PHASE TRANSITION FOR THE LATE POINTS OF RANDOM WALK

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Abstract

Let X be a random walk on the torus of side length N in dimension $d \geqslant 3$ with uniform starting point, and t_{cov} be the expected value of its cover time, which is the first time that X has visited every vertex of the torus at least once. For $\alpha > 0$, the set $\mathcal{L}^\alpha$ of α-late points consists of those points not visited by X at time αt_{cov}. We prove the existence of a value $\alpha_* \in (\frac{1}{2}, 1)$ across which $\mathcal{L}^\alpha$ trivialises as follows: for all $\alpha > \alpha_*$ and $\varepsilon \geqslant N^{-c}$ there exists a coupling of $\mathcal{L}^\alpha$ and two occupation sets $\mathcal{B}^{\alpha\pm}$ of i.i.d. Bernoulli fields having the same density as $\mathcal{L}^{\alpha\pm\varepsilon}$, which is asymptotic to $N^{-(\alpha\pm\varepsilon)d}$, with the property that the inclusion $\mathcal{B}^{\alpha+} \subseteq \mathcal{L}^\alpha \subseteq \mathcal{B}^{\alpha-}$ holds with high probability as $N \to \infty$. On the contrary, when $\alpha \leqslant \alpha_*$ there is no such coupling. Corresponding results also hold for the vacant set of random interlacements at high intensities. The transition at α_* corresponds to the (dis-)appearance of 'double-points' (i.e. neighboring pairs of points) in $\mathcal{L}^\alpha$. We further describe the law of $\mathcal{L}^\alpha$ for $\alpha > \frac{1}{2}$ by adding independent patterns to $\mathcal{B}^{\alpha\pm}$. In dimensions $d \geqslant 4$ these are exactly all two-point sets. When $d = 3$ one must also include all *connected* three-point sets, but no other.

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1 Introduction

This article studies two models, the random walk on the d-dimensional torus $\mathbf{T} = (\mathbb{Z}/N\mathbb{Z})^d$ of large side length N, for $d \geqslant 3$, at time scales close to the typical time it takes the walk to cover the whole torus, and random interlacements on $\mathbb{Z}^d$ at corresponding intensities. Let $\mathbf{P}$ denote the canonical law of the walk on the graph $\mathbf{T}$ started from uniform distribution, and $X = (X_n)_{n \geqslant 0}$ be the corresponding (discrete-time) canonical process. It is well-known that the cover time C_N of X, which is the first time X has visited every vertex of $\mathbf{T}$ at least once, satisfies

$$(1.1) \qquad t_{\mathrm{cov}} \stackrel{\mathrm{def.}}{=} \mathbf{E}[C_N] \sim g(0) N^d \log(N^d), \text{ as } N \to \infty,$$

where $g(0)$ denotes the Green's function of the simple random walk on $\mathbb{Z}^d$ at the origin, see Section 2 for notation, and $\sim$ means that the ratio of the two quantities tends to one in the given limit. In view of (1.1), letting

$$(1.2) \qquad \mathcal{V}_N^u = \mathbf{T} \setminus X_{[0, uN^d]}, \text{ for } u > 0,$$

where $X_{[0,t]} = \{x \in \mathbf{T} : X_n = x \text{ for some } n \leqslant t\}$, it is natural to introduce

$$(1.3) \qquad u_N(\alpha) = \alpha g(0) \log(N^d), \text{ for } \alpha > 0, \ N \geqslant 1$$

(whence $u_N(\alpha)N^d \sim \alpha t_{\mathrm{cov}}$) and consider the vacant set

$$(1.4) \qquad \mathcal{L}^\alpha \stackrel{\mathrm{def.}}{=} \mathcal{V}_N^{u_N(\alpha)} \text{ (under } \mathbf{P}).$$

The elements of $\mathcal{L}^\alpha$ will be referred to as α-late points. Note that $\mathcal{L}^\alpha$ is decreasing in α, and the choices (1.2)-(1.3) imply that $\mathcal{L}^\alpha$ has density (see (6.5))

$$(1.5) \qquad \mathbf{P}(0 \in \mathcal{L}^\alpha) \sim N^{-\alpha d} \text{ as } N \to \infty.$$

The parametrisation in (1.2)-(1.4) is a matter of convenience; our results do in fact remain true for any choice of 'α-late time scale' such that (1.5) holds (for instance, $u = \alpha N^{-d} t_{\mathrm{cov}}$ in (1.2)), see Remark 7.3,5) for this and more; see also Remark 7.3,8) regarding natural (on account of (1.5)) extensions to *random* timescales such as the first time the vacant set of the walk contains exactly $\lceil N^{(1-\alpha)d} \rceil$ points.

We are interested in global (i.e. macroscopic) properties of $\mathcal{L}^\alpha$ as a subset of $\mathbf{T}$. One difficulty in addressing questions of this type stems from the long-range correlations inherent to $\mathcal{L}^\alpha$.

For the sake of clarity, we focus on (1.4) in this introduction. As will turn out, all results presented below allow for either of two generalisations. First, we can deal with late points $\mathcal{L}_F^\alpha$ in arbitrary (large) (sub-)regions $F \subseteq \mathbf{T}$ at appropriate timescales, ensuring in essence that $\mathcal{L}_F^\alpha$ has asymptotic density $|F|^{-\alpha}$, cf. (1.5). Second, the conclusions of all the results presented in this introduction continue to hold if one replaces (1.4) by $\mathcal{L}^\alpha = \mathcal{V}^{u_N(\alpha)} \cap ([0, N) \cap \mathbb{Z})^d$ (identifying the vertices of $([0, N) \cap \mathbb{Z})^d$ with those of $\mathbf{T}$), where $\mathcal{V}^u$ refers to the vacant set of random interlacements at level u; see Remark 7.3,1) for details. The set $\mathcal{V}^u$ is characterised by the property that

$$(1.6) \qquad \mathbb{P}^I(K \subset \mathcal{V}^u) = \exp\{-u\mathrm{cap}(K)\}, \text{ for finite } K \subset \mathbb{Z}^d,$$

and corresponds to a local limit of $\mathcal{V}_N^u$ in (1.2) as $N \to \infty$. We refer to Corollary 5.2 for an explicit coupling between $\mathcal{V}^u$ and $\mathcal{V}_N^u$. This coupling acts as a powerful transfer mechanism. For instance it allows to lift the formula (1.6) from $\mathcal{V}^u$ to $\mathcal{V}_N^u$, up to small error, see (6.4). We now present our results.

1.1. Phase transition at α_*. The transition we establish as part of our first main result exhibits a sharp regime of parameters α in which $\mathcal{L}^\alpha$ in (1.4) completely 'trivialises,' i.e. resembles an i.i.d. sample of appropriate density. This question has, directly and indirectly, already received considerable attention in the past, as we now briefly review. In view of (1.5), the case $\alpha > 1$ is readily dispensed with since $\mathcal{L}^\alpha$ is empty with high probability for such values of α. In the course of proving Gumbel fluctuations for C_N, Belius obtained [5, (1.4)-(1.5) and Corollary 3.4] that the suitably rescaled process of points 'around' $\mathcal{L}^1$ converges to a Poisson point process on $(\mathbb{R}/\mathbb{Z})^d$. Intuitively, this means that the 'very late' points, i.e. the last few vertices to be covered around $\alpha = 1$, are roughly independent and uniform. The proof relies on similar results for interlacements [4], and a coupling of the two objects. We return to this below.

Matters become all the more delicate in the regime $\alpha < 1$, notably because $|\mathcal{L}^\alpha|$ is no longer tight in N, cf. (1.5). Let $(\mathcal{B}^\alpha)_{\alpha \geqslant 0}$ denote a family of i.i.d. Bernoulli (site) percolation processes on $\mathbf{T}$ with respective density $\mathbf{P}(0 \in \mathcal{L}^\alpha)$ and coupled in a monotone fashion in α (e.g. by means of uniform independent random variables). One is naturally led to wonder how $\mathcal{L} = (\mathcal{L}^\alpha)_{\alpha \geqslant 0}$ and $\mathcal{B} = (\mathcal{B}^\alpha)_{\alpha \geqslant 0}$ relate, if at all. This question was taken up in [22], the main contribution of which can be paraphrased as stating that $\alpha_{**} < 1$, where

$$(1.7) \qquad \alpha_{**} \stackrel{\text{def.}}{=} \inf\big\{\alpha > 0 : \lim_N d_{\mathrm{TV}}\big(\mathcal{L}^\beta, \mathcal{B}^\beta\big) = 0 \text{ for all } \beta \geqslant \alpha\big\},$$

where d_{TV} denotes the total variation distance; the result that $d_{\mathrm{TV}}(\mathcal{L}^1, \mathcal{B}^1) \to 0$ had already been shown prior to this by Prata in [25, Part II]. Within the class of vertex transitive graphs, the recent work [6] by Berestycki, Hermon and Teyssier actually gives an optimal characterization in terms of the diameter for the appearance of Gumbel fluctuations and the uniformity of $\mathcal{L}^1$.

The method of [22] originally gave $\alpha_{**} \leqslant \alpha_1$ for an explicit value of $\alpha_1 = \alpha_1(d) < 1$ satisfying $\alpha_1 \to 1$ as $d \to \infty$. This deficiency was later removed in [28], where it is shown that

$$(1.8) \qquad \alpha_{**} \leqslant \alpha_2 (< \alpha_1 < 1),$$

with $\alpha_2 = \frac{3}{4}(d - \frac{2}{3})/(d-1) \to \frac{3}{4}$ as $d \to \infty$, see Remark 7.3,7). To be precise, all afore mentioned results in [22, 25, 28] deal with a slightly different set of late points in place of $\mathcal{L}^\alpha$, operating at time scales αt_*, where t_* is carefully chosen, see e.g. [22, (4.3)], and satisfies $t_* = t_{\mathrm{cov}}(1 + o(1))$.

A natural barrier for proximity of $\mathcal{L}$ and $\mathcal{B}$ arises as follows. Let

$$(1.9) \qquad D^\alpha \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{x \sim y} 1\{x \in \mathcal{L}^\alpha, y \in \mathcal{L}^\alpha\},$$

where $x \sim y$ denote neighbouring vertices in $\mathbf{T}$, which counts 'double-points' in $\mathcal{L}^\alpha$, and define

$$(1.10) \qquad \alpha_* = \alpha_*(d) \stackrel{\text{def.}}{=} \sup\big\{\alpha > 0 : \lim_N \mathbf{E}[D^\alpha] = 0\big\},$$

(with the convention $\sup \varnothing = 0$). The quantity $\mathbf{E}[D^\alpha]$ appearing in the limit as $N \to \infty$ above depends implicitly on N via the choice of the side length for the torus $\mathbf{T}$ on which $\mathcal{L}^\alpha$ is defined; this will in fact be the case for all our limits in N. One can show, see Lemma 6.3 below, that

$$(1.11) \qquad \alpha_* = \tfrac{1}{2}\big(1 + P_0(\widetilde{H}_0 < \infty)\big) \; \big(\in \big(\tfrac{1}{2}, 1\big)\big),$$

where P_0 denotes the law of simple random walk on $\mathbb{Z}^d$ with starting point $X_0 = 0$ and $\widetilde{H}_0 = \inf\{n \geqslant 1 : X_n = 0\}$. The inequality $\alpha_* > \frac{1}{2}$ implied by (1.11) is important and signals a different qualitative behaviour of $\mathcal{L}^\alpha$ and $\mathcal{B}^\alpha$. For, the quantity corresponding to $\mathbf{E}[D^\alpha]$ with $\mathcal{B}^\alpha$ in place of $\mathcal{L}^\alpha$ in (1.9) diverges for all $\alpha > \frac{1}{2}$. The threshold α_* already appears in [22], where it is shown (for slightly different $\mathcal{L}^\alpha$) that $\alpha_{**} \geqslant \alpha_*(> \frac{1}{2})$, which together with (1.8) implies that the threshold α_{**} in (1.7) is non-degenerate.

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Our first result establishes a sharp transition for the set $\mathcal{L}^\alpha$ in (1.4) across the threshold α_*, for a slightly different measure of distance between random sets than (1.7), allowing the introduction of a small sprinkling parameter $\varepsilon > 0$ (the case of d_{TV} corresponds to setting $\varepsilon = 0$). We note in passing that such a measure of proximity has a long history in the context of problems involving strong correlations, see [5, 8, 32, 34] for instance. In the sequel we denote by $\mathcal{Q}_{\alpha,\varepsilon,N}$ the family of all couplings between $\mathcal{L}^\alpha$ and $(\mathcal{B}^{\alpha-\varepsilon}, \mathcal{B}^{\alpha+\varepsilon})$, for $\alpha, \varepsilon > 0$ and integer $N \geqslant 1$.

Theorem 1.1 $(d \geqslant 3)$. *With α_* as in (1.10), for suitable $\varepsilon_0 = \varepsilon_0(\alpha, d) > 0$, the following hold:*

  i) For all $\alpha > \alpha_*$, $\varepsilon \in (0, \varepsilon_0)$ and $N \geqslant 1$, there exists a coupling $\mathbb{Q} \in \mathcal{Q}_{\alpha,\varepsilon,N}$ such that
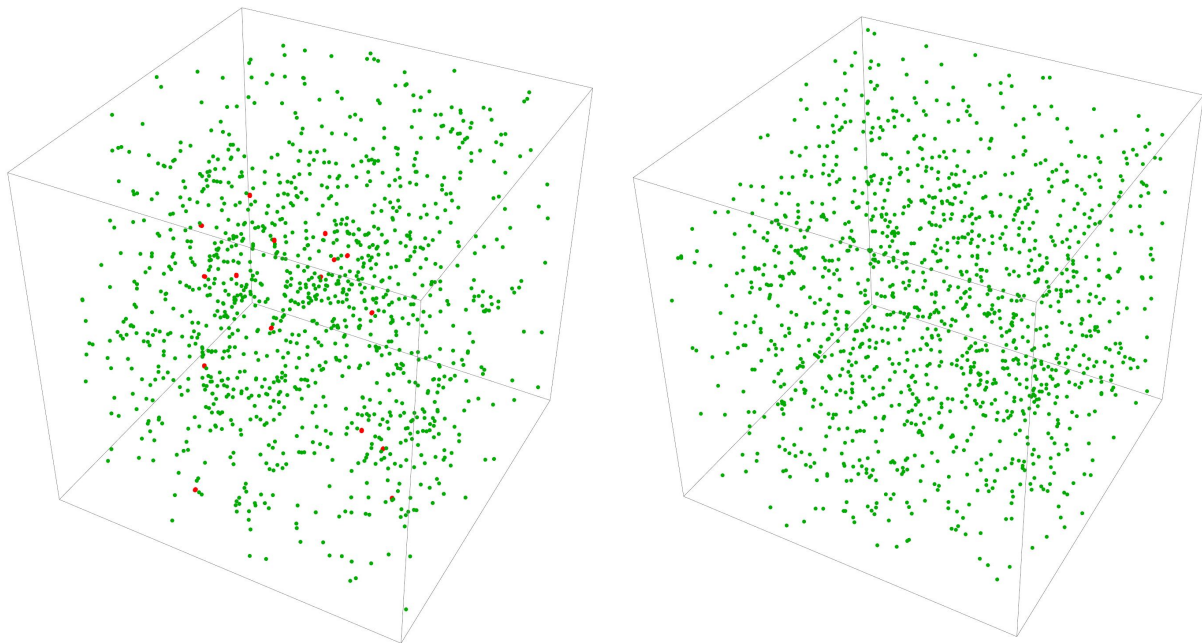
$$(1.12) \qquad \lim_N \mathbb{Q}\big(\mathcal{B}^{\alpha+\varepsilon} \subset \mathcal{L}^\alpha \subset \mathcal{B}^{\alpha-\varepsilon}\big) = 1.$$

  ii) If $\alpha = \alpha_*$, then for all $\varepsilon \in (0, \varepsilon_0)$, with the supremum ranging over $\mathcal{Q}_{\alpha_*,\varepsilon,N}$ below,

$$(1.13) \qquad \limsup_N \sup_{\mathbb{Q}} \mathbb{Q}\big(\mathcal{B}^{\alpha*+\varepsilon} \subset \mathcal{L}^{\alpha*} \subset \mathcal{B}^{\alpha*-\varepsilon}\big) = e^{-d}.$$

  iii) For all $\alpha < \alpha_*$, $\varepsilon \in (0, \varepsilon_0)$ and any coupling $\mathbb{Q} \in \mathcal{Q}_{\alpha,\varepsilon,N}$ one has that

$$(1.14) \qquad \lim_N \mathbb{Q}\big(\mathcal{B}^{\alpha+\varepsilon} \subset \mathcal{L}^\alpha \subset \mathcal{B}^{\alpha-\varepsilon}\big) = 0.$$

(a) Simulation of $\mathcal{L}^{0.6}$ for $N = 400$ (b) Simulation of $\mathcal{B}^{0.6}$ for $N = 400$

Figure 1: In red, the points of $\mathcal{L}^\alpha$ which have a neighbor in $\mathcal{L}^\alpha$. In dimension three for $\alpha = 0.6$, there are many such "double points" in $\mathcal{L}^\alpha$, but they disappear at $\alpha_* \approx 0.67$. There are no such red points in $\mathcal{B}^\alpha$, $\alpha > 0.5$.

In words, (1.12)-(1.14) indicate that $\mathcal{L}^\alpha$ is close to an i.i.d. Bernoulli field, up to a sprinkling parameter ε, if and only if $\alpha > \alpha_*$. The constant e^{-d} appearing in (1.13) corresponds to the probability not to have any neighbours in $\mathcal{L}^{\alpha*}$, i.e. the probability that there does not exist $x \sim y$ in $\mathbf{T}$ such that $\{x, y\} \subset \mathcal{L}^{\alpha*}$, see Remark 7.3,3) for details. Theorem 1.1 is proved in Section 7 and allows for various extensions, see Remark 7.3, which among other things, include an analogue of Theorem 1.1 for interlacements and give in either case quantitative bounds in N

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and ε for the probability of the event in (1.12), see (7.23). In fact the quantitative bounds allow to take $\varepsilon = \varepsilon_N$ polynomially small in N in (1.12). Up to a small sprinkling, Theorem 1.1 thus answers Question 1 in Section 7 of [22]. Moreover, all afore mentioned results of [22, 25, 28] (in particular, (1.8)) as well as [4, 5] can all be easily recovered from Theorem 1.1, see Remark 7.3,7) for details. We return to the challenges in proving Theorem 1.1 in §1.2 below. We further note that the process of covering in dimension $d = 2$ is very different, owing to the recurrence of the walk, which causes late points to cluster; see [13] for more on this.

In order to support the intuition conveyed by (1.12)-(1.14) that $\mathcal{L}^\alpha$ consists a.s. of asymptotically independent points if and only if $\alpha > \alpha_*$, one could instead also examine the convergence of the rescaled point process $\sum_{x \in \mathcal{L}^\alpha} \delta_{x/N^{\alpha d}}$ to a Poisson point process on $\mathbb{R}^d$, as done for instance in [4, Corollary 0.2], [5, (1.5)], or [10, Theorem 1.1] in the context of the high points of the Gaussian free field. However, the resulting scaling limit is not fine enough to capture the phase transition of Theorem 1.1: the limit is actually Poissonian for *any* $\alpha \in (0, 1]$, cf. Remark 8.4,3).

In view of Theorem 1.1, a natural question is to describe the law of $(\mathcal{L}^\alpha)_{\alpha > 0}$ as a process in α. This was first investigated in [25, Theorem 3] for α very close to 1 – more precisely for $\alpha = 1 + \frac{\beta}{\log N}$, $\beta > 0$, at which $|\mathcal{L}^\alpha|$ remains tight – and we will prove a similar result for $\alpha \in (\alpha_*, 1]$ in §8.1. More precisely, defining

$$(1.15) \qquad \alpha_x = \sup\{\alpha > 0 : x \in \mathcal{L}^\alpha\}, \quad x \in \mathbf{T},$$

we show in Theorem 8.1 that

$(1.16) \quad (\alpha_x - \alpha_*)_{x \in \mathcal{L}^{\alpha*}}$ is 'close' to a family of i.i.d. exponential variables with mean $d \log N$,

where proximity is again measured in terms of a coupling such that both processes differ by at most ε.

1.2. Localization. We now discuss one of the main difficulties permeating this work, which is that of 'localizing' the dependence of $\mathcal{L}^\alpha$. Attending to it leads to Theorem 1.2 below (and its more elaborate version, Theorem 5.1), which is of independent interest; see also [12] for other localization phenomena for Brownian motion in dimensions $d \geqslant 3$. To provide some context, we now comment on the proof of Theorem 1.1, and focus on the case $\alpha > \alpha_*$, which already highlights the essence of the issue. Following [28], see also [10] in the context of the Gaussian free field, one may seek to apply the Chen-Stein method, e.g. in the form presented in [2, Theorem 3], directly to $\mathcal{L}^\alpha$, which requires individual control on three terms, commonly referred to as b_1, b_2 and b_3, that each need to be small. The problematic term is b_3, which carries the long-range information and leads to limitations of this method, giving rise e.g. to the threshold α_2 in (1.8).

To overcome this difficulty, we follow a modified scheme, see Lemma 3.1 below, which consists of introducing an intermediate (localized) family $\widetilde{\mathcal{L}} = (\widetilde{\mathcal{L}}^\alpha)_{\alpha \geqslant 0}$, coupled to $\mathcal{L}$ in a way that the two are close (up to sprinkling), and to which [2] applies with $b_3 = 0$. The latter requires a (pointwise) short-range property: $\{x \in \widetilde{\mathcal{L}}^\alpha\}$ needs to be independent of $\widetilde{\mathcal{L}}^\alpha \cap Q(x, R)^c$, for a suitable localization scale R. Here $Q(x, R)$ denotes the box around x with size length N in $\mathbf{T}$, see §2 for a precise definition. Our next result asserts that a process $\widetilde{\mathcal{L}}$ with these features can indeed be constructed. In fact, this is not specific to the high intensity regime (1.3) at all, and best stated in terms of local times. Thus let $(\ell_{x,u})_{x \in \mathbf{T}}$ denote the local times of X_{uN^d} under $\mathbf{P}$, cf. (2.2), so that $\mathcal{V}_N^u = \{x \in \mathbf{T} : \ell_{x,u} = 0\}$, cf. (1.2). The following result is proved in Section 5 (see Theorem 5.1 for a more general statement) and has an analogue for random interlacements.

Theorem 1.2 (Localization)**.** *For all $N \geqslant 1$, $R \in [1, \frac{N}{10}]$ and $u_0 > 0$, there exists a family $(\widetilde{\ell}_{x,u})_{x \in \mathbb{Z}^d, u > 0}$ such that the following holds. For all $u \in (0, u_0]$ and $\varepsilon > 0$, there exists a coupling $\widetilde{\mathbf{P}}$ of $\ell_{\cdot,u}$ with $\widetilde{\ell}_{\cdot,u(1 \pm \varepsilon)}$ such that:*

$$(1.17) \qquad \widetilde{\ell}_{x,u(1\pm\varepsilon)} \text{ is independent from } \sigma(\widetilde{\ell}_{y,u(1\pm\varepsilon)} : y \notin Q(x, R)), \text{ for all } x \in \mathbf{T}, \text{ and}$$

$$(1.18) \qquad \widetilde{\mathbf{P}}\big(\widetilde{\ell}_{x,u(1-\varepsilon)} \leqslant \ell_{x,u} \leqslant \widetilde{\ell}_{x,u(1+\varepsilon)}, \ \forall\, x \in \mathbf{T} \big) \geqslant 1 - CN^d R^{2d} \exp\big(- c\varepsilon\sqrt{uR^{d-2}} \big),$$

for some $c = c(d) > 0$, $C = C(d) < \infty$.

The set $\widetilde{\mathcal{L}}^\alpha$ alluded to above is then obtained by applying Theorem 1.2 with $u = u_N(\alpha)$ upon defining $\widetilde{\mathcal{L}}^\alpha = \{x \in \mathbf{T} : \widetilde{\ell}_{x,u_N(\alpha)} = 0\}$ with R appropriately chosen, e.g. $R \approx (\varepsilon^{-2}\log(N^d))^{\frac{1}{d-2}}$, to make the error in (1.18) small. From this, (1.12) eventually follows from [2, Theorem 3].

The two localisation features (1.17) and (1.18) are difficult to engineer simultaneously. The requirements (1.17)-(1.18) can be seen to correspond to a kind of 'approximate spatial Markov property' (at one point) for the local times. Indeed for the related Gaussian free field, a version of Theorem 1.2 can be obtained by exploiting the field's Markov property. In a follow-up article, we will exploit this to establish a phase transition for the high points of the field, studied previously in [10, 28], as well as other long-range correlated models.

The 'approximate' spatial Markov property implied by Theorem 1.2 is however much harder to obtain, and the proof of Theorem 1.2 is at the heart of this article. One of our main tools is the soft local times method from [24, 1], with a twist. The method has been introduced to compare random interlacements trajectories on fixed well-separated sets A_1 and A_2, to a version of random interlacements independent on A_1 and A_2, whose definition depend on the choice of A_1 and A_2. In order to derive Theorem 1.2 though, we need to compare random interlacements with a version of interlacements defined on the whole set $\mathbf{T}$ at once, and which is independent on *any* pair of well-separated sets A_1 and A_2, when A_1 is a singleton. We achieve this by introducing an inverse soft local times technique, see Section 4. Actually, our techniques also lead to a new, simpler, more explicit but slightly less general proof of the coupling between the random walk and random interlacements from [8], see the discussion below Theorem 5.1 and Corollary 5.2.

1.3. Law of the late points for $\alpha \in (\frac{1}{2}, \alpha_*]$. For $\alpha \leqslant \alpha_*$, Theorem 1.1 asserts that $\mathcal{L}^\alpha$ and the i.i.d. set $\mathcal{B}^\alpha$ are not close anymore, which, in view of (1.9)-(1.10), is due to the emergence of neighboring pairs of points in $\mathcal{L}^\alpha$ (but not in $\mathcal{B}^\alpha$). But could proximity for α smaller but close to α_*, perhaps be restored by adding such pairs, and, if so, in an independent fashion? What about other 'clusters' of late points, e.g. sets of two points at distance $k \geqslant 2$, or even finite sets of larger cardinality? Do such 'clusters' appear in $\mathcal{L}^\alpha$ as α is reduced? Is their occurrence Poissonian? The next result sheds light on these questions. For finite $K \subset \mathbb{Z}^d$, denote by $\mathrm{cap}(K)$ the capacity of K on the transient graph $\mathbb{Z}^d$, $d \geqslant 3$. We also extend this definition to any set $K \subset \mathbf{T}$ with ℓ^∞-diameter $\delta(K)$, (see Section 2 for a definition) at most $N - 1$ by essentially identifying $\mathbf{T}$ with the cube $Q(0, N)$ via an adapted bijection so that $K \subset Q(0, N-1)$, where $Q(0, R)$ is the box in $\mathbb{Z}^d$ around the origin 0 of side length R for all $R > 0$. For sets $K \subset \mathbf{T}$ with diameter N, we simply take the convention $\mathrm{cap}(K) = \infty$. We refer to below (2.3) for precise definitions. Now for K either in $\mathbb{Z}^d$ or in $\mathbf{T}$ with diameter at most $N - 1$, let

$$(1.19) \qquad \alpha_*(K) \stackrel{\text{def.}}{=} (g(0)\mathrm{cap}(K))^{-1}$$

and note that $\alpha_*(\{x\}) = 1$ for all $x \in \mathbb{Z}^d$ or $\mathbf{T}$ by (2.6), while in fact $\alpha_*(\{x,y\}) = \alpha_*$ as in (1.10), for $x \sim y$, see Lemma 6.3. As will become clear, the parameter $\alpha_*(K)$ corresponds to the largest value of α at which one can find a translated version of the set K in $\mathcal{L}^\alpha$ similarly as in the definition (1.10) of α_*; cf. (6.10) or Lemma 6.4 for details.

Let $p^\alpha(A) = \mathbf{P}\big(\mathcal{L}^\alpha \cap Q(A, R_{\mathbf{T}}) = A\big)$ for all $A \subset \mathbf{T}$ and $\alpha > 0$, where $R_{\mathbf{T}} = \log(|\mathbf{T}|)^{\frac{1}{d-2}}$, let $(U_A)_{A \subset \mathbf{T}}$ be an i.i.d. family of uniform random variables on $[0, 1]$, and for $K \subset\subset \mathbb{Z}^d$, define the family $\mathcal{B}_K = (\mathcal{B}_K^\alpha)_{\alpha \geqslant 0}$ of sets

$$(1.20) \qquad \mathcal{B}_K^\alpha = \bigcup_{\substack{A \subset \mathbf{T}:\, \alpha_*(A) > \alpha_*(K) \\ U_A \leqslant p^\alpha(A)}} A.$$

Note that $K \subset \mathbb{Z}^d$ in (1.20) (whereas $A \subset \mathbf{T}$). In particular K does not depend on N, the side length of the torus, which is important when taking limits as $N \to \infty$, as in (1.22) below. Notice also that sets $A \subset \mathbf{T}$ with diameter N are never considered in the definition of $\mathcal{B}_K^\alpha$ by the above convention. Actually $\mathcal{B}_K^\alpha$ only depends on K through $\mathrm{cap}(K)$. Moreover, if $K = \{x, y\}$ for $x \sim y$, then $\alpha_*(K) = \alpha_* \geqslant \alpha_*(A)$ for all $A \subset \mathbf{T}$ with $|A| \geqslant 2$, whence A in (1.20) only ranges over singletons and $\mathcal{B}_K^\alpha$ is virtually equal in law to $\mathcal{B}^\alpha$, as will later be seen in detail. This is actually not entirely true, notably when $\alpha \leqslant \alpha_*$, see (7.17), but one may pretend for the purposes of this introduction that $(\mathcal{B}_K^\alpha)_{\alpha \geqslant 0} \overset{\text{law}}{=} (\mathcal{B}^\alpha)_{\alpha \geqslant 0}$ when $K = \{x, y\}$, $x \sim y$. Admitting this, the following result can then essentially be viewed as a generalization of Theorem 1.1.

To state it, we introduce one more convenient notation, which allows to capture transitions such as (1.12)-(1.14) in a concise (albeit less explicit) way. For two decreasing families of random subsets $\mathcal{S} = (\mathcal{S}^\alpha)_{\alpha \geqslant 0}$ and $\mathcal{U} = (\mathcal{U}^\alpha)_{\alpha \geqslant 0}$ of $\mathbf{T}$, we define for all $\alpha \in (0, 1)$ and $\varepsilon \in (0, \alpha)$

$$(1.21) \qquad d_\varepsilon(\mathcal{S}, \mathcal{U}; \alpha) = \inf \left\{ \delta \in [0, 1] : \begin{array}{l} \exists \text{ a coupling } \mathbb{Q} \text{ between } \mathcal{S}^\alpha \text{ and } (\mathcal{U}^{\alpha-\varepsilon}, \mathcal{U}^{\alpha+\varepsilon}) \\ \text{such that } \mathbb{Q}(\mathcal{U}^{\alpha+\varepsilon} \subset \mathcal{S}^\alpha \subset \mathcal{U}^{\alpha-\varepsilon}) \geqslant 1 - \delta. \end{array} \right\}$$

Note that $\lim_{\varepsilon \to 0} d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha) = d_{\mathrm{TV}}(\mathcal{L}^\alpha, \mathcal{B}^\alpha)$ by continuity.

Theorem 1.3. *For $\mathcal{L}^\alpha$ as in (1.4), all $K \subset\subset \mathbb{Z}^d$ and all $\alpha \in (0, 1)$,*

$$(1.22) \qquad \lim_{\varepsilon \to 0} \lim_{N \to \infty} d_\varepsilon(\mathcal{L}, \mathcal{B}_K; \alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_*(K) > 1/2 \\ = 1 & \text{if } \alpha < \alpha_*(K) \text{ or } \alpha_*(K) \leqslant 1/2 \end{cases}$$

and if $\alpha_*(K) > 1/2$

$$(1.23) \qquad 0 < \lim_{\varepsilon \to 0} \liminf_{N \to \infty} d_\varepsilon(\mathcal{L}, \mathcal{B}_K; \alpha_*(K)) \leqslant \lim_{\varepsilon \to 0} \limsup_{N \to \infty} d_\varepsilon(\mathcal{L}, \mathcal{B}_K; \alpha_*(K)) < 1.$$

By applying Theorem 1.3 with K any pair of neighbors, e.g. $K = K_0 = \{0, x\}$, $x \sim 0$, one (essentially, cf. above) recovers Theorem 1.1. Now suppose that $K \subset\subset \mathbb{Z}^d$ is another set with $\alpha_*(K) > 1/2$, not isomorphic to K_0 under lattice symmetries; for instance $K = \{0, x\}$, $|x|_1 = 2$. Then (1.22) asserts that $\mathcal{L}^\alpha$ will be close (as measured by d_ε) to $\mathcal{B}_K$ if and only if $\alpha > \alpha_*(K)$. The set $\mathcal{B}_K^\alpha$ comprises independent samples of all allowed 'clusters' (namely, sets $A \subset \mathbf{T}$ with $\alpha_*(A) > \alpha_*(K)$), at the correct intensity, corresponding to their probability to be seen in $\mathcal{L}^\alpha$ and to be isolated, i.e. with no other late points present in their $R_{\mathbf{T}}$-neighbourhood. This can be regarded as a positive answer to Question 2 in [22, Section 7]. Note the consistency of the transitions (1.22) as K varies: for instance, if $\alpha > \alpha_*(= \alpha_*(K_0))$, then, as it turns out, for every $K \in \mathbb{Z}^d$ with $\alpha_*(K) \in (1/2, 1)$, the sets $\mathcal{B}_K^\alpha$ and $\mathcal{B}_{K_0}^\alpha$ are virtually indistinguishable as $N \to \infty$, and actually also indistinguishable of $\mathcal{B}^\alpha$. In fact (1.12) will be deduced from the first line of (1.22) in precisely this way, cf. §7; see also Remark 7.3,2) for alternatives. Note also that the critical case (1.23) is less explicit than (1.13), see Remark 7.3,3) for more on this.

Theorem 1.3 is mainly interesting when $\alpha_*(K) > 1/2$, as it then describes precisely the law of $\mathcal{L}^\alpha$ for $\alpha > \alpha_*(K)$ via $\mathcal{B}_K^{\alpha \pm \varepsilon}$. It is easy to see that the constraint $\alpha_*(K) > 1/2$ is saturated for $K = \{x, y\}$ in the limit when $|x - y| \to \infty$, see (2.6) below. In particular for all $\alpha > 1/2$ there exists $K \subset\subset \mathbb{Z}^d$ with $\alpha > \alpha_*(K) > 1/2$, for which $\mathcal{B}_K^\alpha$ is a good approximation of $\mathcal{L}^\alpha$ by (1.22). On the other hand if $\alpha_*(K) \leqslant 1/2$, then for all $x, y \in \mathbf{T}$ we have $\alpha_*(\{x, y\}) > \alpha_*(K)$ and so the probability that $x \in \mathcal{B}_K^\alpha$, see (1.20), will typically be much larger than the probability that $x \in \mathcal{L}^\alpha$, which explains the last part of (1.22); cf. also (7.16). An important question is thus to determine which kind of sets K are in

$$(1.24) \qquad \mathcal{A}_{\mathbf{T}} = \{K \subset \mathbf{T} : K \neq \varnothing, \, \alpha_*(K) \geqslant \tfrac{1}{2}\},$$

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for these are exactly the sets one needs to consider in Theorem 1.3 to approximate $\mathcal{L}^\alpha$ for $\alpha > \frac{1}{2}$. By (2.6) below, $\mathcal{A}_{\mathbf{T}}$ contains at least all singletons and two-point sets. But does $\mathcal{A}_{\mathbf{T}}$ contain larger sets, for instance containing three points, and, if so, which ones? Put differently, as $N \to \infty$, does $\{\mathcal{L}^\alpha \cap Q(x, R_{\mathbf{T}}), x \in \mathbf{T}\}$ contain sets with three points before containing all possible 2 points sets with diameter at most $R_{\mathbf{T}}$? Viewing $\mathbb{Z}^2$ as $\mathbb{Z}^2 \times \{0\}^{d-2} \subset \mathbb{Z}^d$, let

$$(1.25) \qquad K_1 = \{(0,0), (0,1), (0,2)\}, \ \ K_2 = \{(0,0), (0,1), (1,0)\}.$$

We denote by $\mathbf{K_1}$ and $\mathbf{K_2}$ the projections of K_1 and K_2 onto $\mathbf{T}$ (and generally bold sets will always be the projection on $\mathbf{T}$ of sets on $\mathbb{Z}^d$), which are the only connected sets of cardinality 3, up to torus isomorphisms. We prove in Appendix B that

$$(1.26) \qquad \text{if } \mathbf{K_1}, \mathbf{K_2} \notin \mathcal{A}_{\mathbf{T}}, \text{ then } \mathcal{A}_{\mathbf{T}} \text{ consists only of sets with cardinality at most 2,}$$

and there are sets $A_i \subset \mathbb{Z}^d$, $i \in \{1, \ldots, 8\}$ with cardinality at least three such that, denoting by $\mathbf{A_i}$ the projection of A_i on $\mathbf{T}$

$$(1.27) \qquad \begin{array}{l} \text{if } \mathbf{K_1}, \mathbf{K_2} \in \mathcal{A}_{\mathbf{T}} \text{ and } \mathbf{A_i} \notin \mathcal{A}_{\mathbf{T}} \text{ for all } i \in \{1, \ldots, 8\}, \text{ then } \mathcal{A}_{\mathbf{T}} \text{ consists only of sets} \\ \text{with cardinality at most 2 and images of } \mathbf{K_1} \text{ and } \mathbf{K_2} \text{ by torus isomorphisms.} \end{array}$$

We then deduce in Theorem B.1 that (1.26) applies for all $d \geqslant 4$ and (1.27) when $d = 3$. The proof of Theorem B.1 relies on computer-assisted methods partially inspired by [18, Appendix B], to determine the capacities of the sets K_1, K_2 and A_i, $i \in \{1, \ldots, 8\}$ (the latter only when $d = 3$) as well as $g(0)$, cf. (1.19), which are computed with precision 10^{-30}, see Lemma B.4.

Together, Theorems 1.3 and B.1 readily yield the following:

Corollary 1.4. *For small enough $\eta > 0$, there exists $D = D(\eta, d) < \infty$ such that, if $\mathcal{D}^\alpha \subset \mathbf{T}$, $\alpha = \frac{1}{2} + \eta$, is obtained as the union of all sets $K \subset \mathbf{T}$ which are either*

   i) singletons,
   ii) pairs of points at ℓ^1-distance $\leqslant D$,
   iii) images of $\mathbf{K_1}$ or $\mathbf{K_2}$ by torus isomorphisms, that is connected sets of cardinality 3,

each sampled independently with probability $p^\alpha(K)$, then there exists $\mathbf{Q} = \mathbf{Q}_\eta$ such that if $d = 3$

$$(1.28) \qquad \mathbf{Q}\big(\mathcal{D}^{\frac{1+3\eta}{2}} \subset \mathcal{L}^{\frac{1}{2}+\eta} \subset \mathcal{D}^{\frac{1+\eta}{2}}\big) \to 1 \ \text{ as } N \to \infty$$

Moreover if $d = 3$, there is no such $\mathbf{Q}$ if iii) is omitted, whereas if $d \geqslant 4$, all conclusions remains true upon discarding type iii) from the construction of $\mathcal{D}^\alpha$.

We conclude by explaining the significance of the value $\alpha = \frac{1}{2}$ in (1.24), Theorem 1.3 and Corollary 1.4. The parameter $\alpha = \frac{1}{2}$ has already been identified as the critical parameter for another question in [21], and on the torus the main result from [21] can actually be deduced from Theorem 1.3, see Remark 7.3,4). For each $\alpha > \frac{1}{2}$, one can find a constant $C = C(\alpha)$ such that with high probability, each $x, y \in \mathcal{L}^\alpha$ verifies either $d(x, y) \leqslant C$ or $d(x, y) \geqslant (\log N)^{1/(d-2)}$ (in fact even $d(x, y) \geqslant N^{2\alpha-1-\eta}$ for some $\eta > 0$), see Lemma 6.8. In other words, any value of $\alpha > \frac{1}{2}$ induces a natural localization scale for $\mathcal{L}^\alpha$, which is a union of small sets (with diameter at most C) far away from one another. This well-separatedness fails when $\alpha \leqslant \frac{1}{2}$, and one cannot straightforwardly deduce an approximative version of the law of $\mathcal{L}^\alpha$ in this regime, see Remark 8.4,4). It would of course be interesting to assess whether a result similar to Theorem 6.5, which is a consequence of our localization statement Theorem 1.2, valid throughout the phase $\alpha > \frac{1}{2}$, and one of the driving force behind our main results, still holds for $\alpha \leqslant \frac{1}{2}$ or not.

1.4. Organization of this article. Section 2 introduces the setup and a minimal amount of useful notation. Section 3 is centred around the modified Chen-Stein method, which allows for a sprinkling. Section 4 introduces the soft local time method to couple two different Markov chains from [24], and explains how to invert it. This method is then applied in Section 5 to the excursions of the random walk on $\mathbf{T}$ and random interlacements on $\mathbb{Z}^d$ to prove the localization result, Theorem 1.2. Its strengthened version, Theorem 5.1, also immediately yields a state-of-the-art coupling between random walk and random interlacements with quantitative coupling error, stated in Corollary 5.2. Section 6 discusses the consequences of localization for late points. In §6.1, we introduce the set of late points $\mathcal{L}_F^\alpha$ to be studied (see (6.2)), which contains $\mathcal{L}^\alpha$ defined in (1.4) as a special case, and gather its first properties, including precise estimates on the probability for a generic set to be late. In §6.2, we then compare $\mathcal{L}_F^\alpha$ with a suitably localized version $\widetilde{\mathcal{B}}_F^\alpha$ defined in (6.17). The main result is Theorem 6.5.

