Additional Exercises for ‘Topics in Geometry’.

Connections and Curvature.

Exercise 1. Let V be a vector space over \( \mathbb{R} \) of dimension \( n \). We consider multilinear maps

\[ R : V \times V \times V \times V \rightarrow \mathbb{R} \]

which are ‘algebraic curvature tensors’ in the sense that

\[
R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z) \\
R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.
\]

for all \( x, y, z, w \in V \).

(i) Any such \( R \) satisfies \( R(x, y, z, w) = R(z, w, x, z) \), and if \( R(x, y, x, y) = 0 \) for all \( x, y \in V \), then \( R = 0 \).

(ii) The dimension of the space of algebraic curvature tensors is \( n^2(n^2 - 1)/12 \).

(iii) Assume now that V carries an inner product \( \langle \cdot, \cdot \rangle \). The multilinear map \( Q \) given by \( Q(x, y, z, w) = (x, z)(y, w) - (y, z)(w, x) \) is an algebraic curvature tensor. Suppose that \( R \) is an algebraic curvature tensor, and define \( K(P) = R(p_1, p_2, p_1, p_2) \) for any plane \( P \subset V \) with orthonormal basis \( (p_1, p_2) \). If there exists a constant \( K \) with \( K = K(P) \) for all \( P \), then \( R = KQ \). [Show that for any basis \( (x, y) \) of \( P \) we have \( K(P) = R(x, y, x, y)/Q(x, y, x, y) \) and use (i).]

(iv) Use (iii) to deduce an expression for the Riemann curvature tensor of \( S^n \).

Exercise 2. Let \( \gamma \) be a loop in \( S^2 \) with \( p = \gamma(0) = \gamma(1) \). The parallel transport map \( P_\gamma \) is in \( SO(TS^2_p) \), and hence corresponds to an angle \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \) (‘holonomy angle’).

(i) Compute the holonomy angle in the case where \( \gamma \) is circle of latitude.

(ii) More generally, derive an expression for the holonomy angle for any simple closed loop \( \gamma \). [Use the Gauss-Bonnet formula.]

Chern Classes.

Exercise 3. Suppose that the tangent bundle of real projective \( n \)-space is trivial. Show that \( n + 1 \) is a power of 2. [Compute the total Stiefel-Whitney class of the tangent bundle].
Exercise 4. Let $M$ be a compact oriented smooth manifold of dimension $m$.

(i) Let $a_1, \ldots, a_r$ be a basis of $H^*(M, \mathbb{Q})$, and $b_1, \ldots, b_r$ be the dual basis. Then the Poincare dual of the diagonal $\Delta \subset M \times M$ can be expressed as

$$\delta = \sum_{k=1}^{r} (-1)^{|a_k|} a_k \times b_k,$$

where $\times$ is the cross product in cohomology. [Show that both sides have the same intersection form with $b_i \times a_j$ for all $i, j$ with $|b_i| + |a_j| = m$.]

(ii) Use (i) to deduce the equality

$$\int_M e(T_M) = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{Q}} H^i(M, \mathbb{Q}).$$

How is this related to the Gauss-Bonnet theorem and the Poincare-Hopf theorem? [Let $\Delta : M \to M \times M$ be the diagonal map. Then $T_M \simeq \Delta^* N_{\Delta/M \times M}$ implies $e(T_M) = \Delta^* \delta$.]

Exercise 5. (i) The tangent bundle of a Lie group $G$ is trivial, in particular orientable. Use exercise 5 to conclude that $\chi(G) = 0$ if $G$ is compact.

(ii) Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$. If $G$ is not commutative, then $H^3(G; \mathbb{R}) \neq 0$. [Let $\{-, -\}$ be a bi-invariant Riemannian metric on $G$. The multilinear map $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $(x, y, z) \mapsto ([x, y], z)$ induces a bi-invariant 3-form $\theta$ on $G$. Prove that $\theta$ is closed but not exact (if it were, we would have $\theta = 0$).]

(iii) For which $n$ does $S^n$ admit a Lie group structure? [A commutative compact connected Lie group must be a torus (the exponential map is a surjective morphism of Lie groups).]

(iv) Show that the tangent bundle of $S^7$ is trivial. [Use the octons to define a trivialisation.]

