GAUGE THEORY IN HIGHER DIMENSIONS

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The interaction between geometry in the adjacent dimensions $2, 3$ and 4 is a theme which runs through a great deal of the work by mathematicians on gauge theory over the past few years. In this article we will examine the possibility of developing this theme in higher dimensions. We will find extensions following two intertwining threads. One thread, which we say more about here, replaces real variables by complex variables, and hence operates in *complex* dimensions $2, 3, 4$. The other thread involves, from one point of view, replacing the quaternions by the octonians, and operates in the realm of manifolds with "exceptional holonomy". The picture we will find pulls together various ideas which have been touched on in the literature but the striking analogies which emerge do not seem to be well-known. Our treatment will be informal throughout this article—our main aim is to advertise the potential for research in these directions. A great deal of technical work is needed to develop these ideas in detail, and a more thorough and wide-ranging account of the Calabi-Yau story will appear in the D.Phil. thesis of the second author. The first author would like to emphasise the debt due to other mathematicians in forming parts of the picture described here; particularly Dominic Joyce, Simon Salamon and Christopher Lewis for lessons on exceptional holonomy. A substantial part of this picture is essentially due to Joyce and Lewis, and again futher details will appear in the doctoral thesis of Lewis.

1. The familiar theory.

Let us begin by reviewing very briefly, ignoring many important technicalities, the sort of ideas and structures in gauge theory that we wish to generalise. These involve gauge theory, with structure group a compact Lie group (which we may wish to take to be $SU(2)$ or $SO(3)$ in a detailed development) over differentiable manifolds of dimensions $2, 3, 4$, all of which have definite *orientations*. It is convenient to focus on the intermediate dimension 3, where we have the well-known theories of Casson and Floer. If $Y_{\mathbf{R}}$ is a compact 3-manifold, the Chern-Simons functional gives a map $CS : \mathcal{B}(Y_{\mathbf{R}}) \to \mathbf{R}/\mathbf{Z}$ where $\mathcal{B}(Y_{\mathbf{R}})$ is the infinite-dimensional space of gauge-equivalence classes of connections on a fixed bundle over $Y_{\mathbf{R}}$. The derivative of CS is the given by the curvature of a connection, regarded as a 1-form on $\mathcal{B}(Y_{\mathbf{R}})$, through the formula:

$$\delta CS = \int_{Y_{\mathbf{R}}} \mathrm{Tr}(F \wedge \delta A), \tag{1}$$

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and so the critical points are the *flat* connections. We assume for simplicity that the CS is a generic function, so the critical points are non-degenerate and in particular isolated. The Casson invariant of $Y_{\mathbf{R}}$ is given by counting, with signs, the flat connections. It can be interpreted, formally, as the Euler characterisitic of the infinite-dimenional space $\mathcal{B}(Y_{\mathbf{R}})$. The Floer homology of $Y_{\mathbf{R}}$ is defined by fixing a metric on $Y_{\mathbf{R}}$, and hence on $\mathcal{B}(Y_{\mathbf{R}})$. This allows one to define the gradient vector field of CS; the integral curves of the gradient vector field connecting different critical points give, in Floer's celebrated construction, a chain complex which computes the Floer homology. The homology groups do not depend on the metric and are formally the homology groups of $\mathcal{B}(Y_{\mathbf{R}})$ in "semi-infinite dimensions".

The four-dimensional view-point on these ideas comes from the fact that the gradient flow equation for CS is precisely the Yang-Mills instanton equation on $Y_{\mathbf{R}} \times \mathbf{R}$: thus the pointwise symmetry group $SO(3)$ of the three-dimensional theory has a surprising extension to $SO(4)$ (related to the Lorentzian invariance of Maxwell's equations). More generally, if $Y_{\mathbf{R}}$ is the boundary of a 4-manifold $X_{\mathbf{R}}$ one gets an interaction between the instanton theory on $X_{\mathbf{R}}$, made into a complete manifold by adjoining an infinite cylinder, and the Floer theory on $Y_{\mathbf{R}}$.

To go down to 2-dimensions we consider a splitting of the 3-manifold, $Y_{\mathbf{R}} = Y^+_{\mathbf{R}} \cup_{S_{\mathbf{R}}} Y^-_{\mathbf{R}}$, by a surface $S_{\mathbf{R}} \subset Y_{\mathbf{R}}$. The moduli space $M(S_{\mathbf{R}})$ of flat connections over the surface $S_{\mathbf{R}}$ is, roughly speaking, a finite-dimensional manifold with an intrinsic *symplectic structure* induced by the formula:

$$\Omega(a, b) = \int_{S_{\mathbf{R}}} \mathrm{Tr}(a \wedge b), \tag{2}$$

where a and b are bundle-valued 1-forms over $S_{\mathbf{R}}$ representing tangent vectors to $M(S_{\mathbf{R}})$. Now we consider the subsets $L^+, L^- \subset M(S_{\mathbf{R}})$ given by the connections which extend over $Y^{\pm}_{\mathbf{R}}$. These are Lagrangian submanifolds of $M(S_{\mathbf{R}})$ and the flat connections on $Y_{\mathbf{R}}$ appear as the intersection points $L^+ \cap L^-$. (This is the point of view Casson took in his original definition.) The instantons in 4-dimensions are more elusive in this picture but they are related to the *holomorphic discs* in $M(S_{\mathbf{R}})$ with boundary in the $L^{\pm}$. (Here one chooses a metric on $S_{\mathbf{R}}$, which makes $M(S_{\mathbf{R}})$ into a Kähler manifold.) This is the essence of the "Atiyah-Floer conjecture", versions of which have been proved by D. Salamon and others [**DS**]. The main point is that if T is another surface the "adiabatic limit" of the instanton equations on the product $S_{\mathbf{R}} \times T$, as the metric in the $S_{\mathbf{R}}$ direction is scaled down, can be identified with the holomorphic mapping equation for maps from T to $M(S_{\mathbf{R}})$.