The pieces are put together in Section 7, where we derive Theorems 1.1 and 1.3. Remark 7.3, which appears at the end of §7, deserves highlighting. It concerns various extensions of these results and also explains how to recover existing ones. Thereafter, §8 contains some interesting further results which can be derived using our methods. More precisely, §8.1 deals with $\mathcal{L}_F^\alpha$ as a process in α (see (1.16) and Theorem 8.1), and §8.2 is an outlook to the regime $\alpha \leqslant \frac{1}{2}$ containing a description of the law of large clusters, see Theorem 8.3. Finally, Appendix A contains the proof of two technical ingredients, Lemmas 5.5 and 5.6, relegated from Section 5, which concern certain large deviation estimates for the excursions of random walk and random interlacements. Appendix B revolves around the notion of admissible sets. In particular, Theorem B.1 identifies the elements of $\mathcal{A}_\mathbf{T}$ in (1.24), which leads to Corollary 1.4. The elements of the proof of Theorem B.1 relying on computer assistance are all summarized in Lemma B.4. We stress however that – with the exception of Corollary 1.4 – none of our results make use of Appendix B.

We conclude with our convention regarding constants. In the rest of this article, we denote by $c, c', \ldots$ and $C, C', \ldots$ positive and finite constants changing from place to place. All constants may depend implicitly on the dimension d, and their dependence on any other quantity will be made explicit.

2   Notation and preliminaries

We set up some notation that will be used throughout. We use $\pi : \mathbb{Z}^d \to \mathbf{T} = (\mathbb{Z}/N\mathbb{Z})^d$ to denote the canonical projection from the infinite d-dimensional cubic latice $\mathbb{Z}^d$ to the d-dimensional torus $\mathbf{T}$ of size length N, for $d \geqslant 3$, $N \geqslant 1$. In order to simplify notation, we will often use the notation $\boldsymbol{x}$ to denote the projection $\pi(x)$ of $x \in \mathbb{Z}^d$. We denote by 0 the origin of $\mathbb{Z}^d$, and thus call $\mathbf{0} := \pi(0)$ the origin of $\mathbf{T}$. For each $x \in \mathbb{Z}^d$ and $r > 0$, we let $Q(x, r) = Q_r(x) = x + ([-\lfloor (r-1)/2 \rfloor, \lceil (r-1)/2 \rceil] \cap \mathbb{Z})^d$ the box of size length r around x in $\mathbb{Z}^d$, and we let $Q(\boldsymbol{x}, r) = \pi(Q_r(x))$, which only depends on $\boldsymbol{x}$ and r. Our convention for boxes is tuned so that $Q_N(\mathbf{0}) = \mathbf{T}$, and $\pi_{|Q_N(0)}$ is a bijection into $\mathbf{T}$. Note that the definition of $Q_r(x)$ depends on whether $x \in \mathbb{Z}^d$ or $x \in \mathbf{T}$. We will also introduce the notation Q_N (see above (6.1)), which will only be used in Sections 6-8 (Q_N can correspond to either $Q_N(\mathbf{0})$ or $Q_N(0)$, depending on the model considered, which will allow for a uniform presentation). We also let $Q(A, R)$ be the union of the balls $Q(x, R)$ over all $x \in A$, for any $A \subset \mathbb{Z}^d$ or $A \subset \mathbf{T}$.

For a set $A \subset \mathbb{Z}^d$ or $A \subset \mathbf{T}$ we write ∂A for the internal vertex boundary of A, i.e. $\partial A = \{ y \in A : \exists \, x \notin A \text{ adjacent to } y \}$, $|A|$ for the cardinality of A and $\delta(A)$ for the (ℓ^∞-)diameter of A, that is the smallest $R \in \mathbb{N}$ such that $A \subset Q(x_A, R)$ for some vertex $x_A \in \mathbb{Z}^d$ or $x_A \in \mathbf{T}$. We use

the notation $A \subset\subset \mathbb{Z}^d$ to say that A is a finite subset of $\mathbb{Z}^d$. Let also $d(x, y) = \delta(\{x, y\})$ for all $x, y \in \mathbb{Z}^d$ or $x, y \in \mathbf{T}$, which on $\mathbb{Z}^d$ corresponds to the ℓ^∞-distance between x and y. For subsets K, K' either of $\mathbb{Z}^d$ or $\mathbf{T}$ we write $d(K, K') = \inf_{x \in K, y \in K'} d(x, x')$.

We write P_x for the canonical law of the discrete-time simple random walk on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$. We denote by $X = (X_n)_{n \geqslant 0}$ the corresponding canonical process, and by θ_n, $n \geqslant 0$, the canonical shifts, so that $X \circ \theta_n = (X_{m+n})_{m \geqslant 0}$. For $K \subset \mathbb{Z}^d$ we define $H_K = \inf\{n \geqslant 0 : X_n \in K\}$ and $T_K = H_{\mathbb{Z}^d \setminus K}$ the respective entrance time in K and exit time from K, for $K \subset \mathbb{Z}^d$. We write $\widetilde{H}_K = \inf\{n \geqslant 1 : X_n \in K\}$ for the hitting time of K. Here, we take the convention $\inf \varnothing = +\infty$. For $n \geqslant 0$ we also introduce the time of last visit to K (before time n),

$$(2.1) \qquad L_K(n) = \sup\{m \leqslant n : X_m \in K\}, \; L_K = \lim_{n \to \infty} L_K(n)$$

with the convention $\sup \varnothing = -\infty$. For $x \in \mathbf{T}$ and $y \in \pi^{-1}(\{x\})$, we set $\mathbf{P}_x = \pi \circ P_y$, which is well-defined (i.e. does not depend on the choice of y). With a slight abuse of notation, we also write $(X_n)_{n \geqslant 0}$ for the canonical process under $\mathbf{P}_x$, $x \in \mathbf{T}$, which is the simple random walk on $\mathbf{T}$, starting at x, and extend the definition of $H_K, T_K, \widetilde{H}_K, L_K(n)$ and L_K under $\mathbf{P}_x$. Let also $\mathbf{P} \overset{\text{def.}}{=} N^{-d} \sum_{x \in \mathbf{T}} \mathbf{P}_x$. Under $\mathbf{P}_x$ (or $\mathbf{P}$), one defines for all $t, u > 0$ the (discrete) local times

$$(2.2) \qquad \ell_x(t) = \sum_{0 \leqslant n \leqslant \lfloor t \rfloor} 1\{X_n = x\}, \; t \in [0, \infty), \quad \ell_{x,u} = \ell_x(uN^d).$$

We denote by $g(x, y)$, $x, y \in \mathbb{Z}^d$ the Green's function of X under P_x, which is known to be finite and symmetric. Moreover $g(x, y) = g(x - y, 0) \equiv g(x - y)$ so in particular, $g(0) = g(0, 0)$. For suitable f on $\mathbb{Z}^d$ (e.g. with finite support), we write $Gf(x) = \sum_{y \in \mathbb{Z}^d} g(x, y) f(y)$. For finite $K \subset \mathbb{Z}^d$, we denote by e_K the equilibrium measure of K,

$$(2.3) \qquad e_K(x) = P_x(\widetilde{H}_K = \infty) 1_K(x), \text{ for } x \in \mathbb{Z}^d,$$

which is finite and supported on ∂K, and by $\mathrm{cap}(K)$ its total mass, the capacity of K, which is monotone in K. We write $\overline{e}_K = \frac{e_K}{\mathrm{cap}(K)}$ for the normalized equilibrium measure. If $K \subset \mathbf{T}$ with $\delta(K) < N$, we also define the capacity of K as follows: if $K \subset Q_{N-1}(\mathbf{0})$ and $K' \subset Q_{N-1}(0)(\subset \mathbb{Z}^d)$ is a set such that $\pi(K') = K$, then we let $\mathrm{cap}(K) = \mathrm{cap}(K')$. For other K with $\delta(K) < N$, we define $\mathrm{cap}(K)$ by translation invariance. Moreover, we take $\mathrm{cap}(K) = \infty$ for the subsets of $\mathbf{T}$ which are not topologically trivial, i.e. when $\delta(K) = N$. One knows that

$$(2.4) \quad Ge_K = h_K, \quad \text{where } h_K(x) = P_x(H_K < \infty), \; x \in \mathbb{Z}^d \text{ is the equilibrium potential of } K.$$

The following straightforward (strict) monotonicity result for g will be repeatedly used. Let $|\cdot|_1$ denote the ℓ^1-distance (i.e. the graph distance) on $\mathbb{Z}^d$.

Lemma 2.1. *For all integers $n \geqslant 0$,*

$$(2.5) \qquad \sup_{|x|_1 > n} g(x) < \sup_{|x|_1 = n} g(x).$$

Proof. Set $B_1(x, n) = \{y \in \mathbb{Z}^d : |y - x|_1 \leqslant n\}$. First, one classically knows that there exists $c(n) > 0$ such that $P_x(H_{B_1(0,n)} = \infty) \geqslant c(n)$ for all $x \notin B_1(0, n)$. Hence, by the strong Markov property and translation invariance, one obtains, for all $x \in \mathbb{Z}^d$ with $|x|_1 > n$,

$$\frac{g(0, x)}{g(0)} \overset{(2.4)}{=} P_0(H_x < \infty) = E_0\big[1\{H_{B_1(x,n)} < \infty\} P_{X_{H_{B_1(x,n)}}}(H_x < \infty) \big]$$

$$\leqslant (1 - c(n)) \sup_{y \in \partial B_1(x,n)} P_y(H_x < \infty) \overset{(2.4)}{=} (1 - c(n)) \sup_{|y|_1 = n} \frac{g(0, y)}{g(0)},$$

from which (2.5) follows. $\qquad \square$

By evaluating (2.4) for $x \in K$ and solving the resulting linear system, one can explicitly determine e_K and $\mathrm{cap}(K)$ in terms of g. In case K is a singleton or a two-point set, this gives

$$(2.6) \qquad \mathrm{cap}(\{x,y\}) = \frac{2}{g(0) + g(x-y)}, \text{ for all } x, y \in \mathbb{Z}^d$$

(in particular, $\mathrm{cap}(\{x\}) = g(0)^{-1}$). For $K \subseteq U \subset \mathbb{Z}^d$ we further set $e_K^U(x) = P_x(\widetilde{H}_K > T_U)1_K(x)$, so that (2.3) corresponds to choosing $U = \mathbb{Z}^d$, and write $\mathrm{cap}_U(K) = \sum_x e_K^U(x)$ for its total mass. We now collect some further facts about capacity, which will be useful in the rest of the article. First, as we now explain, combining (2.6) and Lemma 2.1 one obtains that

$$(2.7) \qquad \mathrm{cap}(K) > \mathrm{cap}(\{x,y\}) \text{ for any } K \subset \mathbb{Z}^d \text{ with } \mathrm{diam}_{\ell^1}(K) \geqslant 2 \text{ and } x \sim y.$$

Indeed by monotonicity of $K \mapsto \mathrm{cap}(K)$, see [19, Prop. 2.2.1], it is enough to consider the case $|K| = 2$ with two-points at distance $n \geqslant 2$, from which (2.7) follows using the formula (2.6), translation invariance, and (2.5) for $n \geqslant 2$. Next, by [15, Lemma 1.11] and [3, Proposition 1.2], for any sets $K, K' \subset\subset \mathbb{Z}^d$, one has

$$(2.8) \qquad 0 \leqslant \mathrm{cap}(K) + \mathrm{cap}(K') - \mathrm{cap}(K \cup K') \leqslant c|K'||K|d(K,K')^{-(d-2)},$$

for some constant c depending only on d. An important application of the right-hand side of (2.8) is that if $F \subset\subset \mathbb{Z}^d$ and $K, K' \subset F$ are such that $d(K,K') \geqslant \log(|F|)^{1/(d-2)}/2$, then for all $\alpha \in (0,1]$

$$(2.9) \qquad |F|^{-\alpha g(0)\mathrm{cap}(K \cup K')} \leqslant C|F|^{-\alpha g(0)\mathrm{cap}(K)} \cdot |F|^{-\alpha g(0)\mathrm{cap}(K')},$$

for some constant C depending only on $d, |K|$ and $|K'|$. The following improvement of (2.9) will also be used. If, as will commonly occur in practice, $|F| \to \infty$ and $\frac{d(K,K')}{\log(|F|)^{1/(d-2)}} \to \infty$ whereas $|K|, |K'| \leqslant C'$, one can further replace the constant C appearing in (2.9) by $(1 + o(1))$ (as $|F| \to \infty$). Note that the reverse inequality in (2.9) is also true with $C = 1$ by (2.8). Another interesting application of (2.8) is the following: for each $r > 0$,

$$(2.10) \quad \text{there exists } C = C(r) < \infty \text{ so that each } K \subset\subset \mathbb{Z}^d \text{ with } |K| \geqslant C \text{ satisfies } \mathrm{cap}(K) \geqslant r.$$

Indeed, in order to prove (2.10), it suffices to prove that if K contains $\lceil r/\mathrm{cap}(\{0\}) \rceil + 1$ points x_i so that $d(x_i, x_j)$ is large enough for each $i \neq j$, then $\mathrm{cap}(K) \geqslant r$, which follows easily from (2.8). Finally note that (2.7), (2.8), (2.9) and (2.10) still hold when $K, K' \subset \mathbf{T}$ as long as the capacities in question are not infinite, which is for instance the case when $\delta(K), \delta(K') < N/2$.

We now briefly introduce the random interlacement process and its associated local times to the extent we need them. We denote by ω the random interlacements process on $\mathbb{Z}^d$ with law $\mathbb{P}^I$, as defined in [30], which is a Poisson process of bi-infinite random walk trajectories with positive labels. We will actually never need to describe the full law of ω here, but only its push-forward ω_B^u to the trajectories with label less than u which hit a ball $B \subset \mathbb{Z}^d$, and started after their first hitting time of B, for some $u > 0$, that we now describe. For each finite set $B \subset \mathbb{Z}^d$, there exists a Poisson process $N_B = (N_B^u)_{u \geqslant 0}$ on $[0,\infty)$ with intensity $\mathrm{cap}(B)$ and an independent i.i.d. sequence of random walks $(X^i)_{i \geqslant 1}$, each with law $P_{\bar{e}_B}$, such that for all $u > 0$, under $\mathbb{P}^I$,

$$(2.11) \qquad \omega_B^u = \sum_{i=1}^{N_B^u} \delta_{X^i}.$$

This description of ω_B^u via N_B^u and $(X^i)_{i \geqslant 1}$ entirely characterises its law. Note that the trajectories $(X^i)_{i \geqslant 1}$ depend on the choice of B, but this dependence does not appear in the notation

for simplicity. We write $\mathcal{I}_B^u$ for the interlacements set in B and $\ell_{y,u}$ for the associated local time at $y \in B$ at level u, i.e.

$$(2.12) \qquad \ell_{y,u} = \ell_{y,u}(\omega) = \sum_{i=1}^{N_B^u} \sum_{k=0}^{\infty} 1\{X_k^i = y\},$$

and

$$(2.13) \qquad \mathcal{I}_B^u = \mathcal{I}^u \cap B = \bigcup_{i=1}^{N_B^u} X^i[0, \infty) \cap B \quad (= \{y \in B : \ell_{y,u} > 0\})$$

where $X^i[0, \infty) = \{X_k^i, \ k \in \mathbb{N}\}$. The vacant set at level u is defined as $\mathcal{V}^u = \mathbb{Z}^d \backslash \mathcal{I}^u$ and the formula (1.6) follows as $\mathbb{P}^I(\mathcal{V}^u \subset B) = \mathbb{P}^I(N_B^u = 0)$.

3 Modified Chen-Stein method with sprinkling

We now collect a result, of independent interest, which roughly speaking generates a coupling of some (Bernoulli) process Y of interest –possibly highly correlated– and an independent process W from a given coupling between Y with another process Z in such a way that (Y, W) are 'close' whenever (Y, Z) are. Here, 'distance' will be quantified by the functional d_ε introduced below, which allows for a sprinkling with parameter $\varepsilon > 0$ in the underlying density of the processes. The main result appears in Lemma 3.1. For the applications in this article, Y will be the occupation time field of RW/RI, Z will be a suitable finite-range approximation, in a sense to be made precise, see (3.4), and W the target i.i.d. field. We compare the philosophy underlying Lemma 3.1 with the traditional Chen-Stein method further in Remark 3.2.

In the sequel, given some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a finite set S (typically a finite collection of subsets of $\mathbb{Z}^d$ or $\mathbb{Z}_N^d$), we call *Bernoulli process on S* any random variable $Z = (Z_x)_{x \in S} : \Omega \to \{0, 1\}^S$. The process Z will be referred to as *independent* whenever $\{Z_x : x \in S\}$ constitutes an independent family of random variables. Given $I \subset [0, \infty)$ an interval and two families $Z = (Z^\alpha)_{\alpha \in I}$, $Y = (Y^\alpha)_{\alpha \in I}$ of Bernoulli processes on a countable set S (i.e. for every $\alpha \in I$, Y^α, Z^α are Bernoulli processes on S), we define, for every $\alpha, \varepsilon \geqslant 0$ such that $\alpha \pm \varepsilon \in I$,

$$(3.1) \qquad d_\varepsilon(Y, Z; \alpha) = \inf \left\{ \delta \in [0, 1] \ : \ \begin{array}{l} \exists \text{ a coupling } \widehat{\mathbb{P}} \text{ between } Y^\alpha \text{ and } (Z^{\alpha-\varepsilon}, Z^{\alpha+\varepsilon}) \\ \text{s.t. } \widehat{\mathbb{P}}(Z_x^{\alpha+\varepsilon} \leqslant Y_x^\alpha \leqslant Z_x^{\alpha-\varepsilon} \ \forall x \in S) \geqslant 1 - \delta \end{array} \right\}.$$

Note that $d_0(Y, Z; \alpha) = d_{\mathrm{TV}}(Y^\alpha, Z^\alpha)$. This is consistent with the notation from (1.21) by setting $d_\varepsilon(Y, Z; \alpha) = d_\varepsilon(\mathcal{S}_Y, \mathcal{S}_Z; \alpha)$, i.e. identifying $Y = (Y^\alpha)_{\alpha \in I}$ with the corresponding family of occupations sets $\mathcal{S}_Y = (\mathcal{S}_Y^\alpha)_{\alpha \in I}$, where $\mathcal{S}_Y^\alpha = \{x : Y_x^\alpha = 1\}$ and similarly for Z. We will use d_ε to measure proximity between Y and Z.

The following setup is tailored to our purposes. The process $Y = (Y^\alpha)_{\alpha \in I}$ is called a *decreasing* family of Bernoulli processes on S if Y^α, $\alpha \in I$, are defined on a joint probability space and $Y_x^\beta \leqslant Y_x^\alpha$ a.s. for all $x \in S$ and $\alpha, \beta \in I$ with $\beta \geqslant \alpha$. We consider two families $Y = (Y^\alpha)_{\alpha \in I}$ and $W = (W^\alpha)_{\alpha \in I}$ with the following properties:

$$(3.2) \qquad\qquad Y \text{ is a decreasing family of Bernoulli processes on } S;$$

$$(3.3) \qquad\qquad \begin{array}{l} \text{for each } \alpha \in I, \ W^\alpha \text{ is an independent Bernoulli process on } S, \\ \text{decreasing in } \alpha, \text{ and } \mathbb{P}(W_x^\alpha = 1) = \mathbb{P}(Y_x^\alpha = 1) \text{ for all } \alpha \in I, \ x \in S. \end{array}$$

The goal will typically be to couple Y and W in a manner keeping $d_\varepsilon(Y, W; \cdot)$ small. In practice, we will first approximate Y by a family $Z = (Z^\alpha)_{\alpha \in I}$ of finite-range processes. Note that our

notion of finite range processes, see (3.4), is slightly different from the usual notion of finite range, and is adapted to our context. The following lemma then allows to 'lift' couplings between (Y, Z) controlling the quantity $d_\varepsilon(Y, Z; \cdot)$, to couplings with similar properties between (Y, W).

Lemma 3.1. *Let Y, W satisfy (3.2)-(3.3). If*

$(Z^\alpha)_{\alpha \in I}$ *is a family of Bernoulli processes on S such that*

(3.4)   *(Monotonicity): $Z = (Z^\alpha)_{\alpha \in I}$ is a decreasing family, and*
         (Finite range): for every $x \in S$, there exists $\mathcal{N}_x \subset S$ such that for $\alpha, \beta \in I$, $\alpha \leqslant \beta$,
$$Z_x^\beta \text{ and } Z_x^\alpha - Z_x^\beta \text{ are each independent of } \{Z_y^\alpha, Z_y^\beta : y \notin \mathcal{N}_x\},$$

then for all $\varepsilon > 0$ and $\alpha \in I$ such that $\alpha \pm 3\varepsilon \in I$ one has

$$(3.5) \qquad d_{2\varepsilon}(Y, W; \alpha) \leqslant 400 \sup_{\alpha' \in \{\alpha - 2\varepsilon, \alpha\}} \left\{ b_1(\alpha') + b_2(\alpha') + d_\varepsilon(Y, Z; \alpha') \cdot |S|^2 \right\},$$

where

$$(3.6) \qquad b_1(\alpha) \stackrel{\text{def.}}{=} \sum_{x \in S} \sum_{y \in \mathcal{N}_x} \mathbb{P}(Y_x^\alpha = 1) \mathbb{P}(Y_y^\alpha = 1),$$

$$(3.7) \qquad b_2(\alpha) \stackrel{\text{def.}}{=} \sum_{x \in S} \sum_{y \in \mathcal{N}_x \setminus \{x\}} \mathbb{P}(Y_x^\alpha = 1, Y_y^\alpha = 1).$$

We comment further on the utility of (3.5) in Remark 3.2 at the end of the proof.

Proof. Let $\delta > 0$ and $\mathbb{P}^U$ denote a probability measure carrying a family U_x, $x \in S$, of i.i.d. uniform random variables on $[0, 1]$. With a slight abuse of notation, we realize W on the space $\mathbb{P}^U$ along with an auxiliary process $\widetilde{W}$ by setting

$$(3.8) \qquad W_x^\alpha = 1\{U_x \leqslant \mathbb{P}(Y_x^\alpha = 1)\}, \quad \widetilde{W}_x^\alpha = 1\{U_x \leqslant \mathbb{P}(Z_x^\alpha = 1)\}, \quad x \in S, \ \alpha \in I.$$

Clearly, (3.3) holds and (3.8) defines a coupling of $(W, \widetilde{W})$. We will first work with $\widetilde{W}$, which matches the one-point densities of Z, and return to W towards the end of the proof. Henceforth, fix $\alpha \in I$ and $\varepsilon > 0$ such that $\alpha \pm 3\varepsilon \in I$ and let $\gamma(\alpha, \varepsilon)$ refer to the supremum on the right-hand side of (3.5). Observe that $Z^{\alpha-\varepsilon} - Z^{\alpha+\varepsilon}$ is a (well-defined) Bernoulli process on S by monotonicity of Z and the same holds true for $\widetilde{W}^{\alpha-\varepsilon} - \widetilde{W}^{\alpha+\varepsilon}$ on account of (3.8). We will first construct a coupling $\widetilde{\mathbb{Q}}$ between $(Z^{\alpha-\varepsilon}, Z^{\alpha+\varepsilon})$ and $(\widetilde{W}^{\alpha-\varepsilon}, \widetilde{W}^{\alpha+\varepsilon})$ with the property that

$$(3.9) \qquad \widetilde{\mathbb{Q}}\big(Z^{\alpha-\varepsilon} = \widetilde{W}^{\alpha-\varepsilon}, Z^{\alpha+\varepsilon} = \widetilde{W}^{\alpha+\varepsilon}\big) \geqslant 1 - 384\gamma(\alpha, \varepsilon) - \delta.$$

To this end, let $S' = S \times \{-, +\}$ and consider the Bernoulli processes Z' and $\widetilde{W}'$ on S' defined as

$$(3.10) \qquad Z'_{(x,\sigma)} = \begin{cases} Z_x^{\alpha-\varepsilon} - Z_x^{\alpha+\varepsilon}, & \text{if } \sigma = - \\ Z_x^{\alpha+\varepsilon}, & \text{if } \sigma = + \end{cases}, \qquad \widetilde{W}'_{(x,\sigma)} = \begin{cases} \widetilde{W}_x^{\alpha-\varepsilon} - \widetilde{W}_x^{\alpha+\varepsilon}, & \text{if } \sigma = - \\ \widetilde{W}_x^{\alpha+\varepsilon}, & \text{if } \sigma = + \end{cases}.$$

Due to (3.8), Z' and $\widetilde{W}'$ have the same one-dimensional marginals, i.e. $Z'_{(x,\sigma)} \stackrel{\text{law}}{=} \widetilde{W}'_{(x,\sigma)}$ for all $(x, \sigma) \in S'$. For any point $x' = (x, \sigma) \in S'$, define its neighborhood $\mathcal{N}_{x'} = \mathcal{N}_x \times \{\pm\}$, with $\mathcal{N}_x$ as given by (3.4). We want to bound the total variation distance between Z' and $\widetilde{W}'$. Because $\widetilde{W}'$ is not an independent Bernoulli process, we let W'' be an independent Bernoulli process with the

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same one dimensional marginals as Z'. Using the triangle inequality for total variation distance and as we explain below, applying [2, Theorem 3] (see also [9, 29]), one obtains that

$$(3.11) \qquad d_{\mathrm{TV}}(Z', \widetilde{W}') \leqslant d_{\mathrm{TV}}(Z', W'') + d_{\mathrm{TV}}(W'', \widetilde{W}') \leqslant 24(b_1' + b_2'),$$

where

$$(3.12) \qquad b_1' = \sum_{x' \in S'} \sum_{y' \in \mathcal{N}_{x'}} \mathbb{P}(Z'_{x'} = 1) \mathbb{P}(Z'_{y'} = 1), \quad b_2' = \sum_{x' \in S'} \sum_{y' \in \mathcal{N}_{x'} \setminus \{x'\}} \mathbb{P}(Z'_{x'} = 1, Z'_{y'} = 1).$$

Indeed, the right-hand side of (3.11) a priori includes a third term

$$(3.13) \quad b_3' = \sum_{x' \in S'} \mathbb{E}\Big[\big| \mathbb{E}\big[Z'_{x'} - \mathbb{P}(Z'_{x'} = 1) \,\big|\, Z'_{y'}, y' \notin \mathcal{N}_{x'} \big] \big| + \big| \mathbb{E}\big[\widetilde{W}'_{x'} - \mathbb{P}(\widetilde{W}'_{x'} = 1) \,\big|\, \widetilde{W}'_{y'}, y' \notin \mathcal{N}_{x'} \big] \big| \Big],$$

but the finite-range property in (3.4) and the definitions of $Z'_{x'}$, $\widetilde{W}'_{x'}$ and $\mathcal{N}_{x'}$ imply that $b_3' = 0$ (note to this effect that the σ-algebra generated by $\{Z'_{y'} : y' \notin \mathcal{N}_{x'}\}$ is the same as that generated by $\{Z_y^{\alpha-\varepsilon}, Z_y^{\alpha+\varepsilon} : y \notin \mathcal{N}_x\}$). Note also that the b_1 and b_2 terms we obtain for $d_{\mathrm{TV}}(W'', \widetilde{W}')$ are both smaller than b_1', and hence this justifies the constant 24 appearing in (3.11). The reason we consider the process $\widetilde{W}'$ instead of only working with the independent process W'' here is that we want the Bernoulli process W we obtain at the end to be decreasing in α, see (3.3) (note that we would avoid this problem if we used Poisson processes throughout as in [2, Theorem 2], instead of Bernoulli processes).

We now proceed to bound the right-hand side of (3.11) in terms of $\gamma(\alpha, \varepsilon)$. By means of the defining properties of d_ε, one sees that for any $\delta' > 0$ and $x \in S$,

$$(3.14) \qquad \begin{aligned} \mathbb{P}(Z'_{(x,+)} = 1) &= \mathbb{P}(Z_x^{\alpha+\varepsilon} = 1) \leqslant \mathbb{P}(Y_x^\alpha = 1) + d_\varepsilon(Y, Z; \alpha) + \delta' \\ \mathbb{P}(Z'_{(x,-)} = 1) &\leqslant \mathbb{P}(Z_x^{\alpha-\varepsilon} = 1) \leqslant \mathbb{P}(Y_x^{\alpha-2\varepsilon} = 1) + d_\varepsilon(Y, Z; \alpha - 2\varepsilon) + \delta', \end{aligned}$$

Thus, letting $\delta' \to 0$, abbreviating $d_\varepsilon = d_\varepsilon(Y, Z; \alpha) \vee d_\varepsilon(Y, Z; \alpha - 2\varepsilon)$, observing that $|S'| = 2|S|$ and using monotonicity of Y to have $\mathbb{P}(Y_x^\alpha = 1) \leqslant \mathbb{P}(Y_x^{\alpha-2\varepsilon} = 1)$ for all $x \in S$, it follows from (3.14), in view of (3.12) and (3.6), that

$$(3.15) \qquad b_1' \leqslant 4(b_1(\alpha - 2\varepsilon) + 3d_\varepsilon|S|^2).$$

Similarly, using that $Z'_{(x,\sigma)} \leqslant Z_x^{\alpha-\varepsilon}$ for all $x \in S$ and $\sigma \in \{\pm\}$ by monotonicity in (3.4), one obtains that whenever $x' = (x, \sigma)$ and $y' = (y, \rho)$ with $y \neq x$,

$$\mathbb{P}(Z'_{(x,\sigma)} = 1, Z'_{(y,\rho)} = 1) \leqslant \mathbb{P}(Z_x^{\alpha-\varepsilon} = 1, Z_y^{\alpha-\varepsilon} = 1) \leqslant d_\varepsilon + \mathbb{P}(Y_x^{\alpha-2\varepsilon} = 1, Y_y^{\alpha-2\varepsilon} = 1).$$

Observing that $\mathbb{P}(Z'_{(x,\sigma)} = 1, Z'_{(x,-\sigma)} = 1) = 0$ for all $x \in S$ by monotonicity, this yields that

$$(3.16) \qquad b_2' \leqslant 4(b_2(\alpha - 2\varepsilon) + d_\varepsilon|S|^2).$$

Substituting (3.15) and (3.16) into (3.11) readily implies that $d_{\mathrm{TV}}(Z', \widetilde{W}') \leqslant 384\gamma(\alpha, \varepsilon)$. The existence of a coupling $\widetilde{\mathbb{Q}}$ having the property (3.9) immediately follows from this (in fact, one could even choose $\delta = 0$ by using a maximal coupling but this won't be necessary).

With (3.9) at hand, we now prove (3.5), which entails finding a coupling of $(W^{\alpha-2\varepsilon}, W^{\alpha+2\varepsilon})$ and Y^α with suitable properties. To this end, let $\widehat{\mathbb{Q}}$ denote a coupling of Y^α and $(Z^{\alpha-\varepsilon}, Z^{\alpha+\varepsilon})$ satisfying

$$(3.17) \qquad \widehat{\mathbb{Q}}\big(Z_x^{\alpha+\varepsilon} \leqslant Y_x^\alpha \leqslant Z_x^{\alpha-\varepsilon} \ \forall x \in S \big) \geqslant 1 - d_\varepsilon(Y, Z; \alpha) - \delta,$$

which exists by definition of d_ε. We proceed to define a measure $\widehat{\mathbb{P}}$ on $(\{0,1\}^S)^3$, with canonical coordinates $(\widehat{W}^-, \widehat{Y}, \widehat{W}^+)$ as follows: for all $w^\pm, y \in \{0,1\}^S$,

$$\widehat{\mathbb{P}}\big(\widehat{W}^- = w^-, \widehat{Y} = y, \widehat{W}^+ = w^+\big) \overset{\text{def.}}{=}$$
$$\sum_{\widetilde{w}^+, \widetilde{w}^-} \mathbb{P}^U\big(W^{\alpha \pm 2\varepsilon} = w^\pm \big| \widetilde{W}^{\alpha \pm \varepsilon} = \widetilde{w}^\pm\big) \sum_{z^+, z^-} \widetilde{\mathbb{Q}}\big(\widetilde{W}^{\alpha \pm \varepsilon} = \widetilde{w}^\pm \big| Z^{\alpha \pm \varepsilon} = z^\pm\big) \widehat{\mathbb{Q}}\big(Z^{\alpha \pm \varepsilon} = z^\pm, Y^\alpha = y\big),$$

where $\{W^{\alpha \pm 2\varepsilon} = w^\pm\}$ is short for $\{W^{\alpha - \varepsilon} = w^-, W^{\alpha + \varepsilon} = w^+\}$ and similarly for other events, and $\widetilde{w}^+, \widetilde{w}^-, z^+$ and z^- range over all points in $\{0,1\}^S$ such that the corresponding events appearing in each conditioning have non-zero probability. One readily checks using (3.8) and the definition of the couplings $\widetilde{\mathbb{Q}}$ and $\widehat{\mathbb{Q}}$ that $\widehat{\mathbb{P}}$ is a probability measure with marginals $(\widehat{W}^-, \widehat{W}^+) \overset{\text{law}}{=} (W^{\alpha - 2\varepsilon}, W^{\alpha + 2\varepsilon})$ and $\widehat{Y} \overset{\text{law}}{=} Y^\alpha$, with Y^α and $W^{\alpha \pm 2\varepsilon}$ as prescribed by (3.2) and (3.3). Finally it follows from (3.8) combined with the bounds appearing in (3.14) that $W_x^{\alpha - 2\varepsilon} < \widetilde{W}_x^{\alpha - \varepsilon}$ if and only if $\mathbb{P}(Y_x^{\alpha - 2\varepsilon} = 1) < U_x \leqslant \mathbb{P}(Z_x^{\alpha - \varepsilon} = 1)$, which has probability at most d_ε (similar considerations apply to the event $\widetilde{W}_x^{\alpha + \varepsilon} < W_x^{\alpha + 2\varepsilon}$), whence

$$\mathbb{P}^U\big(W_x^{\alpha + 2\varepsilon} \leqslant \widetilde{W}_x^{\alpha + \varepsilon} \leqslant \widetilde{W}_x^{\alpha - \varepsilon} \leqslant W_x^{\alpha - 2\varepsilon}, \forall\, x \in S\big) \geqslant 1 - 2|S| d_\varepsilon,$$

which, together with (3.9) and (3.17), implies that

$$\widehat{\mathbb{P}}\big(\widehat{W}_x^+ \leqslant \widehat{Y}_x \leqslant \widehat{W}_x^- \,\forall x \in S\big) \geqslant 1 - d_\varepsilon(Y, Z; \alpha) - \delta - 384\gamma(\alpha, \varepsilon) - \delta - 2|S| d_\varepsilon.$$

Since $d_\varepsilon(Y, Z; \alpha) \leqslant d_\varepsilon \leqslant \gamma(\alpha, \varepsilon)$ and $\delta > 0$ was arbitrary, (3.5) follows. $\qquad\square$

Remark 3.2. Of course, the utility of (3.5) as a means to compare Y and W around level α rests in particular on having a good bound on $\sup_{\alpha' \in \{\alpha - 2\varepsilon, \alpha\}} d_\varepsilon(Y, Z; \alpha')$ to begin with, for a process Z satisfying (3.4), which in the case of the random walk on the torus will be provided by Theorem 5.1 below. The presence of $d_\varepsilon(Y, Z; \alpha')$ acts as a surrogate for a certain quantity "b_3" (much like b_3' above, cf. [2]), which would arise when attempting to compare Y and W directly using the Chen-Stein method, as done for instance in [28] in the present context. The issue with this is that b_3 typically turns out to be too large when Y has long-range. In fact, limitations of the method in the presence of long-range correlations are well-known, see for instance the discussion in [2], Sec. 2 "Open problem," pp.12-13.

4   Soft local times and inverse soft local times

In this section we give a brief exposition of the method of soft local times introduced by Popov and Teixeira in [24]. An 'inversion' of this technique, introduced below, see (4.10)-(4.11), leading to Proposition 4.4, will be used to manufacture couplings in the next section. We defer to Remark 4.5 for a discussion of the benefits of this construction and its interplay with the technique of [24].

Consider the measure space (Σ, μ), where Σ is a locally compact Polish metric space endowed with its Borel σ-algebra, carrying a (Radon) measure μ. At its root, the method of [24] is a particular way to sample sequences $Z = (Z_i)_{i \geqslant 0}$ of Σ-valued random variables from a Poisson process on $\Sigma \times \mathbb{R}_+$. Although in principle, any sequence Z such that all Z_i's have a density with respect to μ can be accomodated, cf. [1, Section 3], the following Markovian setup will be enough for our purposes. Note however that all results presented in this section continue to hold at this greater level of generality.

Let $Z = (Z_i)_{i \geqslant 0}$ be a time-inhomogeneous Markov chain on Σ. That is, there exist transition densities $g_i : \Sigma \times \Sigma \to \mathbb{R}_+$, $i \geqslant 1$, with respect to μ (i.e. the functions g_i are measurable and $\int g_i(x, y)\mu(\mathrm{d}y) = 1$ for all $x \in \Sigma$) such that under a probability measure P,

$$(4.1) \qquad P(Z_{i+1} \in \mathrm{d}x \mid Z_i) = g_{i+1}(Z_i, x)\mu(\mathrm{d}x) \text{ for all } i \geqslant 0.$$

Under an auxiliary probability Q, let η be a Poisson point process on $\Sigma \times \mathbb{R}_+$ with intensity measure $\mu \otimes \mathrm{d}v$, where $\mathrm{d}v$ denotes Lebesgue measure on $\mathbb{R}_+$. Our assumptions on (Σ, μ) ensure that the construction of η falls within the realm of standard theory. We assume that Q carries a random variable having the same law as Z_0 under P, independent of η, which we continue to denote by Z_0.

Letting $z_0 = Z_0$, $v_0 = 0$ and $\eta_0 = \eta$, one defines recursively, for $i \geqslant 0$,

$$(4.2) \qquad \xi_{i+1} = \inf_{(z,v) \in \eta_i} \frac{v}{g_{i+1}(z_i, z)},$$

where, in writing e.g. $(z, v) \in \eta$ we tacitly identify the point measure η with its support. Combining Propositions 4.1 and 4.10 of [24], it follows that the infimum in (4.2) is attained Q-a.s. at a unique pair (z_{i+1}, v_{i+1}), and defining

$$(4.3) \qquad \eta_{i+1} = \sum_{(z,v) \in \eta_i \setminus \{(z_{i+1}, v_{i+1})\}} \delta_{(z, v - \xi_{i+1} g_{i+1}(z_i, z))},$$

the following holds:

(4.4) $\xi = (\xi_i)_{i \geqslant 1}$ are i.i.d. exponential random variables with parameter 1;

(4.5) for all $i \geqslant 1$, $(z_0, \ldots, z_i) \overset{\text{law}}{=} (Z_0, \ldots, Z_i)$ and is independent of $(\xi_1, \ldots \xi_i)$;

(4.6) for all $i \geqslant 1$, η_i is a Poisson process of intensity $\mu \otimes \mathrm{d}v$ independent of $(\xi_j, z_j, v_j)_{0 \leqslant j \leqslant i}$

(with $\xi_0 = 0$). We refer to the sequence $z = (z_i)_{i \geqslant 0}$ thereby constructed as obtained from $(\eta, Z_0, (g_i)_{i \geqslant 1})$ via soft local times. The associated soft local time process is defined as

$$(4.7) \qquad G_0(z) = 0, \quad G_{i+1}(z) = G_i(z) + \xi_{i+1} g_{i+1}(z_i, z), \ i \geqslant 0.$$

As the next proposition illustrates, one benefit of this construction is to supply a natural coupling in terms of η of two (or more) chains having densities with respect to μ. The coupling allows for a comparison between the ranges of these chains, which is controlled in terms of the scalar fields $G_i(\cdot)$. To wit, let $\widetilde{Z}$ be another Markov chain on Σ having transition densities $(\widetilde{g}_i)_{i \geqslant 1}$ with respect to the same measure μ, cf. (4.1). With hopefully obvious notation, we write $\widetilde{z}_i$ (along with $\widetilde{\xi}_i$, $\widetilde{\eta}_i$), $i \geqslant 0$, under Q (which is tacitly understood to carry a copy of $\widetilde{Z}_0$ independent of η, Z_0) when referring to the chain obtained from $(\eta, \widetilde{Z}_0, (\widetilde{g}_i)_{i \geqslant 1})$ by soft local times. We denote by $\widetilde{G}_i$, $i \geqslant 0$ the corresponding soft local times, defined analogously to (4.7).

Proposition 4.1. *The processes $(z_i)_{i \geqslant 0}$, resp. $(\widetilde{z}_i)_{i \geqslant 0}$, have the same law under Q as $(Z_i)_{i \geqslant 0}$, resp. $(\widetilde{Z}_i)_{i \geqslant 0}$, and for each $m, n \geqslant 1$, on the event*

$$(4.8) \qquad G_m(z) \leqslant \widetilde{G}_n(z) \text{ for all } z \in \Sigma,$$

one has

$$(4.9) \qquad \{z_1, \ldots, z_m\} \subseteq \{\widetilde{z}_1, \ldots, \widetilde{z}_n\}.$$

Proof. The first part is immediate on account of (4.5). To see that (4.8) implies (4.9), observe that, by construction, cf. (4.2)-(4.3) and by definition of G_i, see (4.7), one has for all, $m, n \geqslant 1$,

$$\{z_1, \ldots, z_m\} = \{z \in \Sigma : \text{ there exists } (z, v) \in \eta \text{ s.t. } G_m(z) \geqslant v\}$$

$$\{\widetilde{z}_1, \ldots, \widetilde{z}_n\} = \{z \in \Sigma : \text{ there exists } (z, v) \in \eta \text{ s.t. } \widetilde{G}_n(z) \geqslant v\}.$$

From this, (4.8) plainly yields the inclusion (4.9). $\qquad\square$

Proposition 4.1 is not entirely adapted to our purpose. In Section 5, see in particular the proof of Proposition 5.4, we will actually need to couple the process $\widetilde{z}$ with the initial chain Z on a suitable extension of P. This is conveniently achieved using an inverse soft local time method, that we now explain.

With P referring to the original measure under which the Markov chain $Z = (Z_i)_{i \geqslant 0}$ is defined, cf. above (4.1), let

(4.10) $\widehat{P}$:   extension of P carrying an independent random variable $\chi = \big((\widehat{\xi}_k)_{k \geqslant 1}, \widehat{\eta}_0\big)$, where $\widehat{\eta}_0$ is a Poisson point process on $\Sigma \times \mathbb{R}_+$ with intensity $\mu \otimes \mathrm{d}v$, and $(\widehat{\xi}_k)_{k \geqslant 1}$ are i.i.d. exponential variables with mean 1, independent of $\widehat{\eta}_0$.

