the $L^2$-boundedness of Fourier integral operators of special forms. Let us begin with the case when the amplitude $a(x, y, \xi)$ is independent of the variable $y$.

**Theorem 2.2.** Let operator $T$ be defined by

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(x, \xi) u(y) dy d\xi,$$

(2.5)
where \( a(x, \xi) \in C^\infty(\mathbb{R}^n_+ \times \mathbb{R}^n_+) \) and \( \varphi(y, \xi) \in C^\infty(\mathbb{R}^n_+ \times \mathbb{R}^n_+) \). Assume that the pseudo-differential operators \( a(X, D) \) defined by

\[
a(X, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x, \xi) u(y) \, dy \, d\xi
\]

and the oscillatory integral operator \( I_\varphi \) defined by

\[
I_\varphi u(\xi) = \int_{\mathbb{R}^n} e^{i\varphi(y, \xi)} u(y) \, dy
\]

are both \( L^2(\mathbb{R}^n) \)-bounded. Then \( T \) is \( L^2(\mathbb{R}^n) \)-bounded, and satisfies

\[
\|T\|_{L^2 \to L^2} \leq (2\pi)^{n/2} \|a(X, D)\|_{L^2 \to L^2} \cdot \|I_\varphi\|_{L^2 \to L^2}.
\]

**Proof.** We remark that \( T = (2\pi)^n a(X, D) \mathcal{F}^{-1} I_\varphi \), where \( \mathcal{F}^{-1} \) is the inverse Fourier transform. The \( L^2(\mathbb{R}^n) \)-boundedness of \( T \) is obtained from the assumptions and Plancherel’s theorem. \( \square \)

As a corollary, we have the result announced in Ruzhansky and Sugimoto [20] (the following Corollaries 2.3 and 2.4). Now we recall the definition of the Besov space \( B^{(s,s')}_{p,q}(\mathbb{R}^{2n}) \) for \( 0 < p, q \leq \infty \) and multi-indices \( (s, s') \), where \( s = (s_1, \ldots, s_N) \) and \( s' = (s'_1, \ldots, s'_{N'}) \). Let \( n = (n_1, \ldots, n_N) \), \( n' = (n'_1, \ldots, n'_{N'}) \) be splittings of \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_+ \), respectively:

\[
n = n_1 + \ldots + n_N = n'_1 + \ldots + n'_{N'}.
\]

Then \( f \in B^{(s,s')}_{p,q}(\mathbb{R}^{2n}) \) if \( f = f(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n}) \) and

\[
\|f\|_{B^{(s,s')}_{p,q}} = \left\{ \sum_{j,k \geq 0} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |2^j s + k s'| \mathcal{F}^{-1} \Phi_{j,k} \Phi f(x, \xi)|^p \, dx d\xi \right)^{q/p} \right\}^{1/q} < \infty.
\]

Here \( j = (j_1, \ldots, j_N) \), \( k = (k_1, \ldots, k_{N'}) \), \( \mathcal{F} \) is the Fourier transform with respect to \( (x, \xi) \), \( \mathcal{F}^{-1} \) is the inverse Fourier transform with respect to the dual variable \( (y, \eta) \), and \( \Phi_{j,k} = \Phi_{j,k}(y, \eta) = \Theta_{j_1}(y_1) \cdots \Theta_{j_N}(y_N) \Theta_{k_1}(\eta_1) \cdots \Theta_{k_{N'}}(\eta_{N'}) \). Here we split variables \( y, \eta \in \mathbb{R}^n_+ \) following the splitting \( n, n' \). Functions \( \Theta_i(z) \in \mathcal{S} \) form the dyadic system of the corresponding dimension: \( \text{supp} \Theta_0 \subset \{z; |z| \leq 2\} \), \( \text{supp} \Theta_i \subset \{z; 2^{-i-1} \leq |z| \leq 2^{i+1}\} \) for \( i \in \mathbb{N} \), \( \sum_{i=0}^{\infty} \Theta_i(z) = 1 \), and \( 2^{|i|} |\partial^i \Theta_i(z)| \leq C_\alpha \) for all \( i \geq 0 \) and all \( z \). A natural modification is needed for \( p, q = \infty \), see [28].

**Corollary 2.3.** Let \( 2 \leq p \leq \infty \). Let operator \( T \) be defined by

\[
Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y+\varphi(y, \xi))} a(x, \xi) u(y) \, dy \, d\xi,
\]

where \( \varphi(y, \xi) \in C^{\infty}(\mathbb{R}^n_+ \times \mathbb{R}^n_+) \) is a real-valued function, and amplitude \( a(x, \xi) \in B^{(1/2-1/p)}_{p,1}(\mathbb{R}^{2n}) \). Assume that

\[
|\det \partial_y \partial^\xi \varphi(y, \xi)| \geq C > 0
\]
and that

\[ |\partial_x^\alpha \partial_\xi \varphi(y, \xi)| \leq C_\alpha, \quad |\partial_y^\beta \partial_\xi \varphi(y, \xi)| \leq C_\beta \]

for \(1 \leq |\alpha|, |\beta| \leq 2n + 2\). Then \(T\) is \(L^2(\mathbb{R}^n)\)-bounded, and satisfies

\[ \|Tu\|_{L^2(\mathbb{R}^n)} \leq C \|a(x, \xi)\|_{B^{1/2-1/p}_{p,1}(\mathbb{R}^{n+1})} \|u\|_{L^2(\mathbb{R}^n)}. \]

**Proof.** The \(L^2\)-boundedness of \(T\) follows from Theorems 2.1, 2.2, and the fact that pseudo-differential operators \(a(X, D)\) with \(a(x, \xi) \in B^{1/2-1/p}_{p,1}(\mathbb{R}^{n+1})\) are \(L^2\)-bounded (see Sugimoto [27]). See also Remark 2.1. \(\square\)

Corollary 2.3 is rather general but its conditions may be hard to check. On the other hand, conditions of the corollary below can be checked in various situations.