2. The complex analogy.

In elementary terms, our procedure for extending the ideas sketched above is to replace ordinary derivatives $\frac{\partial}{\partial x_\alpha}$, where x_α are real co-ordinates, by Cauchy-Riemann operators $\frac{\partial}{\partial \bar{z}_\alpha}$, where z_α are complex co-ordinates. The important role played by orientation in the real case leads to the need for a "complex orientation"—a trivialisation of the canonical line bundle. Thus the geometrical setting for our discussion involves Calabi-Yau manifolds. From the point of view of analysis the crucial thing is that the ordinary derivative $\frac{d}{dx}$ on $\mathbf{R}$ and the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ on $\mathbf{C}$ are both ellipic operators, so behave in rather similar ways.

We begin then in complex dimension 3 with a compact Calabi-Yau 3-fold Y, so there is a nowhere degenerate holomorphic form $\theta \in \Omega^{3,0}(Y)$. We sometimes want to suppose that Y is Kähler, and so admits a Kähler-Einstein metric with holonomy $SU(3)$. Fix a C^∞ complex vector bundle E over Y and let $\mathcal{A}$ be the space of $\overline{\partial}$-operators on E: that is differential operators:

$$\overline{\partial}_\alpha : \Omega^0(E) \to \Omega^{0,1}(E) \tag{3}$$

satisfying the usual Leibnitz rule. Any two elements of $\mathcal{A}$ differ by a tensor in $\Omega^{0,1}(End\, E)$. If E has a fixed Hermitian metric these $\overline{\partial}$-operators may be identified with unitary connections, by projecting to the $(0,1)$ part. We consider the action of the complex gauge group $\mathcal{G}^c$ of general linear automorphisms of E, which act on $\mathcal{A}$ by conjugation. Thus we have a quotient space $\mathcal{C}_E = \mathcal{A}/\mathcal{G}^c$. This definition should not be taken too literally: we know very well from other problems that issues involving "stability" arise in forming such quotients, and we may wish to restrict attention to a suitable set of stable points in $\mathcal{A}$. In particular we may employ the well-known framework involving the comparison of symplectic and complex quotients and, having fixed metrics, work with the quotient of the space of connections whose curvature satisfies a moment map condition $F.\omega = 0$ (where ω is the Kähler form), by the group $\mathcal{G}$ of unitary automorphisms of E (see [**DT**] for example). However for the present we shall ignore such technicalities and imagine that $\mathcal{C}_E$ is an infinite-dimensional complex manifold. Now any operator $\overline{\partial}_\alpha \in \mathcal{A}$ prolongs to the E-valued $(0,q)$ forms, and the composite $\overline{\partial}_\alpha^2$ defines a tensor in $\Omega^{0,2}(EndE)$. If we identify the operators with connections, this is just the $(0,2)$ part of the curvature, so we denote it by $F^{0,2}(\alpha)$. Then we define a complex 1-form U on the space $\mathcal{A}$ by

$$U_\alpha(\delta\alpha) = \int_Y \mathrm{Tr}(\delta\alpha \wedge F^{0,2}(\alpha)) \wedge \theta. \tag{4}$$

Here α is a point in $\mathcal{A}$ and $\delta\alpha$ is a tensor in $\Omega^{0,1}(End\, E)$, regarded as a tangent vector to $\mathcal{A}$ at α. The analogy with the case of connections over 3-manifolds will be clear to the reader. Just as in that case one shows that U descends to the quotient space $\mathcal{C}_E$ and defines a closed 1-form, so is locally the derivative of a complex-valued function Φ. The new feature in our present case is that Φ is a holomorphic function on the complex manifold $\mathcal{C}_E$. We can identify Φ more explicitly: if we regard $\mathcal{A}$ as a space of connections then it is just given by pairing the Chern-Simons invariant with the holomorphic form θ. In any event we get a well-defined function on a covering space of $\mathcal{C}_E$, with covering group at most $H^3(Y;\mathbf{Z})$—for our current exposition we will largely ignore this covering issue. Now the main point is that, just as in the Casson-Floer theory, the critical points of Φ—the zeros of the 1-form U—have a solid geometric meaning. They are just the operators satisfying the integrability condition $\overline{\partial}_\alpha^2 = 0$, that is, those which endow E with a *holomorphic* structure. Clearly then we should hope that "counting" the holomorphic bundles of a fixed topological type over a Calabi-Yau manifold Y will yield an invariant which can be regarded as analogous to the Casson invariant of a 3-manifold.

Some remarks are now in order. First, the point of view above is very close to the discussion by Witten in ([**W**], Sec 4.5), which is aimed more at an analogy with the Chern-Simons theory on 3-manifolds, involving integration of the exponential of the Chern-Simons functional. Second, a lot of work would need to be done to give precision to these ideas. For example one would expect to bring in a suitable notion of stability, as mentioned above. One would have to deal with the problem that the zeros of the 1-form U may be degenerate, i.e. the situation is not generic, and consider suitable perturbations, or appropriate methods of counting degenerate zeros. One would need compactness results to ensure that counting gave a finite answer. However, at least as far as the "local" discussion goes (in the space of connections) one can take over much of the usual Fredholm-theory analysis [**T**] from the usual 3-manifold case to this complex setting. The setting for the local analysis in the ordinary case is the de Rham complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3, \tag{5}$$

coupled to the flat connection on the adjoint bundle. In the holomorphic case we get the Dolbeault complex

$$\Omega^{0,0} \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2} \xrightarrow{\bar\partial} \Omega^{0,3} \tag{6}$$

coupled to the endomorphism bundle. The point is that these are both elliptic complexes. As far as compactness goes, one knows at least that the L^2 norm of the curvature of a Hermitian-Yang-Mills connection over a compact Kähler manifold of any dimension m is fixed by the topology of the bundle. This follows from the Chern-Weil theory and the identity:

$$|F|^2\omega^m = -\mathrm{Tr}(F^2) \wedge \omega^{m-2} \tag{7}$$

for curvature tensors F of type $(1,1)$ with $F.\omega = 0$. A third remark is that, while the tight analogy with the Casson theory is restricted to Calabi-Yau manifolds, one may try more generally to approach the integrabilty equations $F^{0,2} = 0$ (or what is more or less the same, the Hermitian-Yang-Mills equations for unitary connections) from the point of view of nonlinear Fredholm theory. The problem is that the equations are overdetermined in complex dimension 3 or more. This difficulty is not so serious in a situation where one knows that all the higher dimensional cohomology groups $H^i(End_0 E)$, for all the relevant holomorphic bundles E, vanish for $i \geq 3$. Here End_0 denotes the trace-free endomorphisms. From this perspective the good feature of a Calabi-Yau 3-fold is that the cohomology group $H^3(End_0 E)$ is dual to $H^0(End_0 E)$, which vanishes for stable bundles E.