Given a realisation of the time inhomogeneous Markov chain $(Z_i)_{0 \leqslant i \leqslant T}$ up to some deterministic integer time $T < \infty$, we set inductively for $k = 0, \ldots, T-1$ (under $\widehat{P}$)

(4.11) $\widehat{\eta}_{k+1} = \delta_{(Z_{T-k}, \widehat{\xi}_{T-k} g_{T-k}(Z_{T-k-1}, Z_{T-k}))} + \displaystyle\sum_{(z,v) \in \widehat{\eta}_k} \delta_{(z, v + \widehat{\xi}_{T-k} g_{T-k}(Z_{T-k-1}, z))}$

and write $\eta = \widehat{\eta}_T$. Note that, albeit implicit in our notation, all processes $\widehat{\eta}_k$, $1 \leqslant k \leqslant T$, implicitly depend on the choice of T.

Lemma 4.2. *Under $\widehat{P}$, for all integers $T \geqslant 1$, the process η is a Poisson process of intensity $\mu \otimes \mathrm{d}v$, independent of Z_0. Moreover, the sequence obtained by applying soft local times to $(\eta, Z_0, (g_i)_{i \geqslant 1})$ up to time T is $(Z_i)_{0 \leqslant i \leqslant T}$, with corresponding exponential variables $(\widehat{\xi}_i)_{1 \leqslant i \leqslant T}$.*

Before proving Lemma 4.2, we isolate the case $T = 1$.

Lemma 4.3. *Let $\widehat{\eta}, \xi, Z$ be independent random variables, with $\widehat{\eta}$ a Poisson process of intensity $\mu \otimes \mathrm{d}v$, ξ exponentially distributed of parameter 1 and Z having density g with respect to μ. Then*

$$\eta = \delta_{(Z, \xi g(Z))} + \sum_{(z,v) \in \widehat{\eta}} \delta_{(z, v + \xi g(z))}$$

is also a Poisson point process with intensity $\mu \otimes \mathrm{d}v$.

Proof. For $t \in \mathbb{R}$, consider the (measurable) function $f_t : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ with

(4.12) $f_t(z, v) = (z, v + tg(z))$

and note that $f_{-t} = f_t^{-1}$. Given a point measure $\bar{\eta} = \sum_\lambda \delta_{(z_\lambda, v_\lambda)}$ on $\Sigma \times \mathbb{R}$, let $f_t(\bar{\eta}) = \sum_\lambda \delta_{f_t(z_\lambda, v_\lambda)}$. Note that $(t, \bar{\eta}) \mapsto f_t(\bar{\eta})$ is measurable, and thus $\eta = f_\xi(\widehat{\eta})$ is also measurable. Consider the Poisson process η_0 on $\Sigma \times \mathbb{R}_+$ under the (auxiliary) probability Q, cf. above (4.2), which has intensity $\mu \otimes \mathrm{d}v$. In order to be consistent with the previous setup, fix an arbitrary point $z_0 \in \Sigma$ and define g_1 by declaring that $g_1(z_0, z) = g(z)$. Applying (4.2)-(4.3) for $i = 0$, one finds ξ_1 and a point $(z_{\lambda_1}, v_{\lambda_1})$, corresponding to the unique minimizer in (4.2), i.e. with $\xi_1 g(z_{\lambda_1}) = v_{\lambda_1}$. In view of (4.3) and (4.12), one has, for $\eta_0 = \sum_\lambda \delta_{(z_\lambda, v_\lambda)}$,

$$\eta_1 = \sum_{\lambda \neq \lambda_1} \delta_{f_{\xi_1}^{-1}(z_\lambda, v_\lambda)}$$

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which, in particular, yields that

$$(4.13) \qquad \eta_0 = f_{\xi_1}(\eta_1) + \delta_{(z_{\lambda_1}, v_{\lambda_1})} = f_{\xi_1}(\eta_1) + \delta_{(z_{\lambda_1}, \xi_1 g(z_{\lambda_1}))}.$$

Now, by (4.4) one knows that $\xi_1 \overset{\text{law}}{=} \xi$, by (4.5) one has that $z_{\lambda_1} \overset{\text{law}}{=} Z$ is independent of ξ_1 and by (4.6), η_1 is a Poisson process of intensity $\mu \otimes dv$ independent from ξ_1 and z_{λ_1}. Therefore in view of (4.13) the point processes η_0 and η have the same law, and so η is also a Poisson process of intensity $\mu \otimes dv$. $\qquad \square$

One now easily deduces Lemma 4.2 inductively from Lemma 4.3.

Proof of Lemma 4.2. For $k \in \{0, \dots, T-1\}$, assume that $\widehat{\eta}_k$ as defined in (4.11) (see also (4.10) regarding $\widehat{\eta}_0$) is a Poisson point process under $\widehat{P}$ independent of $\mathcal{A}_k := \sigma(Z_i, \widehat{\xi}_i, 0 \leqslant i \leqslant T - k)$, with the convention $\widehat{\xi}_0 = 0$. Note in particular that this is automatically satisfied in case $k = 0$ on account of (4.10). Then by Lemma 4.3, applied with $\widehat{\eta} = \widehat{\eta}_k$, $Z = Z_{T-k}$, $\xi = \widehat{\xi}_{T-k}$ and $g(z) = g_{T-k}(Z_{T-k-1}, z)$, one deduces that, conditionally on $\mathcal{A}_{k+1}$, $\widehat{\eta}_{k+1}$ is a Poisson point process with intensity $\mu \otimes dv$. In particular, $\widehat{\eta}_{k+1}$ is independent of $\mathcal{A}_{k+1}$. By induction, we conclude that $\eta = \widehat{\eta}_T$ is a Poisson point process with intensity $\mu \otimes dv$ independent of $\mathcal{A}_T = \sigma(Z_0)$.

Referring to (4.2)-(4.3), let ξ_i, η_i, v_i and z_i, $0 \leqslant i \leqslant k$, be the variables obtained by applying soft local times to $(\eta, Z_0, (g_i))$ up to time k (so in particular $\xi_0 = v_0 = 0$, $z_0 = Z_0$ and $\eta_0 = \eta(= \widehat{\eta}_T)$). Assume that for some $k \in \{0, \dots, T-1\}$ we have $\xi_i = \widehat{\xi}_i$, $\eta_i = \widehat{\eta}_{T-i}$, $v_i = \widehat{\xi}_i g_i(Z_{i-1}, Z_i)$ and $z_i = Z_i$ for all $0 \leqslant i \leqslant k$ (with $v_0 = 0$). Then

$$\xi_{k+1} \overset{(4.2)}{=} \inf_{(z,v) \in \eta_k} \frac{v}{g_{k+1}(z_k, z)} \overset{\substack{\eta_k = \widehat{\eta}_{T-k} \\ z_k = Z_k}}{=} \inf_{(z,v) \in \widehat{\eta}_{T-k}} \frac{v}{g_{k+1}(Z_k, z)} = \widehat{\xi}_{k+1}$$

and the infimum is a.s. uniquely attained at $(Z_{k+1}, \widehat{\xi}_{k+1} g_{k+1}(Z_k, Z_{k+1}))$ by definition of $\widehat{\eta}_{T-k}$ in (4.11). Moreover one easily checks that $\eta_{k+1} = \widehat{\eta}_{T-k-1}$, and it follows with a simple induction argument that $z_i = Z_i$ for all $i \leqslant T$. $\qquad \square$

Combining Proposition 4.1 and Lemma 4.2, one can couple $Z = (Z_i)_{i \geqslant 0}$ under the extended measure $\widehat{P}$ defined in (4.10) to any other (inhomogenous) Markov chain having transition densities relative to μ, with good control on the ranges in terms of appropriately defined (inverse) soft local times, see Remark 4.5 below regarding the terminology.

Proposition 4.4. *For all $T \geqslant 1$, $\widetilde{z} \in \Sigma$ and any family $(\widetilde{g}_i)_{i \geqslant 1}$ of transition densities with respect to μ, one can define under $\widehat{P}$ two sequences $\widetilde{Z} = (\widetilde{Z}_i)_{i \geqslant 0}$, $\widetilde{\xi} = (\widetilde{\xi}_i)_{i \geqslant 1}$ such that, letting*

$$(4.14) \qquad G_i(z) = \sum_{1 \leqslant k \leqslant i} \widehat{\xi}_k g_k(Z_k, z), \quad \widetilde{G}_i(z) = \sum_{1 \leqslant k \leqslant i} \widetilde{\xi}_k \widetilde{g}_k(\widetilde{Z}_k, z) \quad \text{for } i \geqslant 1, \; z \in \Sigma$$

(see (4.10) regarding $\widehat{\xi}_k$), the following hold:

i) $\widetilde{Z}$ is a Markov chain with $\widetilde{Z}_0 = \widetilde{z}$ and transition densities $(\widetilde{g}_i)_{i \geqslant 1}$;

ii) $\widetilde{\xi}$ are i.i.d. exponential variables with mean one, independent of $\widetilde{Z}$;

iii) For each $p, m, n \in \{1, \dots, T\}$,

$$(4.15) \quad \{\widetilde{G}_p(z) \leqslant G_m(z) \leqslant \widetilde{G}_n(z), \; \text{for all } z \in \Sigma\}$$

$$\subset \{\{\widetilde{Z}_1, \dots, \widetilde{Z}_p\} \subset \{Z_1, \dots, Z_m\} \subset \{\widetilde{Z}_1, \dots, \widetilde{Z}_n\}\}.$$

Proof. Recall $\eta = \widehat{\eta}_T$ from below (4.11) and let $(z_i)_i$ and $(\widetilde{z}_i)_i$ be the Markov chains obtained by applying soft local times respectively to $(\eta, Z_0, (g_i))$ and $(\eta, \widetilde{z}, (\widetilde{g}_i))$. Define $\widetilde{Z}_i = \widetilde{z}_i$, $i \geqslant 0$, and $\widetilde{\xi}$ the corresponding sequence of exponential random variables produced by applying soft local times, cf. (4.2) and (4.4). In particular, this implies that $(\widetilde{G}_i(\cdot))_{i \geqslant 0}$ defined in (4.14) is the corresponding soft local time process.

With these choices for $\widetilde{Z}$ and $\widetilde{\xi}$, i) and ii) follow immediately from the first part of Proposition 4.1 and by Lemma 4.2, which guarantees that η has the correct law. Finally, one notices that, due to Lemma 4.2, $z_i = Z_i$ for all $0 \leqslant i \leqslant T$ and $(G_i(\cdot))_{0 \leqslant i \leqslant T}$ as defined in (4.14) is the corresponding soft local time process (up to time T). From this, iii) follows upon applying Proposition 4.1 twice, swapping the roles of (g_i) and $(\widetilde{g}_i)$, to deduce (4.15) from (4.8)-(4.9). □

Remark 4.5.

1) It is natural to refer to G_i defined in (4.14) as an *inverse* (or backwards) soft local time. Unlike its 'forward' counterpart (4.7), in which the random variables $\xi = (\xi_k)_{k \geqslant 1}$ emerge as minimizers in (4.2), the random variables $\widehat{\xi} = (\widehat{\xi}_k)_{k \geqslant 1}$ involved in (4.14) are given by fiat, see (4.10). Loosely speaking, this corresponds to the fact that, instead of constructing the chain Z (and the variables ξ) from η, one reconstructs η from a given realization of Z (with the help of additional independent randomness, comprising $\widehat{\xi}$). The benefit of doing this (and the gist of Proposition 4.4) is that, with η at hand, one can now apply (forward) soft local times to couple any other chain $\widetilde{Z}$ having transition densities relative to μ to the original chain Z via η.

2) In Proposition 4.4, the random variables $\widetilde{Z}$ and $\widetilde{\xi}$ depend only on the Markov chain Z and on the variable χ from (4.10) (as well as the choice of T, of the probability P and of the densities g and $\widetilde{g}$). This will be important in Section 5, where we will apply Proposition 4.4 several times simultaneously for varying choices of Z and χ. These varying choices are coupled together on a common probability space, thus the corresponding varying processes $\widetilde{Z}$ and $\widetilde{\xi}$ are also naturally defined on the same probability space, see around (5.25).

5 Localization

In this section we prove our main localization result, Theorem 1.2, which will follow from a more general result, Theorem 5.1 below. This result is of independent interest and is not specific to the "late" or "high-intensity" regime, to which it will later be applied. Such applications are discussed separately in Section 6. The proof of Theorem 5.1 is split over §5.1-5.3, and involves inverse soft local times, cf. Proposition 4.4. An overview of the proof appears atop of §5.1.

For the purposes of Theorem 5.1, which couples processes with range in both $\mathbf{T}(= (\mathbb{Z}/N\mathbb{Z})^d)$ and $\mathbb{Z}^d$, it will be important to distinguish clearly between the two. Recall from §2 that $\pi : \mathbb{Z}^d \rightarrow \mathbf{T}$ denotes the canonical projection, that for $x \in \mathbb{Z}^d$ we often abbreviate $\boldsymbol{x} = \pi(x)$, that 0 is the origin of $\mathbb{Z}^d$ whereas $\mathbf{0}$ is the origin of $\mathbf{T}$, and that $Q(x, r) = Q_r(x)$ are boxes around x of side length r either in $\mathbb{Z}^d$ or in $\mathbf{T}$, depending on whether $x \in \mathbb{Z}^d$ or $\mathbf{T}$.

In what follows, a family of (point) processes $(\omega^{(x)})_{x \in Q_N(0)}$ is said to have range R in $\mathbb{Z}^d$, resp. in $\mathbf{T}$, if $\omega^{(x)}$ and $\{\omega^{(x')} : x' \in Q_N(0) \backslash Q_R(x)\}$, resp. $\{\omega^{(x')} : \boldsymbol{x}' \notin Q(\boldsymbol{x}, R)\}$, are independent for each $x \in Q_N(0)$. Intuitively, if one identifies $Q_N(0)$ with $\mathbf{T}$, then $(\omega^{(x)})_{x \in \mathbf{T}}$ has range R in $\mathbf{T}$ if for all $x \in \mathbf{T}$, $\omega^{(x)}$ is independent of $\omega^{(x')}$ for all $x' \notin Q(x, R)$. We refrain from doing such an identification, since it could cause confusion in the next statement, in which the random walk and random interlacements appear jointly. Following is our main localisation result, from which Theorem 1.2 will follow as a special case.

Theorem 5.1 (Localization). *For all $\delta \in (0,1)$, there exist $c = c(\delta) > 0$ and $C = C(\delta) < \infty$ such that the following holds. For every $N \in \mathbb{N}$, $R \in [1, \frac{N}{1+\delta}]$ and $u_0 > 0$, there exists a probability measure $\widetilde{\mathbf{P}}_0$ extending $\mathbf{P}_0$, resp. $\widetilde{\mathbb{P}}^I$, extending $\mathbb{P}^I$, carrying a family of processes $(\omega^{(x)})_{x \in Q_N(0)}$ such that*

(5.1) $\omega^{(x)}$ *has law* $\mathbb{P}^I$ *for each* $x \in Q_N(0)$,

(5.2) $(\omega^{(x)})_{x \in Q_N(0)}$ *has range* $2(1+\delta)R$ *in* $\mathbf{T}$, *resp.* $\mathbb{Z}^d$,

and, writing $(\ell_{y,u}^{(x)})_{y \in \mathbb{Z}^d, u \geqslant 0}$ for the field of local times associated to $\omega^{(x)}$, cf. (2.12), for all $F \subseteq Q_N(0)$, $0 < v < u \leqslant u_0$, and $\varepsilon \in (0,1)$ with $u(1-\varepsilon) > v(1+\varepsilon)$ one has

(5.3) $\widetilde{\mathbf{P}}_0 \left(\ell_{y,u(1-\varepsilon)}^{(x)} - \ell_{y,v(1+\varepsilon)}^{(x)} \leqslant \ell_{\boldsymbol{y},u} - \ell_{\boldsymbol{y},v} \leqslant \ell_{y,u(1+\varepsilon)}^{(x)} - \ell_{y,v(1-\varepsilon)}^{(x)}, \ \forall x \in F, \ y \in Q(x,R) \right)$

$$\geqslant 1 - C|F|R^{2d} \lceil uR^{d-2} \rceil \exp\left(-c \cdot \varepsilon \cdot \sqrt{v \cdot R^{d-2}} \right),$$

resp. for all $u \leqslant u_0$ and $\varepsilon \in (0,1)$,

(5.3') $\widetilde{\mathbb{P}}^I \left(\ell_{y,u(1-\varepsilon)}^{(x)} \leqslant \ell_{y,u} \leqslant \ell_{y,u(1+\varepsilon)}^{(x)}, \ \forall\, x \in F, \ y \in Q(x,R) \right) \geqslant 1 - C|F|R^{2d} \exp\left(-c\varepsilon^2 u R^{d-2} \right).$

We refer to Remark 5.7 at the end of this section for various extensions of the above result, reflecting in particular a certain flexibility for the requirement (5.1), and alternatives to (5.3) (see (5.37)-(5.38)), which do not involve *increments*; these matters are best explained after giving the proof. Applications of Theorem 5.1 specific to our purposes are postponed to Section 6.

Before proceeding any further, let us give the short:

Proof of Theorem 1.2 (assuming Theorem 5.1). Define $\widetilde{\mathbf{P}} = N^{-d} \sum_{x \in \mathbf{T}} \widetilde{\mathbf{P}}_x$ where $\widetilde{\mathbf{P}}_x$ is obtained from $\widetilde{\mathbf{P}}_0$ through translation by $x \in \mathbf{T}$. One then applies Theorem 5.1 for the choices $\delta = 1$, with $2u$ in place of u, $v = u$, $F = Q_N(0)$ and $\frac{R}{4}$ in place of R. Property (5.2) is still verified under $\widetilde{\mathbf{P}}$ since the law of $\omega^{(y)}$ under $\widetilde{\mathbf{P}}_x$ does not depend on $x \in \mathbf{T}$ for each $y \in Q_N(0)$ by (5.1). Upon defining $\widetilde{\ell}_{\boldsymbol{x},u(1\pm\varepsilon)} = \ell_{x,2(1\pm\varepsilon)u}^{(x)} - \ell_{x,(1\mp\varepsilon)u}^{(x)}$ for all $x \in Q_N(0)$, and observing that $\ell_{\cdot,2u} - \ell_{\cdot,u}$ has the same law under $\widetilde{\mathbf{P}}$ as $\ell_{\cdot,u}$ under $\mathbf{P}$ due to stationarity, the desired properties (1.17) resp. (1.18) follow readily from (5.2) resp. (5.3). $\qquad\square$

As explained at length in §1.2, the main upshot of Theorem 5.1 is the combined effect of (5.2) and (5.3)/(5.3'), which yields a "close + local approximation" to the field of interest, the (increments of) local times. By virtue of the additional requirement (5.1), Theorem 5.1 also entails a coupling between random interlacements and random walk, which will prove useful as well; cf. Lemma 6.1. Similar couplings at a mesoscopic scale were first obtained in [34], and then improved in [32, 5]. At a macroscopic scale, that is, outside of an annulus of size δN for some small $\delta > 0$, a similar coupling was proved in [8, Theorem 4.1]. In fact, Theorem 5.1 directly implies the coupling from [8, Theorem 4.1] for fixed $\delta > 0$, as we now explain.

Corollary 5.2. *For all $\delta > 0$, there exist $c, C \in (0, \infty)$ depending only on δ and d such that the following holds. For every $u > 0$, $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ there exists a coupling $\widetilde{\mathbf{P}}$ between $(\ell_{x,u})_{x \in \mathbf{T}}$ under $\mathbf{P}$ and $(\ell_{x,u(1\pm\varepsilon)} : x \in Q_N(0))$ under $\mathbb{P}^I$ so that*

$$\widetilde{\mathbf{P}} \left(\ell_{x,u(1-\varepsilon)} \leqslant \ell_{\boldsymbol{x},u} \leqslant \ell_{x,u(1+\varepsilon)}, \ \forall x \in Q_{N(1-\delta)}(0) \right) \geqslant 1 - CN^{2d} \lceil uN^{d-2} \rceil \exp\left(-c\varepsilon\sqrt{uN^{d-2}} \right).$$

Proof. The desired coupling $\widetilde{\mathbf{P}}$ of $(\ell_{x,u})_{x\in\mathbf{T}}$, $(\ell_{x,u(1-\varepsilon)})_{x\in Q_N(0)}$ and $(\ell_{x,u(1+\varepsilon)})_{x\in Q_N(0)}$ is obtained as the joint law of $(\ell_{x,2u} - \ell_{x,u})_{x\in\mathbf{T}}$, $(\ell_{x,2u(1-\varepsilon/3)}^{(0)} - \ell_{x,u(1+\varepsilon/3)}^{(0)})_{x\in Q_N(0)}$ and $(\ell_{x,2u(1+\varepsilon/3)}^{(0)} - \ell_{x,u(1-\varepsilon/3)}^{(0)})_{x\in Q_N(0)}$ under $N^{-d}\sum_{x\in\mathbf{T}}\widetilde{\mathbf{P}}_x$. Using (5.3) (replacing v by u and u by $2u$) with $u_0 = 2u$, $F = \{0\}$ and $R = \lceil N(1-\delta) \rceil$, the claim immediately follows from the independence and stationarity of the increments of random interlacements. $\qquad\square$

Note that $\widetilde{\mathbf{P}}$ appearing in Corollary 5.2 is in fact the same measure as in Theorem 1.2, hence the identical notation. As opposed to [8, Theorem 4.1], the coupling error obtained here is more explicit in u, ε and N. We believe moreover that our proof is more elementary since it does not involve the coupling appearing in [8, Theorems 3.1-2] of general Markov chains via soft local times. As in [8], one could probably also take $\delta = \delta(N) \to 0$ slowly enough as $N \to \infty$ in Theorem 5.1. Actually, Corollary 5.2 is the only reason we chose to prove Theorem 5.1 for all $\delta > 0$, instead of for all δ large enough, which would be sufficient for our purposes. The proof of Theorem 5.1 would be slightly simpler for δ large, for instance one would not have to use Lemma 5.5 below by proceeding similarly as in [11, Lemma 2.1] (upon adapting the results from Appendix A.1 to obtain Harnack bounds with constants close to 1 as $\delta \to \infty$), and the proof of Lemma A.1 would also have been simpler. But Lemma 5.5 is actually easy to prove given the tools developed in Appendix A.2, which are anyway required to prove Lemma 5.6 below. In other words, proving Theorem 5.1 for all $\delta > 0$ does not require a lot of additional work, and provides a more elementary proof of a version of [8, Theorem 4.1].

In the remainder of this section, we prove Theorem 5.1. We shortly explain the general strategy for the random walk; the case of random interlacements is similar. First, in Section 5.1, for each $B_1 \subset B_2 \subset B_3(\subset \mathbb{Z}^d)$, we approximate the excursions $(Z_i)_{i\geqslant 0}$ of the random walk from (the projection of) ∂B_1 to ∂B_2 before hitting ∂B_3, see (5.6), by a process $(\widetilde{Z}_i)_{i\geqslant 0}$ of excursions which is independent of the walk outside of B_3, see Lemma 5.3, using the inverse soft local times method from Proposition 4.4. Then in Section 5.2 for $r_1 < r_2 < r_3$, we put together the excursions $(\widetilde{Z}_i^{(x)})_{i\geqslant 0}$, $x \in Q_N(0)$, each corresponding to the choice $B_i = Q(x, r_i)$ for each $i \in \{1, 2, 3\}$, and show that they satisfy a short range property, see Proposition 5.4. Moreover for each fixed $m \in \mathbb{N}$, we estimate the probability that $(\widetilde{Z}_i^{(x)})_{i\leqslant m}$ is close to the initial excursions $(Z_i^{(x)})_{i\leqslant m}$ in Lemma 5.5, and show that the number of excursions performed by the random walk or random interlacements at a given time is well concentrated around some deterministic m in Lemma 5.6. The proof of these two lemmas is given in Appendix A. Finally, the different pieces of the proof are put together in Section 5.3 by defining $\omega^{(x)}$ for each $x \in Q_N(0)$ as an interlacement process whose excursions are given by the short-range excursions $(\widetilde{Z}_i^{(x)})_{i\geqslant 0}$ for $r_1 = R$, $r_2 = R\sqrt{1+\delta}$ and $r_3 = R(1+\delta)$.

5.1. Construction of short-range excursions in a fixed set. We start with a realization X of the random walk on $\mathbf{T}$ under $\mathbf{P_0}$, and for $B_1 \subseteq B_2 \subseteq B_3 \subset Q_{2N}(0)$ with diameter smaller than N, we first define a process of excursions (Z_j) in B_3 from B_1 to ∂B_2, and a corresponding clothesline process (ζ_j) in $\partial B_2 \times \partial B_3$, such that for each j, up to projection onto $\mathbf{T}$, X first visits the first coordinate of ζ_j, then after hitting B_1 follows the excursion Z_j until the last hitting time in B_2 before reaching ∂B_3, and reaches ∂B_3 in the second coordinate of ζ_j, similarly as in [1, Section 3] and [8, Section 4]. Then, coupling via the inverse soft local time method of Section 4, conditionally on (ζ_j), we define excursions $(\widetilde{Z}_j)$ close to (Z_j) and independent of the walk X outside of B_3, see Lemma 5.3, which will form the basis of the finite-range process $\omega^{(x)}$ from Theorem 5.1 in the random walk case. Finally, we extend this construction to the case of random interlacements.

Let $B_1 \subseteq B_2 \subseteq B_3 \subset Q_{2N}(0)$ be three concentric boxes with diameter at most $N - 1$. For a sequence $x = (x_n)_{n\geqslant 0}$ in $\mathbb{Z}^d$, we introduce two sequences of successive return and departure

times as $R_0(x, B_2, B_3) = 0$, $D_0(x, B_2, B_3) = H_{\partial B_3^c}$, where $\partial B_3^c = \partial(B_3^c)$ is the exterior boundary of B_3, and inductively for integers $k \geqslant 0$

$$(5.4) \qquad \begin{aligned} R_{k+1}(x, B_2, B_3) &= \inf\{n \geqslant D_k(x, B_2, B_3) : x_n \in \partial B_2\}, \\ D_{k+1}(x, B_2, B_3) &= \inf\{n \geqslant R_{k+1}(x, B_2, B_3) : x_n \in \partial B_3^c\}, \end{aligned}$$

Thus, $D_k \leqslant R_{k+1} \leqslant D_{k+1}$ for all $k \geqslant 0$. We further define, for $k \geqslant 1$,

$$H_k(x, B_1, B_2, B_3) = \inf\{n \in [R_k(x, B_2, B_3), D_k(x, B_2, B_3)] : x_n \in \partial B_1\}$$

(we use the convention $\inf \varnothing = +\infty$). Attached to these stopping times are the process $(\zeta_i(x, B_2, B_3))_{i \geqslant 1}$, where

$$(5.5) \qquad \zeta_i(x, B_2, B_3) = (x_{R_i(x, B_2, B_3)}, x_{D_i(x, B_2, B_3)}) \in \partial B_2 \times \partial B_3^c \quad \text{for} \quad i \geqslant 1,$$

and the excursions $(Z_i(x, B_1, B_2, B_3))_{i \geqslant 1}$ as

$$(5.6) \qquad Z_i(x, B_1, B_2, B_3) = \begin{cases} x_{\left[H_i(x, B_1, B_2, B_3), L_{B_2}(D_i(x, B_2, B_3)) \right]} & \text{if} \quad H_i(x, B_1, B_2, B_3) < \infty, \\ \Theta & \text{otherwise,} \end{cases}$$

where $x_{[s,t]} = \{x_k : k \in [s,t] \cap \mathbb{N}\}$, the time $L_{B_2}(D_i(x, B_2, B_3))$ is defined similarly as in (2.1) but relative to x and Θ is a cemetery state corresponding to excursions that do not hit ∂B_1. In words, $Z_i(x, B_1, B_2, B_3)$ is the part of the i-th excursion from B_2 to B_3 from the first time it hits ∂B_1 until the last time it is in B_2 before hitting ∂B_3^c. The attentive reader will have noticed that the system of excursions defined by (5.6), though designed to keep track of x inside B_1, actually neglects its first part until first exiting B_3 (which may well intersect B_1).

We now introduce B_4, a box of side length N satisfying $B_4 \supset B_3$, which is otherwise arbitrary. With X the random walk in $\mathbf{T}$ under $\mathbf{P_0}$, the restriction $\pi_{|B_4} : B_4 \to \mathbf{T}$ is a bijection, and we let $\widehat{X} = (\widehat{X}_n)_{n \geqslant 0}$ with

$$(5.7) \qquad \widehat{X}_n = (\pi_{|B_4})^{-1}(X_n), \quad \text{for all integers } n \geqslant 0.$$

Even though $\widehat{X}$ in (5.7) depends on the choice of B_4, the subsequent construction (in particular the definition of the processes Z, ζ and λ below) do not (so long as $B_4 \supset B_3$). We will be interested in the system $Z = (Z_i)_{i \geqslant 1}$ of excursions (on $\mathbb{Z}^d$)

$$(5.8) \qquad Z_i \overset{\text{def.}}{=} Z_i(\widehat{X}, B_1, B_2, B_3), \; i \geqslant 1,$$

with corresponding clothesline process $\zeta = (\zeta_i)_{i \geqslant 1}$, where $\zeta_i = \zeta_i(\widehat{X}, B_2, B_3)$. For $u > 0$, we denote by $\mathcal{N}_{\mathrm{RW}}(X, B_2, B_3, u)$ the total number of excursions X performs across $\pi(B_3 \backslash B_2)$ before time uN^d and after time $D_0(\widehat{X}, B_2, B_3)$; that is,

$$(5.9) \qquad \mathcal{N}_{\mathrm{RW}}(X, B_2, B_3, u) = \sup\{k \geqslant 0 : R_k(\widehat{X}, B_2, B_3) < uN^d\}.$$

We now adapt this setup to random interlacements. Let $\omega_{Q_{2N}(0)} = (\omega^u_{Q_{2N}(0)})_{u > 0}$ be the restriction of a random interlacements process ω under $\mathbb{P}^I$ to trajectories hitting $Q_{2N}(0)$. Recalling (2.11), $\omega_{Q_{2N}(0)}$ is defined in terms of the Poisson counting process $u \mapsto N^u = N^u_{Q_{2N}(0)}$ of intensity $\mathrm{cap}(Q_{2N}(0))$ and a family X^j, $j \geqslant 1$, of independent simple random walks having law $P_{\bar{e}_{Q_{2N}(0)}}$ each. Let

$$(5.10) \qquad T^j = T(X^j, B_2, B_3) = \sup\{k \geqslant 0 : R_k(X^j, B_2, B_3) < \infty\}.$$

Note that $T^j = 0$ with positive probability, and that actually $T^j = 0$ if and only if X^j does not visit B_2 in view of (5.4) and since we assumed $B_3 \subset Q_{2N}(0)$. Let us now define the clothesline process $\lambda = (\lambda_j(\omega, B_2, B_3))_{j \geqslant 1}$ and the excursions process $(W_j(\omega, B_1, B_2, B_3))_{j \geqslant 1}$ of random interlacements by taking for all $j \geqslant 1$ and $1 \leqslant i \leqslant T^j$ (cf. (5.5) and (5.6) for notation)

(5.11)
$$\lambda_{i+\sum_{k=1}^{j-1} T^k}(\omega, B_2, B_3) = \zeta_i(X^j, B_2, B_3) \text{ and}$$
$$W_{i+\sum_{k=1}^{j-1} T^k}(\omega, B_1, B_2, B_3) = Z_i(X^j, B_1, B_2, B_3).$$

In analogy with (5.9), we write $\mathcal{N}_{\mathrm{RI}}(\omega, B_2, B_3, u)$ for the total number of excursions across $B_3 \backslash B_2$ performed by the interlacement at level u, i.e.

(5.12)
$$\mathcal{N}_{\mathrm{RI}}(\omega, B_2, B_3, u) = \sum_{j=1}^{N^u} T^j,$$

with T^j as in (5.10) and where N^u is a Poisson random variable of parameter $u\mathrm{cap}(Q_{2N}(0))$, see above (5.10). We will frequently omit various arguments, e.g. the sets B_1, B_2, B_3, from the above notation whenever those are clear from the context.

We now aim to couple the process $Z = (Z_i)_{i \geqslant 1}$ from (5.8) via inverse soft local times, i.e. using Proposition 4.4, with a process $\widetilde{Z}$ independent from X outside B_3, corresponding to excursions of random interlacements. We start by defining the appropriate state space Σ, measure μ and transition densities g, cf. (4.1). Let $\mathcal{K}$ denote the set of finite nearest-neighbor paths in B_3 from ∂B_1 to ∂B_2, i.e.

(5.13)
$$\mathcal{K} = \{z = (z_0, \ldots, z_\ell) : z_0 \in \partial B_1, z_j \in B_3 \ \forall \ j \leqslant \ell, z_\ell \in \partial B_2\}.$$

Recalling Θ from (5.6), which represents a cemetery state corresponding to the excursions from ∂B_2 to ∂B_3^c that do not hit B_1, we set $\Sigma = \mathcal{K} \cup \{\Theta\}$ and define for $S \subseteq \Sigma$

(5.14)
$$\mu(S) = \sum_{x \in \partial B_1, y \in \partial B_2} \mathbf{P}_x\big(\widehat{X}_{[0, L_{B_2}(T_{B_3})]} \in S \,\big|\, \widehat{X}_{L_{B_2}(T_{B_3})} = y\big) + 1\{\Theta \in S\},$$

with the convention $\mathbf{P}_x(\cdot \,|\, A) = 0$ for any event A with $\mathbf{P}_x(A) = 0$, and where L_{B_2} and T_{B_3} ($= H_{\partial B_3^c}$ when starting from B_3) are defined as in Section 2, but for $\widehat{X}$ instead of X (since $B_i \subset \mathbb{Z}^d$). For every $(y, w) \in \partial B_2 \times \partial B_3^c$ and $z = (z_0, \ldots, z_\ell) \in \mathcal{K}$, abbreviating $H_K(\widehat{X}) = H_K$ we let

(5.15)
$$g_{(y,w)}(z) = \mathbf{P}_y\big(T_{B_3} \geqslant H_{B_1}, \widehat{X}_{H_{B_1}} = z_0, \widehat{X}_{L_{B_2}(T_{B_3})} = z_\ell \,\big|\, \widehat{X}_{T_{B_3}} = w\big)$$

and also

(5.16)
$$g_{(y,w)}(\Theta) = \mathbf{P}_y\big(T_{B_3} < H_{B_1} \,\big|\, \widehat{X}_{T_{B_3}} = w\big).$$

It then follows that for all $z \neq \Theta$,

$$\mathbf{P}_y\big(Z_1 = z \,\big|\, \widehat{X}_{T_{B_3}} = w\big)$$
$$= g_{(y,w)}(z)\mathbf{P}_y\big(Z_1 = z \,\big|\, T_{B_3} \geqslant H_{B_1}, \widehat{X}_{H_{B_1}} = z_0, \widehat{X}_{L_{B_2}(T_{B_3})} = z_\ell, \widehat{X}_{T_{B_3}} = w\big) = g_{(y,w)}(z)\mu(\{z\}),$$

where the last equality follows by writing the relevant conditional probability as a ratio and applying the simple(!) Markov property separately to numerator and denominator by summing over all possible values of $L_{B_2}(T_{B_3})$. One readily finds that the equality $\mathbf{P}_y(Z_1 = z \,|\, \widehat{X}_{T_{B_3}} = w) = g_{(y,w)}(z)\mu(\{z\})$ continues to hold for $z = \Theta$. That is, $g_{(y,w)}$ is the density with respect to

μ of the image on B_4 of the random walk path from the first time it hits B_1 until the last time it visits B_2 during one excursion which starts from y and ends at w, cf. (4.1).

By the Markov property of X, conditionally on $\zeta = (\zeta_i)_{i \geq 1}$ where $\zeta_i = \zeta_i(\widehat{X}, B_2, B_3))$, $(Z_i)_{i \geq 1}$ is a Markov process with transition densities $(g_{\zeta_i})_{i \geq 1}$ with respect to μ. Here and in the rest of the section, we identify $z \mapsto g_{\zeta_i}(z)$ with the function $(z', z) \in \Sigma \times \Sigma \mapsto g_{\zeta_i}(z)$, since the transition densities of Z only depend on the second variable. This is owed to the fact that, conditionally on ζ, the random variables $(Z_i)_{i \geq 1}$ are independent. With regards to fitting the setup of (4.1), the latter property also makes specifying Z_0 obsolete (for definiteness, the reader may wish to choose some $z \in \Sigma$ and set $Z_0 = z$).

For $(y, w) \in \partial B_2 \times \partial B_3^c$, the measure μ and the function $g_{(y,w)}$ stay the same when replacing in their definition the image $\widehat{X}$ of the random walk under $\mathbf{P}$. by the random walk X on $\mathbb{Z}^d$ under the probability $P_{\bar{e}_{B_2}}$. In particular, conditionally on $\lambda = (\lambda_i)_{i \geq 1}$, $(W_i)_{i \geq 1}$ is a Markov process with transition densities $(g_{\lambda_i})_{i \geq 1}$ with respect to μ. We refer to [1, Section 3] for a detailed account, with illustrative figures, of clothesline process, excursions and the resulting density in the case of random interlacements.

With this setup, which fits the framework of Section 4, cf. around (4.1), we proceed to explain how to use Proposition 4.4 to approximate the random walk excursions $(Z_i)_{i \geq 1}$ in (5.8) by some random interlacements excursions independent of the walk outside of B_3. To this end, assume $\mathbf{P_0}$ to be suitably extended as to carry, independently of X, a family χ as appearing in (4.10) and an independent clothesline process $\widetilde{\lambda} = (\widetilde{\lambda}_i)_{i \geq 1}$ having the same law as λ in (5.11). Now consider for each sequence $x = (x_n)_{n \geq 0}$ in $\mathbb{Z}^d$ the map

$$(5.17) \qquad Y_i(x, B_2, B_3) = \begin{cases} x_{[0, R_1(x, B_2, B_3)]} & \text{if } i = 0, \\ x_{[D_i(x, B_2, B_3), R_{i+1}(x, B_2, B_3)]}, & \text{if } i \geq 1, \end{cases}$$

assuming $R_{i+1}(x, B_2, B_3) < \infty$; in case $R_{i+1}(x, B_2, B_3) = \infty$ for some $i \geq 0$ the right endpoint is excluded in the corresponding formula for Y_i in (5.17), and by convention $Y_i = \varnothing$ if $D_i(x, B_2, B_3) = \infty$. Thus, $Y_i(x, B_2, B_3)$ represents the part of x occurring before the clothesline $\zeta_1(x, B_2, B_3)$ or between the clotheslines $\zeta_i(x, B_2, B_3)$ and $\zeta_{i+1}(x, B_2, B_3)$. We abbreviate $Y_i = Y_i(\widehat{X}, B_2, B_3)$. Importantly, conditionally on ζ, the processes $(Z_i)_{i \geq 1}$ and $(Y_i)_{i \geq 0}$ are independent. Thus, conditionally on ζ, $\widetilde{\lambda}$ and $(Y_i)_{i \geq 0}$, $(Z_i)_{i \geq 1}$ is still a Markov chain with transition densities (g_{ζ_i}) with respect to μ. Therefore, applying Proposition 4.4 conditionally on $(\zeta_i)_{i \geq 1}$, $(\widetilde{\lambda}_i)_{i \geq 1}$, and $(Y_i)_{i \geq 0}$, we can construct under the extended measure $\mathbf{P_0}$ for every integer $T \geq 1$ and initial state $\widetilde{Z}_0 = \widetilde{z}$ a Markov chain

$$(5.18) \qquad \big(\widetilde{Z}_i\big)_{i \geq 0} = \big(\widetilde{Z}_i(X, \chi, \widetilde{\lambda}, T, B_1, B_2, B_3)\big)_{i \geq 0}$$

with transition densities $\widetilde{g}_i = g_{\widetilde{\lambda}_i}$, $i \geq 1$ and satisfying (4.15). Note that since the transition densities $(z', z) \mapsto g_{\widetilde{\lambda}_i}(z)$ do not depend on the first variable, and since both $(Z_i)_{i \geq 1}$ and $(Y_i)_{i \geq 0}$ do not depend on B_4 (as long as it contains B_3), $(\widetilde{Z}_i)_{i \geq 1}$ does not depend on the choice of the initial excursion $\widetilde{z}$, which we will henceforth omit, nor on the choice of B_4. For later reference, we denote by G^{RW} (corresponding to G in (4.14)) the soft local times associated to Z, that is

$$(5.19) \qquad G_m^{\mathrm{RW}}(z) = \sum_{k=1}^m \widehat{\xi}_k g_{\zeta_k}(z) \text{ for all } z \in \Sigma \text{ and } m \geq 1;$$

cf. (4.10) regarding $\widehat{\xi}_k$ (part of χ), which are independent of Z. In view of (5.15), $G_m^{\mathrm{RW}}(z)$ actually only depends on its argument through the start- and endpoint of z. The clothesline process ζ corresponding to Z below (5.8) is a Markov chain, as follows readily from the strong Markov property. One can show, see Lemma A.5 and (A.12), that the stationary distribution

of ζ is given by $\bar{e}_2^3(x)P_x(X_{T_{B_3}} = y)$ for $(x, y) \in (\partial B_2 \times \partial B_3^c)$, where $\bar{e}_2^3 = \bar{e}_{B_2}^{B_3}$ denotes the normalized equilibrium measure of B_2 relative to B_3 (see below (2.6) for notation). Abbreviating $P_\mu = \sum \mu(x)P_x$ and similarly $\mathbf{P}_\mu$, writing $\bar{\mathbf{e}}_2^3$ for the projection of $\bar{e}_2^3$ onto $\mathbf{T}$ (i.e. the measure such that $\bar{\mathbf{e}}_2^3 \circ \pi(x) = \bar{e}_{B_2}^{B_3}(x)$ for all $x \in B_4$), and letting

$$(5.20) \qquad \bar{g}(z) = \mathbf{E}_{\bar{\mathbf{e}}_2^3}[g_{\zeta_1}(z)] \ (= \mathbf{E}_{\bar{\mathbf{e}}_2^3}[g_{\zeta_0}(z)]),$$

(where ζ_0 is declared as in (5.5) but with $i = 0$), which amounts to the average of the random variable $\xi \mapsto g_\xi(z)$ of interest under the stationary distribution for the process ζ, it then readily follows that $m\bar{g}(z)$ is the expectation of $G_m^{\mathrm{RW}}(z)$ starting from stationarity.