Exercise 6. Assume there exists a polynomial $T_d(T_1, T_2, T_3) = \alpha T_1^3 + \beta T_1 T_2 + \gamma T_3$ such that for every smooth projective 3-fold $X$ we have

$$\chi(T_X) = \int_X T_d(c_1, c_2, c_3)$$

Show that $\alpha = \gamma = 0$ and $\beta = 1/24$. [Consider $X = \mathbb{P}^3, \mathbb{P}^2 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to get a system of three equations in $\alpha, \beta, \gamma$.]

Exercise 7. (i) Let $C \subset S$ be a smooth curve in a smooth projective surface $S$. Prove the formula

$$\int_C c_1(N_{C/S}) = \int_S D[C] c_1(S) - \chi(C).$$

(1) With respect to the intersection form.
where $D[C]$ is the Poincaré dual of $[C]$.

(ii) Use (i) to deduce the degree-genus formula for $C \subset S = \mathbb{P}^2$.

**Exercise 8.** (i) Show that there exists an exact sequence (Euler sequence)

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\mathbb{P}^n} \overset{1}{\rightarrow} \mathbb{P}^n \rightarrow 0.$$  

How is it related to the tautological exact sequence? Compute $c(T_{\mathbb{P}^n})$.

(ii) Compute the Euler characteristic

$$\chi(X) = \int_X c_{n-1}(T_X)$$

of a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d$. [Consider the short exact sequence $0 \rightarrow T_X \rightarrow T_{\mathbb{P}^n} \mid X \rightarrow N_{X/\mathbb{P}^n} \rightarrow 0$.]

**Exercise 9.** (i) Let $\mathcal{E}$ be a vector bundle of rank $e$, and $\mathcal{L}$ a line bundle. Prove

$$c_t(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^e c_j(\mathcal{E})c_t(\mathcal{L})^{e-j}t^j.$$  

(ii) Let $\mathbb{P}^n = \mathbb{P}(V)$ with tautological subbundle $\mathcal{O}$ and quotient bundle $\mathcal{Q}$, and $q$ (resp. $p$) denote the first (resp. second) projection of $\mathbb{P}^n \times \mathbb{P}^n$. Construct a morphism of bundles $q^*\mathcal{O} \rightarrow p^*\mathcal{Q}$ whose zero locus is exactly the diagonal $Δ \subset \mathbb{P}^n \times \mathbb{P}^n$.

[Use the tautological exact sequence; at a point $(x, y) \in \mathbb{P}^n \times \mathbb{P}^n$ corresponding to $L_x, L_y \subset V$ the map on fibres should be $L_x \rightarrow V/L_y$.]

(iii) Use (i) and (ii) to compute the class

$$\delta \in H^n(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$$

Poincaré dual to the diagonal.

**Exercise 10** (Yau). (i) Consider the intersection $Z \subset \mathbb{P}^3 \times \mathbb{P}^3$ of the hypersurfaces

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,$$

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0,$$

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0.$$  

Compute $\chi(Z) = -18$. [View $Z$ as the zero scheme of a section of the vector bundle $\mathcal{E} = \mathcal{O}(3, 0) \oplus \mathcal{O}(0, 3) \oplus \mathcal{O}(1, 1)$. Note that

$$\left(\frac{c(T_{\mathbb{P}^3 \times \mathbb{P}^3})}{c(\mathcal{E})}\right)_3 c_3(\mathcal{E}) = -18\alpha^3\beta^3$$

in $H^n(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$.]

(ii) Let $\zeta$ be a primitive third root of unity, and consider the automorphism $\sigma = \ldots$
ADDITIONAL EXERCISES FOR ‘TOPICS IN GEOMETRY’

\[ \sigma_1 \times \sigma_2 \text{ of } \mathbb{P}^3 \times \mathbb{P}^3 \text{ given by} \]

\[ \sigma_1([x_0 : x_1 : x_2 : x_3]) = [x_1, x_2, x_0, \xi x_3], \]

\[ \sigma_2([y_0 : y_1 : y_2 : y_3]) = [y_1, y_2, y_0, \xi^2 y_3]. \]

Show that the cyclic group \( \Sigma \) generated by \( \sigma \) acts freely on \( Z \), and conclude that \( X = Z/\Sigma \) has \( \chi(X) = -6 \). [If a finite group \( G \) acts on a compact manifold \( M \), then \( \chi(M^G) = \frac{1}{|G|} \sum_{g \in G} \chi(M^g) \).

Complex Manifolds.