We now go up to complex dimension 4, beginning with the product $\mathbf{C} \times Y$. Here we fix a Kähler-Einstein metric on Y and a metric on E, so that $\mathcal{A}$ gets an induced Hermitian L^2-metric. Thus the complex co-tangent vector U is dual to a tangent vector $\widehat{U}$. We would like to consider a "complex gradient equation", in parallel with Floer's theory. Some subtleties are involved, however, because the complex gauge group $\mathcal{G}^c$ does not preserve the vector field $\widehat{U}$, so there are different ways to proceed. On the one hand we could consider a map $\alpha(z)$ from $\mathbf{C}$ to $\mathcal{A}$ which satisfies the equation

$$\frac{\partial \alpha}{\partial \overline{z}} = \widehat{U}. \tag{8}$$

On the other hand we can consider the complex quotient $\mathcal{C} = \mathcal{A}/\mathcal{G}^c$ which (modulo questions of stability) is identified with the symplectic quotient $\{F.\omega = 0\}/\mathcal{G}$, and in this way get an induced Kähler metric. The vector field $\widehat{U}$ on $\mathcal{A}$ induces a vector field $\widetilde{U}$ on $\mathcal{C}$, and we can consider a map $\tilde{\alpha}(z)$ from $\mathbf{C}$ to $\mathcal{C}$ satisfying

$$\frac{\partial \tilde{\alpha}}{\partial \bar{z}} = \widetilde{U}. \tag{8'}$$

We want to interpret these constructions in terms of gauge theory over $\mathbf{C} \times Y$. Observe that on any n-dimensional Calabi-Yau manifold we have complex *antilinear* maps $*_n : \Omega^{0,q} \to \Omega^{0,n-q}$ defined by the condition that the $(0,n)$-form $\alpha \wedge *_n \alpha$ is $|\alpha|^2$ times the conjugate of the complex volume form. The vector field $\widehat{U}$ can be identified with $*_3 F^{0,2}$, defined by combining the above operation on the form component with the antilinear adjoint map on $End\, E$.

Now move to 4 dimensions. If X is any Calabi-Yau 4-fold we have $*_4 : \Omega^{0,2}_X \to \Omega^{0,2}_X$ with $*_4^2 = 1$, so we get a decomposition $\Omega^{0,2}_X = \Omega^{0,2}_+ \oplus \Omega^{0,2}_-$ into ± 1 eigenspaces. (It is important to realise that these are real, not complex, subspaces. In terms of representations, what we are saying is that the representation $\Lambda^{0,2}$ of $SU(4)$ is a *real* representation. Notice also that multiplying the complex volume form θ by a complex scalar gives a different splitting.) This decomposition of the forms means that there is a "complex anti-self-duality" equation for unitary connections over X: $F^{0,2}_+ = 0$. This equation has been found and studied independently by Lewis. Putting the pieces together, in the first setting of a map from $\mathbf{C}$ to $\mathcal{A}$ we define a connection over $\mathbf{C} \times Y$ with zero components in the $\mathbf{C}$ direction and one sees that solutions of (8) correspond to connections over $\mathbf{C} \times Y$ which satisfy the two equations

$$\begin{cases} F^{0,2}_+ = 0, \\ F.\omega_{\mathbf{C}} = 0, \end{cases} \tag{9}$$

where $F.\omega_{\mathbf{C}}$ is the component of the curvature in the $\mathbf{C}$-variable. In the second setting, of a map from $\mathbf{C}$ to $\mathcal{C}$, one finds that the solutions of (8′) correspond to solutions of the two equations

$$\begin{cases} F^{0,2}_+ = 0, \\ F.\omega_Y = 0. \end{cases} \tag{9'}$$

On the other hand, over any Calabi-Yau 4-fold X the natural supplement to the anti-self-duality equation, as studied by Lewis, is $F.\omega_X = 0$. We say that a unitary connection over X is an *$SU(4)$-instanton* if it satisfies the two equations $F^{0,2}_+ = 0$, $F.\omega_X = 0$. These are elliptic equations, modulo unitary gauge equivalence. Notice that (when X is compact) Hermitian-Yang-Mills connections on stable holomorphic bundles give examples of such instantons. Lewis has shown that if a certain characteristic class condition is satisfied then these are the only solutions. Moreover there is in any case an L^2 bound on the curvature of an $SU(4)$-instanton, coming from an identity like (7).

In the case when $X = \mathbf{C} \times Y$ we can write the $SU(4)$-instanton equations as

$$\begin{cases} F^{0,2}_+ = 0, \\ F.\omega_{\mathbf{C}} + F.\omega_Y = 0. \end{cases} \tag{9''}$$

These three equations (9), (9′) and (9″) all fit into the continuous family $F_+^{0,2} = 0$, $tF.\omega_{\mathbf{C}} + (1-t)F.\omega_Y = 0$ $(0 \leq t \leq 1)$, and supposing that this family is well-behaved the solutions of the three equations are more-or-less equivalent (at least in regard to topological "counting" of the solutions). The advantage of the intermediate equation (9″) (or, more generally, the equation for any parameter t with $0 < t < 1$) is that it is elliptic. Clearly the existence of these different equations has to do with different ways of dealing with the gauge invariance of the problem and we can think of (9″) as a regularisation of the extremes (9), (9′).

Now we can carry through the discussion above replacing $\mathbf{C}$ by any Riemann surface with trivial cotangent bundle, and in particular by $\mathbf{R} \times S^1$. The conclusion we are finally lead to is that the $SU(4)$-instantons over $\mathbf{R} \times S^1 \times Y$ which converge to Hermitian-Yang-Mills connections on holomorphic bundles E^+, E^- at $\pm\infty$ in the $\mathbf{R}$-variable play the role of complex gradient curves for the holomorphic Chern-Simons functional, in the same way that instantons over tubes are viewed as gradient curves in Floer's theory. In Section 7 below we shall return to discuss the topological interpretation of these complex gradient curves; we shall see there that the solutions which are invariant under rotations in the S^1-variable have a particularly simple interpretation and it is worth noting now that for rotation-invariant connections the three equations (9), (9′), (9″) are all directly equivalent since $F.\omega_{\mathbf{C}}$=0.