The process $\widetilde{Z}$ introduced in (5.18) by means of Proposition 4.4 will be the basis of the construction of the short range process $\omega^{(x)}$ (and $\ell^{(x)}$) from Theorem 5.1, with x denoting the common centre of B_1-B_3. As one of its central features, which will eventually give rise to the finite-range property (5.2), the process $\widetilde{Z}$ is independent of X outside of B_3, as argued next.

Lemma 5.3. *For all boxes $B_1 \subseteq B_2 \subseteq B_3 \subset Q_{2N}(0)$ with diameter at most $N-1$ and each $T \geqslant 1$, the process $\widetilde{Z} = \big(\widetilde{Z}_i(X, \chi, \widetilde{\lambda}, T, B_1, B_2, B_3)\big)_{i \geqslant 1}$ is independent of $\zeta = (\zeta_i(\widehat{X}, B_2, B_3))_{i \geqslant 1}$ and $Y = (Y_i(\widehat{X}, B_2, B_3))_{i \geqslant 0}$.*

Proof. By Proposition 4.4, conditionally on $\widetilde{\lambda}$, ζ and Y, $\widetilde{Z}$ is a Markov chain with transition densities $(g_{\widetilde{\lambda}_i})_{i \geqslant 1}$ with respect to μ, and so its conditional law depends only on $\widetilde{\lambda}$. Since $\widetilde{\lambda}$ is independent of ζ and Y, it thus follows that $\widetilde{Z}$ is independent of ζ and Y. $\qquad\square$

Thus, up to controlling their respective soft local times, $(\widetilde{Z}_i)_{i \geqslant 1}$ are excursions close to $(Z_i)_{i \geqslant 1}$ by Proposition 4.4, but with an extra independence property, similarly as the processes $\mathcal{I}_k^u$ from [24, Proposition 5.3]. There are however two main differences in our construction: first the soft local times method is used conditionally on the clothesline process ζ instead of unconditionally as in [24], following ideas from [1], and second, using Proposition 4.4, the process $(Z_i)_{i \geqslant 1}$ is defined directly in terms of the Markov chain X under $\mathbf{P_0}$ (and additional independent randomness) instead of a process on some other probability space having the same law as $(Z_i)_{i \geqslant 1}$, as in [24, 1]. These two changes in the method serve the same purpose: we can construct simultaneously the processes $(\widetilde{Z}_i)_{i \geqslant 1}$ for different choices of sets $B_1 \subseteq B_2 \subseteq B_3$ that are not necessarily disjoint and all have the desired independence properties. Moreover, for each three sets $B_1 \subseteq B_2 \subseteq B_3$, they are close to the excursion process $(Z_i)_{i \geqslant 1}$ associated to the same initial chain X. We refer to Proposition 5.4 for the exact statement.

We now adapt the previous construction to the case of random interlacements. Suppose that $\mathbb{P}^I$ is extended with the same independent processes $(\widetilde{\lambda}, \chi)$ as in the case of $\mathbf{P_0}$, cf. above (5.17) regarding their respective laws. Conditionally on λ (see (5.11)) and $Y_i(X^j, B_2, B_3)$ for all $0 \leqslant i \leqslant T^j$ and $j \geqslant 1$, see (5.17), the process $(W_i)_{i \geqslant 1}$ introduced in (5.11) is a Markov chain with transition densities (g_{λ_i}) with respect to μ. Thus, applying Proposition 4.4 with any choice of $\widetilde{W}_0$, one obtains for every $T \geqslant 1$ a Markov chain

$$(5.21) \qquad \widetilde{W} = \big(\widetilde{W}_i\big)_{i \geqslant 1} = \big(\widetilde{W}_i(\omega, \chi, \widetilde{\lambda}, T, B_1, B_2, B_3)\big)_{i \geqslant 1}$$

with transition densities $(g_{\widetilde{\lambda}_i})$ with respect to μ, (independent of $\widetilde{W}_0$). The soft local time associated to $W = (W_i)_{i \geqslant 1}$ is given by

$$(5.22) \qquad G_m^{\mathrm{RI}}(z) = \sum_{k=1}^m \widehat{\xi}_k g_{\lambda_k}(z) \text{ for all } z \in \Sigma \text{ and } m \geqslant 1,$$

(and similarly for $\widetilde{W}$ with $\widetilde{\xi}$, $\widetilde{\lambda}$ in place of $\widehat{\xi}$, λ). By [8, Lemma 6.1], starting from the invariant distribution of λ, see (5.11), the expectation of G_m^{RI} is equal to $m\bar{g}(z)$, see (5.20), i.e. it is equal

to the expectation of G_m^{RW} starting from the invariant distribution of $(\zeta_i)_{i \geqslant 0}$; to see this recall ζ from (5.5) (which can also be defined for $i = 0$) and note that the expectation in (5.20) equals $E_{\bar{e}_2^3}[g_{\zeta_0(X, B_2, B_3)}(z)]$. Moreover one proves similarly as in Lemma 5.3 above that $\widetilde{W}$ is independent of the clothesline process λ and the excursions $\{Y_i(X^j, B_2, B_3) : 0 \leqslant i \leqslant T^j, j \geqslant 1\}$.

5.2. Simultaneous approximation by short range excursions. Towards proving Theorem 5.1, we now apply the construction of §5.1 jointly to the sets

$$(5.23) \qquad B_k = Q(x, r_k), \ k = 1, \ldots, 4, \ \text{where } 0 < r_1 < r_2 < r_3 < r_4 = N$$

(with $Q(x, r) = Q_r(x)$ referring to the boxes introduced at the beginning of Section 2) as $x \in Q_N(0)$ varies. In particular, $B_3 \subset Q_{2N}(0)$ and has diameter at most $N - 1$ as assumed in §5.1. The resulting sequences $\widetilde{Z}^{(x)} = (\widetilde{Z}_j^{(x)})_{j \geqslant 1}$, see (5.18) and (5.25) below, will provide the excursions in terms of which the processes $\omega^{(x)}$, $x \in Q_N(0)$ will later be defined. For each $x \in Q_N(0)$ and $j \geqslant 1$, recalling (5.6) for notation, let

$$(5.24) \qquad Z_j^{(x)}(r_1, r_2, r_3) \overset{\text{def.}}{=} Z_j(\widehat{X}, Q(x, r_1), Q(x, r_2), Q(x, r_3)),$$

where $\widehat{X}$ is as in (5.7) with $B_4 = Q(x, N)$, and write $\zeta^{(x)} = (\zeta_j^{(x)})_{j \geqslant 1}$ for the associated clothesline process, i.e. $\zeta_j^{(x)} = \zeta_j(\widehat{X}, Q(x, r_2), Q(x, r_3))$, cf. (5.5), and $g_{\zeta^{(x)}}^{(x)}$, the transition densities of $(Z_j^{(x)})_{j \in \mathbb{N}}$, cf. (5.15) and (5.16). Let $\widetilde{\mathbf{P}}_0$ be an extension of $\mathbf{P}_0$ carrying the additional independent processes $\widetilde{\omega}^{(x)}$, $\chi^{(x)}$, $x \in Q_N(0)$, having the following distributions. For each x, $\widetilde{\omega}^{(x)}$ is an interlacement process, i.e. it has the same law as ω above (2.11). Its induced clothesline process for the choice (5.23) will be denoted by $(\widetilde{\lambda}_j^{(x)})_{j \geqslant 1}$. The process $\chi^{(x)}$ is specified by (4.10) with underlying measure space $(\Sigma, \mu) \equiv (\Sigma^{(x)}, \mu^{(x)})$ in (5.14) corresponding to (5.23).

Now, applying Proposition 4.4 in the same manner as above (5.18) but simultaneously for each $x \in Q_N(0)$ yields, for each such x and integers $j, T \geqslant 1$, the random variables

$$(5.25) \qquad \widetilde{Z}_j^{(x)}(T, r_1, r_2, r_3) \overset{\text{def.}}{=} \widetilde{Z}_j(X, \chi^{(x)}, \widetilde{\lambda}^{(x)}, T, Q(x, r_1), Q(x, r_2), Q(x, r_3)).$$

As in (5.19), the soft local times associated to $Z^{(x)} = (Z_j^{(x)})_{j \geqslant 1}$ will be denoted by $G_{m,x}^{\mathrm{RW}}(z)$, for $m \geqslant 1$ and $z \in \Sigma^{(x)}$, with inherent i.i.d. exponential random variables $\widehat{\xi}_j^{(x)}$, $j \geqslant 1$, carried by $\chi^{(x)}$, cf. (4.10). The expectation under the stationary distribution of $g_{\zeta_1^{(x)}}^{(x)}(z)$ is written as $\bar{g}^{(x)}(z)$, $z \in \Sigma^{(x)}$, similarly as in (5.20).

Finally, for each $x \in Q_N(0)$, replacing every occurrence of Z_j, ζ_j, $\widetilde{Z}_j$ and G_m^{RW} above by W_j, λ_j, $\widetilde{W}_j$ and G_m^{RI}, one similarly defines under the extended measure $\widetilde{\mathbb{P}}^I$ (with the same extension $(\widetilde{\omega}^{(x)}, \chi^{(x)})_{x \in Q_N(0)}$ as $\widetilde{\mathbf{P}}_0$) the processes $W_j^{(x)}$, $\lambda_j^{(x)}$, $\widetilde{W}_j^{(x)}$ and $G_{m,x}^{\mathrm{RI}}$ for random interlacements, which correspond to the processes introduced in (5.11), (5.21) and (5.22) for the choices of boxes B_k in (5.23). As a result, by Proposition 4.4, for every $x \in Q_N(0)$,

$$(5.26) \qquad (\widetilde{Z}_j^{(x)})_{j \geqslant 1} \text{ has the same law conditionally on } \widetilde{\lambda} \text{ as } (W_j^{(x)})_{j \geqslant 1} \text{ conditionally on } \lambda.$$

In the sequel, mimicking the notion introduced above Theorem 5.1, a collection $\Pi = (\Pi_j^{(x)})_{x \in Q_N(0), j \geqslant 1}$ of $\mathbb{Z}^d$-valued random paths is said to have range R in $\mathbb{Z}^d$ (resp. in $\mathbf{T}$) if $(\Pi_j^{(x)})_{j \geqslant 1}$ is independent of $(\Pi_j^{(y)})_{j \geqslant 1, \, y \in Q_N(0) \setminus Q(x,R)}$ (resp. of $(\Pi_j^{(y)})_{j \geqslant 1, \, \boldsymbol{y} \notin Q(\boldsymbol{x}, R)}$) for each $x \in Q_N(0)$. Here following the convention from the beginning of §2, we used the symbol $\boldsymbol{y} = \pi(y)$, and $\boldsymbol{y} \notin Q(\boldsymbol{x}, R)$ means that y is such that $\boldsymbol{y} \notin Q(\boldsymbol{x}, R)$.

Proposition 5.4 $(N, T \geqslant 1, \; 1 \leqslant r_1 < r_2 < r_3 < N)$.

i) *The excursion process* $\big(\widetilde{Z}_j^{(x)}(T, r_1, r_2, r_3)\big)_{x \in Q_N(0), j \geqslant 1}$, *resp.* $\big(\widetilde{W}_j^{(x)}(T, r_1, r_2, r_3)\big)_{x \in Q_N(0), j \geqslant 1}$, *have range at most* $2r_3$ *in* $\mathbf{T}$, *resp. in* $\mathbb{Z}^d$.

ii) *For all* $F \subset Q_N(0)$, *integer* $m \geqslant 1$ *and* $\varepsilon \in (0,1)$ *such that* $m\varepsilon \geqslant 3$ *and* $m_+ \leqslant T$ *where* $m_{\pm} = \lceil m(1 \pm \varepsilon) \rceil$, *one has with sup ranging over* $x \in F$, $z \in \Sigma^{(x)}$ *and* $n \in [m_-, m_+]$ *below,*

$$\widetilde{\mathbf{P}}_{\mathbf{0}}\big(\{\widetilde{Z}_j^{(x)} : j \leqslant m_-\} \subseteq \{Z_j^{(x)} : j \leqslant m\} \subseteq \{\widetilde{Z}_j^{(x)} : j \leqslant m_+\}, \; \forall \; x \in F\big)$$

$$\geqslant 1 - |F| r_3^{2d} \sup_{x,z,n} \; \Big\{ \widetilde{\mathbf{P}}_{\mathbf{0}}\big(|G_{n,x}^{\mathrm{RW}}(z) - n\bar{g}^{(x)}(z)| \geqslant \tfrac{\varepsilon}{4} n\bar{g}^{(x)}(z)\big)$$

(5.27)
$$+ 2\widetilde{\mathbb{P}}^I\big(|G_{n,x}^{\mathrm{RI}}(z) - n\bar{g}^{(x)}(z)| \geqslant \tfrac{\varepsilon}{4} n\bar{g}^{(x)}(z)\big) \Big\},$$

$$\widetilde{\mathbb{P}}^I\big(\{\widetilde{W}_j^{(x)} : j \leqslant m_-\} \subseteq \{W_j^{(x)} : j \leqslant m\} \subseteq \{\widetilde{W}_j^{(x)} : j \leqslant m_+\}, \; \forall \; x \in F\big)$$

$$\geqslant 1 - 3|F| r_3^{2d} \sup_{x,z,n} \widetilde{\mathbb{P}}^I\big(|G_{n,x}^{\mathrm{RI}}(z) - n\bar{g}^{(x)}(z)| \geqslant \tfrac{\varepsilon}{4} n\bar{g}^{(x)}(z)\big).$$

Proof. We first give the proof for the random walk. We start with item $i)$. For every $x \in Q_N(0)$, the family $(Z_j^{(y)}, \zeta_j^{(y)})_{j \geqslant 1, \boldsymbol{y} \notin Q(\boldsymbol{x}, 2r_3)}$ only depends on the excursions of X from $Q(y, r_2)$ to $Q(y, r_3)$ for $y \notin Q(\boldsymbol{x}, 2r_3)$, and is thus measurable with respect to $\big(Y_j(\widehat{X}, Q(x, r_2), Q(x, r_3))\big)_{j \geqslant 0}$ defined in (5.17), with $\widehat{X}$ as in (5.7) for $B_4 = B(x, N)$. Hence, it is independent of $(\widetilde{Z}_j^{(x)})_{j \geqslant 1}$ by Lemma 5.3. On account of (5.18) and Remark 4.5,2) (the latter implies that $\widetilde{Z}_j$ depends on X only through $(Z_j)_{j \geqslant 1}$ as well $(\zeta_j)_{j \geqslant 1}$ via the densities $(g_{\zeta_j})_{j \geqslant 1}$, the process $(\widetilde{Z}_j^{(y)})_{j \geqslant 1}$ depends by construction only on $(Z_j^{(y)}, \zeta_j^{(y)})_{j \geqslant 1}$, $\chi^{(y)}$ and $\widetilde{\lambda}^{(y)}$, for all $y \in Q_N(0)$. Using independence of $(\chi^{(y)}, \widetilde{\lambda}^{(y)})$ as y varies, it follows overall that $(\widetilde{Z}_j^{(x)})_{j \geqslant 1}$ is independent of $(\widetilde{Z}_j^{(y)})_{j \geqslant 1, \boldsymbol{y} \notin Q(\boldsymbol{x}, 2r_3)}$, as claimed.

We now show $ii)$. For each $x \in Q_N(0)$, let $(\widetilde{G}_{j,x}^{\mathrm{RI}})_{j \geqslant 1}$ denote the soft local times corresponding to $(\widetilde{G}_j)_{j \geqslant 1}$ in (4.14) when constructing $(\widetilde{Z}_j^{(x)})_{j \geqslant 1}$ in (5.18) by means of Proposition 4.4. In view of the choice above (5.17) $(\lambda, \widehat{\xi})$ and $(\widetilde{\lambda}, \widetilde{\xi})$ have the same law, and thus $\widetilde{G}_{n,x}^{\mathrm{RI}}(z)$ has the same law as $G_n^{\mathrm{RI}}(z)$ in (5.22). Proposition 4.4 (see (4.15)) now gives that for all $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$, the event appearing in the first line of (5.27) is implied by

$$\bigcap_{x \in F} \bigcap_{z \in \Sigma^{(x)}} \Big\{ \widetilde{G}_{m_-,x}^{\mathrm{RI}}(z) \leqslant G_{m,x}^{\mathrm{RW}}(z) \leqslant \widetilde{G}_{m_+,x}^{\mathrm{RI}}(z) \Big\}.$$

Moreover, for each $m \in \mathbb{N}$, $\varepsilon \in (0,1)$ such that $m\varepsilon \geqslant 3$, recalling that $m_{\pm} = \lceil m(1 \pm \varepsilon) \rceil$, the latter event (for fixed $x \in F$ and $z \in \Sigma^{(x)}$) is implied by

$$\Big\{|G_{m,x}^{\mathrm{RW}}(z) - m\bar{g}^{(x)}(z)| \leqslant \frac{\varepsilon}{4} m\bar{g}^{(x)}(z)\Big\} \cap \Big\{|\widetilde{G}_{m_+,x}^{\mathrm{RI}}(z) - m_+\bar{g}^{(x)}(z)| \leqslant \frac{\varepsilon}{4} m_+\bar{g}^{(x)}(z)\Big\}$$

$$\cap \Big\{|\widetilde{G}_{m_-,x}^{\mathrm{RI}}(z) - m_-\bar{g}^{(x)}(z)| \leqslant \frac{\varepsilon}{4} m_-\bar{g}^{(x)}(z)\Big\}.$$

The assertion (5.27) now follows by a union bound over x and z, upon noting that $g_{\zeta^{(x)}}^{(x)}(z)$ as defined in (5.15) with B_k as in (5.23) only depends on the first and last point of the excursion z. The same is thus true of $G_{m,x}^{\mathrm{RW}}(z)$, $\widetilde{G}_{m_{\pm},x}^{\mathrm{RI}}(z)$ and $\bar{g}^{(x)}(z)$, leading to the factor r_3^{2d} in (5.27).

The proof for random interlacements is similar, using this time in item $i)$ that the family $(W_j^{(y)}, \lambda_j^{(y)})_{j \geqslant 1, y \in Q(0,N) \setminus Q(x, 2r_3)}$ depends only on the excursions of the trajectories of random interlacements from $Q(y, r_2)$ to $Q(y, r_3)(\subset Q_{2N}(0))$ for $y \in Q(0,N) \setminus Q(x, 2r_3)$ (if a trajectory does not hit $Q(y, r_2)$ its excursions are just the full trajectory), and is thus measurable with respect to $\{Y_i(X^j, B_2, B_3) : 0 \leqslant i \leqslant T^j, j \geqslant 1\}$, see above (5.10) and (5.17). $\qquad\square$

The interlacement processes $\omega^{(x)}$ appearing in Theorem 5.1 will be constructed using the short range excursions $(\widetilde{Z}_j^{(x)})$ (in case of $\widetilde{\mathbf{P}}_{\mathbf{0}}$) or $(\widetilde{W}_j^{(x)})$ (in case of $\widetilde{\mathbb{P}}^I$) from Proposition 5.4. Items i) and ii) above thereby roughly correspond to (5.2) and (5.3)/(5.3'). The latter requires good control on the proximity between the short-range excursion processes $(\widetilde{Z}_j^{(x)})/(\widetilde{W}_j^{(x)})$ and the initial excursion processes $(Z_j^{(x)})/(W_j^{(x)})$, which is the object of the next two lemmas.

In view of (5.27), one central aspect is showing that the relevant soft local times concentrate around their mean. For unconditional soft local times, i.e. without conditioning on the clothesline process as in the construction of $\widetilde{Z}$ in (5.18), this was first achieved for random interlacements in [24, Section 6], and then for random walk if δ is large enough in [11, Lemma 2.1]. For conditional soft local times as in (5.22), concentration around the mean was proved for interlacements in [1, Proposition 4.1]. In the following lemma, the proof of which is relegated to Appendix A, we extend this concentration of conditional soft local times to the random walk case, cf. (5.19). In doing so we also give a shorter proof of [1, Proposition 4.1] when the sets ∂B_2 and ∂B_3 are well-separated, as implied by the parameter $\delta > 0$ below.

In view of (5.23) and with a slight abuse of notation, $G_m^\xi(\cdot)$, $\xi \in \{\mathrm{RW}, \mathrm{RI}\}$, refers in the sequel to the quantities introduced in (5.19) and (5.22) but with the choice $B_k = Q(x, r_k)$ for arbitrary $x \in Q_N(0)$ (implicit below; note however that translation invariance is spoiled under $\mathbf{P}_{\mathbf{0}}$ so one cannot simply set $x = 0$). In particular, $G_m^\xi(\cdot)$ depends on the parameters r_k, $1 \leqslant k \leqslant 3$. Recall the function $\bar{g}$ from (5.20).

Lemma 5.5. *For all $\delta \in (0, 1)$, there exist $c = c(\delta)$, $C = C(\delta)$, such that for all $N \geqslant 1$, all $0 < r_1 < r_2 < r_3 < N$ with $r_{k+1} \geqslant (1 + \delta)r_k$, $k = 1, 2$, and all $\varepsilon \in (0, 1)$, $m \geqslant 1$, $z \in \Sigma$,*

$$\mathbf{P}_{\mathbf{0}} \left(|G_m^{\mathrm{RW}}(z) - m\bar{g}(z)| \geqslant \varepsilon m \bar{g}(z) \right) \leqslant Cm \exp\left(- c\sqrt{\varepsilon^2 m} \right),$$
$$\mathbb{P}^I \left(|G_m^{\mathrm{RI}}(z) - m\bar{g}(z)| \geqslant \varepsilon m \bar{g}(z) \right) \leqslant C \exp(-c\varepsilon^2 m).$$

Proposition 5.4 and Lemma 5.5 deal with a fixed number of excursions (parametrized by m). For this to be successfully deployed, one needs to show that the actual number of excursions, which is random and given by $\mathcal{N}_{\mathrm{RW}}/\mathcal{N}_{\mathrm{RI}}$ see (5.9)/(5.12), suitably concentrates. Recalling the relevant notation from §2, see below (2.6), let

$$(5.28) \qquad\qquad M = M(B_2, B_3) \stackrel{\text{def.}}{=} \mathrm{cap}_{B_3}(B_2);$$

the quantity uM represents an 'asymptotic mean' number of excursions until the terminal time uN^d for the walk; see also (A.32) for an alternative formula for M conveying this intuition.

Lemma 5.6. *For all $\delta \in (0, 1)$, there exist $c, C \in (0, \infty)$ depending on δ so that for all $N \geqslant 1$, $u > 0$, $\varepsilon \in (0, 1)$, $1 \leqslant r_2 < r_3 < N$ with $r_3 \geqslant (1 + \delta)r_2$, and B_2, B_3 as in (5.23),*

$$\mathbf{P}_{\mathbf{0}}\left(|\mathcal{N}_{\mathrm{RW}}(B_2, B_3, u) - uM| \geqslant \varepsilon u M \right) \leqslant C\lceil uM \rceil \exp\left(- c\sqrt{\varepsilon^2 uM} \right),$$
$$\mathbb{P}^I\left(|\mathcal{N}_{\mathrm{RI}}(B_2, B_3, u) - uM| \geqslant \varepsilon u M \right) \leqslant C \exp\left(- c\varepsilon^2 uM \right).$$

Lemma 5.6 is essentially proved in [8, Proposition 9.1] for the random walk and in [8, Proposition 9.3] for random interlacements, but the bounds obtained therein are not explicit. We prove Lemma 5.6 in App. A using general large deviation results for random walk excursions and random interlacements trajectories, see Propositions A.7 and A.9, from which Lemma 5.5 follows as well.

5.3. Proof of Theorem 5.1. With Proposition 5.4 and Lemmas 5.5 and 5.6 at hand, we are now ready to proceed to the:

Proof of Theorem 5.1. We focus on the case of the random walk X under $\mathbf{P_0}$ and discuss the necessary modifications to accommodate random interlacements at the end of the proof. With R as appearing in the statement of Theorem 5.1, let $r_1 = R$, $r_2 = r_1(1 + \delta')$ and $r_3 = r_2(1 + \delta')$, where $\delta' > 0$ is such that $(1 + \delta')^2 = 1 + \delta$. In particular, $r_3 \leqslant N$ by assumption on R and this fixes the value of $M = M(Q(x, r_2), Q(x, r_3))$ in (5.28), which by a standard capacity estimate satisfies

$$(5.29) \qquad M \geqslant \mathrm{cap}(Q(x, r_2)) \geqslant c r_2^{d-2} \geqslant c' R^{d-2}.$$

Throughout the proof we write $\varepsilon' = \frac{\varepsilon}{3}$ for a given $\varepsilon \in (0, 1)$. We first introduce an approximation for (the increments of) the local times of X, which count a fixed number of excursions. With $Z_j^{(x)}$ as introduced below (5.23) (a (finite) excursion in $\mathbb{Z}^d$), writing $l_j^{(x)}$ for its total length, so that $Z_j^{(x)} = \{Z_j^{(x)}(n) : 0 \leqslant n \leqslant l_j^{(x)}\}$, we set

$$(5.30) \qquad \widehat{\ell}_{y,[v,u]}^{(x)} = \sum_{j=\lceil vM \rceil}^{\lceil uM \rceil} \sum_{n=0}^{l_j^{(x)}} 1\{Z_j^{(x)}(n) = y\}, \quad \text{for all } x \in Q_N(0), \ y \in Q_R(x) \text{ and } 0 < v < u,$$

which counts the total number of visits to y by all excursions with index j between $\lceil vM \rceil$ and $\lceil uM \rceil$. Similarly, let $\ell_{y,[v,u]} = \ell_{y,u} - \ell_{y,v}$, for $y \in \mathbf{T}$, which is the relevant quantity appearing in (5.3). By (2.2) and (5.9), $\ell_{y,[v,u]}$ admits a similar representation as (5.30), but counting excursions with label $j \in [\mathcal{N}_{\mathrm{RW}}(B_2, B_3, v), \mathcal{N}_{\mathrm{RW}}(B_2, B_3, u)]$ instead, where $B_i = Q(x, r_i)$ as in (5.23). We now claim that for all $x \in Q_N(0)$, $y \in Q(x, R)$ and $0 < v < u$,

$$(5.31) \quad (\{\mathcal{N}_{\mathrm{RW}}(B_2, B_3, v) \geqslant \lceil v(1 - \varepsilon')M \rceil\}$$
$$\cap \{\mathcal{N}_{\mathrm{RW}}(B_2, B_3, u) \leqslant \lceil u(1 + \varepsilon')M \rceil\}) \subset \{\ell_{\boldsymbol{y},[v,u]} \leqslant \widehat{\ell}_{y,[v(1-\varepsilon'),u(1+\varepsilon')]}^{(x)}\}$$

(with $\boldsymbol{y} = \pi(y)$). Indeed, first notice that a possibly non-vanishing contribution to the local time increment can arise from $X_{\cdot \wedge R_1}$, but only in case where $R_1 \geqslant \lfloor vN^d \rfloor$ which is equivalent to $\mathcal{N}_{\mathrm{RW}}(B_2, B_3, v) = 0$ on account of (5.9). This additional contribution to ℓ compared to $\widehat{\ell}$ is owed to the fact, noted below (5.6), that the process $Z^{(x)}$ defined in (5.6) neglects the very first excursion of the random walk in B_1 before time R_1. To obtain (5.31) one then uses the definition of ℓ, $\widehat{\ell}$ and $\mathcal{N}_{\mathrm{RW}}$ together with the fact that the event in the first line of (5.31) implies $\mathcal{N}_{\mathrm{RW}}(B_2, B_3, v) > 0$, which prevents $\ell_{\boldsymbol{y},[v,u]}$ from counting the very first excursion before time R_1.

Combining (5.31), a similar inclusion yielding a reverse inequality (for which over-counting the first excursion is not an issue), applying a union bound over $x \in F(\subset Q_N(0))$ and using Lemma 5.6, one thus infers that for all $0 < v < u$ and $\varepsilon \in (0, 1)$ with $u(1 - \varepsilon') > v(1 + \varepsilon')$,

$$(5.32) \quad \mathbf{P_0}\Big(\widehat{\ell}_{y,[v(1+\varepsilon'),u(1-\varepsilon')]}^{(x)} \leqslant \ell_{\boldsymbol{y},[v,u]} \leqslant \widehat{\ell}_{y,[v(1-\varepsilon'),u(1+\varepsilon')]}^{(x)}, \quad \text{for all } x \in F, y \in Q(x, R)\Big)$$
$$\geqslant 1 - C\lceil uM \rceil |F| \exp\big(-c\varepsilon\sqrt{vM}\big),$$

for positive constants c, C depending only on δ.

To proceed further, we now work under the extended measure $\widetilde{\mathbf{P}}_0$ introduced at the beginning of §5.2, which will form the basis of the desired coupling. Recall the process $(\widetilde{Z}_j^{(x)})$ introduced in (5.25) which has range at most $2r_3$ in $\mathbf{T}$ by Proposition 5.4,i), and choose $T = \lceil 4u_0 M \rceil$, for a (fixed) $u_0 > 0$ as appearing in the statement of Theorem 5.1. Mimicking (5.30), set

$$(5.33) \quad \widetilde{\ell}_{y,[v,u]}^{(x)} = \sum_{j=\lceil vM \rceil}^{\lceil uM \rceil} \sum_{n=0}^{\widetilde{l}_j^{(x)}} 1\{\widetilde{Z}_j^{(x)}(n) = y\}, \quad \text{for all } x \in Q_N(0), \ y \in Q_x(R) \text{ and } 0 \leqslant v < u \leqslant 4u_0,$$

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with $\widetilde{l}_j^{(x)}$ denoting the length of $\widetilde{Z}_j^{(x)}$. Combining Proposition 5.4,ii), applied for the choices $m = \lceil uM \rceil, \lceil vM \rceil$, with Lemma 5.5 and a union bound, it follows that for all $0 < v < u \leqslant u_0$ and $\varepsilon \in (0,1)$ such that $u(1 - 2\varepsilon') > v(1 + 2\varepsilon')$,

$$(5.34) \quad \widetilde{\mathbf{P}}_0 \left(\begin{array}{c} \widetilde{\ell}_{y,[v(1+2\varepsilon'),u(1-2\varepsilon')]}^{(x)} \leqslant \widehat{\ell}_{y,[v(1+\varepsilon'),u(1-\varepsilon')]}^{(x)}, \\ \widehat{\ell}_{y,[v(1-\varepsilon'),u(1+\varepsilon')]}^{(x)} \leqslant \widetilde{\ell}_{y,[v(1-2\varepsilon'),u(1+2\varepsilon')]}^{(x)}, \\ \text{for all } x \in F \text{ and } y \in Q(x,R) \end{array} \right) \geqslant 1 - Cr_3^{2d} \lceil uM \rceil |F| \exp\left(- c\varepsilon \sqrt{vM} \right).$$

To complete the proof, we now define a random interlacements process $\omega^{(x)}$ (i.e., satisfying (5.1)), which will inherit the finite-range property of $(\widetilde{Z}_j^{(x)})$ (so as to satisfy (5.2)) and whose local times in $Q(x,R)$ are close to $\widetilde{\ell}_{y,[0,\cdot]}^{(x)}$, $y \in Q(x,R)$, up to sprinkling (thus leading to (5.3)).

We first construct a family $(X^{(x),i})_{i \geqslant 1, x \in Q_N(0)}$ of independent random walks, where $X^{(x),i}$ has law $P_{\bar{e}_{B_2}}$, $B_2 = Q(x,r_2)$, for every $i \geqslant 1$. Importantly, for each $x \in Q_N(0)$ the excursions by any of the walks $(X^{(x),i})_{i \geqslant 1}$ between $\partial Q(x,r_1)$ and the last exit time of $Q(x,r_2)$ before exiting $Q(x,r_3)$ will be given precisely by $(\widetilde{Z}_j^{(x)})_{j \geqslant 1}$, and the remaining parts of the random walks will be conditionally independent as $x \in Q_N(0)$ varies.

Recall from above (5.25) that $(\widetilde{\omega}^{(x)})_{x \in \mathbb{Z}^d}$ is a family of independent random interlacements processes, each with corresponding clothesline process $\widetilde{\lambda}^{(x)}$ associated to the choice $B_k = Q(x,r_k)$, $k = 1, \ldots 3$, cf. (5.23). For $B \subseteq \mathbb{Z}^d$ we denote by $\widetilde{\omega}_B^{(x)}$ the restriction of $\widetilde{\omega}^{(x)}$ to forward (unlabeled) trajectories hitting B, and started at their entrance time in B. We call $(\widetilde{X}^{(x),i})_{i \geqslant 1}$, the trajectories thereby obtained from $\widetilde{\omega}_{B_2}^{(x)}$, corresponding to the trajectories in (2.11) when $B = B_2$ and $\omega = \widetilde{\omega}^{(x)}$. Note that by definition, see (5.11), each clothesline $\widetilde{\lambda}_j^{(x)}$, $j \geqslant 1$, arises from a certain trajectory $\widetilde{X}^{(x),k}$. As part of the ranges of $(\widetilde{X}^{(x),i})_{i \geqslant 1}$, we now define the sequence $(\widetilde{Y}_j^{(x)})_{j \geqslant 1}$ as follows. Whenever $\widetilde{\lambda}_j^{(x)}$ and $\widetilde{\lambda}_{j+1}^{(x)}$ correspond to the same trajectory $\widetilde{X}^{(x),k}$, we let $\widetilde{Y}_j^{(x)}$ be the excursion starting from ∂B_3^c until first hitting B_2 between the last time the clothesline $\widetilde{\lambda}_j^{(x)}$ is visited and the first time the clothesline $\widetilde{\lambda}_{j+1}^{(x)}$ is visited. If the clotheslines $\widetilde{\lambda}_j^{(x)}$ and $\widetilde{\lambda}_{j+1}^{(x)}$ correspond to two different trajectories $\widetilde{X}^{(x),k}$ and $\widetilde{X}^{(x),k+1}$ of $\widetilde{\omega}_{B_2}^{(x)}$, then $\widetilde{Y}_j^{(x)}$ is defined to be equal to the part of $\widetilde{X}^{(x),k}$ after last visiting ∂B_3^c. Now define recursively $\widetilde{V}_0^{(x)} = 0$ and $\widetilde{V}_i^{(x)} = \inf\{k > \widetilde{V}_{i-1}^{(x)} : \text{range}(\widetilde{Y}_k^{(x)}) \text{ is unbounded}\}$. Intuitively, $\widetilde{V}_i^{(x)}$ equals the number of excursions from B_2 to ∂B_3^c before the $(i+1)$-st walk from $\widetilde{\omega}_{B_2}^{(x)}$ starts, and $\widetilde{V}_{i+1}^{(x)} - \widetilde{V}_i^{(x)} - 1$ is precisely the number of excursions performed by this walk. Recall from above (5.14) that Θ is the cemetery point of Σ, and intuitively corresponds to excursions which do not hit B_1. Lastly, by suitable extension, assume that $\widehat{\mathbf{P}}_0$ carries for each $x \in Q_N(0)$ and $i \geqslant 1$ independent families $B^{(x),i} = \{B_{y,z;k}^{(x),i} : y \in \partial B_2, z \in \partial B_1 \cup \{\Theta\}, k \geqslant 1\}$ and $\widehat{B}^{(x),i} = \{\widehat{B}_{v,w;k}^{(x),i} : v \in \partial B_2, w \in \partial B_3^c, k \geqslant 1\}$ of independent random variables, whereby $B_{y,z;k}^{(x),i}$ has the same law as $(X_t)_{t \leqslant H_{B_1}}$ under $P_y(\cdot \mid H_{B_1} < T_{B_3}, X_{H_{B_1}} = z)$ if $z \in \partial B_1$ and the same law as $(X_t)_{t \leqslant L_{B_2}(T_{B_3})}$ under $P_y(\cdot \mid H_{B_1} > T_{B_3})$ if $z = \Theta$, and $\widehat{B}_{v,w;k}^{(x),i}$ has the same law as $(X_t)_{t \leqslant T_{B_3}}$ under $P_v(\cdot \mid \widetilde{H}_{B_2} > T_{B_3}, X_{T_{B_3}} = w)$.

We can now define the walk $X^{(x),i}$ for any $x \in Q_N(0)$ and $i \geqslant 1$, as follows. We introduce four sequences $(y_k), (z_k), (v_k), (w_k)$ (all implicitly depending on x and i), each with k ranging from $1 \leqslant k \leqslant \widetilde{V}_i^{(x)} - \widetilde{V}_{i-1}^{(x)}$. One sets $y_1 = \widetilde{X}_0^{(x),i}$ and for each $1 \leqslant k < \widetilde{V}_i^{(x)} - \widetilde{V}_{i-1}^{(x)}$, the vertex y_{k+1} (in ∂B_2) is defined as the endpoint of $\widetilde{Y}_{\widetilde{V}_{i-1}^{(x)}+k}^{(x)}$. The points z_k and v_k are the start- and endpoints of the excursion $\widetilde{Z}_{\widetilde{V}_{i-1}^{(x)}+k}^{(x)}$ when it is not equal to Θ, and we take $z_k = \Theta$ and $v_k = y_k$ when

$\widetilde{Z}^{(x)}_{\widetilde{V}^{(x)}_{i-1}+k} = \Theta$, and w_k is the starting point of $\widetilde{Y}^{(x)}_{\widetilde{V}^{(x)}_{i-1}+k}$. Now $X^{(x),i}$ starts in y_1. Then, for every k, the process $X^{(x),i}$ first follows $B^{(x),i}_{y_k,z_k;k}$, then $\widetilde{Z}^{(x)}_{\widetilde{V}^{(x)}_{i-1}+k}$ if $\widetilde{Z}^{(x)}_{\widetilde{V}^{(x)}_{i-1}+k} \neq \Theta$ (and otherwise stays in place), then $\widehat{B}^{(x),i}_{v_k,w_k;k}$ and finally $\widetilde{Y}^{(x)}_{\widetilde{V}^{(x)}_{i-1}+k}$ and the pieces thereby obtained are concatenated as $k \in \{1,\ldots,\widetilde{V}^{(x)}_i - \widetilde{V}^{(x)}_{i-1}\}$ increases to form the sample path of $X^{(x),i}$ (note in particular that the last piece is indeed unbounded).

The starting point of $X^{(x),i}$ is the same as $\widetilde{X}^{(x),i}$, and thus has law $\bar{e}_{Q(x,r_2)}$ by the representation (2.11) of random interlacements. Using (5.26) and a similar calculation as following (5.16) (in order to witness the correct conditional distributions of the bridges $B^{(x),i}$ and $\widehat{B}^{(x),i}$), one concludes that $(X^{(x),i})_{i\geq 1}$ are i.i.d. random walks with starting distribution $\bar{e}_{Q(x,r_2)}$ each, as desired. Moreover, by construction,

$$(5.35) \qquad (X^{(x),i})_{i\geq 1} \text{ is independent of } \{(X^{(y),i})_{i\geq 1} : \boldsymbol{y} \notin Q(\boldsymbol{x}, 2(1+\delta)R)\},$$

since $\widetilde{Z}^{(x)}$ has range $2r_3 = 2R(1+\delta)$ in $\mathbf{T}$ and $X^{(x),i}$ only involves additional randomness which is independent as x varies: namely, $B^{(x),i}$, $\widehat{B}^{(x),i}$ and $(\widetilde{Y}^{(x)}_j)_{j\geq 1}$ (function of $\widetilde{\omega}^{(x)}$).

To complete the construction of $\omega^{(x)}$, let $(N^{(x),u})_{u\geq 0}$, $x \in Q_N(0)$, be an i.i.d. family of Poisson processes with intensity $\mathrm{cap}(B_2)$ and define

$$\omega^{(x),u}_{B_2} = \sum_{1 \leq i \leq N^{(x),u}} \delta_{X^{(x),i}} \text{ for all } u > 0.$$

Then $\big(\omega^{(x),u}_{B_2}\big)_{u>0}$ has the same law as the restriction of $(\omega^u)_{u>0}$ to forward trajectories hitting B_2 after entering B_2 and one completes it independently to obtain an interlacements process $\omega^{(x)} = (\omega^{(x),u})_{u>0}$ at all levels on $\mathbb{Z}^d$, which has the desired law, see (5.1), and satisfies (5.2) by means of (5.35).

It remains to show (5.3). Denoting by $(\ell^{(x)}_{y,u})_{u\geq 0,y\in\mathbb{Z}^d}$ the field of local times associated to $\omega^{(x)}$, recalling (5.33) and noting that the trace of $\omega^{(x)}$ inside $B_1 = Q(x,R)$ coincides with that of the excursions $(\widetilde{Z}^{(x)}_j)_{j\geq 1}$ which enter it, it then follows by Lemma 5.6 and a union bound that for all $0 < v < u \leq u_0$ and $\varepsilon \in (0,1)$ with $u(1-3\varepsilon') > v(1+3\varepsilon')$,

$$(5.36) \qquad \widetilde{\mathbf{P}}_0 \left(\begin{array}{c} \ell^{(x)}_{y,[v(1+3\varepsilon'),u(1-3\varepsilon')]} \leq \widetilde{\ell}^{(x)}_{y,[v(1+2\varepsilon'),u(1-2\varepsilon')]}, \\ \widetilde{\ell}^{(x)}_{y,[v(1-2\varepsilon'),u(1+2\varepsilon')]} \leq \ell^{(x)}_{y,[v(1-3\varepsilon'),u(1+3\varepsilon')]}, \\ \text{for all } x \in F \text{ and } y \in Q(x,R) \end{array} \right) \geq 1 - C|F|\exp\left(-c \cdot \varepsilon^2 v M\right).$$

Thus, (5.3) follows by combining (5.32), (5.34) and (5.36) with the lower bound (5.29) on M.

