A field theory for the Wiener Sausage

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RSME-SCM-SEMA-SIMAI-UMI
Outline

1. The Wiener Sausage Problem
2. Spattering random walk
3. Field Theory
4. Renormalisation
5. Results on regular lattices
6. Extensions
The other Wiener!

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Wiener process
(named after Norbert Wiener)

Consider a random walker in 2D, leaving a trace:

Think of the random walker (red dot) as the tip of a pen, spilling ink.

What is the area covered in blue (volume of a “Wiener sausage”, traced out in one, two, three dimensions)?
Wiener Sausage
Motivation

- Original problem (average area, 2D) solved by Kolmogoroff and Leontowitsch (1933).
- Famously studied by Spitzer, Kac and Luttinger.
- Applications . . .
- Lots of variants and extensions . . .
Wiener Sausage

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- Applications in
  - Medicine, e.g. tissue “priming”, Dagdug, Berezhkovskii and Weiss (2002).
  - Chemical engineering, e.g. agglomerates forming by “sweeping particles”, Eggersdorfer and Pratsinis (2014).
  - Ecology, e.g. feeding plankton, Visser (2007).
  - ...

- Lots of variants and extensions...
Wiener Sausage

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- Lots of variants and extensions
  - Presence of traps, *e.g.* Oshanin, Bénichou, Coppey, and Moreau (2002).
  - Surface of the sausage, *e.g.* Rataj, Schmidt and Sporadev (2009).
  - Different boundary conditions, *e.g.* Dagdug, Berezhkovskii and Weiss (2002).
  - . . .
Determine the volume of the Wiener using Statistical Field Theory

Keeping track of a walker’s trace is hard.
Easy (-ier, -ish): Walker spatters ink as it walks.

Asymptotic statistics of spatter is that of a continuous trace.
The trajectory of a random walker is self-similar
Wiener Sausage
Poissonian modification

Wiener Sausage observable difficult in a field theory. Therefore:

Poissonian modification

On the lattice: With Poisson rate $H$ walker jumps to a nearest neighbouring site, with rate $\gamma$ attempts to place immobile offspring at current site.

*Deposition suppressed if immobile particle is present already.*

Anticipate regularisation: Add extinction rate $\epsilon'$ and $r$ for immobile species and walkers respectively.

Mean field approach: $\partial_t \rho_s = \rho_a (1 - \rho_s) \gamma$, where $\rho_s$ number of immobile offspring and $\rho_a$ number density of walkers. ($\rho_s$ is a functional of [the entire history of] $\rho_a$)

Perturbation theory: $\rho_a (1 - \rho_s) \gamma = \gamma \rho_a - \gamma \rho_a \rho_s$. 

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Field Theory for the Wiener

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Wiener Sausage

Perturbation theory

**Perturbation theory:** $\rho_a (1 - \rho_s) \gamma = \gamma \rho_a - \gamma \rho_a \rho_s$.

Implementation of the suppressed deposition by

- (to first order) allowing unrestricted deposition
- (to second order) removing excess (deposited) particles

The suppression is difficult to deal with.
Wiener Sausage
Mean field theory in the bulk

If returns (and thus previous deposition) can be ignored, total deposition $a$ is linear in time,

$$\langle a \rangle = \gamma t$$

and Poissonnian moments,

$$P^{(a)}(a) = \frac{(\gamma t)^a}{a!} \exp(-\gamma t).$$

Two intertwined Poisson processes for deposition in the presence of extinction, generating function

$$M^{(a)}(x) = \frac{r/\gamma}{r/\gamma + 1 - \exp(x)}$$
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Motivation for a field theory

Motivation for a \textit{field theoretic} study:

- Benefit: Very flexible regarding boundary conditions, additional interactions \textit{etc.}; Very elegant.
- \textbf{Two species} field theory . . . 
- . . . with \textit{immobile particles} . . .
- . . . and observables that are spatial integrals.
- “Doable” version of a “heavy duty” field theory.
- Guinea pig example of a \textbf{fermionic problem (excluded volume constraint)}.

Excluded volumes are difficult in field theories. May require fermionic treatment (painful).