**Corollary 2.4.** Let operator \(T\) be defined by

\[ Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-x+\varphi(y, \xi))} a(x, \xi) u(y) dy d\xi, \]

where \(\varphi(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) is a real-valued function. Assume that

\[ |\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0 \]

and that

\[ |\partial_y^\alpha \partial_\xi \varphi(y, \xi)| \leq C_\alpha, \quad |\partial_\xi^\beta \varphi(y, \xi)| \leq C_\beta \]

for \(1 \leq |\alpha|, |\beta| \leq 2n + 2\). Also assume one of the following conditions:

1. \(\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) for \(\alpha, \beta \in \{0, 1\}^n\).
2. \(\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) for \(|\alpha|, |\beta| \leq [n/2] + 1\).
3. \(\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) for \(|\alpha| \leq [n/2] + 1, \beta \in \{0, 1\}^n\).
4. \(\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) for \(\alpha \in \{0, 1\}^n, |\beta| \leq [n/2] + 1\).
5. There exist real numbers \(\lambda, \lambda' > n/2\) such that

\[ (1 - \Delta_x)^{\lambda/2}(1 - \Delta_\xi)^{\lambda'/2} a(x, \xi) \in L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi). \]

6. There exist a real number \(\lambda > 1/2\) and a constant \(C\) such that

\[ \|\delta_x^\alpha(h) \delta_\xi^\beta(h') a(x, \xi)\|_{L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)} \leq C \prod_{i,j=1}^{n} |h_i|^{\alpha_i \lambda}|h_j'|^{\beta_j \lambda} \]

holds for all \(\alpha, \beta \in \{0, 1\}^n\) and all \(h = (h_1, \ldots, h_n), h' = (h'_1, \ldots, h'_n) \in \mathbb{R}^n\). Here \(\delta_x^\alpha(h) = \delta_{x_1}^{\alpha_1}(h_1) \cdots \delta_{x_n}^{\alpha_n}(h_n)\) is the difference operator, with

\[ \delta_x^0(h_i) a(x, \xi) = a(x, \xi), \quad \delta_x^1(h_i) a(x, \xi) = a(x + h_i e_i, \xi) - a(x, \xi), \]

where \(e_i\) is the \(i\)-th standard basis vector in \(\mathbb{R}^n\). The definition of \(\delta_\xi^\beta\) is similar.

7. There exists a real number \(2 \leq p < \infty\) such that \(\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^p(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)\) for \(|\alpha|, |\beta| \leq [n(1/2 - 1/p)] + 1\).
Then $T$ is $L^2(\mathbb{R}^n)$-bounded.

Corollary 2.4 with $\varphi(y, \xi) = -y : \xi$ is a refined version of the following results on the $L^2$-boundedness of pseudo-differential operators with non-regular symbols: (1) with $\alpha, \beta \in \{0, 1, 2, 3\}^n$ is due to Calderón and Vaillancourt [6], (2) and (5) are due to Coifman and Meyer [9], (3) with $|\alpha| \leq [n/2] + 1, \beta \in \{0, 1, 2\}^n$, (7) with $\alpha \leq [n(1/2 - 1/p)] + 1, |\beta| \leq 2n$ are due to Cordes [10], the difference condition (6) is due to Childs [8], and conditions (3) with $|\alpha| \leq [n/2] + 1, \alpha \in \{0, 1, 2\}^n$, (7) with $\alpha \leq [n(1/2 - 1/p)] + 1, |\beta| \leq 2n$ are due to Coifman and Meyer [9].

Proof. The $L^2$-boundedness under all conditions follows from Corollary 2.3 with different choices of $p$ and splitting of the spaces. In fact, conditions (1)–(6) are obtained with $p = \infty$ $(N = N' = 1$ in conditions (2), (5); $N = N' = n$ in (1), (6); $N = 1, N' = n$ in (3), and $N = n, N' = 1$ in (4)). Condition (7) is obtained from Corollary 2.3 by taking the same $p$ and $N = N' = 1$. For more relations between symbol classes and Besov spaces, we refer to Sugimoto [27] and Triebel [28].

We note that there are other variants of global $L^2$-boundedness theorems for pseudo-differential operators. For each of these statements, Theorem 2.2 yields the corresponding statement for operators (2.5). Let us now briefly describe such a statement in uniformly local Sobolev–Kato spaces. For $s \in \mathbb{R}$ let $H^s_{ul}(\mathbb{R}^n)$ be the space of all tempered distributions $u$ such that its norm is finite:

$$
||u||_{s,ul} = \sup_{y \in \mathbb{R}^n} ||u \tau_y \chi||_{H^s} < \infty.
$$

Here $\chi \in C_0^\infty(\mathbb{R}^n)$ has nonzero integral and $\tau_y \chi(x) = \chi(x - y)$. Similarly, one can define the space $H^{s'}_{ul}(\mathbb{R}^n)$ by taking all tempered distribution $u$ with

$$
||u||_{s,s',ul} = \sup_{y \in \mathbb{R}^n \times \mathbb{R}^n} ||u \tau_y \chi||_{s,s'} < \infty,
$$

where $||v||^2_{s,s'} = \int_{\mathbb{R}^n \times \mathbb{R}^n} |\langle \xi \rangle^s \langle \xi' \rangle^{s'} \hat{v}(\xi, \xi')|^2 d\xi d\xi'$. We note that the space $L^2_{ul} = H^0_{ul}$ contains $L^\infty$, while spaces $H^s_{ul}$ and $H^{s'}_{ul}$ are subalgebras of $L^\infty$ for $s, s' > n/2$. For the more detailed discussion of these spaces one can consult [5], from which we also have the $L^2$-boundedness of pseudo-differential operators with symbols in $H^{s,s'}_{ul}(\mathbb{R}^n \times \mathbb{R}^n)$ with $s, s' > n/2$. By Theorem 2.2 we immediately see that if the phase function $\varphi$ satisfies assumptions of Corollary 2.4 and $a(x, \xi) \in H^{s,s'}_{ul}(\mathbb{R}^n \times \mathbb{R}^n)$ for $s, s' > n/2$, then operator (2.5) is bounded in $L^2(\mathbb{R}^n)$ with

$$
||Tu||_{L^2(\mathbb{R}^n)} \leq C||a(x, \xi)||_{s,s',ul} ||u||_{L^2(\mathbb{R}^n)}.
$$

We also note that results of [5] do not cover operators (2.5) in our setting directly because we do not assume the boundedness of $\partial_\xi \partial_\xi \varphi$.