3. Exceptional holonomy.

There is a remarkable feature of these $SU(4)$-instanton equation, which leads us to the second thread mentioned at the beginning of this article. This thread gives us, in some ways , a more direct generalisation of the familiar 3 and 4-dimensional picture. (Much of what follows was explained to the first author by Dominic Joyce: a general reference for exceptional holonomy is [**S**].) Consider the 8-dimensional, real, spin representation V of $Spin(7)$, and the standard embedding of $Spin(6)$ in $Spin(7)$. There is an exceptional isomorphism between $Spin(6)$ and $SU(4)$, so we get an embedding of $SU(4)$ in $Spin(7)$, and under this embedding V becomes the fundamental representation of $SU(4)$, of complex dimension 4. This means that a Calabi-Yau 4-fold, with holonomy group $SU(4)$, furnishes an example of an 8-manifold with an integrable $Spin(7)$-structure, where $Spin(7)$ acts via the spin representation. Now the second exterior power $\Lambda^2 V$ splits under the $Spin(7)$ action into two irreducible pieces, a copy of the adjoint representation (given by the infinitesimal action on V), and a complement, H say, of dimension 7. Restricting to $SU(4) \subset Spin(7)$, the representation H splits into

$$H = \mathbf{R}\omega \oplus \Lambda_+^{0,2}. \tag{10}$$

So we conclude that the $SU(4)$-instanton equation $F_+^{0,2} = F.\omega = 0$ has a further symmetry under $Spin(7)$: it is just the condition that the H component of the curvature vanishes, which makes sense on any manifold with a $Spin(7)$-structure. (So we will also refer to the equations as the $Spin(7)$-instanton equations.) This is rather similar, although the similarity does not seem to fit into the general pattern of this article, to the way in which the Hermitian-Yang-Mills equation on a Kähler surface (with holonomy $U(2)$) is just the ordinary instanton equation, and as such makes sense on any oriented Riemannian 4-manifold (with holonomy $SO(4)$). Higher dimensional versions of the instanton equation have been discussed by various authors (see [**C**], [**Wa**] for example), and these $Spin(7)$-instanton equations are part of a general theory developed by Salamon and Reyes-Carrion [**RC**].

We may also take these ideas over to 7 real dimensions, and the exceptional holonomy group G_2. Let us start on a different tack, and consider a compact oriented n-manifold N furnished with a closed $(n-3)$-form σ. Then we may consider a functional on the space of unitary connections on a bundle over N defined by the closed 1-form

$$\int_N \mathrm{Tr}(F \wedge \delta A) \wedge \sigma, \tag{11}$$

following the notation of (1). The critical points of this functional are the connections with $F \wedge \sigma = 0$. Linearising the theory leads to a complex which is a bundle-valued version of:

$$\Omega^0_N \xrightarrow{d} \Omega^1_N \xrightarrow{\sigma\wedge d} \Omega^{n-1}_N \xrightarrow{d} \Omega^n_N. \tag{12}$$

The condition that this complex be *elliptic* is an algebraic condition on σ at each point which (choosing a metric) essentially comes down to the condition that $*\sigma$ defines a non-degenerate skew-symmetric cross product $TM \times TM \to TM$. As is well-known, this leads to an almost complex structure on the $(n-1)$ sphere, and can only exist in dimensions $n = 3, 7$, where the algebraic models are the cross-product on the imaginary quaternions and octonians. The first case is the ordinary Floer theory. The second case operates best when we have a 7-manifold with holonomy G_2, and $*\sigma$ is the fundamental covariant constant 3-form, which can be taken as the definition of the structure. In this situation one gets an L^2 bound on the curvature, by an identity similar to (7). In sum then, on a manifold N with a G_2-structure, we have the basis for a Casson/Floer-type theory, in which the role of the flat connections is played by the solutions of the equation $F\wedge\sigma = 0$. The corresponding gradient lines are just the solutions of the $Spin(7)$ instanton equation on the manifold $N \times \mathbf{R}$ (which has a $Spin(7)$ structure, corresponding to the natural inclusion $G_2 \subset Spin(7)$). This picture interacts with the previous one, when we consider $N = Y \times S^1$, where Y is our Calabi-Yau manifold with holonomy $SU(3)$. For a general G_2 manifold, and a bundle $E \to N$, one could expect to have a Casson invariant, and a Floer homology theory which bears the same relation to the $Spin(7)$-instantons on a $Spin(7)$ manifold which is asymptotic to a tube $N \times [0, \infty)$, as in the ordinary case of three and four dimensions.

4. The two-dimensional picture.

We now return to the complex geometry strand of Section 2, and view things from the standpoint of complex dimension 2. Let S be a compact Calabi-Yau surface (so either a torus or a $K3$ surface), and let $M(S)$ be a moduli space of vector bundles over S (as usual, we ignore niceties involving stability). In favourable cases at least, this is a complex manifold with a complex symplectic structure due to Mukai [**M**]. The tangent space of $M(S)$ at a bundle E is the cohomology group $H^1(End\, E)$ and the symplectic form is given by the cup product $H^1 \otimes H^1 \to H^2$, composed with the trace pairing $End\, E \otimes End\, E \to \mathbf{C}$ and the evaluation map $H^2(\mathcal{O}) = H^2(K_S) \to \mathbf{C}$. More explicitly, if tangent vectors are represented by bundle-valued $(0,1)$ forms, the symplectic form is given by a formula analogous to (2):

$$\langle a, b \rangle = \int_S \mathrm{Tr}(a \wedge b) \wedge \theta_S. \tag{13}$$