Exercise 11. (i) Let \( X \) be a smooth projective variety of dimension \( d \). Show that \( H^{2k}(X; \mathbb{C}) \neq 0 \) for \( k \leq d \). [Embed \( X \) into some projective space and consider the intersection of \( X \) with linear subspace.]

(ii) Which spheres \( S^n \) can be the underlying topological space of a smooth projective variety?

Exercise 12. (i) Show that if \( S^n \) admits an almost complex structure, then \( S^{n+1} \) is parallelisable. [Let \( e_1, \ldots, e_{n+2} \) be the standard basis of \( \mathbb{R}^{n+2} \), view \( S^n \) as the unit sphere in \( \mathbb{R}^{n+1} = \langle e_1, \ldots, e_{n+1} \rangle \). Use the almost complex structure \( J \) on \( S^n \) to define for every \( p \in S^{n+1} \) a linear map \( \sigma_p : \mathbb{R}^{n+1} \to TS^{n+1}_p \) such that the vector bundle map \( \sigma : S^{n+1} \times \mathbb{R}^{n+1} \to TS^{n+1}, (p, v) \mapsto (p, \sigma_p(v)) \) is an isomorphism. Note that any \( p \in \mathbb{R}^{n+2} \) can uniquely be written as \( p = \alpha e_{n+2} + \beta s \) with \( s \in S^n, \alpha \in \mathbb{R}, \beta \geq 0 \).

(ii) View \( S^6 \) as the purely imaginary octonions of norm one, and use octonionic multiplication to define an almost complex structure on \( S^6 \). Compute the Nijenhuis tensor to show that it is not integrable.

Exercise 13 (Borel-Serre). Show that if a sphere \( S^{2n} \) admits an almost complex structure, then \( n \leq 3 \). [If \( S^{2n} \) has an almost complex structure, then the tangent bundle \( T \) of \( S^{2n} \) is a complex vector bundle. Compute the Chern character of \( T \) to see that the top-dimensional part is \( c_n(T)/(n-1)! \). Assume that \( c_n(T) \) is divisible by \( (n-1)! \) in integral cohomology (this is nontrivial), and use exercise 5 to conclude that \( 2 \) is divisible by \( (n-1)! \).

Hodge Theory.

Exercise 14. (i) Compute the Hodge numbers of \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \).

(ii) Compute the Chern and Hodge numbers of \( \mathbb{P}^3 \) and a quadric threefold.

Exercise 15. Let \( X \) be a compact Kaehler manifold, and \( Z \) a complex submanifold of codimension \( c \). Show that the Poincaré dual of \( [Z] \) lies in \( H^{c,c}(X) \).

Exercise 16 (H.-C. Wang). Let \( X \) be a compact Kaehler manifold. Then \( T_X \) is trivial if and only if \( X \) is a torus. [Show that the Albanese map is an etale covering.]
Exercise 17. Let $X$ be a compact connected Kaehler manifold with vanishing Ricci curvature.

(i) If $\omega$ is a holomorphic $p$-form, then $\nabla \omega = 0$. [Compute $\Delta_d \omega = \nabla^* \nabla \omega$. Notice that $\Delta_d \omega = 0$, and integrate over $X$ to conclude.]

(ii) Let $x \in X$. Use (i) to deduce that the map $H^0(X, \Omega^p) \to \left(\wedge^p (T_{1,0}^*X)^\vee\right)^{\text{Hol}_X(X)}$ given by $\omega \mapsto \omega(x)$ is an isomorphism.

(iii) Assume that $\text{Hol}_X(X) = SU(\dim X)$. Show that $H^0(X, \Omega^p) = 0$ for $0 < p < \dim X$. [Use (i), and show that the representation $\wedge^p \sigma^\vee$ (where $\sigma$ is the standard representation of $SU(\dim X)$) is irreducible.]

Geometric Invariant Theory.

Exercise 18. We consider the action of $\mathbb{C}^*$ on $\mathbb{C}^4$ with weight $(1, 1, -1, -1)$.