Now we present a theorem for amplitudes which are independent of the variable $x$. 

Theorem 2.5. Let $\Gamma_y, \Gamma_\xi \subset \mathbb{R}^n$ be open cones. Let operator $T$ be defined by

$$ (2.6) \quad Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(y, \xi) u(y) dyd\xi, $$

where $a(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$, $\text{supp } a \subset \Gamma_y \times \Gamma_\xi$, and $\varphi(y, \xi) \in C^\infty(\Gamma_y \times \Gamma_\xi)$ is a real-valued function. Assume that

$$ |\partial_\alpha^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{\alpha\beta}, $$

for $|\alpha|, |\beta| \leq 2n + 1$. Also assume that

$$ |\partial_\xi \varphi(x, \xi) - \partial_\xi \varphi(y, \xi)| \geq C|x - y| \quad \text{for } x, y \in \Gamma_y, \xi \in \Gamma_\xi, $$

$$ |\partial_y \varphi(y, \xi) - \partial_y \varphi(y, \eta)| \geq C|\xi - \eta| \quad \text{for } y \in \Gamma_y, \xi, \eta \in \Gamma_\xi, $$

and that

$$ |\partial_\xi^\alpha \partial_\xi^\beta \varphi(y, \xi)| \leq C_{\alpha}, \quad |\partial_\xi^\alpha \partial_\xi^\beta \varphi(y, \xi)| \leq C_{\beta} \quad \text{on } \text{supp } a $$

for $1 \leq |\alpha|, |\beta| \leq 2n + 2$. Then the operator $T$ is $L^2(\mathbb{R}^n)$-bounded, and satisfies

$$ \|T\|_{L^2 \rightarrow L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n + 1} \left\| \partial_\xi^\alpha \partial_\xi^\beta a(y, \xi) \right\|_{L^\infty(\Gamma_y \times \Gamma_\xi)}. $$

Note that Remark 2.1 after Theorem 2.1 applies here as well.

Proof. We remark that $T = (2\pi)^n \mathcal{F}^{-1} I_\varphi$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform and $I_\varphi$ is the oscillatory integral operator defined by

$$ I_\varphi u(\xi) = \int e^{i\varphi(y, \xi)} a(y, \xi) u(y) dy. $$

The result is obtained from Theorem 2.1 and Plancherel’s theorem. \qed

As a corollary of Theorems 2.2 and 2.5, we have a result for amplitudes which are of the product type.

Corollary 2.6. Let operator $T$ be defined by

$$ Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(x, y, \xi) u(y) dyd\xi, $$

$$ a(x, y, \xi) = a_1(x, \xi) a_2(y) \quad \text{or } a(x, y, \xi) = a_2(x) a_1(y, \xi), $$

where $a_1 \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $a_2 \in L^\infty(\mathbb{R}^n)$, and $\varphi(y, \xi) \in C^\infty(\Gamma_y \times \Gamma_\xi)$ is a real-valued function. Assume that

$$ |\partial_\xi^\alpha \partial_\xi^\beta a_1(x, \xi)| \leq C_{\alpha\beta} $$

for $|\alpha|, |\beta| \leq 2n + 1$. Also assume that

$$ |\det \partial_\xi \partial_\xi \varphi(y, \xi)| \geq C > 0, $$

and that

$$ |\partial_\xi^\alpha \partial_\xi^\beta \varphi(y, \xi)| \leq C_{\alpha}, \quad |\partial_\xi^\alpha \partial_\xi^\beta \varphi(y, \xi)| \leq C_{\beta} $$

Note that Remark 2.1 after Theorem 2.1 applies here as well.
for $1 \leq |\alpha|, |\beta| \leq 2n + 2$. Then $T$ is $L^2(\mathbb{R}^n)$-bounded, and satisfies

$$\|T\|_{L^2 \rightarrow L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n+1} \left\| \partial_\gamma^\alpha \partial_\xi^\beta a(y, \xi) \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$ 

Proof. Note that $T$ is a product of the multiplication of the function $a_2$ and the operator defined by (2.5) or (2.6), which are all $L^2$-bounded by the assumption. □

3. Weighted $L^2$-estimates

Asada and Fujiwara [1] proved Corollary 2.6 without the product type assumption for $a(x, y, \xi)$, but assumed the boundedness of all the derivatives of $a(x, y, \xi)$ and that of each entry of the matrix $\partial_\xi \partial_\gamma \varphi$. The following theorem, which is a generalized version of Theorem 1.1, says that we do not need the boundedness assumption for $\partial_\xi \partial_\gamma \varphi$ if $a(x, y, \xi)$ has a certain decaying property. In this case, we have weighted estimates as well. Again, to allow more flexibility for applications, we formulate it in cones. Recall that for $m \in \mathbb{R}$, we use the notation

$$\langle x \rangle^m = (1 + |x|^2)^{m/2},$$

and let $L^2_m(\mathbb{R}^n)$ be the set of measurable functions $f$ such that the norm

$$\|f\|_{L^2_m(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2}$$

is finite.