Idea: Introduce \textbf{carrying capacity} $C$, whereby deposition rate drops linearly in the occupation, $1 - \rho_s/C$. \textbf{Cheating?}
Wiener Sausage
Implementation of the carrying capacity

- One dimensional lattice, length $L$, carrying capacity $C$.
- Sites within each column equivalent (particles per column).
- When jumping, probability to hit a neighbouring, occupied site is its occupation over carrying capacity $C$.
- Field-theory now easy (fermionicity is “spurious”).
- Carrying capacity $C$ in system of size $L$ corresponds to carrying capacity 1 on $L \times C$ lattice.
Wiener Sausage
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Using a field theory

Step by step:

1. Write down master equation (with carrying capacity).
2. Rewrite in terms of operators (Doi-Pelitti).
3. Extract propagators and vertices to create diagrams.
4. Dimensional analysis, extract relevant couplings, demonstrate renormalisability.
5. Calculate relevant diagrams, renormalise, extract exponents and other universal quantities.
Wiener Sausage

Master equation: Bilinear parts — Easy

\[ \partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_x \]

\[ -rn \mathcal{P}(\ldots, n, m, \ldots) + r(n + 1) \mathcal{P}(\ldots, n + 1, m, \ldots) \]

\[ -\epsilon' m \mathcal{P}(\ldots, n, m, \ldots) + \epsilon'(m + 1) \mathcal{P}(\ldots, n, m + 1, \ldots) \]

\[ -\frac{H}{q} \sum_e n(x) \mathcal{P}(\ldots, n(x), \ldots, n(x + e), \ldots) \]

\[ +\frac{H}{q} \sum_e n(x + e) \mathcal{P}(\ldots, n(x) - 1, \ldots, n(x + e) + 1, \ldots) \]

extinction

extinction

hoping away

hoping here

non-linear terms
Wiener Sausage

Master equation: Non-linear parts — Difficult

\[
\partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_x \text{bilinear terms} + \\
-\gamma n \left(1 - \frac{m}{C}\right) \mathcal{P}(\ldots, n, m, \ldots) + \gamma n \left(1 - \frac{m-1}{C}\right) \mathcal{P}(\ldots, n, m - 1, \ldots)
\]

deposition

deposition
1) Introduce raising and lowering operators

\[ a^\dagger |n\rangle = |n + 1\rangle \quad \text{and} \quad a |n\rangle = n |n - 1\rangle \]

\[ b^\dagger |n\rangle = |n + 1\rangle \quad \text{and} \quad b |n\rangle = n |n - 1\rangle \]

2) Introduce state-vector / generating function

\[ |\Psi\rangle (t) = \sum_{\{n,m\}} \mathcal{P}(\ldots, n, m, \ldots) \prod_x a^\dagger^n(x) \prod_x b^\dagger^m(x) |0\rangle \]

Expectation \( \langle \bullet \rangle = \langle \Psi_0 | \bullet | \Psi \rangle \) with suitable left vector \( \langle \Psi_0 | \).
Field Theory

Doi-Pelitti technique

3) Doi-shift operators to simplify diagrammatic expansion:

\[ a^\dagger = 1 + \tilde{a} \quad \text{and} \quad b^\dagger = 1 + \tilde{b} \]

4) Rewrite master equation

\[ \partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_x \text{bilinear terms} \ldots + \]

\[ -\gamma n \left( 1 - \frac{m}{C} \right) \mathcal{P}(\ldots, n, m, \ldots) + \gamma n \left( 1 - \frac{m-1}{C} \right) \mathcal{P}(\ldots, n, m - 1, \ldots) \]

as (term-by-term messy):

\[ \partial_t \langle \Psi(t) \rangle = \text{bilinear terms} + \]

\[ \sum_x \gamma \tilde{b}(x) a^\dagger(x) a(x) - \frac{\gamma}{C} \tilde{b}(r) b^\dagger(r) b(r) a^\dagger(r) a(r) \]
5) Introduce Liouvillian:

\[ \partial_t |\Psi(t)\rangle = \sum_x \text{bilinear terms} \ldots \]

\[ + \gamma \tilde{b}(x) a^\dagger(x) a(x) \]

\[ - \frac{\gamma}{C} \tilde{b}(r) b^\dagger(r) b(r) a^\dagger(r) a(r) \]

\[ \mathcal{L}_1 = - \gamma \tilde{\psi} \phi^* \phi \]

\[ + \frac{\gamma}{C} \tilde{\psi} \psi^* \phi \phi^* \phi \]