The proof in the case of random interlacements follows a similar three-step pattern: first one shows using Lemma 5.6 for random interlacements that $\ell_{y,u}$ under $\mathbb{P}^I$ is well-approximated by a process $\widehat{\ell}^{(x)}_{y,u(1\pm\varepsilon')}$, for $y \in Q(x,R)$ and $x \in F$, having a fixed excursion count, thus yielding an analogue of (5.32). This step is somewhat streamlined since there is no subtlety regarding the first excursion, as opposed to X. In the second step, one uses the interlacement parts of Proposition 5.4 and Lemma 5.5 to approximate $\widehat{\ell}^{(x)}_{y,u(1\pm\varepsilon')}$ by a short-range process $\widetilde{\ell}^{(x)}_{y,u(1\pm2\varepsilon')}$, similarly as in (5.34). Finally one reconstructs a short-range family of interlacement processes $(\omega^{(x)})_{x\in Q_N(0)}$ such that their associated local times $\ell^{(x)}_{y,u(1\pm3\varepsilon')}$ are good approximations of $\widetilde{\ell}^{(x)}_{y,u(1\pm2\varepsilon')}$. The second and third of these steps are virtually identical as above upon setting $v = 0$. $\qquad\square$

Remark 5.7 (Extensions of Theorem 5.1).

1) (Flexibility with (5.1)-(5.2)). One could relax (5.1)-(5.2) by requiring instead that $\ell^{(x)} = (\ell_{y,u}^{(x)})_{y \in Q(x,R), u \geq 0}$ be *some* field having a finite-range property, satisfying (5.3)/(5.3') and whose law is translation invariant (that is $(\ell_{y,\cdot}^{(x)})_{y \in Q(x,R)}$ has the same law as $(\ell_{y-x,\cdot}^{(0)})_{y \in Q(x,R)}$), see the proof of Theorem 1.2 as to why this is necessary. Under these less stringent conditions, one can afford to simply choose $\ell_{y,u}^{(x)} = \widetilde{\ell}_{y,[0,u]}^{(x)}$ as in (5.33) and finish the proof with (5.34) in two steps instead of three; note that the law of $\ell^{(x)}$ is translation invariant by (5.26). This weaker result is in fact sufficient to deduce Proposition 6.6 below, which will be the driving force behind the proof of Theorems 1.1 and 1.3 in §7 (see also Theorem 6.5). Various parts of the coupling also simplify in the process. Indeed, one can define the reference values M in (5.28) and $\bar{g}(z)$ in (5.20) without identifying the relevant stationary distribution of the clothesline process, see (A.32) for M and the proof of Lemma 5.5 for $\bar{g}(z)$, thus bypassing the use of exact identities such as (A.12) and (A.13), see also [8, Lemma 6.1 and eq. (9.4)] which, albeit instructive, are not trivial.

Apart from giving a concrete idea as to what $\ell^{(x)}$ is, the conditions (5.1)–(5.2) present the additional benefit of producing an independent proof of the coupling from [8] between the random walk and random interlacements, cf. Corollary 5.2, for which $\ell^{(x)}$ crucially needs to have the correct law. Moreover, knowing that $\ell^{(x)}$ are interlacement local times is also essential in the proof of Lemma 6.1 below.

In a similar vein, one may require as part of Theorem 5.1 that $\ell^{(x)}$ be the local times associated to a short range family of random walks on $\mathbf{T}$ (instead of interlacements). This is essentially a matter of replacing the interlacement clothesline $\widetilde{\lambda}^{(x)}$ in the construction of $\widehat{Z}^{(x)}$ in (5.25) by a random walk clothesline $\widetilde{\zeta}^{(x)}$. The proof suffers very minor modifications (mostly trading one of the estimates in either of Lemmas 5.5 or 5.6 for the other). In particular, in the context of (5.36), the increments of $\ell^{(x)}$ (now associated to a random walk) will not overcount the first excursion for similar reasons as in (5.31). Moreover, the law of $\ell^{(x)}$ is still translation invariant when starting the corresponding random walks from the uniform measure on $\mathbf{T}$.

2) (Increments in (5.3)). The choice of observable $\ell_{y,[v,u]} = \ell_{y,u} - \ell_{y,v}$, for $y \in \mathbf{T}$ is a means to avoid potential issues with the very first excursion of X, see the discussion leading to (5.31): the excursion process $Z^{(x)}$ does not count the first excursion of the random walk in $Q(\boldsymbol{x}, r_1)$ before time $D_0(X, Q(\boldsymbol{x}, r_2), Q(\boldsymbol{x}, r_3)) = H_{\partial Q(x, r_3)^c}(X)$, cf. (5.4), (5.6) and (5.24), hence this excursion does not appear in (5.30) either. Note that this issue does not arise for interlacements since trajectories arrive "from infinity," whence (5.3') rather than (5.3). As we now explain, instead of the increment $\ell_{y,[v,u]}$ one could consider the field

$$\bar{\ell}_{y,u}^{(x)} \stackrel{\text{def.}}{=} \sum_{n \geq 0} 1\{X_n = y, \, n \geq H_{\partial Q(\boldsymbol{x},R')^c}\}, \text{ for } y \in Q(\boldsymbol{x}, R), \, x \in Q(0, N),$$

with $R' = r_3 = (1 + \delta)R$, and replace (5.3) by

$$(5.37) \quad \widetilde{\mathbf{P}}_{\mathbf{0}} \left(\ell_{y,u(1-\varepsilon)}^{(x)} \leq \bar{\ell}_{\boldsymbol{y},u}^{(x)} \leq \ell_{y,u(1+\varepsilon)}^{(x)} \, \forall x \in F, \, y \in Q(x, R) \right)$$
$$\geq 1 - C|F|R^{2d}\lceil uR^{d-2}\rceil \exp\left(-c\varepsilon\sqrt{uR^{d-2}} \right).$$

Observe in particular that $\bar{\ell}_{\boldsymbol{y},u}^{(x)} = \ell_{\boldsymbol{y},u}$ for all $y \in Q(x, R)$ under $\mathbf{P}_{\mathbf{0}}$ whenever $x \notin Q_{R'}(0)$, for then $X_n \in Q(\boldsymbol{x}, R)$ implies $n \geq H_{\partial Q(\boldsymbol{x},R')^c}$. Thus (5.37) yields a true analogue of (5.3') if one restricts to $x \in F \backslash Q_{R'}(0)$. The proof of (5.37) does not require any amendments to the above argument: the restriction on n inherent to $\bar{\ell}_{\boldsymbol{y},u}^{(x)}$ allows to carry out the proof of

Theorem 5.1 with $v = 0$ (and $\bar{\ell}^{(x)}_{\boldsymbol{y},u}$ in place of $\ell_{\boldsymbol{y},[0,u]}$), which in particular does not create issues in (5.31). Alternatively, one replaces X by $X^{(x)} = X \circ \theta_{H_{\partial Q(\boldsymbol{x},R')^c}}$ in the definition (5.24) of $Z^{(x)}_j$, which leaves the associated clothesline $\zeta^{(x)}$ unchanged. The field $\bar{\ell}^{(x)}_{\boldsymbol{y},u}$ is measurable in terms of this modified $Z^{(x)}$. The issue with the first excursion disappears in this context in essence because $D_0(X^{(x)}, Q(x,r_2), Q(x,r_3)) = 0$, cf. (5.4).

In fact (5.37) also implies directly an approximation of $\ell_{\boldsymbol{y},u}$ by $\ell^{(x)}_{\boldsymbol{y},u(1\pm\varepsilon)} + \ell_{\boldsymbol{y},u} - \bar{\ell}^{(x)}_{\boldsymbol{y},u}$ for all $x \in F$ and $y \in Q(x,R)$, which is also a short-range field under $\widetilde{\mathbf{P}}_{\mathbf{0}}$ since it is equal to $\ell^{(x)}_{\boldsymbol{y},u(1\pm\varepsilon)}$ outside of $Q_{R'}(0)$. However its law is not translation invariant, and thus does not necessarily have range $2(1+\delta)R$ under the probability measure $\widetilde{\mathbf{P}}$ from Theorem 1.2.

3) (Coupling in u). The following extension of (5.37) is tailored to later purposes (see the proof of Theorem 8.1), but noteworthy in its own right. Let $0 < u_1 < u_0$. Then, applying (5.37) (for $\frac{\varepsilon}{3}$ instead of ε) at levels $u = u_1 + ku_1\varepsilon/3$ for each $k \in \{0, 1, \ldots, \lceil \frac{3(u_0-u_1)}{\varepsilon u_1} \rceil\}$, using a union bound and monotonicity of all the relevant fields in u, one deduces (as alternative to (5.3) in the statement of Theorem 5.1) that

$$(5.38) \quad \widetilde{\mathbf{P}}_{\mathbf{0}} \left(\ell^{(x)}_{\boldsymbol{y},u(1-\varepsilon)} \leqslant \ell_{\boldsymbol{y},u} \leqslant \ell^{(x)}_{\boldsymbol{y},u(1+\varepsilon)}, \text{ for all } x \in F\backslash Q_{R'}(0), \, y \in Q(x,R), \, u \in [u_1, u_0] \right)$$

$$\geqslant 1 - C|F|R^{2d}\tfrac{u_0}{\varepsilon u_1}\lceil u_0 R^{d-2}\rceil \exp\left(-c\varepsilon\sqrt{u_1 R^{d-2}}\right)$$

(with $R' = (1+\delta)R$). Note here that we used that the field $\ell^{(x)}$ from (5.37) does not depend on the choice of $u \in (0, u_0]$, similarly as in Theorem 5.1. An obvious analogue of (5.38) holds for random interlacements, with $\ell_{y,u}$ in place of $\ell_{\boldsymbol{y},u}$ and without further restriction on $x \in F$. In closer analogy to (5.3), one could also formulate a version of (5.38) for increments.

6 Consequences of localization

Our main localization result, Theorem 5.1, derived in the previous section, has two main applications in the context of late points. First, as asserted in Proposition 6.6 below, it allows to introduce a (localized) family $\widetilde{\mathcal{L}} = (\widetilde{\mathcal{L}}^\alpha)_{\alpha \geqslant 0}$, coupled to $\mathcal{L}$ (recall (1.4)) in a way that i) the two are close up to sprinkling (see (6.20) below) and ii) $\widetilde{\mathcal{L}}$ is amenable to Chen-Stein (due to its finite-range property). Second, as alluded to below (1.6), it allows us by means of Corollary 5.2 to compute various key quantities of interest related to the random walk with sufficient precision using interlacements, see Lemmas 6.1 and 6.3. Combining these two ingredients with Lemma 3.1 (the modified Chen-Stein scheme) then leads to Theorem 6.5, which is the main result of this section, and will be one of the driving forces behind our main results, proved in §7. By exploiting $\widetilde{\mathcal{L}}$ as an intermediate link, Theorem 6.5 gives quantitative control on the difference between the true set of late points $\mathcal{L}$ and its 'Poissonized' version $\widetilde{\mathcal{B}}$, comprising a suitable class of *independently* sampled shapes, see (6.16). Our arguments hint at a generic phenomenon, which ought to be valid for a variety of models of interest, see Remark 8.4,5).

6.1. The set $\mathcal{L}^\alpha_F$ and first properties. We start by introducing a setup that fits all needs. As announced in the introduction, see above (1.6), this includes treating both late-point/high-intensity regimes for random walk/random interlacements, each in a subset F of (but not necessarily equal to) the full torus/box (of side length N), at appropriate timescales. Recall from §2 that 0 denotes the origin of $\mathbb{Z}^d$ and $\mathbf{0} = \pi(0)$ where $\pi : \mathbb{Z}^d \to \mathbf{T}$ is the canonical projection, and that $Q_R(x)$ is the cube of side length R centred at x, either in $\mathbb{Z}^d$ if $x \in \mathbb{Z}^d$ or in $\mathbf{T}$ if $x \in \mathbf{T}$.

In order to allow for a unified presentation, we introduce the following notation, valid from here on and throughout Sections 6-8. In writing $\mathbb{P}$ in the sequel, we tacitly allow for either choice $\mathbb{P} \in \{\mathbb{P}^I, \mathbf{P}\}$, i.e. all statements made under the measure $\mathbb{P}$ hold for either model (recall that $\mathbb{P}^I$ denotes the canonical law of random interlacements on $\mathbb{Z}^d$ and $\mathbf{P}$ the law of the random walk on $\mathbf{T}$ with uniform starting point). We further define the set Q_R for any $R \leqslant N$ (where N denotes the side length of $\mathbf{T}$) as $Q_R = Q_R(0)(\subset \mathbb{Z}^d)$ when $\mathbb{P} = \mathbb{P}^I$ and $Q_R = Q_R(\mathbf{0})(\subset \mathbf{T})$ when $\mathbb{P} = \mathbf{P}$.

With the above notation, we introduce for finite $F \subset Q_N$ and $\alpha > 0$,

$$(6.1) \qquad\qquad u_F(\alpha) = \alpha g(0) \log(|F|),$$

whence $u_N(\alpha) = u_F(\alpha)|_{F=Q_N}$ in view of (1.3) (when $\mathbb{P} = \mathbf{P}$). The scaling (6.1) is explained in Remark 6.2,1) below. With $\mathcal{V}_N$ as in (1.2) and $\mathcal{V}^{\cdot}$ as below (2.13), we now define under $\mathbb{P}$ the random set $\mathcal{L}_F = (\mathcal{L}_F^\alpha)_{\alpha \geqslant 0}$ for arbitrary finite $F \subset Q_N$ to be

$$(6.2) \qquad\qquad \mathcal{L}_F^\alpha = \begin{cases} \mathcal{V}_N^{u_F(\alpha)} \cap F, & \text{if } \mathbb{P} = \mathbf{P} \\ \mathcal{V}^{u_F(\alpha)} \cap F, & \text{if } \mathbb{P} = \mathbb{P}^I . \end{cases}$$

We simply write $\mathcal{L}^\alpha$ when $F = Q_N$, which is consistent with (1.4); the results of the introduction thus deal with the case $\mathbb{P} = \mathbf{P}$ in (6.2) for the specific choice $F = Q_N(= Q_N(\mathbf{0}))$. As will become clear, all of these results can be generalized (with suitable amendments) to the more general framework of (6.2). We start by gathering a few key properties of $\mathcal{L}_F^\alpha$.

Lemma 6.1.

i) *For all $K \subset F \subset\subset \mathbb{Z}^d$ and $\alpha > 0$,*

$$(6.3) \qquad\qquad \mathbb{P}^I(K \subset \mathcal{L}_F^\alpha) = |F|^{-\alpha g(0)\mathrm{cap}(K)}.$$

ii) *For all $N \geqslant 1$, $K \subset F \subset \mathbf{T}$, and all $\beta_0 > 0$, the bound*

$$(6.4) \qquad\qquad \left| \frac{\mathbf{P}(K \subset \mathcal{L}_F^\alpha)}{|F|^{-\alpha g(0)\mathrm{cap}(K)}} - 1 \right| \leqslant C(\beta_0) \frac{\log(N)^{3/2}}{N^{(d-2)/2}}$$

holds whenever $\alpha \in (0, 2]$ and $\mathrm{cap}(K) \leqslant \beta_0$.

Proof. The equality (6.3) follows directly from (1.6), (6.1) and (6.2). To deduce (6.4) first note that the condition $\mathrm{cap}(K) \leqslant \beta_0$ implies $|K| \leqslant C(\beta_0)$ by virtue of (2.10). Since $\delta(K) \leqslant N(1 - \frac{1}{|K|})$ for any $K \subset \mathbf{T}$, using translation invariance we may therefore assume that $K \subset Q_{N(1-\delta)}(\mathbf{0})$ for some $\delta = \delta(\beta_0) > 0$. We then apply Corollary 5.2 for this δ with the choice $\varepsilon = \lambda N^{-(d-2)/2} \frac{\log N}{\sqrt{\alpha \log(|F|)}}$ for $\lambda > 0$ to find that

$$\mathbf{P}(K \subset \mathcal{L}_F^\alpha) \leqslant \mathbb{P}^I(K' \subset \mathcal{L}_F^{\alpha(1-\varepsilon)}) + C\alpha \log(|F|) N^{3d} \exp(-c(\delta)\lambda \log N),$$

where $K' \subset Q_{N(1-\delta)}(0)$ is such that $\pi(K') = K$, similarly as in the definition of $\mathrm{cap}(K)$ below (2.3). From this, one of the two bounds implied by (6.4) readily follows using (6.3) upon taking λ large enough in a manner depending on β_0. The other bound is obtained similarly. $\qquad\square$

Remark 6.2.

1) (Asymptotic density of $\mathcal{L}_F^\alpha$). Applying Lemma 6.1 for $K = \{0\}$ and using (2.6) yields for any $F = F_N \subset Q_N$ with $|F| \to \infty$ as $N \to \infty$ that

$$(6.5) \qquad\qquad \mathbb{P}(0 \in \mathcal{L}_F^\alpha) \sim |F|^{-\alpha} \text{ as } N \to \infty,$$

which accounts for the scaling in (1.3) and (6.1).

2) Throughout §6.2, the following consequence of (6.3) and (6.4) will be sufficient, cf. Remark 8.4,5): for all $\beta_0 > 0$, all $N \geqslant 1$, $F \subset Q_N$, $K \subset F$ with $\mathrm{cap}(K) \leqslant \beta_0$, and $\alpha \in (0, 2]$,

$$(6.6) \qquad \mathbb{P}(K \subset \mathcal{L}_F^\alpha) \leqslant C(\beta_0) |F|^{-\frac{\alpha}{\alpha_*(K)}}$$

(recall the definition of $\alpha_*(K)$ from (1.19)). In Lemmas 6.3 and 6.4 as well as in §7 and §8 below, we will also use the following lower bound implied by Lemma 6.1: under the same assumptions as those of (6.6),

$$(6.7) \qquad \mathbb{P}(K \subset \mathcal{L}_F^\alpha) \geqslant c(\beta_0) |F|^{-\frac{\alpha}{\alpha_*(K)}}.$$

Actually the constants $C(\beta_0)$ from (6.6) and $c(\beta_0)$ from (6.7) could be replaced by $1 + o(1)$ as $|F| \to \infty$, but we will not need this fact except in the proof of (1.13) to obtain the exact constant $1 - e^{-d}$. In the proof of Lemma 6.4 below, we will also need the following decoupling formula, which is easily implied by Lemma 6.1 together with (2.9) in the improved form discussed immediately below it (along with its analogue on the torus): for all $\beta_0 > 0$ and $\alpha > 0$, if $K, K' \subset F$ are such that $\frac{d(K,K')}{\log(|F|)^{1/(d-2)}} \to \infty$,

$$(6.8) \qquad \mathbb{P}((K \cup K') \subset \mathcal{L}_F^\alpha) = (1 + o(1))\mathbb{P}(K \subset \mathcal{L}_F^\alpha)\mathbb{P}(K' \subset \mathcal{L}_F^\alpha) \text{ as } |F| \to \infty,$$

where $o(1)$ is uniform in K, K' verifying $|K|, |K'| \leqslant \beta_0$.

Before constructing our coupling between $\mathcal{L}$ and Bernoulli random variables, let us collect some interesting consequences of (6.6), (6.7) and (6.8), which further elucidate the role of the parameters α_* from (1.10) and $\alpha_*(K)$ from (1.19). In view of (6.2), the quantity D^α introduced in (1.9) is naturally declared under $\mathbb{P}$ upon summing over all $x \sim y$ with $x, y \in Q_N$. For any set $\mathcal{S} \subset Q_N$ and $K \subset Q_N(0)$, we introduce similarly

$$(6.9) \qquad D_K(\mathcal{S}) = \begin{cases} \sum_{x \in Q_N} 1\{x + K \subset \mathcal{S}\}, & \text{if } \mathbb{P} = \mathbb{P}^I \\ \sum_{x \in Q_N} 1\{x + \pi(K) \subset \mathcal{S}\}, & \text{if } \mathbb{P} = \mathbf{P} \end{cases}$$

the number of times a translate of K (or its projection on the torus) by $x \in Q_N$ lies in $\mathcal{S}$. We will often abbreviate $D_K^\alpha = D_K(\mathcal{L}^\alpha)$. Note that D^α, see (1.9), is half the sum of D_K^α over all $K = \{0, x\}$ with $x \sim 0$. The following result shows that D_K^α is small on average if and only if $\alpha > \alpha_*(K)$.

Lemma 6.3 (Representations of α_*)**.** *For each $\varnothing \neq K \subset\subset \mathbb{Z}^d$, with $\alpha_*(K)$ as in (1.19),*

$$(6.10) \qquad \alpha_*(K) = \sup\{\alpha > 0 : \lim_N \mathbb{E}[D_K^\alpha] = 0\}.$$

In particular for all $x \sim y$,

$$(6.11) \qquad \alpha_* = \alpha_*(\{x, y\}) = 1 - \frac{1}{2g(0)} = \frac{1}{2}\big(1 + P_0(\widetilde{H}_0 < \infty)\big).$$

Proof. First, observe that (6.6) and (6.7) applied with $F = Q_N$ immediately yields, for all $\alpha > 0$,

$$(6.12) \qquad cN^{d(1-\frac{\alpha}{\alpha_*(K)})} \leqslant \mathbb{E}[D_K^\alpha] \leqslant CN^{d(1-\frac{\alpha}{\alpha_*(K)})},$$

from which (6.10) follows. The first equality in (6.11) then follows by rotational invariance of the capacity. Applying the simple Markov property, one obtains, for all $x \sim 0$,

$$(6.13) \qquad P_0(\widetilde{H}_0 < \infty) = \frac{1}{2d}\sum_{y \sim 0} P_y(H_0 < \infty) \overset{(2.4)}{=} \frac{1}{2d}\sum_{y \sim 0} \frac{g(y)}{g(0)} = \frac{g(x)}{g(0)} = 1 - \frac{1}{g(0)},$$

where the two last steps follow by invariance of P_x under lattice rotations and translations. On account of (1.11), (1.19) and (2.6), this gives the two last equalities in (6.11). $\square$

We now collect a lower bound on D_K^α, resp. D^α, in the 'supercritical' phase $\alpha < \alpha_*(K)$, resp. $\alpha < \alpha_*$, which will be useful in due course. For D^α, a similar but weaker estimate was derived in [22, p. 1040], for a very specific choice of timescale asymptotic to αt_{cov}, see also Remark 7.3,5) below. In addition to yielding a stronger bound valid for any set K, the proof we present below is considerably simpler, which highlights the strength of Lemma 6.1.

Lemma 6.4. *For all* $K \subset\subset \mathbb{Z}^d$, $\alpha \in (0, \alpha_*(K)]$ *and* $\varepsilon_N > 0$ *with* $\lim_N \varepsilon_N = 0$, *one has*

$$
(6.14) \qquad \liminf_N \mathbb{P}\big(D_K^\alpha \geqslant \varepsilon_N N^{d(1-\frac{\alpha}{\alpha_*(K)})}\big) \begin{cases} = 1 & \text{if } \alpha < \alpha_*(K), \\ > 0 & \text{if } \alpha = \alpha_*(K), \end{cases}
$$

and the same holds true with α_* *in place of* $\alpha_*(K)$ *and* D^α *in place of* D_K^α.

Proof. It follows from (1.19) and (2.8) that there exists $C = C(K) < \infty$ such that $\alpha_*((K+x) \cup (K+x')) \leqslant 2\alpha_*(K)/3$ for all $x, x' \in Q_N$ with $d(x, x') \geqslant C$. Applying (6.6) (when $d(x,x') \leqslant C$ or $C \leqslant d(x,x') \leqslant \log(N)^{\frac{2}{d-2}}$) and (6.8) (when $d(x,x') \geqslant \log(N)^{\frac{2}{d-2}}$), one has for all $\alpha > 0$

$$
(6.15) \quad \begin{aligned} \mathbb{E}\big[(D_K^\alpha)^2\big] &= \sum_{x,x' \in Q_N} \mathbb{P}\big(K+x, K+x' \subset \mathcal{L}^\alpha\big) \\ &\leqslant C N^{d-\frac{\alpha d}{\alpha_*(K)}} + C \log(N^d)^{\frac{2d}{d-2}} N^{d-\frac{3\alpha d}{2\alpha_*(K)}} + \big(1 + o(1)\big) \mathbb{E}[D_K^\alpha]^2 \end{aligned}
$$

as $N \to \infty$. If $\alpha < \alpha_*(K)$, then in view of (6.12) the second moment on the right-hand side of (6.15) dominates. Combining (6.12), (6.15) and a standard second-moment argument, it follows that

$$
\mathbb{P}\big(D_K^\alpha \geqslant \varepsilon_N N^{d(1-\frac{\alpha}{\alpha_*(K)})}\big) \geqslant \mathbb{P}\big(D_K^\alpha \geqslant c\varepsilon_N \mathbb{E}[D_K^\alpha]\big) \geqslant \frac{(1-c\varepsilon_N)^2}{1+o(1)},
$$

for all $\alpha < \alpha_*(K)$, from which the claim follows since $\varepsilon_N \to 0$. If $\alpha = \alpha_*(K)$ the proof is similar, except (6.15) is now only smaller than $C\mathbb{E}[D_K^\alpha]^2$. The statement for D^α then follows readily from (6.11) and (6.14) for $K = \{0, x\}$, $x \sim 0$. $\qquad\square$

6.2. Main approximation result for $\mathcal{L}_F^\alpha$. We now combine the modified Chen-Stein result (with sprinkling), Lemma 3.1, with our main localization result, Theorem 5.1 and the asymptotic bounds (6.6) (a consequence of Lemma 6.1) to derive our main approximation result for $\mathcal{L}_F^\alpha$, see Theorem 6.5 below. In a nutshell, we first apply Theorem 5.1 to replace (up to sprinkling) the family $\mathcal{L}_F$ by an approximation $\widetilde{\mathcal{L}}_F$ with a certain finite-range property (see Proposition 6.6 below), to which we then apply Lemma 3.1. The above estimate (6.6) will serve to bound quantities such as b_1 and b_2 in (3.6)-(3.7).

Theorem 6.5, stated below, is of independent interest. In the next section, it will serve as a driving force to derive our main results from §1. The approximation result it entails supplies a coupling between $\mathcal{L}_F$ and a suitably defined process $\widetilde{\mathcal{B}}$ of i.i.d. 'patterns,' which we now introduce. Let $(U_K)_{K \subset Q_N}$ be an i.i.d. family of uniform random variables on $[0,1]$ and for $F \subset \mathbb{Z}^d$ or $\mathbf{T}$ (depending on whether $\mathbb{P}$ equals $\mathbb{P}^I$ or $\mathbf{P}$) define the set of admissible patterns by

$$
(6.16) \qquad \widetilde{\mathcal{A}}_F = \big\{ K \subset F : K \neq \varnothing, \mathrm{cap}(K) \leqslant \tfrac{2}{g(0)}, \delta(K) \leqslant R_F \big\}, \text{ where } R_F = \log(|F|)^{\frac{1}{d-2}}
$$

(note that compared to (1.24) we additionally ask for $\delta(K) \leqslant R_F$). We return to the choice of $\widetilde{\mathcal{A}}_F$ in Remark 6.9 below. We also define $p_F^\alpha(K) = \mathbb{P}(\mathcal{L}_F^\alpha \cap Q(K, R_F) = K)$ for each $K \in \widetilde{\mathcal{A}}_F$ and $\alpha > 0$, and note in passing that $p^\alpha(K)$ introduced above (1.20) corresponds exactly to $p_F^\alpha(K)$ for the choice $F = Q_N$ when $\mathbb{P} = \mathbf{P}$. We then define the family $\widetilde{\mathcal{B}}_F = (\widetilde{\mathcal{B}}_F^\alpha)_{\alpha \geqslant 0}$ as

$$
(6.17) \qquad \widetilde{\mathcal{B}}_F^\alpha = \bigcup_{K \in \widetilde{\mathcal{A}}_F : U_K \leqslant p_F^\alpha(K)} K.
$$

The following result gives quantitative control on the proximity of $\mathcal{L}_F$ and $\widetilde{\mathcal{B}}_F$ above level $\frac{1}{2}$, as measured in terms of d_ε, cf. (1.21).

Theorem 6.5 ($\mathcal{L}_F$ as in (6.2)). *There exist $C, C' < \infty$ such that, for all $\alpha \in (\frac{1}{2}, 1]$, $N \geqslant 1$, $F \subset Q_N$ and $\varepsilon \in (0, \alpha)$, one can couple $\mathcal{L}_F^\alpha$ with $(U_K)_{K \in \widetilde{\mathcal{A}}_F}$ so that with probability one minus*

$$(6.18) \qquad 1 \wedge C \log(|F|)^{C'} |F|^{1-2(\alpha-\varepsilon)} \varepsilon^{-\frac{2d}{d-2}},$$

one has the inclusions

$$(6.19) \qquad \{U_K \leqslant p_F^{\alpha+\varepsilon}(K)\} \subset \{\mathcal{L}_F^\alpha \cap Q(K, R_F) = K\} \subset \{U_K \leqslant p_F^{\alpha-\varepsilon}(K)\}, \text{ for all } K \in \widetilde{\mathcal{A}}_F.$$

Moreover, $d_\varepsilon\big(\mathcal{L}_F, \widetilde{\mathcal{B}}_F; \alpha\big)$ is bounded from above by (6.18).

The proof of Theorem 6.5 occupies the remainder of this section. We begin with two preliminary results. The first of these is a consequence of our main localization result, Theorem 5.1, applied to $\mathcal{L}_F$ in (6.2). In what follows, a family of sets $(\widetilde{\mathcal{L}}_F^{\alpha,(x)})_{\alpha \in (0,2], x \in Q_N}$ is said to be decreasing if $\widetilde{\mathcal{L}}_F^{\alpha,(x)} \subset \widetilde{\mathcal{L}}_F^{\beta,(x)}$ for all $\alpha \geqslant \beta$ and $x \in Q_N$.

Proposition 6.6 (Short-range approximation for $\mathcal{L}_F$). *There exist $c, C, C' \in (0, \infty)$ such that the following holds. For all $N \geqslant 1$, $F \subset Q_N$, $\varepsilon \in (0, 1)$, $R = \left(\frac{\lambda}{\varepsilon}\right)^{\frac{2}{d-2}} R_F$ with $\lambda \geqslant C$, there exists a decreasing family $\widetilde{\mathcal{L}}_F = (\widetilde{\mathcal{L}}_F^{\alpha,(x)})_{\alpha \in (0,2], x \in Q_N}$ such that*

  i) $\widetilde{\mathcal{L}}_F^{\alpha,(x)} \subset \big(Q(x, R) \cap F\big)$ for each $x \in Q_N$ and $\alpha \in (0, 2]$,

  ii) $(\widetilde{\mathcal{L}}_F^{\alpha,(x)})_{\alpha \in (0,2]}$ and $(\widetilde{\mathcal{L}}_F^{\alpha,(y)})_{\alpha \in (0,2], y \in Q(x,3R)^c}$ are independent for each $x \in Q_N$, and

  iii) for each $\alpha \in (0, 2]$, there exists a coupling $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}_\alpha$ of $\mathcal{L}_F^\alpha$ and $\widetilde{\mathcal{L}}_F$ such that for all $\varepsilon \in (0, \alpha)$,

$$(6.20) \qquad \widetilde{\mathbb{Q}}\big(\widetilde{\mathcal{L}}_F^{\alpha+\varepsilon,(x)} \subset \big(\mathcal{L}_F^\alpha \cap Q(x, R)\big) \subset \widetilde{\mathcal{L}}_F^{\alpha-\varepsilon,(x)} \text{ for all } x \in F\big) \geqslant 1 - C' \varepsilon^{-C'} |F|^{-c\lambda}.$$

Proof. We first consider the case $\mathbb{P} = \mathbf{P}$ of the walk. Consider the processes $\ell^{(x)}$ under $\widetilde{\mathbf{P}} = \sum_{x \in \mathbf{T}} \widetilde{\mathbf{P}}_x$, where $\widetilde{\mathbf{P}}_x$ is the translation by x of the probability $\widetilde{\mathbf{P}_0}$ from Theorem 5.1 for $\delta = \frac{1}{2}$, R as above and $u_0 = 3u_F$, where $u_F = u_F(\alpha = 1) = g(0) \log(|F|)$, cf. (6.1). One then defines for each $\alpha \in (0, 4]$ and $x \in Q_N(0)$,

$$\widetilde{\mathcal{L}}_F^{\alpha,(\boldsymbol{x})} = \pi\big(\big\{y \in Q(x, R) : \ell_{y, (2+\frac{\alpha}{2})u_F}^{(x)} - \ell_{y, (2-\frac{\alpha}{2})u_F}^{(x)} = 0\big\}\big) \cap F.$$

From this, i) plainly follows and ii) is a consequence of (5.2) since $\ell^{(x)}$ and hence $\widetilde{\mathcal{L}}_F^{\alpha,(x)}$ is a function of $\omega^{(x)}$ alone, whose law does not depend on x in view of (5.1). Moreover, by (5.3) (applied with $\frac{\varepsilon}{6}$ instead of ε) one obtains, for all $\alpha \in (0, 2]$ $\varepsilon \in (0, \alpha)$, $x \in Q_N$ and $y \in Q(x, R)$

$$\ell_{y, (2+\frac{\alpha-\varepsilon}{2})u_F}^{(x)} - \ell_{y, (2-\frac{\alpha-\varepsilon}{2})u_F}^{(x)} \leqslant \ell_{\boldsymbol{y}, (2+\frac{\alpha}{2})u_F} - \ell_{\boldsymbol{y}, (2-\frac{\alpha}{2})u_F} \leqslant \ell_{y, (2+\frac{\alpha+\varepsilon}{2})u_F}^{(x)} - \ell_{y, (2-\frac{\alpha+\varepsilon}{2})u_F}^{(x)},$$

with probability at least $1 - C|F|^{-c\lambda} \varepsilon^{-C}$ by choice of R, upon taking λ large enough. By definition of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, this yields, for all $\alpha \in (0, 2]$ and $\varepsilon \in (0, \alpha)$, that

$$(6.21) \qquad \widetilde{\mathcal{L}}_F^{\alpha+\varepsilon,(x)} \subset \big((\mathcal{L}_F^{2+\frac{\alpha}{2}} \backslash \mathcal{L}_F^{2-\frac{\alpha}{2}}) \cap Q(x, R)\big) \subset \widetilde{\mathcal{L}}_F^{\alpha-\varepsilon,(x)} \text{ for all } x \in F$$

with probability at least $1 - C|F|^{-c\lambda} \varepsilon^{-C}$, and iii) follows since $\mathcal{L}_F^{2+\frac{\alpha}{2}} \backslash \mathcal{L}_F^{2-\frac{\alpha}{2}}$ has the same law as $\mathcal{L}_F^\alpha$ for each $\alpha > 0$. The proof for $\mathbb{P} = \mathbb{P}^I$ is similar, except that, in view of (5.3') (applied with $\frac{\varepsilon}{\alpha}$ instead of ε) one simply defines $\widetilde{\mathcal{L}}_F^{\alpha,(x)} = \{y \in Q(x, R) : \ell_{y, u_F(\alpha)}^{(x)} = 0\}$ under $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{P}}$. $\square$

Remark 6.7.

1) It follows from Theorem 5.1 and the independence and stationarity of the increments of random interlacements that for each $F \subset Q_N$ and $x \in F$, the process $(\widetilde{\mathcal{L}}_F^{\alpha,(x)})_{\alpha \in (0,2]}$ introduced in Proposition 6.6 has the same law as $(\mathcal{L}_{F \cap Q(x,R)}^{\alpha})_{\alpha \in (0,2]}$ under $\mathbb{P}^I$ when $\mathbb{P} = \mathbb{P}^I$, and the same law as $(\pi(\mathcal{L}_{Q_N(0)}^{\alpha}) \cap F \cap Q(x,R))_{\alpha \in (0,2]}$ under $\mathbb{P}^I$ when $\mathbb{P} = \mathbf{P}$. While noteworthy, we will not need this fact in the sequel.

2) For the case $\mathbb{P} = \mathbb{P}^I$, one can afford to take $R = \left(\frac{\lambda}{\varepsilon}\right)^{\frac{2}{d-2}}$ in Proposition 6.6. This can be traced back to the additional square root present in the bound (5.3) compared to (5.3'). Moreover as can be seen plainly in the above proof, one actually has a coupling $\widetilde{\mathbb{Q}}$ uniform in all $\alpha \in (0,2]$ in that case, rather than one depending on α as in the random walk case. The reason the coupling $\widetilde{\mathbb{Q}}_\alpha$ depends on α when $\mathbb{P} = \mathbf{P}$ is that the sets $(\mathcal{L}_F^{2+\frac{\alpha}{2}} \setminus \mathcal{L}_F^{2-\frac{\alpha}{2}})_{\alpha \in (0,2)}$ does not have the same law (as a process in α) as $(\mathcal{L}_F^\alpha)_{\alpha \in (0,2)}$ under $\mathbf{P}$ (only its one-dimensional α-marginals do) since the corresponding random walks have different starting points for different values of α. We refer to Remark 5.7,2) for a variant of the approximation supplied by Theorem 5.1 by which this increment problem can be partially circumvented.

Next, returning to $\widetilde{\mathcal{A}}_F$ in (6.16), we prove a separation property for $\mathcal{L}_F^\alpha$. Namely, for suitably large R, the set $\mathcal{L}_F^\alpha \cap Q(0,R)$ either belongs to $\widetilde{\mathcal{A}}_F$ or is empty with high probability.

Lemma 6.8. *For all $N \geqslant 1$, $F \subset Q_N$, $\alpha \in (\frac{1}{2}, 2]$ and $R \in [R_F, \frac{N}{4})$,*

$$\mathbb{P}\big(\exists\, x \in F : \big(\mathcal{L}_F^\alpha \cap Q(x,R)\big) \notin (\widetilde{\mathcal{A}}_F \cup \{\varnothing\})\big) \leqslant CR^d \log(|F|)^C |F|^{1-2\alpha}.$$

Proof. By (6.6) one has, for all $r \geqslant 1$,

$$\mathbb{E}\big[|\{x, y \in \mathcal{L}_F^\alpha : d(x,y) \in [r,R]\}|\big] \leqslant |F|(2R+1)^d \sup_{d(x,y) \geqslant r} |F|^{-\alpha g(0)\mathrm{cap}(\{x,y\})}.$$

In particular, by (2.9) and since $\mathrm{cap}(\{x\}) = g(0)^{-1}$ for all $x \in F$ (see (2.6)), one obtains with the choice $r = R_F$ by Markov's inequality that

$$(6.22) \qquad \mathbb{P}\big(\exists\, x, y \in \mathcal{L}_F^\alpha : R \geqslant d(x,y) \geqslant R_F\big) \leqslant CR^d |F|^{1-2\alpha}.$$

The bound (6.22) takes care of the contribution to the relevant event appearing in Lemma 6.8 when violating the diameter constraint inherent to $\widetilde{\mathcal{A}}_F$. It remains to address the possibility to violate the capacity constraint in (6.16). To this effect, observe that by (2.10), (6.6) and a union bound we have

$$\mathbb{P}\big(\mathrm{cap}\big(\mathcal{L}_F^\alpha \cap Q_R\big) \geqslant \tfrac{2}{g(0)}, \delta\big(\mathcal{L}_F^\alpha \cap Q_R\big) \leqslant R_F\big)$$
$$\leqslant \mathbb{P}\big(\exists K \subset (\mathcal{L}_F^\alpha \cap Q_R) : |K| \leqslant C, \alpha_*(K) \leqslant \tfrac{1}{2}, \delta(K) \leqslant R_F\big) \leqslant CR^d \log(|F|)^{C'} |F|^{-2\alpha}.$$

Combining this with a union bound over $x \in F$ and (6.22), the claim follows. $\qquad\square$

Remark 6.9. We now briefly comment on the choice of $\widetilde{\mathcal{A}}_F$ in (6.16). In view of Lemma 6.8, it was selected so that $\mathcal{L}_F^\alpha \cap Q(x,R)$ belongs to $\widetilde{\mathcal{A}}_F$ or is empty, with high probability for all $x \in F$ when $R \geqslant R_F$ – of course, the bound obtained in Lemma 6.8 also implicitly entails an upper bound $R \leqslant |F|^{(2\alpha-1)/d} \log(|F|)^{-C}$, above which the estimate is useless. Moreover, as implicitly used in the proof, the definition of $\widetilde{\mathcal{A}}_F$ also ensures that for all $x \in F$ and $R \in [R_F, \frac{N}{4})$,

$$(6.23) \qquad \Big|\big\{K \subset Q(x,R) : K \in \widetilde{\mathcal{A}}_F\big\}\Big| \leqslant C|Q(x,R) \cap F| \log(|F|)^{C'},$$

which is an easy consequence of (2.10) and the bounds on both capacity and diameter in (6.16).

Combining Proposition 6.6 and Lemma 6.8, we are now ready to give the

Proof of Theorem 6.5. All subsequent considerations implicitly hold for all $N \geqslant 1$ and $F \subset Q_N$. We may assume that $\varepsilon > |F|^{-C}$ for suitably large C, for otherwise (6.18) is larger than 1, and also that ε is small enough so that $\varepsilon^{-\frac{2}{d-2}} > 2$ and $(\frac{1}{2}, 1] \pm 3\varepsilon \in (\frac{1}{4}, 2]$, which can be arranged without loss of generality by monotonicity of $\varepsilon \mapsto d_\varepsilon$, cf. (1.21).

We now apply Proposition 6.6 (for N, F, ε as appearing in Theorem 6.5) with $R = \left(\frac{\lambda}{\varepsilon}\right)^{\frac{2}{d-2}} R_F$ and a choice of $\lambda \geqslant 1$ large enough such that the right-hand side of (6.20) is at least $1 - C|F|^{-3}$ uniformly in $\alpha \in (0, 2]$. In view of the above lower bound on ε, λ can be chosen uniformly in N, F and ε. Thus, defining for $K \in \widetilde{\mathcal{A}}_F$ and arbitrary (fixed) $x_K \in K$ the random fields

$$Y_K^\alpha = 1\{\mathcal{L}_F^\alpha \cap Q(K, R_F) = K\}$$
$$Z_K^\alpha = 1\{\widetilde{\mathcal{L}}_F^{\alpha,(x_K)} \cap Q(K, R_F) = K\},$$

with $\widetilde{\mathcal{L}}$ as supplied by Proposition 6.6, (6.20) implies that

$$(6.24) \qquad d_\varepsilon(Y, Z; \alpha) \leqslant C|F|^{-3} \text{ for all } \alpha \in I \overset{\text{def.}}{=} (\tfrac{1}{4}, 2],$$

and all N, F and ε as above.