**Theorem 3.1.** Suppose $m, m_1, m_2 \in \mathbb{R}$. Let $\Gamma_y, \Gamma_\xi \subset \mathbb{R}^n$ be open cones. Let operator $T$ be defined by

$$(3.1) \quad Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(x, y, \xi) u(y) \, dy \, d\xi,$$

where $a(x, y, \xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\xi)$, $\bigcup_{x \in \mathbb{R}^n} \text{supp}_y \xi a(x, \cdot, \cdot) \subset \Gamma_y \times \Gamma_\xi$, and $\varphi(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$ is a real-valued function. Assume that

$$(3.2) \quad |\partial_\xi \varphi(x, \xi) - \partial_\xi \varphi(y, \xi)| \geq C|x - y| \quad \text{for} \quad x, y \in \Gamma_y, \xi \in \Gamma_\xi,$$

$$(3.3) \quad |\partial_\gamma \varphi(y, \xi) - \partial_\gamma \varphi(y, \eta)| \geq C|\xi - \eta| \quad \text{for} \quad y \in \Gamma_y, \xi, \eta \in \Gamma_\xi.$$

Let $N$ be the minimal integer such that $N \geq \max \{|m| + n + 1, 2n + 2\}$, and also assume one of the following:

1. For all $|\alpha|, |\beta| \leq 2n + 1$ and $|\gamma| \leq N$,

$$\left| \partial_\gamma^\alpha \partial_\xi^\beta a(x, y, \xi) \right| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1 - |\alpha|} \langle y \rangle^{m_2},$$
and for all $1 \leq |\alpha| \leq 2n + 2$ and $1 \leq |\beta| \leq N + 1$,
\[
|\partial_\xi^\beta \varphi(y, \xi)| \leq C_\beta \langle y \rangle, \quad |\partial_\xi^\alpha \partial_\eta^\beta \varphi(y, \xi)| \leq C_{\alpha\beta} \quad \text{on} \quad \bigcup_{x \in \mathbb{R}^n} \text{supp}_{y, \xi} a(x, \cdot, \cdot).
\]

(2) For all $|\alpha|, |\beta| \leq 2n + 1$ and $|\gamma| \leq N$,
\[
|\partial_\xi^\alpha \partial_\eta^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2-|\beta|},
\]
and for all $|\alpha| \leq 2n + 2$ and $1 \leq |\beta| \leq N + 1$,
\[
|\partial_\eta^\beta \varphi(y, \xi)| \leq C_{\alpha\beta} \langle y \rangle^{1-|\alpha|} \quad \text{on} \quad \bigcup_{x \in \mathbb{R}^n} \text{supp}_{y, \xi} a(x, \cdot, \cdot).
\]

Then $T$ is bounded from $L^2_{m+m_2}(\mathbb{R}^n)$ to $L^2_{m-m_1}(\mathbb{R}^n)$.

**Remark 3.1.** In the case $\Gamma_y = \Gamma_\xi = \mathbb{R}^n$, we may replace condition (3.2) by
\[
|\det \partial_x \partial_\eta \varphi(x, y)| \geq C > 0, \quad \text{for all} \quad x, y, \in \mathbb{R}^n.
\]
Hence we obtain Theorem 1.1 in Introduction as a special case of Theorem 3.1. See Remark 2.1 after Theorem 2.1.

**Remark 3.2.** We can keep control on the number of derivatives with respect to $\xi$ in assumptions (1) and (2) of Theorem 3.1, but these numbers may depend on the orders of the weights of the spaces on which $T$ is acting. For example, for operator $T$ to be bounded from $L^2_{m+m_1+m_2}(\mathbb{R}^n)$ to $L^2_{m}(\mathbb{R}^n)$, condition $N \geq \max \{ |m| + n + 1, 2n + 1 \}$ must be replaced by $N \geq \max \{ |m+m_1| + n + 1, 2n + 2 \}$.

**Remark 3.3.** From the assumptions on phase functions $\varphi$ in Theorem 3.1, we obtain the estimate
\[
(3.3) \quad C_1 \langle y \rangle \leq \langle \partial_\xi \varphi(y, \xi) \rangle \leq C_2 \langle y \rangle \quad \text{on} \quad \bigcup_{x \in \mathbb{R}^n} \text{supp}_{y, \xi} a(x, \cdot, \cdot)
\]
for some $C_1, C_2 > 0$. In fact, the estimate $\langle \partial_\xi \varphi(y, \xi) \rangle \leq C_2 \langle y \rangle$ is obtained from any of the assumptions (1) or (2). As for the estimate $C_1 \langle y \rangle \leq \langle \partial_\xi \varphi(y, \xi) \rangle$, we may assume that $\Gamma_y$ is a proper cone, and fix an appropriate $y_0 \in \Gamma_y$ such that $|\partial_\xi \varphi(y_0, \xi)| \leq C \langle y_0 \rangle$. From condition (3.2) we obtain the estimate $|y-y_0| \leq C |\partial_\xi \varphi(y, \xi) - \partial_\xi \varphi(y_0, \xi)|$, hence $\langle y \rangle \leq C_{y_0} \langle \partial_\xi \varphi(y, \xi) \rangle$.

**Proof.** We show the $L^2$-boundedness of the operator $T_b$ defined by
\[
T_b u(x) = \int \int e^{i(x \xi + \varphi(y, \xi))} b(x, y, \xi) u(y) dy d\xi,
\]
where
\[
b(x, y, \xi) = \langle x \rangle^{m-m_1} a(x, y, \xi) \langle y \rangle^{-(m+m_2)}.
\]
By using the cut-off function $\chi(x) \in C^\infty_0(|x| \leq 1/2)$ which is equal to one near the origin, we decompose $b$ into two parts:

$$b^I(x, y, \xi) = b(x, y, \xi)\chi((x + \partial_\xi \varphi(y, \xi))/\langle \partial_\xi \varphi(y, \xi) \rangle),$$

$$b^{II}(x, y, \xi) = b(x, y, \xi)(1 - \chi)((x + \partial_\xi \varphi(y, \xi))/\langle \partial_\xi \varphi(y, \xi) \rangle).$$

The corresponding decomposition of the operator $T_b$ is denoted by $T^I$ and $T^{II}$ respectively.