(i) Show that the algebra of invariants can be identified with

$$A_0 = \mathbb{C}[X, Y, Z, W]/(XW - YZ).$$

(ii) To form a GIT quotient, one also needs a linearisation. In our case this is nothing but a $\mathbb{Z}$-grading on $A[Q]$, where $A$ is the polynomial ring $\mathbb{C}[X, Y, Z, W]$ with $\mathbb{Z}$-grading corresponding to the action of $\mathbb{C}^*$ (i.e., $X, Y \in A_1, Z, W \in A_{-1}$, and $A_0$ is as in (i)); the GIT quotient is then $\text{Proj}(A[Q]_0)$. Consider the three gradings on $A[Q]$ determined by $Q \in A[Q]_{-1}, Q \in A[Q]_0, Q \in A[Q]_1$, and denote by $X_-, X_0, X_+$ the corresponding GIT quotients. Identify $X_0$ with $\text{Spec}(A_0)$, and $X_-$ (resp. $X_+$) with the blow up of $X_0$ along along $(X, Z)$ (resp. $(Y, W)$). The induced rational map $X^- \to X^+$ is the Atiyah flop.

Equivariant Cohomology.

Exercise 19. Let $G = \text{Gr}(2, V)$ be the Grassmannian of lines in $\mathbb{P}^3 = \mathbb{P}(V)$, with tautological bundles $\mathcal{E}$ and $\mathcal{Q}$. The torus $T = (\mathbb{C}^*)^4$ acts on $\mathbb{P}^3$ by

$$(t_0, t_1, t_2, t_3)[x_0 : x_1 : x_2 : x_3] = [t_0^{-1}x_0 : t_1^{-1}x_1 : t_2^{-1}x_2 : t_3^{-1}x_3].$$

(i) Show that there is an induced action of $T$ on $G$, and that the fixed locus $G^T$ consists of the 6 lines $L_\lambda$ (where $\lambda = (\lambda_1, \lambda_2)$ satisfies $0 \leq \lambda_1 < \lambda_2 \leq 3$) given by $x_i = 0, i \neq \lambda_1, \lambda_2$. For each $\lambda$, compute the $T$-equivariant Chern classes of the $T$-equivariant vector bundles $\mathcal{F}_{L_\lambda} = L_\lambda$ and $N_{L_{1,0}/G} = T_{G, L_{1,0}} = \text{Hom}(L_{1,0}, V/L_{1,0})$ over $\text{Spec}(\mathbb{C})$. (These are nothing but linear representations of $T$; the $T$-equivariant Chern classes are elements of $H^*_T(*) \simeq \text{Sym}^*(T^\vee \otimes \mathcal{Q})$, where $T^\vee$ is the group of characters of $T$.)

(ii) Use (i) and the Atiyah-Bott integration formula to compute

$$\chi(G) = \int_G c_4(T_G).$$
(iii) Use (i) and the Atiyah-Bott integration formula to compute
\[ \int_G c_1(\mathcal{F})^4 = 2. \]
and give a geometric interpretation of the result. [For the interpretation make use of the definition of Chern classes via degeneracy loci; it is convenient to consider \( c_1(\mathcal{Q}) = c_1(\mathcal{F}) \), since \( H^0(G, \mathcal{F}) = 0 \) and \( H^0(G, \mathcal{Q}) = V \).]

(iv) Use and the Atiyah-Bott integration formula to compute
\[ \int_G c_4(\text{Sym}^3(\mathcal{F}^\vee)) = 27 \]
and give a geometric interpretation of the result.

**Deformation Theory.**

*Exercise 20.* Let \( \mathcal{A} \) be the category of Artin local \( \mathbb{C} \)-algebras with \( \mathbb{C} \).

(i) Let \( X \) be an algebraic scheme, and \( x \in X \) a closed point. Consider the functor
\[ F = h_{x,x} : \mathcal{A} \to \text{Set} \]
given by letting \( F(\mathcal{A}) \) be the set of morphisms of schemes \( f : \text{Spec}(\mathcal{A}) \to X \) whose underlying map of spaces takes \( \text{Spec}(\mathcal{A}) \) to \( \{x\} \). Show that \( F \) is functorially isomorphic to \( \text{Hom}(\hat{\theta}_{x,x}, -) \).

(ii) Take \( X = \text{Spec}(\mathbb{C}[U,V]/(UV)), x = (U,V) \). Show that \( t_F = F(\mathbb{C}[T]/(T^2)) \) is a \( \mathbb{C} \)-vector space of dimension 2, with basis \( e, f \) given by \( e(U) = T, e(V) = 0, f(U) = 0, f(V) = T \).

(iii) Show that an element \( v = ae + bf \in t_F \) lifts to a morphism
\[ V : \mathbb{C}[[U, V]]/(UV) \to \mathbb{C}[T]/(T^3) \]
if and only if \( a = 0 \) or \( b = 0 \).