Dr Robert Nürnberg, Imperial College London

Distance from a Point to an Ellipse

Let $\vec{p} = (x, y) \in \mathbb{R}^2$. Let the ellipse, which is assumed to be centred at the origin, be given by the parameterization $\vec{x}(\theta) := r\,(\alpha\,\cos\theta, \beta\,\sin\theta)^T$, or equivalently by the equation

$$(\tfrac{x}{\alpha})^2 + (\tfrac{y}{\beta})^2 = r^2.$$

A necessary condition for $\vec{x}$ to be the closest point to $\vec{p}$ is that $\vec{p} - \vec{x}$ is perpendicular to the tangent vector in $\vec{x}$, i.e. $(\vec{p} - \vec{x}(\theta))\,.\,\vec{x}'(\theta) = 0$.
This gives

$$f(\theta) := (\alpha^2 - \beta^2)\,r\,\cos\theta\sin\theta - x\,\alpha\,\sin\theta + y\,\beta\cos\theta = 0\,.$$

The above equation can be solved with a Newton method, where one notes that

$$f'(\theta) = (\alpha^2 - \beta^2)\,r\,(\cos^2\theta - \sin^2\theta) - x\,\alpha\,\cos\theta - y\,\beta\,\sin\theta.$$

With the initial guess $\theta^{(0)} = \tan^{-1}(\frac{\alpha\,y}{\beta\,x}) \equiv \texttt{atan2}(\alpha\,y,\beta\,x)$, the iteration always converges in practice, usually taking only a couple of iterations. Of course, the desired distance is then given by $|\vec{p} - \vec{x}(\theta)|$.

Distance from a Point to an Ellipsoid

The same idea generalizes to $3d$. Let $\vec{p} = (x, y, z) \in \mathbb{R}^3$ and let the ellipsoid, which is again assumed to be centred at the origin, be given by the parameterization

$$\vec{x}(\theta, \phi) := r\begin{pmatrix}\alpha\,\cos\phi\cos\theta\\ \beta\,\cos\phi\sin\theta\\ \gamma\,\sin\phi\end{pmatrix},$$

or equivalently by the equation

$$(\tfrac{x}{\alpha})^2 + (\tfrac{y}{\beta})^2 + (\tfrac{z}{\gamma})^2 = r^2.$$

A necessary condition for $\vec{x}$ to be the closest point to $\vec{p}$ is that $\vec{p} - \vec{x}$ is perpendicular to the tangent plane in $\vec{x}$, i.e.

$$(\vec{p} - \vec{x})\,.\,\frac{\partial\vec{x}}{\partial\theta} = 0, \qquad (\vec{p} - \vec{x})\,.\,\frac{\partial\vec{x}}{\partial\phi} = 0\,.$$

This gives

$$\begin{aligned}0 &= (\alpha^2 - \beta^2)\,r\,\cos\theta\sin\theta\cos\phi - x\,\alpha\,\sin\theta + y\,\beta\,\cos\theta,\\ 0 &= (\alpha^2\cos^2\theta + \beta^2\sin^2\theta - \gamma^2)\,r\,\sin\phi\cos\phi - x\,\alpha\,\sin\phi\cos\theta - y\,\beta\,\sin\phi\sin\theta + z\,\gamma\,\cos\phi.\end{aligned}$$

The above system $0 = F(\theta, \phi)$ can be solved with a Newton method, where one notes that $DF(\theta, \phi) = (a_{ij}) \in \mathbb{R}^{2\times2}$ is given by

$$\begin{aligned}a_{11} &= (\alpha^2 - \beta^2)\,r\,(\cos^2\theta - \sin^2\theta)\cos\phi - x\,\alpha\,\cos\theta - y\,\beta\,\sin\theta,\\ a_{12} &= -(\alpha^2 - \beta^2)\,r\,\cos\theta\sin\theta\sin\phi,\\ a_{21} &= -2\,(\alpha^2 - \beta^2)\,r\,\cos\theta\sin\theta\sin\phi\cos\phi + x\,\alpha\,\sin\phi\sin\theta - y\,\beta\,\sin\phi\cos\theta,\\ a_{22} &= (\alpha^2\cos^2\theta + \beta^2\sin^2\theta - \gamma^2)\,r\,(\cos^2\phi - \sin^2\phi) - x\,\alpha\,\cos\phi\cos\theta - y\,\beta\,\cos\phi\sin\theta - z\,\gamma\,\sin\phi\end{aligned}$$

With the initial guess $(\theta^{(0)}, \phi^{(0)}) = (\tan^{-1}(\frac{\alpha\,y}{\beta\,x}), \tan^{-1}(z/[\gamma\,((\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2)^{\frac{1}{2}}])) \equiv (\texttt{atan2}(\alpha\,y,\beta\,x),$ $\texttt{atan2}(z,\gamma\,((\frac{x}{\alpha})^2 + (\frac{y}{\beta})^2)^{\frac{1}{2}}))$, the iteration always converges in practice, usually taking only a couple of iterations to converge, with the desired distance given by $|\vec{p} - \vec{x}(\theta, \phi)|$. This is a significant advantage over other published methods.
See e.g. www.geometrictools.com/Documentation/DistancePointToEllipsoid.pdf by David Eberly, where a possibly ill-conditioned sixth degree polynomial needs to be solved.