We now aim to apply Lemma 3.1 for $S = \widetilde{\mathcal{A}}_F$ and I as above. First observe that $Y = (Y^\alpha)_{\alpha \in I}$ is indeed a family of decreasing Bernoulli processes, as required by (3.2). Thus (3.3) defines a process $W = (W^\alpha)_{\alpha \in I}$ with $W^\alpha = (W_K^\alpha)_{K \in \widetilde{\mathcal{A}}_F}$, which has the same law as $(1_{\{U_K \leqslant p_F^\alpha(K)\}})_{K \in \widetilde{\mathcal{A}}_F, \alpha \in I}$. In particular, recalling (6.17), this means that

$$(6.25) \qquad \widetilde{\mathcal{B}}_F^\alpha \overset{\text{law}}{=} \bigcup_{K \in \widetilde{\mathcal{A}}_F : W_K^\alpha = 1} K.$$

Our aim is to control $d_{2\varepsilon}(Y, W; \alpha)$ for $\alpha \in (\frac{1}{2}, 1]$ by means of (3.5). With regards to the relevant condition (3.4), Proposition 6.6 implies that $Z = (Z^\alpha)_{\alpha \in I}$ has the requested monotonicity and the finite-range property follows from item *ii)* of the same proposition upon choosing the neighborhood $\mathcal{N}_K$ of $K \in \widetilde{\mathcal{A}}_F$ as $\mathcal{N}_K = \{K' \in \widetilde{\mathcal{A}}_F : K' \cap Q(K, 3R) \neq \varnothing\}$. Indeed, for all $K' \notin \mathcal{N}_K$ we have that $x_{K'} \notin Q(x_K, 3R)$, and hence $(\widetilde{\mathcal{L}}_F^{\alpha,(x_{K'})})_{\alpha \in (0,2], K' \notin \mathcal{N}_K}$ is independent of $(\widetilde{\mathcal{L}}_F^{\alpha,(x_K)})_{\alpha \in (0,2]}$, and Z inherits this property. Thus, Lemma 3.1 is in force and the desired bound hinges on suitably estimating b_1 and b_2 in (3.6)–(3.7). Combining (6.6) and (6.23) gives

$$(6.26) \qquad b_1(\alpha') \leqslant \frac{C|S|}{|F|^{2\alpha'}} \sup_{K \in \widetilde{\mathcal{A}}_F} |\mathcal{N}_K| \leqslant \frac{C \log(|F|)^{C'}}{|F|^{2(\alpha - 2\varepsilon) - 1} \varepsilon^{\frac{2d}{d-2}}}, \text{ for all } \alpha \in (\tfrac{1}{2}, 1] \text{ and } \alpha' \geqslant \alpha - 2\varepsilon.$$

Moreover for all $K, K' \in \widetilde{\mathcal{A}}_F$ with $K \neq K'$ and such that $Y_K^{\alpha'} = Y_{K'}^{\alpha'} = 1$ occurs with positive probability, we must have $d(K', K) \geqslant R_F/2$ by definition of $Y.$, and thus using (2.9) and (6.6), one readily shows that $b_2(\alpha')$ verifies the same bound as $b_1(\alpha')$ in (6.26) over the given range of parameters α'. Therefore combining (3.5) with (6.23), (6.24) and (6.26), one obtains that

$$d_{2\varepsilon}(Y, W; \alpha) \leqslant \frac{C \log(|F|)^{C'}}{|F|^{2(\alpha - 2\varepsilon) - 1} \varepsilon^{\frac{2d}{d-2}}} + \frac{C|F|^2 \log(|F|)^{C'}}{|F|^3} \leqslant \frac{C'' \log(|F|)^{C'}}{|F|^{2(\alpha - 2\varepsilon) - 1} \varepsilon^{\frac{2d}{d-2}}},$$

for all $\alpha \in (\frac{1}{2}, 1]$. It readily follows from the definition of d_ε, see (3.1) (and substituting 2ε by ε), that a coupling exists such that (6.19) occurs except on an event with probability bounded

by the expression in (6.18). Moreover by Lemma 6.8, since all sets in $\tilde{\mathcal{A}}_F$ have diameter at most R_F, one has

$$(6.27) \quad \mathbb{P}\Big(\mathcal{L}_F^\alpha \neq \bigcup_{K \in \tilde{\mathcal{A}}_F : Y_K^\alpha = 1} K\Big) \leqslant \mathbb{P}\big(\exists x \in F : \mathcal{L}_F^\alpha \cap Q(x, 2R_F) \notin \tilde{\mathcal{A}}_F \cup \{\varnothing\}\big) \leqslant \frac{C \log(|F|)^{C'}}{|F|^{2\alpha-1}}.$$

To conclude, one simply notes comparing (6.25) and (6.27) that $d_\varepsilon\big(\mathcal{L}_F, \tilde{\mathcal{B}}_F; \alpha\big)$ is bounded by the sum of $d_\varepsilon(Y, W; \alpha)$ and the probability on the left-hand side of (6.27), which is smaller than (6.18) up to increasing the constants C, C'. $\qquad\square$

7 Denouement

Using the findings of §3 and §6, with the latter drawing heavily from Theorem 5.1, we now prove our main results for the set of late points, Theorems 1.1 and 1.3. Recall $\alpha_*(K)$ from (1.19), the measure $\mathbb{P}$ introduced atop §6.1, and abbreviate $\tilde{\mathcal{A}}_N = \tilde{\mathcal{A}}_{Q_N}$, see (6.16), as well as $\tilde{\mathcal{B}}^\alpha = \tilde{\mathcal{B}}_{Q_N}^\alpha$, see (6.17). For any set $\mathcal{S} \subset Q_N$ and $K \subset\subset \mathbb{Z}^d$ we introduce, with $R_N = R_{Q_N} = \log(N^d)^{\frac{1}{d-2}}$ (see (6.16)), the event

$$(7.1) \qquad\qquad E_K(\mathcal{S}) = \big\{\exists\, K' \subset \mathcal{S} : K' \in \mathcal{A}_N \text{ and } \alpha_*(K') \leqslant \alpha_*(K)\big\}$$

corresponding to the existence of 'admissible' sets K' in $\mathcal{S}$ with capacity larger than the capacity of K. Note that $E_K(\mathcal{S})$ depends implicitly on N via the choice of $\mathcal{S} \subset Q_N$. The constant $\alpha_*(K)$ is chosen so that the following result holds.

Lemma 7.1. *For all $\eta \in (0,1)$, all sequences (α_N) with $\alpha_N \in (\frac{1}{2} + \eta, 2]$ for all $N \geqslant 1$, all $K \subset\subset \mathbb{Z}^d$ with $\alpha_*(K) > \frac{1}{2}$, abbreviating $E_K^\alpha = E_K(\tilde{\mathcal{B}}^\alpha)$ one has*

$$(7.2) \qquad\qquad \lim_{N\to\infty} \mathbb{P}\big(E_K^{\alpha_N}\big) = \begin{cases} = 0 & \text{if } \lim_N \big(1 - \frac{\alpha_N}{\alpha_*(K)}\big) \log N = -\infty, \\ = 1 & \text{if } \lim_N \big(1 - \frac{\alpha_N}{\alpha_*(K)}\big) \log N = \infty, \end{cases}$$

and

$$(7.3) \qquad 0 < \liminf_{N\to\infty} \mathbb{P}\big(E_K^{\alpha_N}\big) \leqslant \limsup_{N\to\infty} \mathbb{P}\big(E_K^{\alpha_N}\big) < 1, \text{ if } \lim_N \big[\big(1 - \tfrac{\alpha_N}{\alpha_*(K)}\big) \log N\big] \in (-\infty, \infty).$$

We refer to Remark 7.2 below for a refinement of (7.3).

Proof. Recall that the set $\tilde{\mathcal{B}}^\alpha$ is defined entirely in terms of the family $(U_{K'})_{K' \subset Q_N}$ of i.i.d. uniform random variables from above (6.16). The set K' with the properties postulated by E_K^α may arise in $\tilde{\mathcal{B}}^\alpha$ for two reasons: either because the uniform variable $U_{K'}$ was triggered, i.e. it is at most $p_{Q_N}^\alpha(K')$, or because the uniform random variables corresponding to at least two disjoint subsets whose union is included in K' were triggered. Accordingly, let $\mathcal{U}_{K'}^\alpha = \{U_{K'} \leqslant p_{Q_N}^\alpha(K')\}$ and consider the events

$$(7.4) \qquad F^\alpha = \big\{\exists\, K_1, K_2 \in \tilde{\mathcal{A}}_N : K_1 \neq K_2, \delta(K_1 \cup K_2) \leqslant R_N \text{ and } \mathcal{U}_{K_1}^\alpha \cap \mathcal{U}_{K_2}^\alpha \text{ occurs}\big\},$$

$$G_K^\alpha = \big\{\exists\, K' \in \tilde{\mathcal{A}}_N : K' \subset Q_N, \alpha_*(K') \leqslant \alpha_*(K) \text{ and } \mathcal{U}_{K'}^\alpha \text{ occurs}\big\}.$$

With these definitions, one has

$$(7.5) \qquad\qquad\qquad \mathbb{P}(G_K^\alpha) \leqslant \mathbb{P}(E_K^\alpha) \leqslant \mathbb{P}(F^\alpha) + \mathbb{P}(G_K^\alpha),$$

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for all $\alpha > 0$, $N \geqslant 1$. By a union bound and using (6.6) for singletons and (6.23), one gets

(7.6) $$\mathbb{P}(F^{\alpha_N}) \leqslant CN^d \log(N^d)^C \cdot N^{-2\alpha_N d} \to 0 \text{ as } N \to \infty,$$

since $\alpha_N > 1/2 + \eta$ for all N. In view of (7.5) and (7.6), it is enough to find the limit of $\mathbb{P}(G_K^{\alpha_N})$ as $N \to \infty$. To this end, first note that $\mathrm{cap}(\{x, y\}) \nearrow 2\mathrm{cap}(\{0\}) = 2/g(0)$ as $d(x, y) \to \infty$ by (2.6), and so for each $\alpha' \in (\frac{1}{2} + \eta, \alpha_*(K)]$ and $x \in Q_N$, the number of sets $K' \subset Q_N$ containing x such that $\alpha_*(K') \geqslant \alpha'$ is bounded uniformly in N and α'. In particular, it follows that $|\{K' \subset Q_N : \alpha_*(K') = \alpha_*(K)\}| \leqslant CN^d$ and that there exists $\delta = \delta(K) > 0$ such that $\alpha_*(K') \leqslant \alpha_*(K) - \delta$ for each $K' \subset Q_N$ with $\alpha_*(K') < \alpha_*(K)$. By (6.6) and (6.23) it thus follows that

(7.7)
$$\mathbb{P}\big((G_K^\alpha)^c\big) = \prod_{\substack{K' \subset Q_N : K' \in \tilde{\mathcal{A}}_N \\ \alpha_*(K') \leqslant \alpha_*(K)}} \big(1 - \mathbb{P}(\mathcal{L}^\alpha \cap Q(K', R_N) = K')\big)$$
$$\geqslant \big(1 - CN^{-\frac{d\alpha}{\alpha_*(K)}}\big)^{CN^d} \big(1 - CN^{-\frac{d\alpha}{\alpha_*(K)-\delta}}\big)^{CN^d \log(N)^C}.$$

We now derive an upper bound on $\mathbb{P}((G_K^\alpha)^c)$. To this end, we first introduce a set $\overline{K} \supseteq K$, or $\overline{K} \supset \pi(K)$ if $Q_N = Q_N(\mathbf{0})$, as follows. First, applying (2.8) one finds $r = r(K) < \infty$ such that, whenever $x \notin Q(K, r)$, one has $\mathrm{cap}(K \cup \{x\}) > \mathrm{cap}(K)$. Then, considering all sets of the form $K \cup U$ for $U \subset Q(K, r)$ one finds for N large enough a set $\overline{K} \subset Q_N$ having this form (or a projection on the torus of a set having this form when $Q_N = Q_N(\mathbf{0})$) and such that both $\mathrm{cap}(\overline{K}) = \mathrm{cap}(K)$ and $\mathrm{cap}(K') > \mathrm{cap}(K)$ for all $K' \subset Q_N$ with $\overline{K} \subsetneq K'$. It follows from this construction that

(7.8) $$\inf_{x \in \overline{K}^c} \mathrm{cap}(\overline{K} \cup \{x\}) \geqslant \mathrm{cap}(K) + \delta',$$

for some $\delta' = \delta'(K) > 0$. The desired upper bound on $\mathbb{P}\big((G_K^\alpha)^c\big)$ will follow from a lower bound on the probability of the event $\{\mathcal{L}^\alpha \cap Q(\overline{K}, R_N) = \overline{K}\}$. Combining the upper bound (6.6) and (7.8), one obtains that for all $\alpha \in (0, 2]$,

$$\mathbb{P}\big(\overline{K} \subset \mathcal{L}^\alpha\big) - \mathbb{P}\big(\mathcal{L}^\alpha \cap Q(\overline{K}, R_N) = \overline{K}\big)$$
$$\leqslant \mathbb{P}\big(\exists\, x \in Q(\overline{K}, R_N) \backslash \overline{K} : \overline{K} \cup \{x\} \subset \mathcal{L}^\alpha\big) \leqslant C \log(N)^{C'} N^{-\alpha dg(0)(\mathrm{cap}(K)+\delta')}.$$

If N is large enough, one deduces from this and the lower bound (6.7) that there exists $c = c(K)$ such that for all $\alpha \in (0, 2]$,

(7.9) $$\mathbb{P}\big(\mathcal{L}^\alpha \cap Q(\overline{K}, R_N) = \overline{K}\big) \geqslant cN^{-\frac{\alpha d}{\alpha_*(K)}}.$$

Now observe that the set $K' = \overline{K} + x$ satisfies $\alpha_*(K') = \alpha_*(K)$ by construction, see above (7.8). Thus, by (7.9) and translation invariance we have that for all $\alpha \in (0, 2]$,

(7.10)
$$\mathbb{P}\big((G_K^\alpha)^c\big) \leqslant \prod_{x \in Q_N : \overline{K}+x \subset Q_N} \big(1 - \mathbb{P}(\mathcal{L}^\alpha \cap Q(\overline{K}+x, R_N) = \overline{K}+x)\big) \leqslant \big(1 - cN^{-\frac{d\alpha}{\alpha_*(K)}}\big)^{cN^d}.$$

Combining (7.7) and (7.10) with (7.6) and (7.5) for $\alpha = \alpha_N$ readily yields (7.2)–(7.3). $\qquad \square$

Remark 7.2. We now explain how to refine (7.3), which is of interest for the purposes of obtaining the exact constant e^{-d} in (1.13). For simplicity, we focus on the case $K = K_0 = \{x, y\}$ for some $x \sim y$, whence $\alpha_*(K_0) = \alpha_*$ in view of (6.11). The key is to observe that

(7.11) $$\{K' \subset Q_N : \alpha_*(K') = \alpha_*\} = \{\{z, z'\} \subset Q_N : z' \sim z\};$$

indeed, recalling $\alpha_*(\cdot)$ from (1.19), it follows immediately with the help of (2.6) that $\alpha_*(\{z\}) = 1$, which is larger than α_* on account of Lemma 6.3, so the set in question in (6.3) does not contain singletons. If K' is not a pair of neighbors then K' contains at least two points at ℓ^1-distance $\geqslant 2$ and it follows that $\alpha_*(K') < \alpha_*$ using (2.7). From (7.11), one deduces in turn that

$$(7.12) \qquad \big|\{K' \subset Q_N : \alpha_*(K') = \alpha_*\}\big| \sim d|Q_N|, \text{ as } N \to \infty.$$

Note also that as explained in Remark 6.2,2), the constants $C(\beta_0)$ and $c(\beta_0)$ in (6.6) and (6.7) for $F = Q_N$ and $\beta_0 = 2$ can be respectively replaced by $1 + o(1)$ and $1 - o(1)$ as $N \to \infty$, and thus the constant c in (7.9) can also be replaced by $1 - o(1)$. Now, inspecting the above proof, substituting a suitable upper bound implied by (7.12) into (7.7) and a corresponding lower bound into (7.10), noting that $\overline{K_0} = K_0$ for the same reasons as those yielding (7.11), one obtains

$$(7.13) \qquad \lim_{N \to \infty} \mathbb{P}\big(E_{K_0}^{\alpha_N}\big) = 1 - e^{-\gamma d},$$

if $d \cdot \lim_N \big[\big(1 - \frac{\alpha_N}{\alpha_*}\big) \log N\big] = \log \gamma$ for some $\gamma > 0$.

We now turn to the proof of our main results. These are formulated entirely within the framework of (6.2), which subsumes the setup of the introduction (corresponding to $F = Q_N$ for the choice $\mathbb{P} = \mathbf{P}$), thus lending themselves to immediate generalizations, notably to the case of random interlacements, i.e. the choice $\mathbb{P} = \mathbb{P}^I$, see Remark 7.3,1). With this in mind, we extend the definitions of the sets $\mathcal{B}$ and $\mathcal{B}_K$ from above (1.7) and (1.20) to being subsets of Q_N, rather than just subsets of $\mathbf{T} = Q_N(\mathbf{0})$ as in §1 in accordance with the notation introduced above (6.1), this amounts to simply replacing $\mathbf{T}$ by Q_N in their definition. Combining Theorem 6.5 and Lemma 7.1, one now readily obtains the following.

Proof of Theorem 1.3. Let $\varepsilon_N = \varepsilon_N(\alpha) = N^{-\frac{1}{4}(d-2)(2\alpha-1)}$ for all $N \geqslant 1$. It follows from Theorem 6.5 applied with $F = Q_N$ that for each $\alpha \in (\frac{1}{2}, 1]$ and $N \geqslant 1$, with $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_{Q_N}$ and $\mathcal{L} = \mathcal{L}_{Q_N}$,

$$(7.14) \qquad d_{\varepsilon_N}\big(\mathcal{L}, \widetilde{\mathcal{B}}; \alpha\big) \leqslant C \log(N)^{C'} N^{-d(2\alpha-1)/2 + 2\varepsilon_N d} \xrightarrow[N \to \infty]{} 0.$$

Fix a set $K \subset Q_N$ with $\alpha_*(K) > \frac{1}{2}$. Theorem 1.3 deals with $\mathcal{B}_K$ introduced in (1.20) rather than $\widetilde{\mathcal{B}}$ as defined below (6.17). We proceed to compare the two sets using Lemmas 6.8 and 7.1. Indeed recalling the event $E_K^\alpha = E_K(\widetilde{\mathcal{B}}^\alpha)$ from (7.1), one has for N large enough that $\{\widetilde{\mathcal{B}}^\alpha \neq \mathcal{B}_K^\alpha\} \subset E_K^\alpha$. We used here that the inclusion $\mathcal{B}_K^\alpha \subset \widetilde{\mathcal{B}}^\alpha$ is always satisfied for $N \geqslant N(K)$ large enough, since the condition $\delta(A) \leqslant R_N$ in (6.16) holds for all $A \subset \mathbf{T}$ with $\alpha_*(A) > \alpha_*(K)(> 1/2)$ when N is large enough by (1.19), (2.6) and (2.7).

Now, first assume that $\alpha \in (\alpha_*(K), 1]$. By (7.2) applied to the sequence $\alpha_N = \alpha \pm \varepsilon_N$, which satisfies $\lim_N \big(1 - \frac{\alpha_N}{\alpha_*(K)}\big) \log N = -\infty$, one deduces that $\lim_N \mathbb{P}(\widetilde{\mathcal{B}}^{\alpha \pm \varepsilon_N} \neq \mathcal{B}_K^{\alpha \pm \varepsilon_N}) = 0$. This implies in turn with (7.14) that $d_{\varepsilon_N}\big(\mathcal{L}, \mathcal{B}_K; \alpha\big) \to 0$ as $N \to \infty$, from which the first line in (1.22) directly follows, using that $\varepsilon \mapsto d_\varepsilon$ is decreasing. A similar reasoning using the last bound in (7.3) instead, with the choice $\alpha_N = \alpha_*(K) \pm \varepsilon_N$, yields the last bound in (1.23).

Let us now assume that $\alpha \leqslant \alpha_*(K)$. To obtain the second line in (1.22) as well as the first bound in (1.23), it suffices to show that for $\varepsilon < \alpha(1 - \frac{1}{2\alpha_*(K)})$ and for any coupling $\mathbb{Q}$ between $\mathcal{L}^\alpha$ and $\mathcal{B}_K^{\alpha-\varepsilon}$ one has

$$(7.15) \qquad \liminf_{N \to \infty} \mathbb{Q}(\mathcal{L}^\alpha \nsubseteq \mathcal{B}_K^{\alpha-\varepsilon}) \begin{cases} = 1 & \text{if } \alpha < \alpha_*(K), \\ > 0 & \text{if } \alpha = \alpha_*(K). \end{cases}$$

Recall $D_K^\alpha \equiv D_K(\mathcal{L}^\alpha)$ and $D_K(\mathcal{B}_K^\alpha)$ from (6.9); that is, $D_K^\alpha(\mathcal{B}_K^\alpha)$ counts the number of times a translated version of K (or its projection on the torus) appears in $\mathcal{B}_K^\alpha$. Now fix ρ such that

$1 - 2(\alpha - \varepsilon) < \rho < 1 - \frac{\alpha}{\alpha_*(K)}$ (the midpoint for instance), which exists by our assumption on ε. Then clearly since $\mathcal{B}_K$ is an independent field one has by a similar reasoning as in (7.6) the bound $\mathbb{E}_\mathbb{Q}\big[D_K(\mathcal{B}_K^{\alpha-\varepsilon})\big] \leqslant C \log(N)^{C'} N^{d(1-2(\alpha-\varepsilon))}$. Hence by Markov's inequality one immediately infers that $\mathbb{Q}\big(D_K(\mathcal{B}_K^{\alpha-\varepsilon}) \geqslant N^{d\rho}\big) \to 0$ as $N \to \infty$. On the other hand, applying Lemma 6.4 with $\varepsilon_N = N^{d(\rho-1+\frac{\alpha}{\alpha_*(K)})}$, one finds that $\liminf_N \mathbb{Q}\big(D_K(\mathcal{L}^\alpha) \geqslant N^{d\rho}\big)$ is equal to 1 if $\alpha < \alpha_*(K)$, and is positive if $\alpha = \alpha_*(K)$. Together, these imply (7.15).

Let us finally assume that $\alpha_*(K) \leqslant 1/2$. For each $x, y \in Q_N$, it follows from (2.6) that $\alpha_*(\{x,y\}) > 1/2 \geqslant \alpha_*(K)$ and so by (1.20) $\mathcal{B}_K^\alpha$ stochastically dominates $\{x \in Q_N : \exists y \in Q_N, U_{\{x,y\}} \leqslant p^\alpha(\{x,y\})\}$. Let us define independent Bernoulli random variables $X_{x,y}$, $x, y \in Q_N$, with parameter $p^\alpha(\{x,y\})/2$ so that $X_{x,y} = X_{y,x} = 0$ if $U_{\{x,y\}} > p^\alpha(\{x,y\})$. If $Y_x = 1\{\exists y \in Q_N : X_{x,y} = 1\}$, then $(Y_x)_{x \in \mathbb{Z}^d}$ are i.i.d. Bernoulli random variables and $Y_x = 1$ implies $x \in \mathcal{B}_K^\alpha$. Moreover by (7.9)

$$(7.16) \quad \mathbb{P}(x \in \mathcal{B}_K^\alpha) \geqslant \mathbb{P}(Y_x = 1) \geqslant 1 - \prod_{y \in Q_N} \mathbb{P}(X_{x,y} = 0) \geqslant 1 - (1 - cN^{-2\alpha d})^{N^d} \geqslant c'(N^{d-2\alpha d} \wedge 1).$$

where in the last inequality we used $1 - cN^{-2\alpha d} \leqslant e^{-cN^{-2\alpha d}}$ and $e^{-x} \leqslant 1 - e^{-1}(x \wedge 1)$ for all $x \geqslant 0$. Therefore $|\mathcal{B}_K^\alpha|$ stochastically dominates a binomial random variable with parameters $(N^d, c'(N^{d-2\alpha d} \wedge 1))$, and is thus larger than $c''(N^{2d(1-\alpha)} \wedge N^d)$ with probability going to 1 as $N \to \infty$ by Chebyshev's inequality. Moreover $\mathbb{E}[|\mathcal{L}^\alpha|] \leqslant CN^{d(1-\alpha)}$ by (6.6), and so by Markov's inequality we have $|\mathcal{L}^\alpha| \leqslant N^{d-\alpha d} \log N$ with probability going to 1 as N goes to infinity, and we can conclude since $\alpha < 1$. $\qquad\square$

We postpone further comments for a few lines and first give the:

Proof of Theorem 1.1. Fix two neighbors $x \sim y$, $x, y \in \mathbb{Z}^d$, and let $K_0 = \{x,y\}$. As we now explain, *i)* is in essence an application of Theorem 1.3 for this choice of $K = K_0$. First recall that $\alpha_* = \alpha_*(K_0)$ by Lemma 6.11, and that the only sets K' with $\alpha_*(K') > \alpha_*(K_0)$ are singletons on account of (1.19) and (2.7). Therefore, the set $\mathcal{B}_{K_0}^\alpha$ in (1.20) only has contributions from sets A which are singletons, and recalling $\mathcal{B}^\alpha$ from the beginning of §1.1 which we can define via the same uniform random variables $(U_{\{x\}})_{x \in Q_N}$ as $\mathcal{B}_{K_0}^\alpha$ in (1.20) (that we also define under the probability measure $\mathbb{P}$), it follows that for all $\alpha > \alpha_*$,

$$(7.17) \quad \mathbb{P}\big(\mathcal{B}_{K_0}^\alpha \neq \mathcal{B}^\alpha\big) \leqslant \sum_{z \in Q_N} \mathbb{P}\big(z \in \mathcal{L}^\alpha, Q(z, R_{Q_N}) \cap \mathcal{L}^\alpha \neq \{z\}\big) \leqslant CN^{d(1-\frac{\alpha}{\alpha_*})},$$

where the last inequality relies on (6.6). Combining this with the first line of (1.22) immediately yields item *i)* of Theorem 1.1 for all $\alpha \in (\alpha_*, 1)$ and $\varepsilon \in (0, \varepsilon_0)$, for ε_0 small enough. When $\alpha = 1$, the first line of (1.22) is still valid, as should be clear from the proof of Theorem 1.3, and so item *i)* of Theorem 1.1 is also fulfilled when $\alpha = 1$, as well as when $\alpha > 1$ as both $\mathcal{B}^\alpha$ and $\mathcal{L}^\alpha$ are then empty with high probability by (6.5). Note that we can actually take $\varepsilon_0(\alpha) = \alpha$ for all $\alpha > \alpha_*$ by monotonicity of $\varepsilon \mapsto d_\varepsilon$.

We now turn to the proof of *ii)*. Let $\widetilde{E}(\mathcal{L}^{\alpha_*})$ refer to the event appearing in the statement of Lemma 6.8 for the choices $F = Q_N$, $R = R_N (= R_{Q_N})$ and $\alpha = \alpha_*$. Thus, if $\widetilde{E}(\mathcal{L}^{\alpha_*})$ does not occur, for any $x \in Q_N$ one has that $\mathcal{L}^{\alpha_*} \cap Q(x, R_N)$ is either empty or an element of $\widetilde{\mathcal{A}}_N$. If in addition $E_{K_0}(\mathcal{L}^{\alpha_*})$ does not occur, then assuming that $\mathcal{L}^{\alpha_*} \cap Q(x, R_N) = K$ for some $x \in Q_N$ and $K \neq \varnothing$, one has that $\alpha_*(K) > \alpha_*(K_0)$ by (7.1), and so $|K| = 1$. In particular, applying this to $x \in \mathcal{L}^{\alpha_*}$, one obtains that on the complement of $\widetilde{E}(\mathcal{L}^{\alpha_*}) \cup E_{K_0}(\mathcal{L}^{\alpha_*})$, the set $\mathcal{L}^{\alpha_*}$ is the union of all the $x \in Q_N$ such that $\mathcal{L}^{\alpha_*} \cap Q(x, R_N) = \{x\}$. Therefore if $\widetilde{E}(\mathcal{L}^{\alpha_*}) \cup E_{K_0}(\mathcal{L}^{\alpha_*})$ does not occur and (6.19) at $\alpha = \alpha_*$ is verified for all $K \subset Q_N$ with $|K| = 1$, it follows that

$\mathcal{B}_{K_0}^{\alpha_*+\varepsilon} \subset \mathcal{L}^{\alpha_*} \subset \mathcal{B}_{K_0}^{\alpha_*-\varepsilon}$. Combining now Theorem 6.5 with the trivial inclusion $\mathcal{B}_{K_0}^{\alpha_*-\varepsilon} \subset \mathcal{B}^{\alpha_*-\varepsilon}$ and (7.17) at $\alpha = \alpha_* + \varepsilon$, and using Lemma 6.8 to bound $\mathbb{P}(\widetilde{E}(\mathcal{L}^{\alpha_*}))$, one obtains that

$$(7.18) \qquad d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha_*) \leqslant \mathbb{P}\big(E_{K_0}(\mathcal{L}^{\alpha_*})\big) + CN^{-\frac{d\varepsilon}{\alpha_*}} + C\log(N)^{C'} N^{d-2(\alpha_*-\varepsilon)d}.$$

for some constant $C, C' < \infty$ depending only on ε and d. An asymptotically matching lower bound is

$$(7.19) \qquad d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha_*) \geqslant \mathbb{P}\big(E_{K_0}(\mathcal{L}^{\alpha_*})\big) - \mathbb{P}\big(E_{K_0}(\mathcal{B}^{\alpha_*-\varepsilon})\big),$$

which simply follows from the fact that $\mathcal{L}^{\alpha_*} \subset \mathcal{B}^{\alpha_*-\varepsilon}$ cannot occur when $\mathcal{L}^{\alpha_*}$ contains sets $K \in \widetilde{\mathcal{A}}_N$ and $\alpha_*(K) \leqslant \alpha_*(K_0)$ but not $\mathcal{B}^{\alpha_*-\varepsilon}$. Moreover the event $E_{K_0}(\mathcal{B}^{\alpha_*-\varepsilon})$ is included in the event $F^{\alpha_*-\varepsilon}$ from (7.4) by definition, and so by (7.6) we have for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0(\alpha_*) = \alpha_* - \frac{1}{2}$,

$$(7.20) \qquad \mathbb{P}\big(E_{K_0}(\mathcal{B}^{\alpha_*-\varepsilon})\big) \leqslant C\log(N)^{C'} N^{d-2(\alpha_*-\varepsilon)d} \xrightarrow[N\to\infty]{} 0.$$

Combining this with (7.18) and (7.19), we deduce that for each $\varepsilon \in (0, \varepsilon_0)$, $d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha_*)$ is asymptotically equivalent to $\mathbb{P}\big(E_{K_0}(\mathcal{L}^{\alpha_*})\big)$ as $N \to \infty$. Using (7.13), and combining with (7.14) for $\alpha = \alpha_*$ (recall that $\varepsilon_N = N^{-\frac{1}{4}(d-2)(2\alpha_*-1)}$ therein and so $\varepsilon_N \log(N) \to 0$), we can conclude since

$$(7.21) \qquad \lim_{N\to\infty} \mathbb{P}\big(E_{K_0}(\mathcal{L}^{\alpha_*})\big) = \lim_{N\to\infty} \mathbb{P}\big(E_{K_0}(\widetilde{\mathcal{B}}^{\alpha_*\pm\varepsilon_N})\big) = 1 - e^{-d}.$$

Finally, *iii)* can be shown in exactly the same way as the case $\alpha < \alpha_*(K)$ in the proof of Theorem 1.3, replacing throughout the proof $\alpha_*(K)$ by α_*, $\mathcal{B}_K$ by $\mathcal{B}$, D_K by D, see (1.9), and K by K_0, upon taking $\varepsilon_0(\alpha) = \alpha(1 - \frac{1}{2\alpha_*})$ for $\alpha < \alpha_*$ in Theorem 1.1. $\qquad \square$

Remark 7.3 (Extensions of Theorems 1.1 and 1.3).

1) Although stated in §1 for $\mathcal{L}^\alpha$ as defined in (1.4), the conclusions of Theorems 1.1 and 1.3 hold for either of the choices for $\mathbb{P} = \mathbb{P}^I$ or $\mathbb{P} = \mathbf{P}$ above (6.1), i.e. by (6.2) they have an analogue for random interlacements in $Q_N(0)$. Indeed, the proofs of these theorems are actually written so that they are also valid for random interlacements.

2) The proof of Theorem 1.1,*i)* given above uses Theorem 1.3 to first compare $\mathcal{L}^\alpha$ to $\mathcal{B}_K$ with K a set of neighbors (which in turn follows via comparison of $\mathcal{L}^\alpha$ and $\widetilde{\mathcal{B}}$ using Theorem 6.5 and Lemma 7.1 to relate $\widetilde{\mathcal{B}}$ and $\mathcal{B}_K$) and then $\mathcal{B}_K$ to $\mathcal{B}$. If $\alpha > \alpha_*$, for the sole purpose of deducing the relevant conclusions *i)* in Theorem 1.1, one can actually bypass the intermediate use of $\mathcal{B}_K$ (and $\widetilde{\mathcal{B}}$) completely. Indeed, item *i)* can be deduced directly using Lemma 3.1, combined with Theorem 6.6 and (6.6), similarly as in the proof of Theorem 6.5 itself, thus yielding that $d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha) \to 0$ if $\alpha > \alpha_*$. Note that the proof of Theorem 1.1,*iii)* above also does not require Theorems 1.3 and 6.5, and that if one only wants to prove that the the supremum in Theorem 1.1,*ii)* is bounded away from 0 and 1 uniformly in N, one could bypass the use of $\widetilde{\mathcal{B}}$ by proceeding similarly as in the proof of (1.23). It seems however difficult to obtain the exact constant e^{-d} at criticality in (1.13) without using the more general Theorem 6.5 (or Theorem 1.3). In a nutshell, this is because $\widetilde{\mathcal{B}}$ provides us with more precise information about $\mathcal{L}^{\alpha_*}$, see (7.21), than direct moment methods, see Lemma 6.4. More generally, proceeding similarly as in (7.19), (7.20) and the first equality in (7.21), one could see Theorems 1.1 and 1.3 when $\alpha > 1/2$ as direct consequences of Theorem 6.5 and Lemma 7.1, without using the moment methods of Lemma 6.4 (which is still required for $\alpha \leqslant 1/2$). Lastly we note that if $\alpha_*(K) > 1/2$ there is some flexibility

in defining $\mathcal{B}_K^\alpha$ in (1.20) without spoiling the conclusions of Theorem 1.3, e.g. by adding a constraint on the diameter of A in (1.20) of the form $\delta(A) \leqslant R_N$, similarly as in (6.16)-(6.17). However if $\alpha_*(K) \leqslant 1/2$, it is not clear if (1.22) is still verified when adding such a diameter constraint, see (7.16), and it is thus an interesting question if a change in (1.20) might be relevant when studying $\mathcal{L}^\alpha$ in the phase $\alpha < \frac{1}{2}$.

3) As explained in the proof of Theorem 1.1,ii), the constant e^{-d} in (1.13) corresponds to $\mathbb{P}\big(E_{K_0}(\mathcal{L}^{\alpha*})^c\big)$, which is essentially the probability that there do not exist two neighbors in $\mathcal{L}^{\alpha*}$; we refer to (7.13) and (7.21) for how to compute this probability. On the other hand, the limit in the critical case (1.23) of Theorem 1.3 is not explicit (nor is it clear whether this limit actually exists). This is due to the fact that we only proved (7.13) for $K = \{x, y\}$, $x \sim y$, and in fact for this choice of K the limit in (1.13) is also e^{-d} by a similar reasoning. To obtain the limit in (1.13) for other choices of K with $\alpha_*(K) > \frac{1}{2}$, one would need to extend (7.12) when replacing α_* by $\alpha_*(K)$. The limit in (1.23) would then be $1 - e^{-\beta}$, where $\beta = \beta(K)$ would be the constant multiplying $|Q_N|$ on the right-hand side of this new version of (7.12).

4) Theorem 1.3 focuses on sets K with $\alpha_*(K) > \frac{1}{2}$, and we refer to (6.26) for the main reason why this condition appears. The pertinence of the first line in (1.22) at values $\alpha \in (\alpha_*(K), \frac{1}{2}]$ for any K with $\alpha_*(K) \leqslant \frac{1}{2}$ is another matter entirely. We refer to §8.2 for some results in this direction. The parameter $\frac{1}{2}$ is also the critical parameter from [21]. For the random walk on $\mathbf{T}$, they prove that if $B \subset \mathbf{T}$ is an independent set chosen uniformly at random, then the total variation distance between $(\mathcal{L}^\alpha)^c \cap B$ and B goes to 0 as $N \to \infty$ if $\alpha > 1/2$, and to 1 if $\alpha < 1/2$. Actually, this can be directly deduced from Theorem 1.3 for the case $\alpha > 1/2$ (which is the more difficult case), and also when considering random interlacements instead of the random walk on the torus. Indeed, by [21, Lemma 3.1 and Proposition 3.2] (applied to $\mathbb{P}(\cdot \mid A)$), it is enough to find an event A so that

(7.22) $$\mathbb{E}\Big[2^{|\mathcal{L}^\alpha \cap \widehat{\mathcal{L}}^\alpha|} 1_A\Big] \xrightarrow[N \to \infty]{} 1 \quad \text{and} \quad \mathbb{P}(A) \xrightarrow[N \to \infty]{} 1,$$

where $\widehat{\mathcal{L}}^\alpha$ is an independent copy of $\mathcal{L}^\alpha$. Let $\alpha > 1/2$ and $x \in \mathbb{Z}^d$ be such that $\alpha_*(K) \in (1/2, \alpha)$, with $K = \{0, x\}$, which exists since $\alpha_*(\{0, x\}) \to 1/2$ as $x \to \infty$ by (2.6). Let A be the event that $\{\mathcal{L}^\alpha \subset \mathcal{B}_K^{\alpha-\varepsilon}\}$, with $\varepsilon < \alpha - \frac{1}{2}$, intersected with $\{\widehat{\mathcal{L}}^\alpha \subset \widehat{\mathcal{B}}_K^{\alpha-\varepsilon}\}$, where $\widehat{\mathcal{B}}_K^{\alpha-\varepsilon}$ are independent copy of $\mathcal{B}_K^{\alpha-\varepsilon}$. Then by Theorem 1.3 we have $\mathbb{P}(A) \to 1$ (up to changing the probability space), and one can easily verify that the left-hand side of (7.22) is satisfied since $\mathcal{B}_K^{\alpha-\varepsilon} \cap \widehat{\mathcal{B}}_K^{\alpha-\varepsilon}$ consists of CN^d independent Bernoulli variables each with parameter smaller than $CN^{-2d(\alpha-\varepsilon)}$, and $2(\alpha - \varepsilon) > 1$.

5) One could also modify the relevant timescale in the definition (6.2) of $\mathcal{L}^\alpha$, and the results of Theorems 1.1 and 1.3 remain true as long as (6.6)-(6.7) hold. For instance one could take $\mathcal{L}^\alpha$ in (6.2) as the vacant set at time αt_{cov}, where t_{cov} is the expected cover time of Q_N for either random walk or random interlacements, or in fact replace α by any sequence (α_N) with $\alpha_N \log(N) \sim \alpha \log(N)$ as $N \to \infty$. Indeed if $\gamma = \lim_N (1 - \frac{\alpha_N}{\alpha_*}) \log N$ exists in $[-\infty, +\infty]$, using (7.2), as long as $\alpha_N > \frac{1}{2} + \eta$, for any $\eta > 0$, one can use Theorem 6.5, (7.2), (7.13) and (7.17) for α_N instead of α, to show that for ε small enough $\lim_N d_\varepsilon(\mathcal{L}, \mathcal{B}; \alpha_N)$ is equal to 0 if $\gamma = -\infty$, is equal to $1 - e^{-d}$ if $\gamma \in (-\infty, \infty)$, and is equal to 1 if $\gamma = +\infty$, similarly as in the proof of Theorem 1.1. Thus, proceeding similarly for Theorem 1.3, our results are robust with respect to small changes of time- reparametrization, contrary to those of [22] or [28] where one had to consider the walk at timescales αt_* for a specific choice of $t_* \sim t_{\text{cov}}$.

6) As we now explain, inspection of the proof of (1.12) and of the first line of (1.22) (used in the proof of (1.12)) reveals that item i) of Theorem 1.1 can be quantified as follows; for all $\alpha \in (\alpha_*, 1]$, $N \geqslant 1$, $F \subset Q_N$, $\varepsilon \in (0, \alpha)$, and $\mathcal{L}_F^\alpha$ as in (6.2), one has

$$(7.23) \qquad d_\varepsilon\big(\mathcal{L}_F, \mathcal{U}_F; \alpha\big) \leqslant C|F|\Big(|F|^{-2(\alpha-\varepsilon)} \log(|F|)^{C'} \varepsilon^{-\frac{2d}{(d-2)}} + |F|^{-\frac{\alpha}{\alpha_*}}\Big),$$

where $\mathcal{U}_F^\alpha \subset F$ and $\{x \in \mathcal{U}_F^\alpha\} \stackrel{\text{def.}}{=} \{U_x \leqslant \mathbb{P}(x \in \mathcal{L}_F^\alpha)\}$ for all $x \in F$, so that in particular, $\mathcal{U}_{Q_N} = \mathcal{B}$. The first term on the right-hand side of (7.23) can be traced back to (6.18), which appears in the course of proving (1.22), and the second term to (7.17). Note also that this second term is also a bound on the first line of (7.2) when $K = \{x, y\}$, $x \sim y$, by (7.5), (7.6) and (7.7). Similarly, for all $\alpha > \alpha_*(K)$ one could prove a bound on the first line of (1.22) similar to (7.23) when replacing α_* by $\alpha_*(K)$. The extension to general $F \subset Q_N$ in (7.23) comes for free since all the results of §6 utilized in the proof (namely, Theorem 6.5 and the bound (6.6)) hold at this level of generality.