Now suppose that S is embedded in a compact complex 3-fold Y^+, and that it is cut out as the zero-set of a section of the anticanonical bundle $K_{Y^+}^{-1}$ (i.e. there is a meromorphic 3-form on Y^+ with no zeros and with a simple pole along S—in this situation the adjunction formula forces S to be a Calabi Yau surface). We consider a moduli space L^+ of holomorphic bundles over Y^+ of the appropriate topological type, and the map from L^+ to $M(S)$ given by restriction of bundles. It is an observation of Tyurin [**Ty**], that the image is (roughly speaking) a complex Lagrangian submanifold of $M(S)$, with respect to the complex symplectic structure. At the level of tangent vectors, the situation is described by a portion of the long exact sequence associated with the restriction map, for a bundle E over Y^+:

$$H^1(Y^+, End\, E \otimes K_{Y^+}) \to H^1(Y^+, End\, E) \to H^1(S, End\, E|_S) \to$$

$$\to H^2(Y^+, End\, E \otimes K_{Y^+}) \to H^2(Y^+, End\, E).$$

The key thing is that this exact sequence is its own transpose with respect to Serre duality. This is a standard general fact about duality in complex geometry: if one works with Dolbeault cohomology it essentially comes down to the fact that $1/z$ is a fundamental solution for the $\overline{\partial}$-operator over $\mathbf{C}$. We suppose that the final term is zero: which is roughly speaking the assumption that L^+ is smooth and of the proper dimension. Then the first term also vanishes and we get a short exact sequence

$$0 \to TL^+ \to TM(S) \to T^*L^+ \to 0,$$

self-dual with respect to the symplectic form on $TM(S)$, and this just expresses the fact that TL^+ maps to a Lagrangian subspace under the restriction map.

We want next to consider the analogue of the Lagrangian intersection picture for the Casson invariant in the real case. Suppose that the same surface S is embedded as an anticanonical divisor in another compact 3-fold Y^-. Then we can form a singular space with a normal crossing singularity by gluing together the two copies of S: $Y_0 = Y^+ \cup_S Y^-$. Then we may consider deformations of Y_0 in a family Y_t such that Y_t is smooth. This is a standard kind of deformation problem, to which an extensive theory can be applied [**F**]. Locally around S, the picture can be modelled within the total space of the bundle $\nu_+ \oplus \nu_-$ over S, where $\nu_\pm$ are the normal bundles of S in $Y^\pm$ (which are the restrictions of the anti-canonical bundles). For any section ϵ of the line bundle $\nu_+ \otimes \nu_-$ over S the equation $s_+ s_- = \epsilon$ cuts out a 3-dimensional subvariety V_ϵ of the total space. Here $s^\pm$ are tautological sections of the lifts of $\nu^\pm$ to the total space. The double-crossing space Y_0 is modelled on V_0. If we can choose ϵ to have transverse zeros (forming a smooth curve Z in S) then V_ϵ will be smooth. If appropriate obstruction spaces vanish we can extend this local model to a 1-parameter family of deformations of the whole space, modelled near S on $V_{t\epsilon}$. Even if Z has simple nodes, so that V_ϵ has double points, we can get a family of smooth manifolds by making small resolutions (which will change the topology). A particular example of this comes if ϵ is the product of sections ϵ^+, ϵ^- of ν^+, ν^-, so that $Z = Z^+ \cup Z^-$ is a reducible curve. Then we can proceed in another way, in which the topology of the construction is rather transparent. We blow up Y^+ along Z^+ and Y^- along Z^- to get new 3-folds $\overline{Y}^+, \overline{Y}^-$. The proper transforms of S, which we denote by the same letter, in these manifolds have trivial normal bundles, and the smooth 3-fold we seek to construct is given, topologically, by cutting out tubular neighbourhoods of S in $\overline{Y}^+, \overline{Y}^-$ and gluing together the resulting boundaries, each of which is a product of S with a circle. It is particularly easy to see in this case that the canonical bundle of the manifold Y we make by deforming will be trivial, and in fact this is true for any ϵ. (A useful example in lower dimensions to have in mind comes by taking the rational elliptic surface—the projective plane blown up in nine points of intersection of two cubics. A fibre of the elliptic fibration is an anticanonical divisor and the fibre sum of two copies of this manifold yields a K3 surface, with trivial canonical bundle.)

Now a holomorphic bundle over the singular space Y_0 is given by a pair of bundles E^+, E^- over Y^+, Y^-, which are isomorphic over S. Thus, ignoring questions of stability etc., these holomorphic bundles correspond to intersection points of L^+, L^- in $M(S)$. The general idea should now be clear: we want to regard these Lagrangian intersection points as a limit of the holomorphic bundle moduli space on the smooth Calabi-Yau manifolds Y_t as the complex structure degenerates. So, for example, we would hope that the putative "holomorphic Casson invariant" should go over to the intersection number of L^+, L^- in $M(S)$.

5. Adiabatic limits and dimension reduction.

Here we consider briefly the analogue in the complex case of the link between holomorphic maps into the real moduli space $M(S_{\mathbf{R}})$ and ordinary instantons in 4 dimensions. To do this we take the product $X = T \times S$ of two Calabi-Yau surfaces (tori or K3 surfaces). This is a manifold with holonomy $Sp(1) \times Sp(1) \subset SU(4)$. We have simultaneous actions of the quaternions I, J, K on the tangent spaces of T and S. It is well-known that the moduli space $M(S)$—viewed as a moduli space of instantons—has an induced hyperkähler structure, so the quaternions also act on tangent vectors to $M(S)$. Now consider the $SU(4)$-instanton equations over the product, but with the metric on S scaled by a small factor ϵ. We can decompose the curvature F of a connection over the product into three pieces $F_S \in \Lambda^2(T^*S)$, $F_T \in \Lambda^2(T^*T)$, $F_{ST} \in \Lambda^1 T^*S \otimes \Lambda^1 T^*T$. The $SU(4)$-instanton equations then break up into two parts: one part is

$$F_S^+ = \epsilon F_T^+, \tag{14}$$

which makes sense since the bundles of self-dual forms on S, T are trivialised, and the other part is

$$\pi(F_{ST}) = 0, \tag{15}$$

where π is the projection from $\Lambda^1 T^*S \otimes \Lambda^1 T^*T$ to $\Lambda^{0,2}_+$. The second equation (14) does not involve ϵ. If we naively put $\epsilon = 0$, the first equation (13) tells us that the connection is an ordinary instanton on each copy of S in the product, so yields a map f from T to $M(S)$. The second equation goes over to a linear condition on the derivative of f. To see what this linear equation is, let U, V be quaternionic vector spaces and consider the real vector space $Hom_R(U,V)$ of $\mathbf{R}$-linear maps. This can be decomposed into four subspaces

$$Hom_R(U,V) = H_1 \oplus H_I \oplus H_J \oplus H_K,$$

where H_1 consists of the quaternion linear maps, H_I consists of the maps which are I-linear but J and K antilinear, and so on. So we get four projection maps $\pi_1, \pi_I, \pi_J, \pi_K$ to the different factors. Now we can apply this when U is the tangent space of T and V is the tangent space of $M(S)$: the condition on the map f which arises from the $SU(4)$ instanton equations is $\pi_J(df) = 0$. This is an elliptic equation which is the natural quaternionic analogue of the holomorphic mapping equation in the complex case. (Notice that the choice of this equation breaks the symmetry between I, J, K: this just comes from the fact that the $SU(4)$ equations depend on a particular complex structue on the product, and a particular holomorphic 2-form. We get a family of similar equations by interchanging I, J, K, which corresponds to different choices of $SU(4)$-structure on the product.)