6) Path integral re-formulation

\[ \int \mathcal{D} \tilde{\phi} \mathcal{D} \phi \mathcal{D} \tilde{\psi} \mathcal{D} \psi \exp \left( - \int d^d k d\omega \left( \mathcal{L}_0 + \mathcal{L}_1 \right) \right) \]
Extract bare propagators:

\[ \langle \phi(k, \omega) \bar{\phi}(k', \omega') \rangle_0 = \]
\[ \langle \psi(k, \omega) \bar{\psi}(k', \omega') \rangle_0 = \]
\[ \langle \psi(k, \omega) \bar{\phi}(k', \omega') \rangle_0 = \]

Allow for different renormalisation of initially identical couplings.

Dimensional analysis: upper critical dimension \(d_c = 2\).
Field Theory

Interaction vertices

Different couplings to allow different renormalisation

\[ \mathcal{L}_1 = -\tau \bar{\psi} \phi - \sigma \bar{\psi} \bar{\phi} \phi + \lambda \bar{\psi} \psi \phi + \kappa \bar{\phi} \bar{\psi} \psi \phi + \chi \bar{\psi}^2 \psi \phi + \xi \bar{\phi} \bar{\psi}^2 \psi \phi \]

Diagrams:
Field Theory

Interaction vertices

Different couplings to allow different renormalisation

\[ \mathcal{L}_1 = -\tau \tilde{\psi} \phi - \sigma \tilde{\psi} \tilde{\phi} \phi + \lambda \tilde{\psi} \psi \phi + \kappa \tilde{\phi} \tilde{\psi} \psi \phi + \chi \tilde{\psi}^2 \psi \phi + \xi \tilde{\phi} \tilde{\psi}^2 \psi \phi \]

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Different couplings to allow different renormalisation

\[ \mathcal{L}_1 = -\tau\psi\phi - \sigma\tilde{\psi}\tilde{\phi}\phi + \lambda\tilde{\psi}\psi\phi + \kappa\tilde{\phi}\tilde{\psi}\psi\phi + \chi\tilde{\psi}^2\psi\phi + \xi\tilde{\phi}\tilde{\psi}^2\psi\phi \]

Diagrams:
Field Theory

Tree level in the bulk \((d > 2)\)

Deposition is suppressed in the presence of deposits. *Without* that, deposits could be found all along the walker’s trajectory (multiple deposits at revisited sites):

This diagram is present at tree level. Although it cannot be integrated out, its contribution to correlation functions can be determined easily.
Field Theory
Tree level in the bulk ($d > 2$)

- Tree level = no loops (return asymptotically irrelevant)
- Non-linearities present at tree level.
- $n$'th moment of the sausage volume $a$ dominated\(^1\) by trees with $n$ branches:

\[
\langle a \rangle = \tau
\]
\[
\langle a^2 \rangle = \tau \sigma
\]
\[
\langle a^3 \rangle = \tau \sigma \sigma
\]

Reproduces Poissonian results above…

\(^1\)Lower order terms from other trees.
Field Theory

Tree level in finite systems

In finite systems,

- Fourier integrals turn into sums.
- Loss of translational invariance results in vertices becoming sums.
- Example

\[
\langle a \rangle = \frac{8 \tau}{\pi^4 D} L^2 \sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\tau L^2}{12 D}
\]

- Higher orders increasingly messy, e.g.

\[
2\pi \sum_{nml \text{ odd}} \frac{1}{n^3 m l} \frac{1}{2\pi} \left( \frac{1}{n+m-l} + \frac{1}{n-m+l} + \frac{1}{-n+m+l} - \frac{1}{n+m+l} \right) = \frac{1}{6} \left( \frac{\pi}{2} \right)^6
\]

- Ignoring return, sausage volume is linear in residence time, whose moments can be extracted from recurrence relations of moment generating functions.
Field Theory

