Calculating the area and centroid of a polygon in 2d

Let \( \{(x_i, y_i)\}_{i=0}^{N-1} \subset \mathbb{R}^2 \) be a closed polygon in the plane, and let the vertices be ordered counter clockwise. Then it is well-known that the polygon encloses the area

\[
A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i y_{i+1} - x_{i+1} y_i),
\]

and its centroid is given by

\[
\frac{1}{6A} \left( \sum_{i=0}^{N-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{N-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i) \right)^T \in \mathbb{R}^2;
\]

see e.g. paulbourke.net/geometry/polygonmesh.

Calculating the volume and centroid of a polyhedron in 3d

Similar formulas exist for the enclosed volume and centroid of a polyhedron \( P \) in \( \mathbb{R}^3 \), but these appear to be less well-known. In the following we assume without loss of generality that the boundary of the polyhedron is given by a union of triangles. (More general facets can easily be subdivided into triangles.) We stress that \( P \) need not be convex.

Let \( A_i, i = 0, \ldots, N-1 \), be the \( N \) triangular faces of the polyhedron, with vertices \( (a_i, b_i, c_i) \), which are assumed to be ordered counter clockwise on \( A_i \). This means that we can define the outer unit normal \( n \) to \( P \) on each \( A_i \) as \( n_i = \hat{n}_i/|\hat{n}_i| \), where \( \hat{n}_i = (b_i - a_i) \otimes (c_i - a_i) \). Then the volume of \( P \) is given by

\[
V = \int_P 1 = \frac{1}{3} \int_{\partial P} x \cdot n = \frac{1}{3} \sum_{i=0}^{N-1} \int_{A_i} a_i \cdot n_i = \frac{1}{6} \sum_{i=0}^{N-1} a_i \cdot \hat{n}_i,
\]

where we have used the divergence theorem, the fact that \( x \cdot n_i \) is constant on each \( A_i \), and the fact that the area of \( A_i \) is given by \( \frac{1}{2} |\hat{n}_i| \).

Let \( c \in \mathbb{R}^3 \) denote the centroid of \( P \), i.e. \( c = \frac{1}{V} \int_P x \). Applying the divergence theorem once again, and on denoting the standard basis in \( \mathbb{R}^3 \) by \( \{e_1, e_2, e_3\} \), we obtain for the three coordinates of the centroid that

\[
c \cdot e_d = \frac{1}{V} \int_{\partial P} \frac{1}{2} (x \cdot e_d)^2 (n \cdot e_d) = \frac{1}{2V} \sum_{i=0}^{N-1} \int_{A_i} (x \cdot e_d)^2 (n_i \cdot e_d), \quad d = 1, 2, 3.
\]

It remains to compute that

\[
\int_{A_i} (x \cdot e_d)^2 (n_i \cdot e_d) = \frac{1}{6} \hat{n}_i \cdot e_d \left( \left[ \frac{1}{2} (a_i + b_i) \cdot e_d \right]^2 + \left[ \frac{1}{2} (b_i + c_i) \cdot e_d \right]^2 + \left[ \frac{1}{2} (c_i + a_i) \cdot e_d \right]^2 \right)
\]

\[
= \frac{1}{24} \hat{n}_i \cdot e_d \left( \left[ (a_i + b_i) \cdot e_d \right]^2 + \left[ (b_i + c_i) \cdot e_d \right]^2 + \left[ (c_i + a_i) \cdot e_d \right]^2 \right),
\]

where we have observed that the integrand is a quadratic function on \( A_i \), so that the standard midpoint sampling quadrature formula for triangles yields the integral exactly, see e.g. [1].

References