We will now outline analogues of Hitchin's theory in [**H**], studying translation invariant solutions of the instanton equation. Let S^+ be the positive spin space of $\mathbf{R}^4$. Then $W = \mathbf{R}^4 \times S^+$ is a real 8-dimensional vector space with a $Spin(7)$-structure: that is the obvious action of $Spin(4)$ on W extends to the spin representaion of $Spin(7)$, under a certain embedding $Spin(4) \subset Spin(7)$. (Another way of saying this is to exhibit a certain natural 4-form on W, [**S**].) Let us consider solutions of the $Spin(7)$ instanton equation on the flat space W which are invariant under translations in the S^+ directions. These connections correspond to pairs consisting of a connection A on a bundle E over $\mathbf{R}^4$ and a Higgs field Φ, which is a section of $S^+ \otimes_{\mathbf{R}} ad(E) = S^+ \otimes_{\mathbf{C}} ad(E)^c$, where $ad(E)^c$ is the complexification of the bundle associated to the adjoint representation. The $Spin(7)$-instanton equations go over to equations of the shape:

$$F^+(A) = [\Phi, \Phi^*], \quad D_A\Phi = 0, \tag{16}$$

where D_A is the Dirac operator in 4-dimensions, coupled to the connection A, and the bracket in the first equation denotes the combination of the bracket on the Lie algebra with the map $S^+ \otimes \overline{S}^+ = S^+ \otimes S^+ \to \Lambda^+$. Having written the equations for flat space $\mathbf{R}^4$, we see that they make sense over any spin 4-manifold.

This is clearly a 4-dimensional analogue of Hitchin's theory, but also has obvious similarities with the renowned Seiberg-Witten equations in four dimensions (in a similar mould to the equations studied recently by Pidstragatch and Tyurin, Okonek and Teleman and others). If we look for reducible solutions, where the bundle E is $L \oplus L^{-1}$, the connection A is induced from a $U(1)$ connection on L, and Φ takes values in $L^2 \otimes S^+$, then we essentially arrive at the standard Seiberg-Witten equations.

We can play the same game with the G_2 equations $F \wedge \sigma = 0$ in 7-dimensions. Here there are two standard models. We can either look at $\mathbf{R}^3 \times S$, where S is the spin space of $\mathbf{R}^3$, or $\mathbf{R}^4 \times \Lambda^+$, where Λ^+ is the bundle of self-dual 2-forms. The first model leads to the 3-dimensional Seiberg-Witten equations, and non-abelian versions of these. The second leads to the Vafa-Witten equations for a pair (A, ϕ) consisting of a connection A over a 4-manifold and a section ϕ of $\Lambda^+ \otimes ad(E)$ [**VW**].

6. An example: quadrics in P^5.

We will now discuss an example which illustrates the ideas of Section 4 above. We consider a Calabi-Yau manifold $Y_1 \subset \mathbf{CP}^5$ which is the complete intersection of a quadric Q_0 and a quartic hypersurface V. We can degenerate V into a union of two quadrics $Q_1 \cup Q_2$, and in this way embed Y_1 in a family Y_t with Y_0 the union of two pieces $Y_0 = Y^+ \cup_S Y^-$, where

$$Y^+ = Q_0 \cap Q_1, \; Y^- = Q_0 \cap Q_2, \; S = Q_0 \cap Q_1 \cap Q_2.$$

So S is the intersection of 3 quadrics in $\mathbf{CP}^5$; a $K3$ surface. There is a wonderful explicit construction of a certain moduli space of holomorphic bundles over such a surface S, an example of Mukai duality. To explain this we must review some basic facts about quadrics in $\mathbf{P}^5$. If $Q \subset \mathbf{P}^5$ is any nonsingular quadric there are two families of planes in Q, the "α-planes" and "β-planes" in the language of twistor theorists. The α-planes through each point are parametrised by a copy of $\mathbf{P}^1$ and similarly for the β-planes. So we get two $\mathbf{P}^1$-bundles P_α, P_β over Q. (These can be lifted to vector bundles, but it easier to work with $\mathbf{P}^1$ bundles here.) Another way to view this is to identify Q with the Klein quadric $Gr_2(\mathbf{C}^4)$. Then P_α and P_β are the projectivisations of the tautological bundle U and the quotient bundle $\mathbf{C}^4/U$. However it is important to realise that in general there is a complete symmetry between P_α and P_β, with no preferred way to choose which one is which. (This happens because the orthogonal group has two components.) If we take instead a singular quadric Q', with one singular point, then there is just one family of planes in Q': so roughly speaking the bundles P_α and P_β coalesce as the quadric becomes singular.

Now consider the sextic curve $B \subset \mathbf{P}^2$ given by the equation

$$\det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2) = 0.$$

Each point $(\lambda_0, \lambda_1, \lambda_2)$ of $\mathbf{P}^2 \backslash B$ defines a nonsingular quadric containing the surface S. We get a double cover of this complement by fixing a choice of which of two bundles is to be P_α. For each point of this double cover we get a bundle over S, the restriction of the bundle P_α, for that quadric, to S. This construction extends over the curve B, where the quadrics become singular, but the cover becomes branched there (because the bundles coalesce, as mentioned above). The upshot is that the moduli space $M(S)$ of bundles of the relevant topological type over the K3 surface S is the double cover of the plane branched along the sextic curve B, which is another $K3$ surface.