Full theory for the bulk ($d < 2$)

- Walker walks: $\text{Walker walks: } = \frac{1}{-i\omega + Dk^2}$

- ... and leaves behind a trace in the form of branched-off particles $\text{... and leaves behind a trace in the form of branched-off particles } = b^\dagger(x) a^\dagger(x) a(x)$

- No deposition if a particle is there already $\text{No deposition if a particle is there already } = b^\dagger(x) b(x) a^\dagger(x) a(x)$

- Substrate particles stuck on the lattice: $\text{Substrate particles stuck on the lattice: } = \frac{1}{-i\omega + \epsilon'}$
Field Theory

Meaning of vertices

This diagram probes the lattice for deposits (and suppresses further deposition):

Without it, no loops can be formed $\rightarrow$ tree level theory.

**Interaction** of the walker with its past trace.
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Calculate features of the Wiener sausage using renormalisation. Deposit along the trajectory

\[ \text{Loop} = \text{interaction} = \text{signature of collective phenomenon} \]
Calculate features of the Wiener sausage using renormalisation. Deposit along the trajectory

... is reduced by suppressed deposition

Loop = interaction = signature of collective phenomenon
Field Theory
Interaction diagrams

Calculate features of the Wiener sausage using renormalisation. Deposit along the trajectory

\[ \text{Loop = interaction = signature of collective phenomenon} \]
Calculate features of the Wiener sausage using renormalisation.
Deposit along the trajectory

\[ \ldots \text{is reduced by suppressed deposition} \]

\[ = \int d\omega' d^d k' \frac{1}{-i\omega' + Dk'^2} \frac{1}{i\omega' + \epsilon'} \]

Physical origin of UV divergence: Time spent\(^2\) per volume element diverges at \( d \geq 2 = d_u \), upper critical dimension. Above: interaction irrelevant, size of sphere enters.

\(^2\)Lingering, not returning, \( \int dt (4Dt\pi)^{-d/2} \exp\left(-\frac{(x-x')^2}{4Dt}\right) \).
At the heart of the theory is the renormalisation of the following process:
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\[
\begin{align*}
\bullet &= 1 + \frac{1}{1 - } + \frac{1}{1 - } + \frac{1}{1 - } + \ldots \\
&= \frac{1}{1 - }
\end{align*}
\]
At the heart of the theory is the *renormalisation* of the following process:

\[
\frac{1}{1 - \text{Diagram}} = 1 + \text{Diagram} + \text{Diagram} + \text{Diagram} + \ldots
\]
Renormalisation: What are the loops

What physical process do the loops correspond to? Trajectory intersecting itself (contract along wriggly line):
Field Theory
Renormalisation: What are the loops

What physical process do the loops correspond to? Trajectory intersecting itself twice (contract along wriggly line):
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Results

Focus on first moment of sausage volume as a function of time.

- In one dimensions: Length covered proportional to square root of time, \( \langle a \rangle = \frac{\pi}{4} \sqrt{\frac{tD}{\pi}} \). Exact amplitude!
- In two dimensions: Area covered linear in time, \( t \) (modulo logarithmic corrections, \( t/\ln(t) \)).
- In general: \( \langle a^m \rangle \propto t^{m/d/2} \).

Next: Finite size scaling

- In three dimensions and higher: Volume linear in time, \( t \).
- . . . random walker may never return.

Well known results (Leontovich and Kolmogorov, Berezhkovskii, Makhnovskii and Suris). . .

. . . but, hey, what a nice playground for field theory (fermionicity, renormalisation, calculating moments easily . . . sort of).
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Extension: Regular lattice with open boundary conditions

