Distance from a Point to an Ellipse

Let \( \vec{p} = (x, y) \in \mathbb{R}^2 \). Let the ellipse, which is assumed to be centred at the origin, be given by the parameterization \( \vec{x}(\theta) := r (\alpha \cos \theta, \beta \sin \theta)^T \), or equivalently by the equation
\[
\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = r^2.
\]
A necessary condition for \( \vec{x} \) to be the closest point to \( \vec{p} \) is that \( \vec{p} - \vec{x} \) is perpendicular to the tangent vector in \( \vec{x} \), i.e. \( (\vec{p} - \vec{x}(\theta)) \cdot \vec{x}'(\theta) = 0 \).

This gives
\[
f(\theta) := (\alpha^2 - \beta^2) r \cos \theta \sin \theta - x \alpha \sin \theta + y \beta \cos \theta = 0.
\]
The above equation can be solved with a Newton method, where one notes that
\[
f'(\theta) = (\alpha^2 - \beta^2) r (\cos^2 \theta - \sin^2 \theta) - x \alpha \cos \theta - y \beta \sin \theta.
\]
With the initial guess \( \theta^{(0)} = \tan^{-1}\left(\frac{\alpha y}{\beta x}\right) \equiv \text{atan}2(\alpha y, \beta x) \), the iteration always converges in practice, usually taking only a couple of iterations. Of course, the desired distance is then given by \( |\vec{p} - \vec{x}(\theta)| \).

Distance from a Point to an Ellipsoid

The same idea generalizes to 3d. Let \( \vec{p} = (x, y, z) \in \mathbb{R}^3 \) and let the ellipsoid, which is again assumed to be centred at the origin, be given by the parameterization
\[
\vec{x}(\theta, \phi) := r \begin{pmatrix} \alpha \cos \phi \cos \theta \\ \beta \cos \phi \sin \theta \\ \gamma \sin \phi \end{pmatrix},
\]
or equivalently by the equation
\[
\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 + \left(\frac{z}{\gamma}\right)^2 = r^2.
\]
A necessary condition for \( \vec{x} \) to be the closest point to \( \vec{p} \) is that \( \vec{p} - \vec{x} \) is perpendicular to the tangent plane in \( \vec{x} \), i.e.
\[
(\vec{p} - \vec{x}) \cdot \frac{\partial \vec{x}}{\partial \theta} = 0, \quad (\vec{p} - \vec{x}) \cdot \frac{\partial \vec{x}}{\partial \phi} = 0.
\]
This gives
\[
0 = (\alpha^2 - \beta^2) r \cos \theta \sin \theta \cos \phi - x \alpha \sin \theta + y \beta \cos \theta,
0 = (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta - \gamma^2) r \sin \phi \cos \phi \cos \theta - x \alpha \sin \phi \cos \theta - y \beta \sin \phi \sin \theta + z \gamma \cos \phi,
\]
The above system \( 0 = F(\theta, \phi) \) can be solved with a Newton method, where one notes that \( DF(\theta, \phi) = (a_{ij}) \in \mathbb{R}^{2 \times 2} \) is given by
\[
a_{11} = (\alpha^2 - \beta^2) r (\cos^2 \theta - \sin^2 \theta) \cos \phi - x \alpha \cos \theta - y \beta \sin \theta,
\]
\[
a_{12} = -(\alpha^2 - \beta^2) r \cos \theta \sin \theta \sin \phi,
\]
\[
a_{21} = -2 (\alpha^2 - \beta^2) r \cos \theta \sin \theta \sin \phi \cos \phi + x \alpha \sin \phi \cos \theta - y \beta \sin \phi \sin \theta,
\]
\[
a_{22} = (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta - \gamma^2) r (\cos^2 \phi - \sin^2 \phi) - x \alpha \cos \phi \cos \theta - y \beta \cos \phi \sin \theta - z \gamma \sin \phi.
\]
With the initial guess \( (\theta^{(0)}, \phi^{(0)}) = (\tan^{-1}\left(\frac{\alpha y}{\beta z}\right), \tan^{-1}\left(z/\sqrt{\gamma ((\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2)}\right)) \equiv (\text{atan}2(\alpha y, \beta z), \text{atan}2(z, \gamma ((\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2) \frac{1}{2})) \), the iteration always converges in practice, usually taking only a couple of iterations to converge, with the desired distance given by \( |\vec{p} - \vec{x}(\theta, \phi)| \). This is a significant advantage over other published methods.

See e.g. www.geometric-tools.com/Documentation/DistancePointToEllipsoid.pdf by David Eberly, where a possibly ill-conditioned sixth degree polynomial needs to be solved.