Now consider the manifold $Y^+ = Q_0 \cap Q_1$. The quadrics through Y^+ are given by setting $\lambda_2 = 0$, and we can repeat the construction above to get a moduli space $L^+ \subset M(S)$ of bundles over Y^+ which is the double cover of the line $\{\lambda_2 = 0\}$ in $\mathbf{P}^2$, branched along the six intersection points with B. This is the description by Newstead of the moduli space of bundles of odd degree over a curve of genus 2 as the intersection of two quadrics, but seen in the opposite way: the universal bundle can be seen either as a family of bundles over the curve L^+ parametrised by Y^+ or vice-versa. In just the same way the subset L^-, parametrising bundles over Y^-, is the double cover of the line $\{\lambda_1 = 0\}$. The intersection $L^+ \cap L^-$ consists of two points and, following through the definitions, one sees that these just correspond to the restriction to Y_0 of the two bundles P_α and P_β over the original quadric Q_0. This suggests that the only stable bundles of the relevant topological type over the smooth Calabi-Yau manifold Y_1 are the restrictions of P_α, P_β. (Note that P_α, P_β have different topological type over Q_0 since they have different characteristic classes in $H^4(Q_0) = \mathbf{Z} \oplus \mathbf{Z}$, but this difference is not seen over Y_1 since $H^4(Y_1) = \mathbf{Z}$.)

7. Vanishing cycles and pseudoholomorphic curves.

Here we return to discuss the analogue of the Floer theory in our complex setting. The starting point for the ordinary Floer theory is the Morse theory of a real valued function, and it is well-known that the Picard-Lefschetz theory of monodromy and vanishing cycles is the complex analogue of the Morse theory, so it is not surprising that these ideas emerge from our discussion. If we have any Kähler manifold Z and a holomorphic function $\phi : Z \to \mathbf{C}$ with nondegenerate critical points we can define the complex gradient flow equation, for a map $\Gamma : S^1 \times \mathbf{R} \to Z$,

$$\overline{\partial}\Gamma = \widehat{d\phi}. \tag{17}$$

Here $\widehat{d\phi}$ is the tangent vector obtained from the derivative of ϕ using the metric. This is a deformation of the holomorphic mapping equation, and fits into a class which has been studied quite extensively in that setting. Indeed if H is any real valued function on Z there is a standard deformation of the holomorphic mapping equation given, from one point of view, by adding

$$\int_{S^1} H \circ \gamma$$

to the symplectic action functional for maps $\gamma : S^1 \to Z$. The equation (17) is just this standard deformation for the function $H = Re(\phi)$. Pursuing this line, one sees that there will generically be no solutions of our equation (17) which interpolate between different critical points of ϕ (which are also the critical points of H), essentially because the Morse indices of H at all these critical points are the same. However we can bring in the fact that H came from a holomorphic function by considering the family of equations like (17) parameterised by a circle, just multiplying ϕ by a complex number λ of unit modulus. The simplest solutions are those which are invariant under the rotations acting on $S^1 \times \mathbf{R}$: these are just the ordinary gradient lines of H. Standard Morse Theory arguments give the following picture. For each pair of critical points p^+, p^- we count the S^1-invariant solutions of the family of deformed equations, as λ varies over the circle (but dividing of course by the obvious action of translations) to give a number $n(p^+, p^-)$. Now for each critical point p there is a *vanishing cycle* $W(p)$, well-defined up to isotopy, in any nearby fibre. We take the straight line in $\mathbf{C}$ between $\phi(p^+)$ and $\phi(p^-)$, which generically does not contain any other critical value. Parallel transport along this line allows us to regard the vanishing cycles $W(p^+), W(p^-)$ as submanifolds of the same fibre, and $n(p^+, p^-)$ is the intersection number of these two vanishing cycles. (The data from the more general moduli spaces of solutions, not rotation invariant, describes the action of the "quantum multiplication" in the fibres of ϕ on the vanishing cycles.) The set of numbers $n(p^+,p^-)$, as p^+,p^- range over the different critical points, together with knowledge of the location of the critical values in $\mathbf{C}$, and the reflection formula for the monodromy around a single critical value, gives a complete description of the monodromy action of the fundamental group of $\mathbf{C}\setminus\{\text{critical values}\}$ on the part of the homology of the fibre generated by the vanishing cycles. This is the complex analogue of the boundary operator in the familar Morse/Floer picture, defined by gradient lines, which gives a complete description of the homology of the space. Just as in that case, the numbers $n(p^+,p^-)$ themselves can change under continuous deformations of the set up (Z,ϕ). In the complex case this change comes about when a critical point p' moves across the line segment between p^-,p^+, so one is essentially changing the choice of homotopy class of path used to identify fibres around different critical points.

Now we can take these ideas over to the gauge theory case, where Z becomes the space $\mathcal{C}_E$ of equivalence classes of $\overline{\partial}$-operators on a bundle E over our Calabi-Yau manifold Y (or, perhaps better, a suitable covering space), and ϕ becomes our holomorphic Chern-Simons functional Φ. Multiplying the map Φ by a scalar just corresponds to changing the choice of holomorphic volume form on Y, and hence on $X=\mathbf{R}\times S^1\times Y$. What we expect then is that for each pair of holomorphic bundles E^+,E^- of the same topological type over Y we can define a number $n(E^+,E^-)$ by counting the rotation-invariant solutions of the 1-parameter family of $SU(4)$-instanton equations over X, asymptotic to $E^\pm$ at $\pm\infty$, and that these numbers can be interpreted as giving the monodromy action on the semi-infinite dimensional cohomology of the fibres of Φ. The data from other moduli spaces (not rotation-invariant) should describe the quantum multiplication between the semi-infinite and finite dimensional cohomology of the fibres.