7) In particular, the quantitative bound (7.23) (applied to $F = Q_N$) allows one to choose $\varepsilon = N^{-c}$ for suitable $c = c(\alpha)$ and $\alpha > \alpha_*$, for which

$$(7.24) \qquad d_{\varepsilon_N}(\mathcal{L}, \mathcal{B}; \alpha) \to 0 \text{ as } N \to \infty.$$

As we now argue, this yields a non-trivial regime of parameters $\alpha \leqslant 1$, for which ε_N is so small that (7.24) can be boosted to $d_{\text{TV}} = d_0$ in place of d_{ε_N}. Let $\eta \in (0, 2\alpha - 1)$. Then in fact (7.23) implies that (7.24) holds for $\varepsilon_N = N^{-c(\alpha)}$ with $c(\alpha) = \frac{1}{2}(d-2)(2\alpha - 1 - \eta)$. Moreover, by virtue of Lemma 6.1 one has that if $\alpha \in (0, 1)$

$$(7.25) \quad \mathbb{P}(\mathcal{B}^{\alpha - \varepsilon_N} \neq \mathcal{B}^{\alpha + \varepsilon_N}) \leqslant N^d \mathbb{P}(0 \in \mathcal{L}^{\alpha - \varepsilon_N} \backslash \mathcal{L}^{\alpha + \varepsilon_N})$$

$$\leqslant \frac{N^d}{N^{d(\alpha - \varepsilon_N)}}\Big(1 - \frac{1}{N^{2d\varepsilon_N}} + CN^{-\frac{d-2}{2}} \log(N)^{3/2}\Big) \leqslant C\varepsilon_N N^{d - \alpha d} \log N,$$

where in the last inequality we used that $N^{2d\varepsilon_N} \to 1$ and the inequality $N^{-\frac{d-2}{2}} \log(N)^{3/2} \leqslant C\varepsilon_N$ valid by our choice of ε_N and α. Therefore, if $d - \alpha d - \frac{1}{2}(d-2)(2\alpha - 1) < 0$, that is if $\alpha > \frac{3}{4}(d - \frac{2}{3})/(d - 1)$, upon choosing $\eta > 0$ small enough one deduces that $\mathbb{P}(\mathcal{B}^{\alpha - \varepsilon_N} = \mathcal{B}^{\alpha + \varepsilon_N}) \to 1$ as $N \to \infty$, which together with (7.24) and when $\alpha > \alpha_*$ yields that $d_{\text{TV}}(\mathcal{L}^\alpha, \mathcal{B}^\alpha) \to 0$. Interestingly the parameter $\alpha_2 = \alpha_* \vee \frac{3}{4}(d - \frac{2}{3})/(d - 1)$ thereby emanating coincides with the parameter from [28], as one can see by carefully inspecting [28] (see in particular the term b_2 p.10 therein). In view of 4) and 5) above, our findings thus recover the results for the random walk on the torus of [21, 22, 25, 28] in full (and also extend results such as (1.8) to an arguably more natural choice of time-parametrization). It is an intriguing question to determine whether α_2 and α_{**}, see (1.7), coincide, or whether the (multiple) occurrences of α_2 are an artefact of the methods and in reality $\alpha_{**} = \alpha_*$. Note that $\frac{3}{4}(d - \frac{2}{3})/(d - 1) > \frac{3}{4}$ and α_*, which is decreasing in d by (6.11) and [18, Lemma C.1], verifies $\alpha_* < 0.68$ by computer-assisted methods, see (6.11) and Lemma B.4. Overall this yields $\alpha_2 = \frac{3}{4}(d - \frac{2}{3})/(d - 1) > \alpha_*$.

8) Let us define the α-cover time $C^\alpha(\mathcal{L}) = \inf\big\{\beta > 0 : |\mathcal{L}^\beta| \leqslant N^{(1-\alpha)d}\big\}$ for $\alpha \in (0, 1]$. Recalling $\mathcal{B}$ and $\mathcal{B}_K$ from above (1.7) and (1.20), define $C^\alpha(\mathcal{B})$ and $C^\alpha(\mathcal{B}_K)$ similarly but replacing $\mathcal{L}$ by $\mathcal{B}$ and $\mathcal{B}_K$, respectively. With $C^\alpha = C^\alpha(\mathcal{L})$, one could also show results akin to Theorems 1.1 when $\alpha \neq \alpha_*$ and to Theorem 1.3 when $\alpha \neq \alpha_*(K)$, when replacing the set $\mathcal{L}^\alpha$ by $\mathcal{L}^{C^\alpha}$, which is the set of late points which contains exactly $N^{d(1-\alpha)}$ points (or $\lceil N^{d(1-\alpha)} \rceil$ in case $N^{d(1-\alpha)}$ is not an integer), and the sets $\mathcal{B}_K^\alpha$ and $\mathcal{B}^\alpha$ by the sets $\mathcal{B}_K^{C^\alpha(\mathcal{B}_K)}$ and $\mathcal{B}^{C^\alpha(\mathcal{B})}$. We refer to [22, Theorem 1.2] for a similar result without sprinkling when α

is close enough to one. The proof relies on Theorem 1.1, which in particular implies for all $\alpha \in (\alpha_*, 1]$ and $\varepsilon \in (0, \alpha)$ the inequality

$$\mathbb{P}\big(\alpha - \tfrac{\varepsilon}{3} \leqslant C^\alpha \leqslant \alpha + \tfrac{\varepsilon}{3}\big) \geqslant \mathbb{P}\big(|\mathcal{L}^{\alpha + \varepsilon/3}| < N^{d(1-\alpha)} < |\mathcal{L}^{\alpha - \varepsilon/3}|\big)$$
$$\geqslant \mathbb{P}\big(|\mathcal{B}^{\alpha + \varepsilon}| < N^{d(1-\alpha)} < |\mathcal{B}^{\alpha - \varepsilon}|\big) + o(1) \to 1$$

as $N \to \infty$, where the last bound is an easy consequence of concentration bounds for binomial variables and (6.5), along with similar concentration estimates for $C^\alpha(\mathcal{B})$ (and analogues in the context of Theorem 1.3 when $\alpha > \alpha_*(K) > 1/2$, replacing $\mathcal{B}$ by $\mathcal{B}_K$ throughout).

8 Extensions

We now discuss two extensions of our main results, one concerning the set $\mathcal{L}_F^\alpha$ from (6.2) viewed as a process in $\alpha > 0$, the other regarding a partial description of the behaviour of $\mathcal{L}_F^\alpha$ valid in the regime $\alpha \leqslant \frac{1}{2}$; see Theorems 8.1 and 8.3, respectively, along with the subsequent remarks.

8.1. The process $\alpha \mapsto \mathcal{L}_F^\alpha$. Recall the process $(\alpha_x)_{x \in Q_N}$ from (1.15) (see also our convention in (6.2), by which $(\alpha_x)_{x \in Q_N}$ implicitly refers to either of two choices). Note also that $\alpha_x \geqslant \alpha_*$ for any $x \in \mathcal{L}^{\alpha_*}$.

Theorem 8.1. *For all $N \geqslant 1$, there exists a coupling of $(\alpha_x)_{x \in Q_N}$ with a family $(\widehat{\alpha}_x)_{x \in Q_N}$ of i.i.d. exponential random variables of mean $d \log(N)$ each, such that for all $\varepsilon > 0$,*

$$(8.1) \qquad \lim_{N \to \infty} \mathbb{P}\big(\widehat{\alpha}_x - \varepsilon \leqslant \alpha_x - \alpha_* \leqslant \widehat{\alpha}_x + \varepsilon \text{ for all } x \in \mathcal{L}^{\alpha_*}\big) = 1.$$

The intuition behind Theorem 8.1 is roughly the following. By similar considerations as in the proof of Lemma 7.1, one argues that, for each $\alpha > \alpha_*$, all the vertices in $\mathcal{L}^\alpha$ are at distance at least $R_N = \log(N)^{\frac{1}{d-2}}$ from each other with high probability. Applying our localization results of Section 5 at this scale then implies that the hitting time of each late point is roughly independent and distributed as exponential random variable with the above mean.

Proof. We first consider the case $\mathbb{P} = \mathbf{P}$. Let $t_* = u_N(\alpha_*)N^d$ with $u_N(\alpha) = \alpha g(0) \log(N^d)$ as in (1.3) and define $\mathcal{F}_* = \sigma(X_n : 0 \leqslant n \leqslant t_*)$. Remark 5.7,3) can be applied for the random walk $(X_{t+t_*} - X_{t_*})$ under $\mathbf{P}_0(\cdot \,|\, \mathcal{F}_*)$ instead of X under $\mathbf{P}_0$ since it has law $\mathbf{P}_0$ by Markov's property, and we denote by $(\ell_{y,u}^{(x)})_{y \in \mathbb{Z}^d, x \in Q_N, u \in [u_1, u_0]}$ the associated short-range field of local times when $F = Q_N$, $R = \log(N)^{\frac{2}{d-2}}$, $\delta = 1$, $\varepsilon/3$ instead of ε, $u_1 = u_N(\varepsilon/2)$ and $u_0 = u_N(2)$, defined on some extended probability space $\widetilde{\mathbf{P}}_0$. Let

$$(8.2) \qquad \mathcal{K} = \big\{x \in \mathcal{L}^{\alpha_*} : \mathcal{L}^{\alpha_*} \cap Q(x, 2R) = \{x\}, \ell_{x - X_{t_*}, u_N(2)}^{(x - X_{t_*})} > 0\big\}$$

denote the set of points visited according to the short-range field by the terminal time $u_N(2)N^d$. Conditionally on $\mathcal{F}_*$, we then define

$$\widehat{\alpha}_x = \sup\big\{\alpha \in [0, 2] : \ell_{x - X_{t_*}, u_N(\alpha)}^{(x - X_{t_*})} = 0\big\} \text{ for each } x \in \mathcal{K};$$

similarly, for each $x \in Q_N \backslash \mathcal{K}$ such that $\mathcal{L}^{\alpha_*} \cap Q(x, 2R) = \{x\}$, we define independently $\widehat{\alpha}_x$ as 2 plus an $\mathrm{Exp}(d \log(N))$-distributed random variable, and for each $x \in Q_N \backslash \mathcal{K}$ such that $\mathcal{L}^{\alpha_*} \cap Q(x, 2R) \neq \{x\}$, define independently $\widehat{\alpha}_x$ as some $\mathrm{Exp}(d \log(N))$-distributed random variable. In view of (5.1), (6.3), and the memorylessness property of the exponential random variable, one

checks that $\widehat{\alpha}_x$ is $\mathrm{Exp}(d\log(N))$-distributed for each $x \in Q_N$. Moreover, it follows from the short-range property of $\ell^{(x)}$, see (5.2), that conditionally on $\mathcal{F}_*$, $\widehat{\alpha}_x$ is independent of $\sigma(\widehat{\alpha}_y, y \in \mathcal{K}\backslash\{x\})$ for each $x \in K$, since $d(x,y) \geqslant 2R$ for all $x \neq y \in \mathcal{K}$. Therefore $(\widehat{\alpha}_x)_{x \in Q_N}$ is an i.i.d. family of $\mathrm{Exp}(d\log(N))$-distributed random variables.

It thus remains to prove (8.1). First (5.38) for our choice of parameters implies that a.s.

$$(8.3) \qquad \widetilde{\mathbf{P}}_{\mathbf{0}}\left(\widehat{\alpha}_x - \varepsilon \leqslant \alpha_x - \alpha_* \leqslant \widehat{\alpha}_x + \varepsilon \text{ for all } x \in \mathcal{K} \cap Q(X_{t_*}, 2R)^c \,\middle|\, \mathcal{F}_*\right) \underset{n \to \infty}{\longrightarrow} 1.$$

Indeed for each $x \in \mathcal{K} \cap Q(X_{t_*}, 2R)^c$, if $\alpha_x - \alpha_* \geqslant \varepsilon/2$ then the event $\widehat{\alpha}_x - \varepsilon \leqslant \alpha_x - \alpha_* \leqslant \widehat{\alpha}_x + \varepsilon$ is directly implied by the event in (5.38) for $u_1 = u_N(\varepsilon/2)$ as above and the inequalities $\widehat{\alpha}_x/(1 - \varepsilon/3) \leqslant \widehat{\alpha}_x + \varepsilon$ (for ε small enough) as well as $\widehat{\alpha}_x/(1 + \varepsilon/3) \geqslant \widehat{\alpha}_x - \varepsilon$. If $\alpha_x - \alpha_* \leqslant \varepsilon/2$ the inequalities $\widehat{\alpha}_x - \varepsilon \leqslant 0 \leqslant \alpha_x - \alpha_*$ are similarly implied by (5.38) for $u = u_N(\varepsilon/2)$, and the inequality $\alpha_x - \alpha_* \leqslant \widehat{\alpha}_x + \varepsilon$ is trivial, which concludes the proof of (8.3). Moreover

$$(8.4) \quad \mathbf{P}_{\mathbf{0}}\left(\mathcal{L}^{\alpha_*} \cap Q(X_{t_*}, 2R) \neq \varnothing\right) \leqslant \mathbf{P}_{\mathbf{0}}\left(\mathcal{L}^{\frac{1}{2}} \cap Q(X_{t_*}, 2R) \neq \varnothing\right)$$

$$\leqslant \sup_{x \in \mathbf{T}} \mathbf{P}_{\mathbf{0}}\left(\mathcal{L}^{\frac{1}{2}} \cap Q(x, 2R) \neq \varnothing\right) + \exp(-cN^{d-2}) \leqslant \frac{CR^d}{N^{d/4}},$$

where in the second inequality we applied the Markov property at time $t_{\frac{1}{2}} = u_N(\frac{1}{2})N^d$, observed that $t_* - t_{\frac{1}{2}} \geqslant cN^d$ since $\alpha_* > \frac{1}{2}$ (see (6.11)) and applied a classical bound on the mixing time of X, see for instance [20, Theorem 5.6], to deduce that $X_{t_* - t_{1/2}}$ conditionally on $\mathcal{L}^{\frac{1}{2}}$ is suitably close to being uniformly distributed on a sub-lattice; the last inequality then follows by (6.4) combined with a union bound. Let us now consider vertices $x \in \mathcal{L}^{\alpha_*}\backslash\mathcal{K}$. By (6.3), (6.4) and a union bound and Markov's inequality, we have for large enough N

$$(8.5) \quad \widetilde{\mathbf{P}}_{\mathbf{0}}\left(\widehat{\alpha}_x \leqslant \varepsilon, \alpha_x - \alpha_* \leqslant \varepsilon \text{ for all } x \in \mathcal{L}^{\alpha_*}\backslash\mathcal{K}\right)$$

$$\geqslant \widetilde{\mathbf{P}}_{\mathbf{0}}(\widehat{\alpha}_x \leqslant 2 \text{ for all } x \in Q_N) - 3e^{-\varepsilon d \log(N)} \mathbf{E}_0\left[|\{x \in \mathcal{L}^{\alpha_*} : \mathcal{L}^{\alpha_*} \cap Q(x, 2R) \neq \{x\}\}|\right].$$

Moreover, by (2.6), (2.7), (6.6) and (6.11) we have

$$(8.6) \quad \mathbf{E}_0\left[|\{x \in \mathcal{L}^{\alpha_*} : \mathcal{L}^{\alpha_*} \cap Q(x, 2R) \neq \{x\}\}|\right]$$

$$\leqslant \mathbf{E}_0\left[|\{x, y \in \mathcal{L}^{\alpha_*} : d(x, y) \in [1, 2R]\}|\right] \leqslant CN^d R^d N^{-d} \leqslant C\log(N)^{\frac{2d}{d-2}}$$

since $R = \log(N)^{\frac{2}{d-2}}$, and by (6.6) again and a union bound we know that

$$(8.7) \qquad \widetilde{\mathbf{P}}_{\mathbf{0}}(\widehat{\alpha}_x \leqslant 2 \text{ for all } x \in Q_N) \geqslant 1 - CN^{-d}.$$

The claim now readily follows by combining (8.3), (8.4), (8.5), (8.6) and (8.7). Finally when $\mathbb{P} = \mathbb{P}^I$ the proof is similar except that one conditions on $\mathcal{F}_* = \sigma(\omega_{Q_N}^{u_N(\alpha_*)})$ instead, see (2.11), and uses a version of (5.3') for the process consisting of the trajectories of interlacements above level $u_N(\alpha_*)$, which has the same law conditionally on $\mathcal{F}_*$ as an interlacement process since the increments are stationary and independent. $\qquad\square$

Remark 8.2.   1) In much the same way as in Remarks 7.3,1) and 6), Theorem 8.1 implicitly applies to both random walk on $\mathbf{T}$ and random interlacements in the box Q_N. It further naturally generalises to $F \subset Q_N$, i.e. to the process $\alpha_x^F \overset{\text{def.}}{=} \sup\{\alpha > 0 : x \in \mathcal{L}_F^\alpha\}$, $x \in F$ (so that $\alpha_x = \alpha_x^{Q_N}$, see (1.15)). The conclusions of Theorem 8.1 remain true upon replacing the reference process $\widehat{\alpha}$ by i.i.d. exponential random variables with mean $\log|F|$. Note however that the proof of Theorem 8.1 relies heavily on the Markov property of the random walk, or the independence of the increments of interlacements, and thus might be harder to generalize to other models than our other results, see Remark 8.4,5).

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2) It might be at first surprising that the proof of Theorem 8.1, giving a description of $\mathcal{L}^\alpha$ as a *process* in α for $\alpha > \alpha_*$, does not rely on the Chen-Stein method from Section 3, contrary to the proof of Theorem 1.1, which gives a description of $\mathcal{L}^\alpha$ at *fixed* $\alpha > \alpha_*$. The reason is that Theorem 8.1 only describes the law of the hitting times of the points in $\mathcal{L}^\alpha$, $\alpha > \alpha_*$, and not their position on the lattice. In particular, one cannot deduce Theorem 1.1 from Theorem 8.1. In the proof of Theorem 8.1, the use of Chen-Stein is essentially bypassed by our localization result, Theorem 1.2, applied to the random walk after time $u_N(\alpha_*)$, which gives a short-range field $\widetilde{\ell}$ independent of $\mathcal{L}^{\alpha_*}$. The short-range property of $\widetilde{\ell}$ then manifests itself as independence property on the set $\mathcal{K}$ of well-separated points in $\mathcal{L}^{\alpha_*}$, which is almost equal to $\mathcal{L}^{\alpha_*}$, see (8.5) and (8.6). However, if $\widetilde{\ell}$ is now the process from Theorem 1.2 applied to the random walk after time 0, then it is not independent of $\mathcal{L}^{\alpha_*}$, and so it is not clear at all that the process $\widetilde{\ell}_{x,u}$, $x \in \mathcal{K}$, is independent, hence our use of the Chen-Stein method to overcome this issue in the proof of Theorem 1.1.

3) One can also generalize the description of $\mathcal{L}_\alpha$ as a processus in α for $\alpha \geqslant \alpha_*$ from Theorem 8.1 to a description of $\mathcal{L}_\alpha$ as a processus in α for $\alpha > 1/2$ as follows. For simplicity we focus on the case $\mathbb{P} = \mathbf{P}$, cf. (6.2). Informally, for each $\eta > 0$, on an event E occurring with high probability, the set $\mathcal{L}^{\frac{1}{2}+\eta}$ is a union of islands drawn from $\widetilde{\mathcal{A}}_{\mathbf{T}}$ in (6.16), each at distance at least $\log(N)^{\frac{2}{d-2}}$ (say) from one another. Then $(\alpha_x)_{x \in Q_N}$ behaves almost independently on each island as the hitting time of this island by interlacements.

We now formulate this precisely. For $R > 0$, we say that $\mathcal{K}$ is an R-well-separated partition of $S \subset Q_N$ if $\mathcal{K}$ is a partition of S such that $\delta(K) < R$ for all $K \in \mathcal{K}$ and $d(K, K') \geqslant R$ for all $K \neq K' \in \mathcal{K}$. Note that there is at most one R-well separated partition of S. An example is the set $\mathcal{K}$ from (8.2), which forms an R-well-separated partition of $S = \mathcal{L}^{\alpha_*}$ into singletons $K = \{x\}$, $x \in \mathcal{L}^{\alpha_*}$, with high probability as $N \to \infty$, as shown above.

In a similar vein, let now $R = \log(N)^{\frac{2}{d-2}}$ and fix $\eta \in (0, \frac{1}{2})$. Define $E = E(\mathcal{L}^{\frac{1}{2}+\eta})$ the event that $\mathcal{L}^{\frac{1}{2}+\eta}$ has a $4R$-well-separated partition $\mathcal{K}$, where $t_\eta = u_N(\frac{1}{2} + \eta)N^d$. Then similarly as in Theorem 8.1 one can define on a suitable extension of $\mathbb{P}$ an independent family $(\widehat{\alpha}^K)_{K \in \mathcal{K}}$, such that $\widehat{\alpha}^K$ has the same law as $(\alpha_x)_{x \in K}$ under $\mathbb{P}^I$ (corresponding to (1.15) for interlacements) for every $K \in \mathcal{K}$ and in addition, for all $\varepsilon > 0$,

$$(8.8) \quad \mathbb{P}\big(E, \widehat{\alpha}_x^K - \varepsilon \leqslant \alpha_x - \tfrac{1}{2} - \eta \leqslant \widehat{\alpha}_x^K + \varepsilon \text{ for all } x \in K \text{ and } K \in \mathcal{K}\big) \to 1 \text{ as } N \to \infty.$$

In essence, (8.8) asserts that, with high probability, the set $\mathcal{L}_{\frac{1}{2}+\eta}$ consists of 'islands' K (corresponding to the elements of $\mathcal{K}$) which are far away from one another and such that the law of $\alpha_{|K} - \frac{1}{2} - \eta$ is close, up to sprinkling, to the hitting level of each island by independent random interlacements. The proof of (8.8) follows similar lines as that of (8.1) and relies on our localization result (5.38). Note that the law of $\alpha_{|K}$ for random interlacements, i.e. the law of $\widehat{\alpha}^K$ in (8.8), can be explicitly described as follows: first wait a time $\mathrm{Exp}(d\mathrm{cap}(K)g(0)\log N)$, at which a first trajectory in the random interlacements hits K. This trajectory has law $P_{\bar{e}_K}$ and visits a subset K' of K. One can then let $K_1 = K \backslash K'$, and similarly obtain a set K_2 by repeating the previous procedure but with K_1 instead of K. Iterating this procedure until K_n is empty, the law of α_x, $x \in K$, is then the same as the law of the first time at which $x \notin K_n$, $x \in K$.

4) One can readily deduce Theorem 8.1 from (8.8). To this end, one takes $\eta = \alpha_* - 1/2$ and defines $\widehat{\alpha}_x = \widehat{\alpha}_x^{\{x\}}$ for each $x \in Q_N$ such that $\{x\} \in \mathcal{K}$, on the event $E(\mathcal{L}^{\alpha_*})$, and for each other vertex $x \in Q_N$ samples $\widehat{\alpha}_x$ as independent $\mathrm{Exp}(d\log(N))$ random variable. One can control $\mathbb{E}\left[|\{x \in \mathcal{L}_{\alpha_*} : \{x\} \notin \mathcal{K}\}|\right]$ in effectively the same way as (8.6), and conclude as in

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(8.5). Using the explicit description of the law of $\widehat{\alpha}^K$ above, one checks that $(\widehat{\alpha}_x)_{x \in Q_N}$ are indeed i.i.d. $\mathrm{Exp}(d \log(N))$-distributed.

5) It would also be interesting to prove a version of (8.1) without sprinkling, at least for α close enough to 1, that is to show that with high probability the hitting times of x, $x \in \mathcal{L}^\alpha$, are close in total variation to i.i.d. exponentials with mean $d \log(N)$ for α large enough. This does not seem to follow easily from Theorem 8.1, as the method from Remark 7.3,7) only shows that $\mathcal{B}^{\alpha - \varepsilon_n} = \mathcal{B}^{\alpha + \varepsilon_n}$ for an adapted choice of $\varepsilon_n \to 0$ at a fixed level α, and not as a process in α.

8.2. Outlook: the regime $\alpha \leq \frac{1}{2}$. Let us finish this section with a partial description of the behaviour of the late points $\mathcal{L}^\alpha_F$ for $\alpha \leq 1/2$. For each $F, K \subset Q_N$ with $\delta(K) \in (0, R_F]$ let $S_{F,K} = \{x \in Q_N : x + K \subset F\}$, recall from (6.16) that $R_F = \log(|F|)^{\frac{1}{d-2}}$ and leet

$$(8.9) \qquad \mathcal{L}^\alpha_{F,K} = \big\{x \in S_{F,K} : \mathcal{L}^\alpha_F \cap Q(x + K, R_F) = x + K\big\},$$

be the set of $x \in S_{F,K}$ such that $x + K$ is exactly the set of α-late points in F in a small neighborhood around $x + K$, and we take $\mathcal{L}^\alpha_{F,K} = \varnothing$ if $\delta(K) \geq R_F$. Correspondingly, we also define

$$\mathcal{B}^\alpha_{F,K} = \big\{x \in S_{F,K} : U_{x+K} \leq \mathbb{P}\big(\mathcal{L}^\alpha_F \cap Q(x + K, R_F) = x + K\big)\big\}.$$

Theorem 8.3. *Fix $\beta_0 \in (0, \infty)$. There exists $C = C(\beta_0) < \infty$, such that for all $N \in \mathbb{N}$, $F, K \subset Q_N$ with $\mathrm{cap}(K) \leq \beta_0$, $\delta(K) \in (0, R_F]$, $\alpha \in (\frac{\alpha_*(K)}{2}, 1]$ and $\varepsilon \in (0, \frac{\alpha}{2})$,*

$$(8.10) \qquad d_\varepsilon\big(\mathcal{L}_{F,K}, \mathcal{B}_{F,K}; \alpha\big) \leq C R_F^d |F|^{-\left(\frac{2(\alpha - 2\varepsilon)}{\alpha_*(K)} - 1\right)} \varepsilon^{-\frac{2d}{d-2}}.$$

We refer to Remark 8.4 below for further comments on the above theorem.

Proof. Consider $\alpha, F, K, \varepsilon$ as in the statement of Theorem 8.3. By translation invariance we may assume that $0 \in K$. We may also assume that $\varepsilon \geq |F|^{-\frac{(d-2)(2\alpha-1)}{2d}}$, since otherwise the right-hand side of (8.10) is always larger than 1 (up to taking $C \geq 1$ therein). Consider the field $(\widetilde{\mathcal{L}}_F^{\alpha',(x)})_{\alpha' \in (0,2], x \in Q_N}$ from Theorem 6.6, where $\lambda \geq 2$ is a large enough constant chosen so that (6.20) with $R = \left(\frac{\lambda}{\varepsilon}\right)^{\frac{2}{d-2}} R_F$ is larger than $1 - C/|F|^3$, uniformly in ε as before. We now define

$$\widetilde{\mathcal{L}}_{F,K}^{\alpha'} = \big\{x \in S_{F,K} : \widetilde{\mathcal{L}}_F^{\alpha',(x)} \cap Q(x + K, R_F) = x + K\big\}$$

and aim to apply Lemma 3.1 with the choices $S = S_{F,K}$, $I = [0, 2]$, $Y_x^{\alpha'} = 1\{x \in \mathcal{L}_{F,K}^{\alpha'}\}$, $Z_x^{\alpha'} = 1\{x \in \widetilde{\mathcal{L}}_{F,K}^{\alpha'}\}$ and $\mathcal{N}_x = S_{F,K} \cap Q(x, 3R)$, $x \in S_{F,K}$. Assumption (3.4) is verified by our choice of $\widetilde{\mathcal{L}}$ in Theorem 6.6. Moreover, by (6.20) and since $x + K \subset Q(x, R)$ for each $x \in S_{F,K}$ under our assumptions we have that

$$(8.11) \qquad d_\varepsilon(Y, Z; \alpha') \leq C|F|^{-3} \text{ for all } \alpha' \in I.$$

We thus only need to bound the constants b_1 and b_2 from Lemma 3.1. By (6.6), we have for all $\alpha \in (\frac{\alpha_*(K)}{2}, 1]$ and $\alpha' \geq \alpha - 2\varepsilon$,

$$(8.12) \qquad b_1(\alpha') \leq C|S_{F,K}| \cdot \big(\sup_{x \in S_{F,K}} |\mathcal{N}_x|\big) \cdot |F|^{-\frac{2\alpha'}{\alpha_*(K)}} \leq R_F^d |F|^{1 - \frac{2(\alpha - 2\varepsilon)}{\alpha_*(K)}} \varepsilon^{-\frac{2d}{d-2}},$$

for some constant $C = C(\beta_0) < \infty$. Moreover for $x \in S_{F,K}$ and $y \in \mathcal{N}_x \backslash \{x\}$ with $Y_x^{\alpha'} = Y_y^{\alpha'} = 1$ we have $d(x, y) \geq R_F/2$, and so by (2.9), (2.10) and (6.6) we readily see that $b_2(\alpha')$ satisfies a bound similar to $b_1(\alpha')$ in (8.12). We can now conclude by combining (3.5) with (8.11) and (8.12). $\square$

Remark 8.4.   1) Theorem 8.3 indicates that for each $K \subset\subset \mathbb{Z}^d$, when considering only the subsets of the late points $\mathcal{L}_F^\alpha$ which look locally exactly a translation of K, or its projection on the torus, these sets can be well-approximated up to a sprinkling by independent translations of K, as long as $\alpha > \frac{\alpha_*(K)}{2}$. For $K = \{0\}$, since $\alpha_*(\{0\}) = 1$, this corresponds to an approximation of the isolated vertices of $\mathcal{L}_F^\alpha$ by independent vertices as long as $\alpha > \frac{1}{2}$, which is essentially contained in Theorem 1.3. But when $|K| \geqslant 2$, Theorem 8.3 describes the behaviour of sets which are translations of K for some $\alpha \leqslant \frac{1}{2}$ as well.

2) Actually, Theorem 8.3 is mainly interesting when $\alpha \leqslant \alpha_*(K)$. Indeed, for $\alpha > \alpha_*(K)$, using a first-moment bound and (6.6), one easily sees that $\mathcal{B}_{F,K}^\alpha$ and $\mathcal{L}_{F,K}^\alpha$ are both empty with high probability as $|F| \to \infty$, so $\mathcal{L}_{F,K}^\alpha$ is trivially well-approximated by $\mathcal{B}_{F,K}^\alpha$. In particular, for any sets $K \subset \mathbb{Z}^d$ with $|K| \geqslant 2$, we thus have that both $\mathcal{L}_{F,K}^\alpha$ and $\mathcal{B}_{F,K}^\alpha$ are empty with high probability for all $\alpha > \alpha_*$, a fact which is already implicit in Theorem 1.1.

3) Another result which remains true in the regime $\alpha \leqslant 1/2$ is the convergence of the empirical process associated to $\mathcal{L}^\alpha$ to a Poisson point process on $[0, 1]^d$. More precisely, for each $\alpha \in (0, 1)$ the point process $\sum_{x \in \mathcal{L}^\alpha} \delta_{x/N^\alpha}$ converges in law to a point process on $\mathbb{R}^d$ with intensity the Lebesgue measure. This can be proved using (6.5) in exactly the same way as in [4, Corollary 0.2] for random interlacements and as in [5, Corollary 3.4] for the random walk.

4) It is an interesting open question to obtain a description for the asymptotic law of the full set $\mathcal{L}_F^\alpha$ for $\alpha \leqslant \frac{1}{2}$, and not only of $\mathcal{L}_{F,K}^\alpha$ for large enough K as in Theorem 8.3. The main obstacle in order to do so is the lack of clustering for $\alpha \leqslant \frac{1}{2}$. Indeed, Theorem 6.5, see also Lemma 6.8 and (7.20), indicates that for $\alpha > \frac{1}{2}$, $\mathcal{L}^\alpha$ consists with high probability of 'islands' with capacity smaller than $\frac{2}{g(0)}$, each with diameter smaller than R_N and at distance at least $N^{2\alpha-1-\eta}$ for any $\eta > 0$ from one another. Adapting Lemma 6.8, one could even show that for each $\alpha \in (\frac{1}{2}, 1]$, these islands have diameter at most $C = C(\alpha, \eta)$ with high probability, and are thus asymptotically independent, as highlighted in Theorem 6.5.

However, when $\alpha \leqslant \frac{1}{2}$ a reasoning similar to the proof of (6.22) shows that for each $p \in \mathbb{N}$ the average number of points in $\mathcal{L}^\alpha$ at distance at least p from one another, but less than $\log(N^d)$, diverges to infinity as $N \to \infty$. In other words, $\mathcal{L}^\alpha$ cannot be decomposed in bounded islands at infinite asymptotic distance from one another, which is the main conceptual obstacle in extending Theorem 6.5 to $\alpha \leqslant \frac{1}{2}$. Nevertheless, for large enough sets K, $\mathcal{L}_{K,Q_N}^\alpha$ still consists of bounded islands at infinite asymptotic distance from one another, which are thus independent as highlighted in Theorem 8.3.

5) With future applications in mind, let us briefly explain which properties of random interlacements and random walk are used to obtain all main results from Sections 6-7, including Theorem 8.3. First Theorem 6.5 only uses the bound (6.6) and Proposition 6.6. From this, one can also obtain the case $\alpha > \alpha_*$ from Theorem 1.1, the case $\alpha > \alpha_*(K) > \frac{1}{2}$ from Theorem 1.3, the last bound in (1.23), as well as Theorem 8.3. One additionally needs the lower bound (6.7) and the decoupling (6.8) to obtain Lemma 6.4 and its consequences, namely the case $\alpha < \alpha_*$ from Theorem 1.1, the case $\alpha < \alpha_*(K)$ or $\alpha_*(K) \leqslant \frac{1}{2}$ from Theorem 1.3 and the first bound in (1.23). Except for the critical case (1.23), one could afford weaker versions of the bounds (6.6) and (6.7) with some additional subpolynomial term, that is only the polynomial order of $\mathbb{P}(0 \in \mathcal{L}^\alpha)$ is important, i.e. the limit of $\frac{\log \mathbb{P}(0 \in \mathcal{L}^\alpha)}{\log N}$ as $N \to \infty$. To obtain the precise asymptotic in the critical case $\alpha = \alpha_*$, see (1.13), one needs the asymptotics of $\mathbb{P}(0 \in \mathcal{L}^\alpha)$ as $N \to \infty$, by which the constants $C(\beta_0)$ and $c(\beta_0)$ from (6.6)-(6.7) are replaced by $1 + o(1)$.

This hints at a universal phenomenon, valid for essentially any model satisfying estimates like (6.6), (6.7) (possibly up to some subpolynomial factor except at criticality) and (6.8), and allowing for a 'finite-range' approximation with properties akin to *i)*-*iii)* of Proposition 6.6. We hope to return to this elsewhere. Another possible extension is to consider other graphs than the d-dimensional torus under suitable hypotheses (e.g. polynomial decay of the Green function and polynomial volume growth as in [14]), for which our method should be stable, see for instance Remark A.11. In a related direction, we refer to [6] for recent work characterizing the 'universality class' of Gumbel fluctuations for cover times.

A   Appendix: proofs of Lemmas 5.5 and 5.6

In this appendix, we prove Lemmas 5.5 and 5.6 using some large deviations estimates for excursions of random walks or random interlacements, see Propositions A.7 and A.9. In order to finish the proof of Lemma 5.5, we are also going to need some Harnack-type estimate to show that the function $g_\zeta(z)$, see (5.15) and (5.16), does not depend, up to constants, on the choice $\zeta \in \partial B_2 \times \partial B_3^c$.

A.1. Harnack-type estimates. The following results are tailored to our purposes. Throughout this section we only deal with the process X under P_x, but the results immediately transfer to the walk on $\mathbf{T}$ as long as the events in question are measurable in terms of $X_{\cdot \wedge T_Q}$ with $Q = Q(x, r)$ for some $r < N$ under P_x (this typically means $r_3 < N$ below). We refer to Section 2 for notation. A function $f : \mathbb{Z}^d \to \mathbb{R}$ is called harmonic in $K \subset \mathbb{Z}^d$ if $f(x) = E_x[f(X_1)]$ for all $x \in K$ (which only requires knowing f in the 1-neighborhood of K). By [19, Theorem 1.7.2], one knows that for all $\delta \in (0, 1)$, $r \geqslant 1$ and f non-negative and harmonic in $Q(0, r(1 + \delta))$,

(A.1) $$f(x) \leqslant C(\delta) f(y), \text{ for all } x, y \in Q(0, r)$$

(note that the ref. [19] states (A.1) for Euclidean balls but (A.1) can be deduced from it via a straightforward chaining argument). In the sequel we abbreviate $B_k = Q(0, r_k)$, for $k = 1, 2, 3$ with $1 \leqslant r_1 < r_2 < r_3$; similarly as in (5.23). Let g_K denote the Green's function killed on the set $K \subset \mathbb{Z}^d$, so that $g_\varnothing(x, y) = g(x, y)$, cf. above (2.3). We start with a control which involves killing in nearby ℓ^∞-boxes. Note that the following result is completely standard for large δ (larger than $C \in (1, \infty)$) but the case of small δ requires some care.

Lemma A.1. *For all $\delta \in (0, 1)$, $r_3 \geqslant r_1(1 + \delta)^2$, $K \subset B_1$ a box (possibly $K = \varnothing$) and $x, y \in Q(0, r_3/(1 + \frac{\delta}{2}))\backslash Q(K, \frac{\delta}{2} r_1)$, one has*

(A.2) $$c(\delta)|x - y|^{2-d} \leqslant g_{K \cup B_3^c}(x, y) \leqslant C|x - y|^{2-d}.$$

Moreover, with r_1, r_3 as above and for all $x \in Q(0, (1 + \delta)r_1)\backslash Q(0, (1 + \frac{\delta}{2})r_1)$,

(A.3) $$c(\delta) \leqslant P_x(H_{B_1} < T_{B_3}) \leqslant P_x(H_{B_1} < \infty) \leqslant 1 - c(\delta).$$

Proof. First we observe that the first bound in (A.3) is an easy consequence of (A.2). Indeed, by a last-exit decomposition similar to (2.4), one obtains that for all $x \in Q(0, (1 + \delta)r_1)$,

(A.4) $$P_x(H_{B_1} < T_{B_3}) \geqslant \inf_{x' \in B_1} g_{B_3^c}(x, x') \text{cap}_{B_3}(B_1) \geqslant c(\delta),$$

where the last step uses (A.2) for $K = \varnothing$ along with the capacity estimate $\text{cap}_{B_3}(B_1) \geqslant \text{cap}(B_1) \geqslant cr_2^{d-2}$. In order to prove the last bound in (A.3), first notice that $P_x(H_{B_1} \geqslant T_{Q(0, \lambda r_1)}) \geqslant c(\lambda, \delta)$ for any $\lambda > 0$ by projecting onto a coordinate and using a Gambler's ruin

estimate. Moreover if λ is chosen large enough, it follows from (2.4) similarly as in (A.4) that $P_y(H_{B_1} < \infty) \leqslant \frac{1}{2}$ for all $y \in Q(0, \lambda r_1)^c$. The upper bound follows by the Markov property.

We now prove (A.2). For any $K \subset \mathbb{Z}^d$ one has $g_K \leqslant g$ and the upper bound in (A.2) follows immediately from standard estimates on the Green kernel, see for instance [19, Theorem 1.5.4]. We now show the lower bound, and note that by monotonicity of $B_3 \mapsto g_{K \cup B_3^c}(x, y)$ and symmetry in x and y we may simply assume that $|y|_\infty \geqslant |x|_\infty$ and $y \in \partial Q(0, r_3/(1 + \frac{\delta}{2}))$. Noting that by the Markov property $g_{K \cup B_3^c}(x, y) \geqslant P_x(H_{B_1} = \infty) \inf_{z \in \partial Q(0, r_3/(1 + \frac{\delta}{2}))} g_{K \cup B_3^c}(z, y)$ and using the last bound in (A.3), we may further assume that $x \in \partial Q(0, r_3/(1 + \frac{\delta}{2}))$. Let us first fix $\lambda = \lambda(d) \in (1, \infty)$ large enough such that for all $x \in \mathbb{Z}^d$,

$$
(A.5) \qquad \sup_{|z|_\infty \geqslant \lambda |x|_\infty} g(z) \leqslant \frac{1}{2} g(x)
$$

(recall that $g(x) = g(0, x)$); the bound (A.5) is obtained again using e.g. [19, Theorem 1.5.4]. We now distinguish two cases. Suppose first that $x, y \in \partial Q(0, r_3/(1 + \frac{\delta}{2}))$ and $|x - y|_\infty \leqslant \frac{\delta r}{10\lambda}$, where we abbreviate $r = r_3$. Then applying the strong Markov property at time $H_{K \cup B_3^c}$, it follows that

$$
g_{K \cup B_3^c}(x, y) = g(x, y) - E_x\big[g(X_{H_{K \cup B_3^c}}, y)\big] \overset{(A.5)}{\geqslant} \frac{1}{2} g(x, y),
$$

where in the last step, we used that $|X_{H_{K \cup B_3^c}} - y|_\infty \geqslant \frac{1}{2}\delta r \geqslant \lambda |x - y|_\infty$. Along with the standard bounds on g, this completes the verification of (A.2) in that case.

Now suppose that $|x - y|_\infty \geqslant \frac{\delta r}{10\lambda}$. Then since $x, y \in \partial Q(0, r/(1 + \frac{\delta}{2}))$ the boxes $B_y = Q(y, \frac{\delta r}{100\lambda})$ and $B_x = Q(x, \frac{\delta r}{100\lambda})$ can be joined using a chain of $C(\delta)$ many boxes B_i, each having radius $\frac{\delta r}{100\lambda}$, in such a manner that $i)$ any two consecutive boxes overlap (i.e. $B_i \cap B_{i+1} \neq \varnothing$) and $ii)$ if $\widetilde{B}_i \supset B_i$ refers to the concentric box having radius $\frac{\delta r}{10}$, then $\widetilde{B}_i$ does not intersect $B_1 \cup B_3^c$. It follows that for all $x \in B_i$,

$$
(A.6) \qquad P_x(H_{B_{i+1}} < H_{K \cup B_3^c}) \overset{ii)}{\geqslant} P_x(H_{B_{i+1}} < T_{\widetilde{B}_i}) \geqslant c,
$$

where the last bound uses monotonicity and (A.3), which is in force due to the first inequality in (A.4) and the lower bound on the killed Green's function at 'short' distances already obtained, see also $i)$ and the choice of radius for $\widetilde{B}_i$. Iterating (A.6) using the Markov property yields

$$
g_{K \cup B_3^c}(x, y) \geqslant P_x\big(H_y < H_{K \cup B_3^c}\big) \geqslant c \inf_{y' \in B_y} P_{y'}\big(H_y < H_{K \cup B_3^c}\big) \geqslant c'(\delta) r^{2-d} \geqslant c''(\delta)|x - y|^{2-d},
$$

where the penultimate step follows by bounding $P_{y'}(H_y < H_{K \cup B_3^c}) \geqslant g_{K \cup B_3^c}(y', y)/g(0)$ and using the lower bound already derived, and the last step because $|x - y| \geqslant c(\delta)r$ by assumption. $\qquad \square$

We are now ready to prove that the function $g_\zeta(\Theta)$, see (5.16), is of constant order for suitable choice of the radii r_k for B_k.