Nonlinearity changes in finite systems from

\[ \kappa \int \bar{\omega}_{1,2,3,4} \int \bar{\omega}^d k_{1,2,3,4} \phi^\dagger(k_1) \psi^\dagger(k_2) \phi(k_3) \psi(k_4) \]

\[ \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) \delta^d(k_1 + k_2 + k_3 + k_4) \]

which originates from

\[ \delta^d(k_1 + k_2 + k_3 + k_4) = \int d^d r e^{-\mu k_1} e^{-\mu k_2} e^{-\mu k_3} e^{-\mu k_4} \]

to ...
Field Theory

Extension: Regular lattice with open boundary conditions

\[ \kappa \int d\omega_{1,2,3,4} \int d^{d-1}k_{1,2,3,4} \sum_{nmkl} \phi^{\dagger}(k_1)\psi^{\dagger}(k_2)\phi(k_3)\psi(k_4) \]
\[ \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)\delta^{d-1}(k_1 + k_2 + k_3 + k_4)U_{nmkl} \]

with

\[ U_{nmkl} = \frac{2}{L} \int dz \sin z q_n \sin z q_m \sin z q_k \sin z q_l \]
Field Theory

Extension: Regular lattice with open boundary conditions

\[ = \kappa^2 \left( \frac{2}{L} \right)^2 \sum_{ab} \int d\omega' \, d^{d-1}k' \]

\[ \times \frac{1}{-\omega' + Dk'^2 + Dq_a^2} \frac{1}{\omega' + \epsilon'} U_{nmab} U_{ablk} \]

where \( q_n = n\pi/L, \) \( n = 1, 2, \ldots \) are modes in the finite direction.

\( U_{nmlk} = (2/L) \int_0^L dx \sin(q_n x) \sin(q_m x) \sin(q_l x) \sin(q_k x) \) accounts for lack of translational variance.

**Problem:** Renormalisation scheme requires the RHS to be expressed as a multiple of \( \kappa U_{nmlk}. \)

**Solution:** Deviation of RHS from multiple of \( \kappa U_{nmlk} \) sub-leading (as found in Casimir systems).
Field Theory

Extension: Regular lattice with open boundary conditions

\[ \kappa^2 \left( \frac{2}{L} \right)^2 \sum_{ab} \int d\omega' d^{d-1} k' \]

\[ \times \frac{1}{-\omega' + Dk'^2 + Dq_a^2} \frac{1}{\omega' + \epsilon} U_{nmab} U_{ablk} \]

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\( U_{nmhk} = (2/L) \int_0^L dx \sin(q_n x) \sin(q_m x) \sin(q_l x) \sin(q_k x) \) accounts for lack of translational variance.

General result in \( d < 2 \): \( \langle a^m \rangle \propto m! \tau \sigma^{m-1} \left( \frac{L}{\pi} \right)^{md} \kappa^{-m} \).

Finite size \( L \) has the effect of a lowest mode, \( q_1 = \pi/L \).

Large \( L \) like \( d \to d - 1 \) for periodic BC (crossover).
Field Theory

More exotic extension: Challenges for dealing with “exotic” lattices

- Lack of conservation ($U_{nmkl}$ instead of $\delta()$)
- New interaction ($U_{nmkl}$ possibly not renormalising to $U_{nmkl}$)
- Different spectrum
Field Theory
More exotic extension: Fractal lattices

What is the minimal adjustment to go from regular lattices to networks and fractals?

$$\int d^dk \frac{1}{-i\omega' + Dk'^2 + Dq_n^2} \cdots$$

Eigenvalues \( \mathbf{k} \) of \( d \) dimensional lattice are themselves a \( d \) dimensional lattice. Spectral dimension \( d_s = 2df/dw \) (regular lattice \( d_s = d \)).

Works only if (bare) propagator itself does not renormalise (\( \eta = 0 \)).

So: Wiener sausage volume \( \propto t^{d_s/2} \).

Note: Known return time distribution in networks \( \propto t^{-d_s/2} \).
Wiener Sausage
More exotic extension: Numerics for fractals

Good support for Wiener sausage on fractals

<table>
<thead>
<tr>
<th>Lattice</th>
<th>fractal $d_f$</th>
<th>spectral $d_s$</th>
<th>$d_s/2$</th>
<th>measured</th>
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<td>0.76</td>
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</tbody>
</table>

What about Networks?
Wiener Sausage on Networks
What is needed

- Field theory on networks: Spectrum and structure of eigenvectors for any network.
- At least spectral dimension.
- Exact solution of the Wiener sausage on any network.
- At least numerics for that.
Thank you!