On the support of $b^I(x, y, \xi)$, we have $|x + \partial_\xi \varphi(y, \xi)| \leq (1/2)\langle \partial_\xi \varphi(y, \xi) \rangle$, hence we have the estimates

$$|x| \leq |\partial_\xi \varphi(y, \xi)| + \frac{1}{2} \langle \partial_\xi \varphi(y, \xi) \rangle, \quad |\partial_\xi \varphi(y, \xi)| \leq |x| + \frac{1}{2} \langle \partial_\xi \varphi(y, \xi) \rangle.$$

From the first estimate and estimate (3.3), we obtain $\langle x \rangle \leq C(y)$. From the second estimate, we obtain $\langle \partial_\xi \varphi(y, \xi) \rangle \leq 2\langle x \rangle + (1/2)\langle \partial_\xi \varphi(y, \xi) \rangle$, hence $\langle \partial_\xi \varphi(y, \xi) \rangle \leq 4\langle x \rangle$, which implies $\langle y \rangle \leq C\langle x \rangle$ by (3.3) again. Thus we have the equivalence of $\langle y \rangle$ and $\langle x \rangle$, and obtain

(3.4) $$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b^I(x, y, \xi) \right| \leq C_{\alpha\beta\gamma} \langle x \rangle^{-|\alpha|}$$

or

(3.5) $$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b^{II}(x, y, \xi) \right| \leq C_{\alpha\beta\gamma} \langle y \rangle^{-|\beta|}$$

from the assumptions (1) and (2) respectively.

We assume estimate (3.4). Otherwise, assume (3.5) and just change the role of $x$ and $y$ below. Let real-valued positive functions $\Phi_0(x)$, $\Phi_k(x) = \Phi(x/2^k) (k \in \mathbb{N})$ form a partition of unity which satisfy $\text{supp} \Phi_0 \subset \{x; |x| < 2\}$, $\text{supp} \Phi \subset \{x; 1/2 < |x| < 2\}$. We decompose $b^I$ into the sum of $b^I_k(x, y, \xi) = \Phi_k(x)b^I(x, y, \xi)$. By the equivalence of $\langle x \rangle$ and $\langle y \rangle$ on the support of $b^I$, we can write

$$b^I_k(x, y, \xi) = \Phi_k(x)b^I(x, y, \xi)\tilde{\Psi}_k(y)$$

with functions $\tilde{\Psi}_k \in C^\infty_0(\mathbb{R}^n)$ which are of the from $\tilde{\Psi}_k(y) = \tilde{\Psi}(y/2^k)$, $\tilde{\Psi} \in C^\infty_0(\mathbb{R}^n \setminus 0)$ with large $k$. Furthermore, we have

$$b^I_k(2^kx, y, \xi) = \Psi_k(2^kx) \sum_{l \in \mathbb{Z}^n} e^{il \cdot x} b_{kl}(y, \xi) \tilde{\Psi}_k(y),$$

where $\Psi_k$ is the characteristic function of the support of $\Phi_k$, and

$$b_{kl}(y, \xi) = \int e^{-il \cdot x} b^I_k(2^kx, y, \xi) \, dx$$

$$= (1 + |l|^2)^{-n} \int e^{-il \cdot x}(1 - \Delta_x)^n \{\Phi_k(2^kx)b^I(2^kx, y, \xi)\} \, dx$$

is the Fourier coefficients of the function $b^I_k(2^kx, y, \xi)$ in the variable $x$. Then, by estimate (3.4), we have

$$\left| \partial_y^\alpha \partial_\xi^\beta b_{kl}(y, \xi) \right| \leq C_{\alpha\beta}(1 + |l|^2)^{-n},$$
where \( C_{\alpha\beta} \) is independent of \( k, l \in \mathbb{Z}^n \). Thus we have the decomposition

\[
T^I = \sum_{l \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} e^{i l \cdot x / 2 k} \Psi_k T_{kl} \tilde{\Psi}_k,
\]

where

\[
T_{kl} v(x) = \int \int e^{i(x \cdot \xi + \varphi(y, \xi))} b_{kl}(y, \xi) v(y) dy d\xi.
\]

We remark that

\[
\left\| \sum_{k \in \mathbb{Z}^n} e^{i l \cdot x / 2 k} \Psi_k T_{kl} \tilde{\Psi}_k u \right\|_{L^2}^2 \leq C \sum_{k \in \mathbb{Z}^n} \left\| \Psi_k T_{kl} \tilde{\Psi}_k u \right\|_{L^2}^2 \leq C \sup_{k \in \mathbb{Z}^n} \| T_{kl} \|_{L^2 \to L^2}^2 \sum_{k \in \mathbb{Z}^n} \| \tilde{\Psi}_k u \|_{L^2}^2 \leq C (1 + |l|^2)^{-2n} \| u \|_{L^2}^2
\]

by Theorem 2.5. Hence we have

\[
\| T^I \|_{L^2 \to L^2} \leq C \sum_{l \in \mathbb{Z}^n} (1 + |l|^2)^{-n} \leq C,
\]

that is, the \( L^2 \)-boundedness of \( T^I \).

Next, we show the boundedness of \( T^{II} \). Let \( \rho \in C^\infty_0 \) be a real-valued function which satisfies

\[
\sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1.
\]

We decompose \( b^{II}(x, y, \xi) \) into the sum of

\[
b_k^{II}(x, y, \xi) = b^{II}(x, y, \xi) \rho(\xi - k)
\]

and set

\[
T_k u(x) = \int \int e^{i(x \cdot \xi + \varphi(y, \xi))} b_k^{II}(x, y, \xi) u(y) dy d\xi.
\]