We finish this discussion with one more remark. In a finite-dimensional situation the index associated to the Cauchy-Riemann equations for mappings from a closed Riemann surface Σ of genus g to a complex manifold Z is

$$\langle c_1(Z),[\Sigma]\rangle-(2g-2)\dim Z. \tag{18}$$

We get the same index for any deformed equation, which merely adds lower order terms. Now in our case we take Σ to be a 2-torus, and Z to be the infinite-dimensional space $\mathcal{C}_E$. For simplicity we take the gauge group of our connections to be $SU(2)$. From what we have seen above, $SU(4)$-instantons on a bundle E' over $\Sigma\times Y$ can be interpreted as solutions of a deformation of the holomorphic mapping equation. The familiar slant product construction gives an isomorphism $\mu: H_2(Y)\to H^2(\mathcal{C})$. Under this isomorphism the homology class of Σ in $\mathcal{C}$ can be indentified with the component of $c_2(E')$ in $H^2(Y)=H^2(Y)\otimes H^2(\Sigma)\subset H^4(Y\times\Sigma)$. On the other hand the index associated to the elliptic $SU(4)$-instanton equation in this situation is just $\langle\frac{1}{6}c_2(Y)c_2(E')-\frac{2}{3}c_2(E')^2,\Sigma\times Y\rangle$. The upshot is that the formula (18) is true in the infinite dimensional case if we *define*

$$c_1(\mathcal{C}_E)=\tfrac{1}{6}\mu(PD(c_2(Y)-8c_2(E)),$$

where $PD(c_2(Y)-8c_2(E))\in H_2(Y)$ is the Poincaré dual. The point here is that this Chern class is not defined in any conventional sense for general infinite dimensional complex manifolds (since the infinite general linear group is contractible), but our index theory allows us to give a meaning to it. Notice also that whereas in finite dimensions the deformations of the holomorphic mapping equation do not play any role in the index theory, the analogous deformation is crucial to the discussion above. The condition for a genuine holomorphic mapping from Σ to $\mathcal{C}_E$ can be interpreted as a gauge theory problem on $\Sigma \times Y$, but leads to a nonelliptic equation which is not governed by any index theorem.

8. Submanifolds.

In this article we have discussed gauge theory over various manifolds of dimensions $4, 6, 7, 8$. We close by pointing out that there are analogues of most of our constructions which bear instead on special *submanifolds*, in the framework of Harvey and Lawson's "calibrations" [**HL**]. This is a very active research area at the moment, in part because of connections with "mirror symmetry". The viewpoint which we arrive at, by analogy with the gauge theory set-up, is perhaps new.

On the one hand, returning to our Calabi-Yau 3-fold Y, it is natural to expect connections between counting holomorphic bundles (at least of rank 2), and counting complex *curves.* Indeed one of the standard procedures in algebraic geometry is to go from a rank 2 bundle over a 3-dimensional variety to a curve by taking the zero set of a generic section. On the other hand, we can forget about bundles, and mimic the general scheme in this article, replacing connections by suitable submanifolds.

So let us consider our Calabi-Yau 3-fold Y, with a fixed Kähler metric ω with holonomy $SU(3)$. Let $\mathcal{S}$ be the space of (real) 2-dimensional submanifolds Σ, representing a given homology class in Y, which are symplectic with respect to ω; i.e. such that $\omega|_\Sigma$ is non-degenerate. Fix a base point Σ_0 in $\mathcal{S}$, and let $\widetilde{\mathcal{S}}$ be the covering of $\mathcal{S}$ consisting of pairs $(\Sigma, [Z])$ where $[Z]$ is an equivalence class of 3-chains Z in Y with boundary $\Sigma - \Sigma_0$, and $Z \sim Z'$ if $[Z - Z']$ is zero in $H_3(Y)$. Then we can define a functional

$$\Psi : \widetilde{\mathcal{S}} \to \mathbf{C}$$

by

$$\Psi(\Sigma, [Z]) = \int_Z \theta.$$

It is an easy exercise to show that the critical points of Ψ correspond to the *holomorphic curves* in Y. Now we can define a (real) gradient equation associated to this functional as follows. If ξ, η are independent tangent vectors to Σ at a point p, let v be a tangent vector corresponding, under the metric on Y, to the 1-form

$$\frac{1}{\omega(\xi, \eta)}\, \xi \lrcorner\, \eta \lrcorner\, \mathrm{Re}(\theta).$$

This does not depend on the choice of basis ξ, η for $T_p\Sigma$, and we denote it by $v_\Sigma(p)$. Thus we get a vector field v_Σ along Σ in Y which can be regarded as a tangent vector in $\mathcal{S}$. Then the gradient equation is $\partial\Sigma/\partial t = v_\Sigma$. Similarly, we can define a complex gradient equation for a map from $\mathbf{C}$ to $\mathcal{S}$:

$$\frac{\partial \Sigma}{\partial t} + i\frac{\partial \Sigma}{\partial s} = v_\Sigma.$$

The surprising thing is that these equations have straightforward interpretation in 7 and 8 dimensions. Consider the model $\mathbf{R}^7 = \mathbf{R}^4 \times \Lambda^+(\mathbf{R}^4)$ for the imaginary octonians. A 3-dimensional subspace of $\mathbf{R}^7$ is called *associative* if it lies in the G_2-orbit of $\Lambda^+(\mathbf{R}^4)$ in the Grassmannian of 3-planes in $\mathbf{R}^7$. This leads to the notion of an *associative* 3-dimensional submanifold of a manifold N^7 with a G_2-structure. In the same way the model $\mathbf{R}^8 = \mathbf{R}^4 \times S_+(\mathbf{R}^4)$ leads to the notion of a *Cayley submanifold*, of dimension 4, in an 8-manifold with a $Spin(7)$ structure.

Suppose Σ_t is a 1-parameter family of surfaces in Y. The family may be considered in an obvious way as a 3-dimensional submanifold Γ of $Y \times \mathbf{R}$, and a little thought shows the real gradient equation precisely goes over to the condition that Γ be an associative submanifold. Similarly for the complex gradient equation and Cayley submanifolds of $Y \times \mathbf{C}$. In the same spirit, we can look at the space of 3-manifolds in a manifold N with a G_2 structure, and get a gradient flow equation whose solutions are the Cayley submanifolds in $N \times \mathbf{R}$. In sum, we might hope that there are Floer theories etc. involving these special submanifolds in close parallel to the gauge theory structures we have discussed in this article. The "special Lagrangian" submanifolds of Y can also be brought into this picture: they are just the associative submanifolds in $Y \times \mathbf{R}$ which lie in a single fibre $Y \times \{t\}$: they can be interpreted as gradient lines taking the empty set to itself. Similarly for the co-associative submanifolds in a seven manifold N with a G_2 structure, which appear as the "vertical" Cayley submanifolds in $N \times \mathbf{R}$. In the analogy between the gauge theory and submanifold theory, the L^2 curvature identities like (7) go over to the familiar volume identities for calibrated submanifolds of Harvey and Lawson.

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