We claim, we may replace \( b_k^{II}(x, y, \xi) \) by the symbol (denoted by \( b_k^{II}(x, y, \xi) \) again) which has the same (or smaller) support and satisfies the estimate

\[
|\partial_x^a \partial_y^b b_k^{II}(x, y, \xi)| \leq C_{\alpha\beta} (x)^{-n} (y)^{-n+1},
\]

where \( C_{\alpha\beta} \) is independent of \( k \in \mathbb{Z}^n \). Indeed, by integration by parts we have

\[
T_k u(x) = \int \int e^{i(x \cdot \xi + \varphi(y, \xi))} L^N b_k^{II}(x, y, \xi) u(y) dy d\xi,
\]

where \( L \) is the transpose of the operator

\[
i^L = \frac{x + \partial_x \varphi}{i|x + \partial_x \varphi|^2} : \partial_x
\]
and $N$ is a positive integer. We have $\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|$ on the support of $b^I(x, y, \xi)$, hence we have

$$\langle x \rangle \leq |x + \partial_\xi \varphi(y, \xi)| + 2\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|,$$

$$\langle y \rangle \leq C\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|$$

by estimate (3.3). Thus $|x + \partial_\xi \varphi|^1$ is dominated by $\langle x \rangle^{-1}$ and $\langle y \rangle^{-1}$, and we can justify our claim by taking $N$ as in the formulation of the theorem.

Let $T_k^*$ be the adjoint of $T_k$, and we have

$$T_kT_k^*v(x) = \int K_{kl}(x, y)v(y) dy, \quad T_k^*T_kv(x) = \int \tilde{K}_{kl}(x, y)v(y) dy,$$

where

$$K_{kl}(x, y) = \int \int e^{i\langle x - y + \varphi(z, \xi) - \varphi(z, \eta) \rangle} b^I_k(x, z, \xi)b^I_k(y, z, \eta) dzd\xi d\eta,$$

$$\tilde{K}_{kl}(x, y) = \int \int e^{i\langle \varphi(y, z) - \varphi(x, \eta) + z(x, \xi) \rangle} b^I_k(z, \xi)b^I_k(x, \eta) dzd\xi d\eta.$$

By integration by parts, we have

$$\int e^{i\langle \varphi(z, \xi) - \varphi(z, \eta) \rangle} b^I_k(x, z, \xi)b^I_k(y, z, \eta) dz$$

$$= \int e^{i\langle \varphi(z, \xi) - \varphi(z, \eta) \rangle} L^{2n+1} (b^I_k(x, z, \xi)b^I_k(y, z, \eta)) dz,$$

where $L$ is the transpose of the operator

$$^tL = \frac{1}{i} \frac{\partial_\varphi(z, \xi) - \partial_\varphi(z, \eta)}{|\partial_\varphi(z, \xi) - \partial_\varphi(z, \eta)|^2} \cdot \partial_z.$$

From the assumptions, we obtain

$$|\partial_\varphi(z, \xi) - \partial_\varphi(z, \eta)| \geq C|\xi - \eta|$$

and

$$|\partial_\varphi(z, \xi) - \partial_\varphi(z, \eta)| \leq C_\beta|\xi - \eta|$$

for all $|\beta| \geq 1$. From this argument and (3.6), we obtain

$$|K_{kl}(x, y)| \leq C(x)^{-n+1} \langle y \rangle^{-n+1} (1 + |k - l|^{2n+1})^{-1},$$

where $C$ is independent of $k, l \in \mathbb{Z}^n$. Then we have

$$\sup_x \int |K_{kl}(x, y)| dy \leq C(1 + |k - l|^{2n+1})^{-1},$$

$$\sup_y \int |K_{kl}(x, y)| dx \leq C(1 + |k - l|^{2n+1})^{-1}$$

which implies, by Lemma 2.1,

$$\|T_kT_k^*\|_{L^2 \to L^2} \leq C(1 + |k - l|^{2n+1})^{-1}.$$
Similarly, we have
\[ \|T^n_k T_l\|_{L^2 \to L^2} \leq C (1 + |k - l|^{2n+1})^{-1} \]
if we take
\[ t' L = \frac{1}{i} \frac{\xi - \eta}{|\xi - \eta|^2} \cdot \partial_z. \]
Then we have
\[ \|T_k T^*_l\|_{L^2 \to L^2}, \|T^*_k T_l\|_{L^2 \to L^2} \leq C \{\gamma(k - l)\}^2, \]
where
\[ \gamma(j) = \left(1 + |j|^{2n+1}\right)^{-1/2} \]
and it satisfies the estimate
\[ \sum_{j \in \mathbb{Z}^n} \gamma(j) < \infty. \]

By Lemma 2.2, we have the \(L^2\)-boundedness of \(T^H\).

Finally we note that we get conditions on the numbers of derivatives in assumptions (1) and (2) of the theorem simply by keeping track on the number of derivatives required for this proof. \(\square\)

### 4. Applications

In this section, we explain how to use Theorem 3.1 to show the smoothing effect of generalized Schrödinger equations. First we establish an auxiliary result about transformation operators which is also of interest on its own since the phase function is not smooth at the origin.

The main tool in the analysis is a class of Fourier integral operators of the form
\[
T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi(\xi))} u(y) dy d\xi,
\]
\[
T_{\psi^{-1}}^{-1} u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi^{-1}(\xi))} u(y) dy d\xi,
\]
where \(\psi, \psi^{-1} : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0\) are \(C^\infty\)-maps satisfying \(\psi \circ \psi^{-1}(\xi) = \psi^{-1} \circ \psi(\xi) = \xi, \psi(\lambda \xi) = \lambda \psi(\xi),\) and \(\psi^{-1}(\lambda \xi) = \lambda \psi^{-1}(\xi)\) for all \(\lambda > 0\) and \(\xi \in \mathbb{R}^n \setminus 0\). We remark that we have
\[ T_\psi u(x) = \mathcal{F}_\xi^{-1}[(\mathcal{F}_x u)(\psi(\xi))](x), \quad T_{\psi^{-1}}^{-1} u(x) = \mathcal{F}_\xi^{-1}[(\mathcal{F}_x u)(\psi^{-1}(\xi))](x), \]
where \(\mathcal{F}_x \) (\(\mathcal{F}_\xi^{-1}\) resp.) denotes the (inverse resp.) Fourier transform. Hence, we have \(T_\psi^{-1} \cdot T_\psi = T_{\psi^{-1}} \cdot T_\psi = id\), and the formula
\[ T_\psi \cdot a(D) \cdot T_{\psi^{-1}}^{-1} = (a \circ \psi)(D), \]
where \( a(D) = \mathcal{F}_\xi^{-1} a(\xi) \mathcal{F}_x \). By (4.2) and Plancherel's theorem, the operators \( T_\psi \) and \( T_\psi^{-1} \) are \( L^2 \)-bounded. Furthermore, as a corollary of Theorem 3.1, we have the following:

**Corollary 4.1.** Suppose \( m \in \mathbb{Z} \) and \(|m| < n/2\). Assume that \(|\det \partial \psi(\xi)| \geq C > 0\). Then the operators \( T_\psi \) and \( T_\psi^{-1} \) defined by (4.1) are \( L^2_m(\mathbb{R}^n) \)-bounded.

**Remark 4.1.** By Corollary 4.1 and the interpolation, we have the \( L^2_m(\mathbb{R}^n) \)-boundedness of \( T_\psi \) and \( T_\psi^{-1} \) with \( m \in \mathbb{R} \) such that \(|m| \leq [n/2]_+\). Here \([k]_+\) denotes the greatest integer less than \( k\).

**Proof.** In this proof we will represent vectors as rows and sometimes use the matrix notation instead of inner product when it is convenient.

We will prove the boundedness of \( T_\psi \), from which the boundedness of \( T_\psi^{-1} \) follows. Let \( \chi(\xi) \in C_0^\infty \) be a cut off function of the origin. By (4.2), we have

\[
(1 - \chi(D))T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x_\xi - y \psi(\xi))}(1 - \chi(\xi))u(y)dyd\xi.
\]

Since \( \psi(\xi) \) is smooth away from the origin, \((1 - \chi(D))T_\psi \) is \( L^2_m \)-bounded by Theorem 3.1. On the other hand, if we note

\[
e^{ix_\xi} = \frac{1 - ix \cdot \partial_\xi}{\langle x \rangle^2} e^{ix_\xi}, \quad e^{-iy_\xi} = \frac{1 + iy \cdot \partial_\xi}{\langle y \rangle^2} e^{-iy_\xi},
\]

we have, by change of variables and integration by parts,

\[
\chi(D)T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x_\xi - y \psi(\xi))} \chi(\xi)u(y)dyd\xi
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x_\xi - y \psi(\xi))} \left( \frac{\chi(\xi) + ix \cdot \partial \chi(\xi) + x\chi(\xi)^t \partial \psi(\xi)^t y}{\langle x \rangle^2} \right) u(y)dyd\xi
\]

\[
= \frac{1}{\langle x \rangle^2} \chi(D)T_\psi u + \frac{x}{\langle x \rangle^2} \cdot \partial \chi(D)T_\psi u + \frac{x}{\langle x \rangle^2} \chi(D)^t \partial \psi(D)T_\psi (\chi u)
\]

and

\[
\chi(D)T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x_\psi^{-1}(\xi) - y_\psi(\xi))} \chi(\psi^{-1}(\xi)) |\det \partial \psi^{-1}(\xi)| u(y)dyd\xi
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x_\psi^{-1}(\xi) - y_\psi(\xi))} \left( \frac{d^{-1}(\xi) + a(\xi) \cdot y + xA(\xi)^t y}{\langle y \rangle^2} \right) u(y)dyd\xi
\]

\[
= \chi(D)T_\psi \left( \frac{u}{\langle x \rangle^2} \right) + |D|^{-1}|D|a(\psi(D))d(D) \cdot T_\psi \left( \frac{x}{\langle x \rangle^2} u \right)
\]

\[
+ xA(\psi(D))d(D)T_\psi \left( \frac{t_\xi}{\langle x \rangle^2} u \right)
\]
where
\[
A(\xi) = \chi(\psi^{-1}(\xi))|\det \partial \psi^{-1}(\xi)|^2 |\partial \psi^{-1}(\xi)|, \quad a(\xi) = -i \partial \{\chi(\psi^{-1}(\xi))|\det \partial \psi^{-1}(\xi)|\},
\]
\[
d(\xi) = |\det \partial \psi(\xi)|, \quad d^{-}(\xi) = \chi(\psi^{-1}(\xi))|\det \partial \psi^{-1}(\xi)|.
\]

We remember here that \(T_\psi\) is \(L^2\)-bounded. Assume that \(T_\psi\) is \(L^2_{+(k-1)}\)-bounded with some \(k < n/2\), \(k \in \mathbb{N}\). We remark that \(\chi(D), d(D)\) and all entries of \(\partial \chi(D), \partial \psi(D), A(\psi(D)), |D|a(\psi(D))\) are \(L^2_{+(k-1)}\)-bounded, and \(|D|^{-1}\) is bounded from \(L^2_{(k-1)}\) to \(L^2_{-k}\). To justify these boundedness, use the results of Kurtz and Wheeden [19], Stein and Weiss [26]. The assumption \(|k| < n/2\) is used to insure that all the Fourier multipliers are bounded in the corresponding weighted spaces. Using these results, we obtain the \(L^2_k\)-boundedness of \(\chi(D)T_\psi\) from (4.4), and \(L^2_{-k}\)-boundedness from (4.5). Then, by induction, we have the desired result for all \(m \in \mathbb{Z}\) satisfying \(|m| < n/2\).

Now we will describe an application of this analysis to smoothing properties of equations of Schrödinger type. Let \(p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)\) be a positive function which is also positively homogeneous (of order one), i.e. it satisfies \(p(\lambda \xi) = \lambda p(\xi)\) for all \(\lambda > 0\) and \(\xi \in \mathbb{R}^n \setminus 0\). Let
\[
L_p = p(D_x)^2 = \mathcal{F}_\xi^{-1} p(\xi)^2 \mathcal{F}_x
\]
be the corresponding Fourier multiplier. Assume that \(\Sigma = \{\xi \in \mathbb{R}^n : p(\xi) = 1\}\) has non-vanishing Gaussian curvature. We consider a generalized Schrödinger equation
\[
(4.6) \quad \begin{cases} (i \partial_t + L_p)u(t, x) = 0, \\ u(0, x) = f(x). \end{cases}
\]
If we take
\[
\psi(\xi) = p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|},
\]
we have the relation
\[
(4.8) \quad T_\psi \cdot (-\triangle_x) \cdot T^{-1}_\psi = L_p
\]
by (4.3), and the \(L^2_1\)-boundedness of the operators \(T_\psi\) and \(T^{-1}_\psi\) by Corollary 4.1. Indeed, the curvature condition on \(\Sigma\) means that the Gauss map
\[
\frac{\nabla p}{|\nabla p|} : \Sigma \to S^{n-1}
\]
is a global diffeomorphism and its Jacobian never vanishes (see Kobayashi and Nomizu [17]). Hence, we can construct the inverse \(C^\infty\)-map \(\psi^{-1}(\xi)\) of \(\psi(\xi)\), and can justify the assumption of Corollary 4.1. This argument together with the detailed analysis of the geometry of the corresponding sets can be found in authors' paper [23].
Applying $T^{-1}_\psi$ defined by (4.1) with (4.7) to equation (4.6), and introducing $v = T^{-1}_\psi u$ and $g = T^{-1}_\psi f$, (4.6) can be transformed to the equation

$$
\begin{cases}
(i\partial_t - \Delta_x)v(t, x) = 0, \\
v(0, x) = g(x),
\end{cases}
$$

by (4.8). It has been already known that classical Schrödinger equation (4.9) has the global smoothing estimate

$$
\|\sigma(X,D)v\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|g\|_{L^2(\mathbb{R}^n_x)},
$$

where $n \geq 3$ and

$$
\sigma(X,D) = \langle x \rangle^{-1}\langle D \rangle^{1/2}.
$$

See Ben-Artzi and Klainerman [3], Simon [24], Kato and Yajima [16], or Walther [29]. From this fact, we can extract a similar estimate for generalized Schrödinger equation (4.6). In fact, we have

$$
\langle D \rangle^{1/2} u = M(1 + p(D)^2)^{1/4}T_\psi v = MT_\psi\langle D \rangle^{1/2} v
$$

where

$$
M = \langle D \rangle^{1/2}(1 + p(D)^2)^{-1/4}.
$$

Here we have used the formula (4.3) with $a(\xi) = (1 + |\xi|^2)^{1/4}$. Hence we have

$$
\sigma(X,D) u = \langle x \rangle^{-1}MT_\psi\langle x \rangle\sigma(X,D)v.
$$

Since $M$ is $L^2_{-1}(\mathbb{R}^n_x)$-bounded by Theorem 1.1, and $T_\psi$ by Corollary 4.1, we obtain

$$
\|\sigma(X,D)u\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|f\|_{L^2(\mathbb{R}^n_x)}
$$

from estimates (4.10) and

$$
\|g\|_{L^2(\mathbb{R}^n_x)} = \|T^{-1}_\psi f\|_{L^2(\mathbb{R}^n_x)} \leq C\|f\|_{L^2(\mathbb{R}^n_x)}.
$$

Thus, we have obtained the following result which was partially proved for a type of (principal type) polynomial $p(\xi)^2$ by Ben-Artzi and Devinatz [2], and fully for radially symmetric (elliptic) $p(\xi)^2$ by Walther [30].

**Theorem 4.2.** Suppose $n \geq 3$. Let $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ be a positive and positively homogeneous of order one function, such that the set $\Sigma = \{ \xi \in \mathbb{R}^n : p(\xi) = 1 \}$ has non-vanishing Gaussian curvature. Let $L_p = p(D_x)^2$. Then the solution $u(t, x)$ to the equation

$$
\begin{cases}
(i\partial_t + L_p)u(t, x) = 0, \\
u(0, x) = f(x)
\end{cases}
$$

satisfies the estimate

$$
\|\langle x \rangle^{-1}\langle D \rangle^{1/2} u\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|f\|_{L^2(\mathbb{R}^n_x)}.
$$
In Theorem 4.2, the order “− 1” for the weight is the best possible one because of the estimate for the low frequency part (Walther [29], [30]). But, if we replace \( \langle D \rangle^{1/2} \) by \( |D|^{1/2} \), we have another type of estimate
\[
\| \langle x \rangle^{-\delta} |D|^{(m-1)/2} u \|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \| f \|_{L^2(\mathbb{R}^n_x)}
\]
for \( \delta > 1/2 \). Chihara [7] obtained this latter type of estimates for rather general \( p(\xi)^2 \). In our paper [23], we use Theorem 3.1 to obtain a refinement of this estimate in the case \( \delta = 1/2 \).

Finally we note that by a similar argument we get an estimate for operators of other orders, for which Theorem 4.2 becomes a special case with \( m = 2 \).

**Theorem 4.3.** Suppose \( n > m > 1 \). Let \( p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0) \) be a positive and positively homogeneous of order one function, such that the set \( \Sigma = \{ \xi \in \mathbb{R}^n : p(\xi) = 1 \} \) has non-vanishing Gaussian curvature. Let \( L_p = p(D_x)^m \). Then the solution \( u(t,x) \) to the equation
\[
\begin{cases}
(\partial_t + L_p)u(t,x) = 0, \\
u(0,x) = f(x)
\end{cases}
\]
satisfies the estimate
\[
\| \langle x \rangle^{-m/2} (D)^{(m-1)/2} u \|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \| f \|_{L^2(\mathbb{R}^n_x)}.
\]

The proof of this theorem is similar to that of Theorem 4.2 but requires more microlocal techniques. In fact, the statement is still true for operators \( L_p \) of principal type under the assumption \( n > m + 1 \). Proofs of these facts are more technical and together with proofs of other smoothing estimates are a subject of authors’ paper [22].

**References**


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