Lemma A.2. *For all $\delta \in (0, 1)$, $r_{k+1} \geqslant r_k(1 + \delta)$, $k = 1, 2$, and all $y \in \partial B_2$ and $w \in \partial B_3$,*

$$
P_y\big(T_{B_3} < H_{B_1} \,\big|\, X_{T_{B_3}} = w\big) \geqslant c(\delta).
$$

Proof. By a last-exit decomposition in B_2, one finds that

$$
(A.7) \qquad P_y\big(T_{B_3} < H_{B_1}, X_{T_{B_3}} = w\big) = \sum_{z \in \partial B_2} g_{B_1 \cup B_3^c}(y, z) P_z(\widetilde{H}_{B_2} > T_{B_3}, X_{T_{B_3}} = w).
$$

Using the lower bound in (A.2) with the choice $K = B_1$ and the upper bound with $K = \varnothing$, it follows that $g_{B_1 \cup B_3^c}(y, z) \geqslant c(\delta) g_{B_3^c}(y, z)$ for all $y, z \in \partial B_2$. Substituting above, it follows that the right-hand side of (A.7) is bounded from below by

$$c(\delta) \sum_{z \in \partial B_2} g_{B_3^c}(y, z) P_z(\widetilde{H}_{B_2} > T_{B_3}, X_{T_{B_3}} = w) = c(\delta) P_y(X_{T_{B_3}} = w),$$

where the last equality follows again by last-exit decomposition. $\qquad\square$

Next, we prove that the function $g_\zeta(z)$, $z \in \mathcal{K}$, see (5.15), does not depend on the choice of $\zeta \in \partial B_2 \times \partial B_3^c$, up to constants. Recall from (2.1) that $L_{B_2}(T_{B_3})$ denotes the time of last visit to B_2 prior to exiting B_3.

Lemma A.3. *For all $\delta \in (0, 1)$, $r_{k+1} \geqslant r_k(1 + \delta)$, $k = 1, 2$, and $x, y, z \in \partial B_2$, $v \in \partial B_1$, $w \in \partial B_3^c$,*

$$(A.8) \qquad c(\delta) \leqslant \frac{P_y\big(X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z \,\big|\, X_{T_{B_3}} = w\big)}{P_x\big(X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z\big)} \leqslant C(\delta).$$

We first isolate the following:

Claim A.4. *For all $\delta \in (0, 1)$, $r_3 \geqslant (1 + \delta)r_2$, all $y, z \in \partial B_2$ and $w \in \partial B_3^c$,*

$$c(\delta) \cdot P_y\big(X_{T_{B_3}} = w\big) \leqslant P_z\big(X_{T_{B_3}} = w \,\big|\, T_{B_3} < \widetilde{H}_{B_2}\big) \leqslant C(\delta) \cdot P_y\big(X_{T_{B_3}} = w\big).$$

Proof. Let $V = \partial Q(0, r_2(1 + \delta'))$, where $(1 + \delta')^2 = 1 + \delta$. Since V separates B_2 from B_3^c, applying the strong Markov property at time H_V, one obtains that

$$(A.9) \quad P_z\big(X_{T_{B_3}} = w, T_{B_3} < \widetilde{H}_{B_2}\big) = \sum_{z' \in V} P_z\big(X_{T_{B_3}} = w, T_{B_3} < \widetilde{H}_{B_2}, X_{H_V} = z'\big)$$

$$= \sum_{z' \in V} P_z\big(H_V < \widetilde{H}_{B_2}, X_{H_V} = z'\big) P_{z'}\big(T_{B_3} < H_{B_2} \,\big|\, X_{T_{B_3}} = w\big) P_{z'}\big(X_{T_{B_3}} = w\big).$$

Using Lemma A.2 we know that the middle term in the second line is at least $c(\delta')$ (and at most 1). Since $z' \mapsto P_{z'}(X_{T_{B_3}} = w)$ is harmonic in B_3, by (A.1) one obtains that the last term is bounded from above and below by $P_y\big(X_{T_{B_3}} = w\big)$, up to constants depending only on δ. Finally one knows by (A.3) that $c(\delta') \leqslant P_{z'}(T_{B_3} < H_{B_2})(\leqslant 1)$. Substituting all of this into (A.9) yields that the left-hand side of (A.9) is bounded up to constants from above and below by

$$P_y\big(X_{T_{B_3}} = w\big) \sum_{z' \in V} P_z\big(H_V < \widetilde{H}_{B_2}, X_{H_V} = z'\big) P_{z'}\big(T_{B_3} < H_{B_2}\big) = P_y\big(X_{T_{B_3}} = w\big) \cdot P_z\big(T_{B_3} < \widetilde{H}_{B_2}\big),$$

which is the claim. $\qquad\square$

It remains to give the

Proof of Lemma A.3. We first reduce the task to the case $x = y$, by applying (A.1) and a chaining argument to the function $x \mapsto P_x(X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z)$, which is harmonic in $B_3 \backslash B_1$. Harmonicity holds crucially because $v \in \partial B_1$, which forces the walk to visit B_1 prior to time $L_{B_2}(T_{B_3})$, whence $L_{B_2}(T_{B_3}) \geqslant 1$ under P_x. This allows to effectively replace the starting point x by y in the denominator appearing in (A.8).

We now show (A.8) for $x = y$, and write

$$(A.10) \quad P_y\big(X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z \,\big|\, X_{T_{B_3}} = w\big)$$

$$= \frac{P_y\big(X_{T_{B_3}} = w \,\big|\, X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z\big)}{P_y\big(X_{T_{B_3}} = w\big)} P_y\big(X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z\big)$$

By a last-exit decomposition in ∂B_2,

$$P_y\big(X_{T_{B_3}} = w \,\big|\, X_{H_{B_1} \wedge T_{B_3}} = v, X_{L_{B_2}(T_{B_3})} = z\big) = P_z\big(X_{T_{B_3}} = w \,\big|\, T_{B_3} < \widetilde{H}_{B_2}\big).$$

Inserting this into (A.10) and using Claim A.4 completes the proof. $\qquad\square$

A.2. Large deviation estimate for excursions. In this section, we prove Lemma 5.5 on the concentration of the soft local times, as well as Lemma 5.6 on the concentration of the number of excursions, both for random walk and random interlacements. As in §A.1, we assume that $B_k = Q(0, r_k)$ for $k = 1, \ldots 3$ with $r_1 < r_2 < r_3 < N$.

We start by collecting some preliminary large-deviation estimates, which concern the random walk on the torus $\mathbf{T}$ of side length $N \geqslant 1$ (in dimension $d \geqslant 3$). Recall the definition of the successive return times $R_k = R_k(\widehat{X}, B_2, B_3)$ from (5.4) and (5.7) (well-defined when $r_3 < N$). In the sequel, we denote by ν the stationary measure of $(\widehat{X}_{R_k})_{k \geqslant 1}$, which is supported on ∂B_2. To avoid clumsy notation, we identify ν with its projection on the torus (which is the invariant distribution of $(X_{R_k})_{k \geqslant 1}$), and abbreviate $X_{[s,t]} = (X_n)_{s \leqslant n \leqslant t}$ in the sequel.

Lemma A.5. *For all $\delta > 0$ and $N > r_3 \geqslant (1 + \delta)r_2 \geqslant 1$, the Markov chain $(Y_k)_{k \geqslant 0}$ with $Y_k = X_{[R_k, R_{k+1}]}$ has invariant distribution $\mathbf{P}_\nu(X_{[0,R_1]} \in \cdot)$. Moreover,*

$$\text{(A.11)} \qquad d_{\mathrm{TV}}\big(\mathcal{L}((Y_{iM})_{i=1}^K), (\mathbf{P}_\nu(X_{[0,R_1]} \in \cdot))^{\otimes K}\big) \leqslant CK e^{-c(\delta)M},$$

for all $M, K \in \mathbb{N} = \{1, 2, \ldots\}$, with $\mathcal{L}$ denoting the joint law of $(Y_{iM})_{i=1}^K$ under $\mathbf{P}_x$, $x \in \mathbf{T}$.

Proof. By definition, ν is the invariant distribution of $(X_{R_k})_{k \geqslant 1}$ and since $(Y_n)_{n \geqslant 1}$ are independent random variables conditionally on $(X_{R_k})_{k \geqslant 1}$, one readily concludes that the invariant distribution of $(Y_k)_{k \geqslant 1}$ is $\mathbf{P}_\nu(X_{[0,R_1]} \in \cdot)$. A claim on the total variation similar to (A.11) but concerning $(X_{R_k})_{k \geqslant 0}$ instead of Y holds by [22, Lemma 2.2], and (A.11) then follows easily. Note that [22, Lemma 2.2] is stated for the exit chain, i.e. $(X_{D_k(\widehat{X}, B_1, B_2)})_{k \geqslant 0}$, but the proof for the entrance points is identical. There it is further assumed that $r_3 \geqslant 10\sqrt{d}r_2$, but this is owed to the fact that excursions from square boxes to round boxes are considered. If instead one considers excursions from square boxes to square boxes as in the present case, the assumption $r_3 \geqslant (1 + \delta)r_2$ for some $\delta > 0$ is sufficient. $\qquad\square$

Remark A.6 (Identifying ν). Recalling that $\overline{\mathbf{e}}_2^3$ denotes the projection of the equilibrium measure $\overline{e}_{B_2}^{B_3}$ onto $\mathbf{T}$, see above (5.20) and below (2.6) for notation, it follows from [8, Lemma 6.1] that

$$\text{(A.12)} \qquad \nu = \overline{\mathbf{e}}_2^3$$

for $r_3 < N$. Moreover, by [8, eq. (9.4)], one has the exact(!) formula

$$\text{(A.13)} \qquad \mathbf{E}_{\overline{\mathbf{e}}_2^3}[R_1(X, B_2, B_3)] = \big(\mathrm{cap}_{B_3}(B_2)\big)^{-1} N^d.$$

The identities (A.12) and (A.13) are needed to precisely match both the number of excursions and the soft local times between random walk and random interlacements, but are otherwise unnecessary; cf. Remark 5.7,1).

Let $W_{2,3}$ denote the set of nearest-neighbors paths in $\mathbf{T}$ starting in ∂B_2, hitting ∂B_3^c, and then ending the next time ∂B_2 is hit. Thus $W_{2,3}$ represents the state space of the excursion process $(Y_k)_{k \geqslant 0}$ appearing in Lemma A.5. We now prove the following large deviations estimate for these excursions, from which Lemmas 5.5 and 5.6 for the random walk will later follow.

The following setup is tailored to our purposes. We consider $(Z_i)_{i \geqslant 0}$ an i.i.d. sequence of random variables with values in a measurable space $(A, \mathcal{A})$ and independent of X (under $\mathbf{P}_x$). For a measurable function $F : W_{2,3} \times A \to [0, \infty)$ we then introduce the random variables $V_i = F(X_{[R_i, R_{i+1}]}, Z_i)$ for $i \geqslant 0$ and for $i_0 \geqslant 0$ and $m \in \mathbb{N}$ the average

$$(A.14) \qquad \overline{V}_m = \overline{V}_{i_0, m} = \frac{1}{m} \sum_{i=0}^{m-1} V_{i_0 + i},$$

Proposition A.7. *For all $i_0 \geqslant 0$, $\delta > 0$, $N > r_3 \geqslant (1 + \delta) r_2 \geqslant 1$, the following holds. If for $\theta \geqslant 1$,*

$$(A.15) \qquad \begin{aligned} &\sup_{x \in \mathbf{T}} \mathbf{E}_x[V_{i_0}] \leqslant \theta \inf_{x \in \mathbf{T}} \mathbf{E}_x[V_{i_0}] < \infty \text{ and} \\ &\mathbf{E}_x[V_{i_0}^k] \leqslant k! \theta^k \mathbf{E}_x[V_{i_0}]^k \text{ for all } x \in \mathbf{T}, \, k \in \mathbb{N}, \end{aligned}$$

then there exist $C = C(\theta, \delta) < \infty$ and $c = c(\theta, \delta) > 0$ such that for all $m \in \mathbb{N}$ and $\eta \in (0, 1)$,

$$\sup_{x \in \mathbf{T}} \mathbf{P}_x\big(|\overline{V}_m - \mathbf{E}_\nu[V_{i_0}]| > \eta \mathbf{E}_\nu[V_{i_0}]\big) \leqslant C m \exp\big\{-c\sqrt{\eta^2 m}\big\}.$$

Proof. Let $W = (W_i)_{i \geqslant 0}$ be i.i.d. centered random variables each having the law of $V_{i_0} - \mathbf{E}_\nu[V_{i_0}]$ under $\mathbf{P}_\nu$ and P^W denote their joint law. Then, using (A.11) and e.g. the characterization of d_{TV} in terms of couplings, one obtains for all $M, K \in \mathbb{N}$ and $1 \leqslant i \leqslant K$ the bound

$$(A.16) \qquad d_{\mathrm{TV}}\big(\mathcal{L}\big((V_{iM} - \mathbf{E}_\nu[V_{i_0}])_{i=1}^K\big), \, \mathcal{L}\big((W_i)_{i=1}^K\big)\big) \leqslant CK e^{-c(\delta)M},$$

with $\mathcal{L}$ governing the V.'s on the left-hand side referring to their joint law under $\mathbf{P}_x$ for any $x \in \mathbf{T}$. Assuming $\frac{m}{M} \geqslant 2$ we have $K \stackrel{\mathrm{def.}}{=} \lfloor m/M \rfloor - 1 \geqslant 1$, and by the triangle inequality and a union bound,

$$(A.17) \qquad \mathbf{P}_x\big(|\overline{V}_m - \mathbf{E}_\nu[V_{i_0}]| > \eta \mathbf{E}_\nu[V_{i_0}]\big) \leqslant a_1 + a_2 + a_3,$$

where

$$a_1 = \mathbf{P}_x\left(\exists\, p \in \{0, \dots, M-1\} : \left|\sum_{i=1}^K (V_{i_0 + p + iM} - \mathbf{E}_\nu[V_{i_0}])\right| > \frac{\eta m \mathbf{E}_\nu[V_{i_0}]}{2M}\right),$$

$$a_2 = \mathbf{P}_x\left(\sum_{i=0}^{M-1} |V_{i_0 + i} - \mathbf{E}_\nu[V_{i_0}]| > \frac{\eta m \mathbf{E}_\nu[V_{i_0}]}{4}\right),$$

$$a_3 = \mathbf{P}_x\left(\sum_{i=M+KM}^{m-1} |V_{i_0 + i} - \mathbf{E}_\nu[V_{i_0}]| > \frac{\eta m \mathbf{E}_\nu[V_{i_0}]}{4}\right).$$

Applying (A.16), the strong Markov property at time R_k for $i_0 \leqslant k < i_0 + M$, a union bound and letting $t = \frac{\eta m}{2M} \mathbf{E}_\nu[V_{i_0}]$, a_1 is bounded by

$$(A.18) \qquad M \sup_{x \in \mathbf{T}} \mathbf{P}_x\left(\left|\sum_{i=1}^K V_{iM} - \mathbf{E}_\nu[V_{i_0}]\right| > t\right) \leqslant M P^W\left(\left|\sum_{i=1}^K W_i\right| > t\right) + CMK e^{-cM}.$$

Under (A.15), it follows from the Bernstein inequality, see for instance [7, Corollary 2.11] for the version we use here, that for some constant $c > 0$,

$$(A.19) \qquad P^W\left(\left|\sum_{i=1}^K W_i\right| > t\right) \leqslant 2 \exp\left\{-\frac{ct^2}{K\theta^4 \mathbf{E}_\nu[V_{i_0}]^2 + \theta^2 \mathbf{E}_\nu[V_{i_0}] t}\right\};$$

here we are implicitly using that the controls on higher moments appearing in the second line of (A.15) hold with θ^2 in place of θ and $\mathbf{E}_\nu[\,\cdot\,]$ in place of $\mathbf{E}_x[\,\cdot\,]$ everywhere. Recalling t and that $K \leqslant \frac{m}{M}$, one sees that the right-hand side of (A.19) is bounded by $\exp\{-c(\theta)\frac{\eta^2 m}{M}\}$, and together with (A.18) this yields that

$$(\text{A.20}) \qquad a_1 \leqslant M e^{-c(\theta)\frac{\eta^2 m}{M}} + CMKe^{-cM}.$$

Next, we bound a_2 for $\eta m \geqslant C(\theta)M$. To this effect, first note that, combining the assumptions in (A.15) and the strong Markov property, one readily obtains that $\theta^{-1}\mathbf{E}_\nu[V_{i_0}] \leqslant \mathbf{E}_x[V_{i_0}] \leqslant \theta\mathbf{E}_\nu[V_{i_0}]$. Feeding this into a_2, a union bound and the strong Markov property (applied at time R_i) then give that if $\eta m \geqslant C(\theta)M$,

$$(\text{A.21}) \qquad a_2 \leqslant M \sup_{x\in\mathbf{T}} \mathbf{P}_x\left(\left|V_{i_0} - \mathbf{E}_x[V_{i_0}]\right| \geqslant \frac{c\eta m\mathbf{E}_x[V_{i_0}]}{4M}\right) \leqslant M\exp\left(-\frac{c'(\theta)\eta m}{M}\right),$$

where the second inequality is obtained by Bernstein's inequality similarly as in (A.20) for $K = 1$. Similarly, since $M + KM \geqslant m - M + 1$ by definition of K, we have that if $\eta m \geqslant C(\theta)M$,

$$(\text{A.22}) \qquad a_3 \leqslant M e^{-\frac{c(\theta)\eta m}{M}}.$$

Choosing $M = \lceil\sqrt{\eta^2 m}\rceil$, noting that $\eta m \geqslant C(\theta)M$ and $m/M \geqslant 2$ hold if $\eta m \geqslant C'(\theta)$, which is no loss of generality, we conclude by combining (A.17), (A.20), (A.21) and (A.22). □

We now have all the tools to give the proof of Lemma 5.5 in the random walk case.

Proof of Lemma 5.5 (Random walk case). One applies Proposition A.7 with the following choices. Let $\eta = \varepsilon$. Recalling G^{RW} from (5.19), one takes $Z_i = \widehat{\xi}_i$, $V_i = \widehat{\xi}_i g_{\zeta_i}(z)$, and notes that ζ_i is $X_{[R_i, R_{i+1}]}$-measurable, cf. (5.4)-(5.5). Then with $\overline{V}_m = \overline{V}_{1,m}$, i.e. $i_0 = 1$ in (A.14), one has $m\overline{V}_m = G_m^{\text{RW}}(z)$, see (5.19), and by (A.12), it follows that $\mathbf{E}_\nu[V_1] = \bar{g}(z)$, see (5.20). Moreover $\mathbf{E}_x[\xi_1^j] = j!$ since ξ_1 is an exponential random variable with parameter one, and one readily deduces that the assumption (A.15) holds for some $\theta = \theta(\delta) < \infty$ by Lemmas A.2 and A.3, see also (5.15) and (5.16), if $r_3 \geqslant (1+\delta)r_2$ and $r_2 \geqslant (1+\delta)r_1$. The claim follows. □

The proof of Lemma 5.6 for the random walk involves another application of Proposition A.7. Verifying the relevant condition (A.15) in that case will rely on the following result.

Lemma A.8. *For any $\delta > 0$ and $r \leqslant N$, abbreviating $Q = Q(\mathbf{0}, r)$, one has*

$$(\text{A.23}) \qquad \sup_{x\in\mathbf{T}} \mathbf{E}_x\big[H_Q\big] \leqslant C \cdot N^2\Big(\frac{N}{r}\Big)^{d-2},$$

$$(\text{A.24}) \qquad \inf_{x\in\mathbf{T}\setminus Q(\mathbf{0},r(1+\delta))} \mathbf{E}_x\big[H_Q\big] \geqslant c(\delta) \cdot N^2\Big(\frac{N}{r}\Big)^{d-2}.$$

Proof. We will often use the classical fact that for all $\delta \in (0,1)$, $x \in Q(\mathbf{0}, r(1-\delta))$ and $r < N$,

$$(\text{A.25}) \qquad c(\delta)r^2 \leqslant \mathbf{E}_x[T_Q] \leqslant Cr^2,$$

which follows e.g. by observing that $\mathbf{E}_x[T_Q] = E_{\widetilde{x}}[T_{Q(0,r)}] = \sum_{y\in Q(0,r)} g_{\mathbb{Z}^d\setminus Q(0,r)}(\widetilde{x}, y)$, where $\widetilde{x} \in Q(0,r) \subset \mathbb{Z}^d$ is such that $\pi(\widetilde{x}) = x$, and performing the sum using (A.2) (the upper bound in (A.2) remains valid without restriction on x and y).

We now first assume that $r \geqslant \frac{N}{100}$. In this case (A.24) is immediate since $\mathbf{E}_x[H_Q] \geqslant \mathbf{E}_x[T_{Q(x,\frac{\delta}{2}r)}] \geqslant c(\delta)N^2$ by (A.25) and assumption on r. As to (A.23), writing the expected value in terms of its tail probabilities, one readily obtains for any $\lambda \geqslant 1$ that

$$(A.26) \qquad \mathbf{E}_x[H_Q] \leqslant \lambda N^2 \big(1 + \sum_{k \geqslant 1} \mathbf{P}_x[H_Q > k\lambda N^2] \big).$$

One then argues that for all $x \in \mathbf{T}$, with $\tilde{x}$ as above and $\tilde{Q} = Q(\tilde{x}, \lambda^{1/4}N)$,

$$\mathbf{P}_x(H_Q \leqslant \lambda N^2) = P_{\tilde{x}}(H_{\pi^{-1}(Q)} \leqslant \lambda N^2) \geqslant P_{\tilde{x}}(H_{\pi^{-1}(Q)} \leqslant T_{\tilde{Q}}) - c\lambda^{-1/2} \geqslant c_0$$

upon choosing λ large enough; here the first lower bound follows from (A.25) and a first-moment estimate and the second one simply by observing that $\pi^{-1}(Q) \cap \tilde{Q}$ always contains at least one translate of $Q(0,r)$ in its bulk, i.e. at distance at most N from $\tilde{x}$, and so since $r \geqslant cN$ it follows from (A.3) and monotonicity that the hitting probability $P_{\tilde{x}}[H_{\pi^{-1}(Q)} \leqslant T_{\tilde{Q}}]$ admits a uniform lower bound. Feeding the resulting estimate into (A.26) and applying the Markov property yields that the sum on the right-hand side is bounded by $\sum_{k \geqslant 1}(1 - c_0)^k < \infty$, and (A.23) follows.

Assume now that $r \leqslant \frac{N}{100}$. Consider the set $V = \partial Q(\mathbf{0}, \frac{N}{20})$. As we now explain, it is enough to argue that

$$(A.27) \qquad c \cdot N^2 \Big(\frac{N}{r} \Big)^{d-2} \leqslant \mathbf{E}_y\big[H_Q\big] \leqslant C \cdot N^2 \Big(\frac{N}{r} \Big)^{d-2}, \quad y \in V;$$

indeed, once (A.27) is shown, the bound (A.23) immediately follows by applying the strong Markov property at time H_V, by which $\mathbf{E}_x[H_Q] \leqslant \mathbf{E}_x[H_V] + \sup_{y \in V} \mathbf{E}_y[H_Q]$, using (A.27) to bound the second term and (A.23) in the case already treated to deduce that $\sup_{x \in \mathbf{T}} \mathbf{E}_x[H_V] \leqslant CN^2$. To obtain (A.24), one writes instead $\mathbf{E}_x[H_Q] \geqslant \mathbf{P}_x(H_Q > H_V) \inf_{y \in V} \mathbf{E}_y[H_Q]$. The desired lower bound now follows from (A.27) and since by (A.3)

$$\mathbf{P}_x(H_Q > H_V) \geqslant \inf_{z \in \mathbb{Z}^d \setminus Q(0,r(1+\delta))} P_z[H_{Q(0,r)} = \infty] \geqslant c(\delta).$$

It thus remains to show (A.27), under the assumption $r \leqslant \frac{N}{100}$. Throughout the rest of this proof we abbreviate $R_k = R_k(X, Q(\mathbf{0}, \frac{N}{10})^c \cup Q, V^c)$ and $D_k = D_k(X, Q(\mathbf{0}, \frac{N}{10})^c \cup Q, V^c)$, see (5.4) whose definition can easily be extended from boxes in $\mathbb{Z}^d$ to general sets in $\mathbf{T}$, the system of successive stopping times corresponding to the excursions from $Q(\mathbf{0}, \frac{N}{10})^c \cup Q$ to V. Let

$$(A.28) \qquad K = \min\{k \geqslant 1 : X_{R_k} \in Q\},$$

which counts the number of excursions of type $X_{[D_k, R_{k+1}]}$ until the first one that visits Q. For all $y \in V$ one obtains by a reasoning similar to (A.4), using the assumption on r, that $\mathbf{P}_y(X_{R_1} \in Q)$ is comparable with $(\frac{r}{N})^{d-2}$, which together with the Markov property, is readily seen to imply that K stochastically dominates/is stochastically dominated by geometric random variables with corresponding parameters. In particular, it follows that for all $y \in V$,

$$(A.29) \qquad c \cdot \Big(\frac{N}{r} \Big)^{d-2} \leqslant \mathbf{E}_y[K] \leqslant C \cdot \Big(\frac{N}{r} \Big)^{d-2},$$

which is all we will use in the sequel. Now, by definition of R_k and D_k, for any $y \in V$, one has that $\mathbf{P}_y$-a.s. $H_Q = R_K$, hence

$$(A.30) \qquad \mathbf{E}_y\big[H_Q\big] = \mathbf{E}_y\Big[\sum_{1 \leqslant k \leqslant K} (R_k - R_{k-1}) \Big]$$

Owing to the strong Markov property, with $\mathcal{F}_k = \sigma(X_{\cdot \wedge R_k})$, since $\{K \geqslant k\} \in \mathcal{F}_{k-1}$, one has that

$$\text{(A.31)} \quad \mathbf{E}_y\Big[\sum_{1 \leqslant k \leqslant K} (R_k - R_{k-1}) \Big] = \sum_{k \geqslant 1} \mathbf{E}_y\big[(R_k - R_{k-1}) \mathbf{1}\{K \geqslant k\} \big]$$

$$= \sum_{k \geqslant 1} \mathbf{E}_y\big[\mathbf{E}_y[(R_k - R_{k-1})|\mathcal{F}_{k-1}] \mathbf{1}\{K \geqslant k\} \big]$$

and for any $y \in V$ and $k \geqslant 1$

$$\mathbf{E}_y[(R_k - R_{k-1})|\mathcal{F}_{k-1}] = \mathbf{E}_y\big[\mathbf{E}_{X_{R_k}}[R_1] \big] \leqslant \sup_{v \in V} \mathbf{E}_v[T_{Q(\mathbf{0}, \frac{N}{10})}] + \sup_{w \in \mathbf{T}} \mathbf{E}_w[H_V] \leqslant C N^2,$$

which follows on account of (A.25) and (A.23) for the choice $r = \frac{N}{20}$ (already treated). One also has a corresponding deterministic lower bound of the same order by (A.25), since by the Markov property, $R_k - R_{k-1}$ under $\mathbf{P}_y(X_{D_{k-1}} = y' \,|\, \mathcal{F}_{k-1})$, $y' \in V$, is stochastically dominated by $T_{Q(y', \frac{N}{50})}$, using the fact that $r \leqslant \frac{N}{100}$. Feeding the above deterministic upper/lower bound on the conditional expectation into (A.31) and using (A.29) to bound the resulting $\mathbf{E}_y[K]$, one deduces (A.27) from (A.30). $\qquad\square$

With Lemma A.8 at hand, we proceed with the:

Proof of Lemma 5.6 (Random walk case). We aim to apply Proposition A.7 with the choices $i_0 = 0$, $V_i = R_{i+1} - R_i$, so that $V_{i_0} = R_1$ and $m \overline{V}_m = R_m$ for any $m \geqslant 1$ in view of (A.14). Now pick $\eta = 1 - \frac{1}{1+\varepsilon}$ and $m = \lceil (1+\varepsilon) u M \rceil$ with

$$\text{(A.32)} \qquad\qquad M = \frac{N^d}{\mathbf{E}_\nu[V_0]},$$

which equals the value defined by (5.28) on account of (A.12)-(A.13). Note in passing that (A.32) is very intuitive (more so than its pendant (5.28)): uM with M as in (A.32) is the total time uN^d for the walk, divided by the 'average' time $\mathbf{E}_\nu[V_0]$ consumed by an excursion, whence M counts the 'average' number of excursions. With the above choices,

$$\mathbf{P}_\mathbf{0}\big(\mathcal{N}_{\mathrm{RW}}(B_2, B_3, u) \geqslant (1+\varepsilon) u M \big) \overset{(5.9)}{\leqslant} \mathbf{P}_\mathbf{0}\big(R_{\lceil (1+\varepsilon) u M \rceil} \leqslant u N^d \big) \leqslant \mathbf{P}_\mathbf{0}\big(\overline{V}_m \leqslant (1-\eta) \mathbf{E}_\nu[V_0] \big),$$

with B_k as in (5.23). Similarly, taking $m' = \lfloor (1-\varepsilon) u M \rfloor + 1$ and $\eta' = -1 + \frac{1}{1 - \varepsilon/2}$, one obtains that if $\varepsilon u M \geqslant 2$

$$\mathbf{P}_\mathbf{0}\big(\mathcal{N}_{\mathrm{RW}}(B_2, B_3, u) \leqslant (1-\varepsilon) u M \big) \leqslant \mathbf{P}_\mathbf{0}\big(\overline{V}_{m'} \geqslant (1 + \eta') \mathbf{E}_\nu[V_0] \big).$$

Since by monotonicity, we may assume that $\varepsilon u M \geqslant 2$ and $\varepsilon \leqslant 1/2$, the claim immediately follows by means of Proposition A.7, provided we show that (A.15) holds, which we proceed to do with the help of Lemma A.8.

Recalling the definition of $R_1(= V_0)$ from (5.4), identifying B_k with its projection onto $\mathbf{T}$, one has that $R_1 \geqslant T_{B_3}$ holds $\mathbf{P}_y$-a.s. for any $y \in \mathbf{T}$. Thus, applying the strong Markov property at time T_{B_3}, and combining (A.23) and the exit time estimate $\mathbf{E}_y[T_{B_3}] \leqslant C r_3^2$, valid for all $x \in \mathbf{T}$ (cf. (A.25)), one sees that

$$\text{(A.33)} \qquad \sup_{y \in \mathbf{T}} \mathbf{E}_y[V_0] \leqslant \sup_{y \in B_3} \mathbf{E}_y[T_{B_3}] + \sup_{z \in \mathbf{T}} \mathbf{E}_z[H_{B_2}] \leqslant C\Big(r_3^2 + \frac{N^d}{r_2^{d-2}} \Big) \leqslant C' \frac{N^d}{r_2^{d-2}}.$$

In particular, this implies that $V_0 \in L^1(\mathbf{P}_y)$ for any $y \in \mathbf{T}$, as required by (A.15). On the other hand, combining the Markov property and (A.24), it follows that

$$\text{(A.34)} \qquad \inf_{y \in \mathbf{T}} \mathbf{E}_y[V_0] \geqslant \inf_{z \notin B_3} \mathbf{E}_z[H_{B_2}] \geqslant c(\delta)\frac{N^d}{r_2^{d-2}}.$$

Combining (A.33) and (A.34), the condition in the first line of (A.15) immediately follows, for all suitably large $\theta \geqslant C(\delta)$. Regarding higher moments, noting that by the Markov property $V_0 - T_{B_3}$ has the same law as H_{B_2} starting from some random point of B_3^c, and applying the bound $(a+b)^k \leqslant 2^k(a^k + b^k)$ valid for all $a, b \geqslant 0$, yields that for all $x \in \mathbf{T}$, and $k \geqslant 2$,

$$\text{(A.35)} \qquad \mathbf{E}_x[V_0^k] \leqslant 2^k\big(\mathbf{E}_x[T_{B_3}^k] + \sup_{y \notin B_3} \mathbf{E}_y[H_{B_2}^k]\big).$$

Applying a similar argument as e.g. in [27, (2.21)], one obtains that for all $x \in \mathbf{T}$,

$$\text{(A.36)} \qquad \mathbf{E}_x[T_{B_3}^k] \leqslant k! \sup_{y \in \mathbf{T}} \mathbf{E}_y[T_{B_3}]^k \overset{\text{(A.25)}}{\leqslant} k! C^k r_3^{2k} \overset{\text{(A.34)}}{\leqslant} k! C'(\delta)^k \inf_{y \in \mathbf{T}} \mathbf{E}_y[V_0]^k.$$

Similarly, using the fact that $V_0 \geqslant H_{B_2}$ holds $\mathbf{P}_y$-a.s. for any $y \in \mathbf{T}$, one finds that for all $y \notin B_3$,

$$\text{(A.37)} \qquad \mathbf{E}_y[H_{B_2}^k] \leqslant k! \sup_{x \in \mathbf{T}} \mathbf{E}_x[H_{B_2}]^k \leqslant k! \sup_{x \in \mathbf{T}} \mathbf{E}_x[V_0]^k \leqslant k! C(\delta)^k \inf_{x \in \mathbf{T}} \mathbf{E}_x[V_0]^k,$$

where the last step follows from the first moment comparison in (A.15) already established. Feeding (A.36) and (A.37) into (A.35) completes the verification of (A.15), for suitably large choice of $\theta = \theta(\delta) \in (1, \infty)$, and with it the proof. $\qquad\square$

We now turn to the proofs of Lemmas 5.5 and 5.6 for random interlacements. The starting point is the following large deviation estimate, similar to Proposition A.7 above, but simpler. For finite $B \subset \mathbb{Z}^d$ let W_B denote the set of infinite nearest-neighbor paths in $\mathbb{Z}^d$ starting in ∂B escaping all finite sets in finite time. Recalling (2.11) with $B = Q_r(0)$, $r \geqslant 1$, $(X^j)_{j \geqslant 1}$ denotes in the sequel the random walks on $\mathbb{Z}^d$ corresponding to the restriction of the interlacement process to $Q_r(0)$, which are i.i.d. with law $P_{\bar{e}_r}$, where $\bar{e}_r \equiv \bar{e}_{Q_r(0)}$, and at level u the number $N_r^u \equiv N_{Q_r(0)}^u$ of trajectories hitting $Q_r(0)$ is a Poisson random variable with parameter $u\mathrm{cap}(Q_r)$, independent of $(X^j)_{j \geqslant 1}$. Similarly as above (A.14), we consider an $(A, \mathcal{A})$-valued sequence $(Z^j)_{j \geqslant 1}$, which we assume to be declared under $\mathbb{P}^I$ and independent of $((X^j)_{j \geqslant 1}, N_r^u)$, and study for measurable $F : W_{Q_r} \times A \to [0, \infty)$, $u > 0$ and $r \geqslant 1$ the averages

$$\text{(A.38)} \qquad \overline{V}_u = \frac{1}{u\mathrm{cap}(Q_r)} \sum_{j=1}^{N_r^u} V_j, \quad V_j = F(X^j, Z^j).$$

(with $\overline{V}_u = 0$ by convention whenever $N_r^u = 0$).

Proposition A.9. *For all $r, \theta \geqslant 1$, there exist $c = c(\theta), C = C(\theta) \in (0, \infty)$ such that, if*

$$\text{(A.39)} \qquad \mathbb{E}^I[V_1] < \infty \text{ and } \mathbb{E}^I[V_1^k] \leqslant k! \theta^k \mathbb{E}^I[V_1]^k, \text{ for all } k \geqslant 2,$$

then for all $u > 0$ and $\eta \in (0, 1)$, one has

$$\text{(A.40)} \qquad \mathbb{P}^I\big(|\overline{V}_u - \mathbb{E}^I[V_1]| > \eta\mathbb{E}^I[V_1]\big) \leqslant C \exp\big\{-c\eta^2 u\mathrm{cap}(Q_r)\big\}.$$

Proof. Under (A.15), it follows from Bernstein's inequality, see for instance [7, Corollary 2.11] for the version we use here, that for all $\delta \in (0,1)$ and integers $m \geqslant 1$ (see around (A.19) for a similar argument)

$$(A.41) \qquad \mathbb{P}^I\Big(\Big|\frac{1}{m}\sum_{j=1}^{m}V_j - \mathbb{E}^I[V_1]\Big| > \delta\mathbb{E}^I[V_1]\Big) \leqslant 2\exp\big\{-c(\theta)\delta^2 m\big\},$$

Using a Chernoff bound for Poisson random variables, see for instance [7, p.21-23] combined with the inequality $(1+\varepsilon)\log(1+\varepsilon) - \varepsilon \geqslant \varepsilon^2/4$ for all $\varepsilon \in (-1,1)$, one has for all $u > 0$ and $\varepsilon \in (0,1)$

$$\mathbb{P}^I(|N_r^u - u\mathrm{cap}(Q_r)| > \varepsilon u\mathrm{cap}(Q_r)) \leqslant \exp\big\{-c\varepsilon^2 u\mathrm{cap}(Q_r)\big\}.$$

Combining this with (A.41), applied with (ε,δ,m) chosen either as $(\eta/2, \eta/(2+\eta), \lfloor u(1+\eta/2)\mathrm{cap}(Q_r)\rfloor)$ or $(\eta/2, \eta/(2-\eta), \lceil u(1-\eta/2)\mathrm{cap}(Q_r)\rceil)$, and assuming that $\eta u\mathrm{cap}(Q_r) \geqslant 2$ and $\eta < 1/4$ which is no loss of generality, (A.40) readily follows by means of a suitable union bound. $\qquad\square$

Remark A.10. Similarly as in Remark A.6 one can derive an exact formula for a key quantity associated to random interlacements. First, defining $T = T(X, B_2, B_3)$ as in (5.10), by [8, eq. (6.9)-(6.11)] we have

$$(A.42) \qquad E_{\overline{e}_{B_2}}\Big[\sum_{i=0}^{T}\mathbb{1}\{X_{R_i(X,B_2,B_3)} = x\}\Big] = \frac{e_{B_2}^{B_3}(x)}{\mathrm{cap}(B_2)} \text{ for all } x \in B_2.$$

Note that compared to [8] we started the sum at $i = 0$ instead of $i = 1$, which is due to the fact that we started the definition of R_i from $R_0 = 0$, see (5.4), while in the paragraph above (4.8) in [8] it starts from $R_1 = 0$. In particular in view of (A.12), the left-hand side of (A.42) is crucially proportional (up to projection on the torus) to the invariant distribution of the stopping times R_i for the random walk on the torus. Note that we only use these exact formulas to prove that the means in Lemmas 5.5 and 5.6 are the same for random interlacements and the random walk, and thus that the process $\omega^{(x)}$ is an interlacement process in (5.1) under $\widetilde{\mathbf{P}}_0$. In particular, if one is only interested in the proof of all our results for random interlacements, these formulas are never required.

Similarly as for the random walk, we are now ready to prove Lemmas 5.5 and 5.6 for random interlacements (starting with the latter), using Proposition A.9 instead of Proposition A.7.

Proof of Lemma 5.6 (Random interlacements case). For $r = r_2$ take $V_j = T^j + 1$ in (A.38), see (5.10), where $T^j = T^j(B_2, B_3)$ is defined as in (5.12) but for the walk X^j now starting in B_2, and $\eta = \varepsilon$. With these choices $\overline{V}_u = (u\mathrm{cap}(Q_r))^{-1}\mathcal{N}_{\mathrm{RI}}(\omega, B_2, B_3, u)$. Note that we consider here $T^j + 1$ instead of T^j since the walks X^j in (5.12) were started outside of B_3, and thus the times R_i, see (5.4), are shifted by 1 compared to the corresponding walk started in B_2. It moreover follows from (A.3) and monotonicity that the random variable T^1 is dominated by a geometric random variable of parameter p for some $p = p(\delta) > 0$, which readily implies that condition (A.39) is verified for some $\theta = \theta(\delta)$. Moreover by the lower bound in (A.3), we have $\mathrm{cap}_{B_3}(B_2) \leqslant C(\delta)\mathrm{cap}(B_2)$. Since $\mathbb{E}^I[V_1] = E_{\overline{e}_{B_2}}[T(X, B_2, B_3)] + 1$ the claim follows by an application of (A.40) and summing (A.42) over $x \in B_2$. $\qquad\square$

Proof of Lemma 5.5 (Random interlacements case). Take $r = r_2$ and denote again by $T^j = T(X^j, B_2, B_3)$ the total number of excursions that the walk X^j performs across the annulus $B_3\backslash B_2$, see (5.10), and let $(\zeta_i(X^j, B_2, B_3))_{i\geqslant 0}$ be the clothesline process associated to X^j, see (5.5) (whose definition can clearly be extended to the case $i = 0$), where $(X^j)_{j\geqslant 1}$ is now the set

of walks in the interlacements process ω_{B_2} that hit B_2, started after first hitting B_2. Let $\widehat{\xi}_i^j$ refers to $\widehat{\xi}_k$ as appearing in (5.22) for the unique choice of k such that $k = i + \sum_{1 \leqslant n < j}(T^n + 1)$, which is an independent and i.i.d. sequence of exponential random variables with mean 1. Define for some fixed $z \in \Sigma$

$$V_j = \sum_{0 \leqslant i \leqslant T^j} \widehat{\xi}_i^j g_{\zeta_i(X^j, B_2, B_3)}(z).$$

As in the random walk case, by assumption on the r_k's and Lemmas A.2 and A.3, one has that $g_\zeta(z) \leqslant C(\delta)g_{\zeta'}(z)$ for all $\zeta, \zeta' \in \partial B_2 \times \partial B_3^c$ and $z \in \Sigma$. One can then deduce that assumption (A.39) holds similarly as in the proof of Lemma 5.6 for random interlacements, for suitable choice of $\theta = \theta(\delta) \in (1, \infty)$. Moreover, it follows from (5.5), (5.20), (A.42) and a small calculation involving the Markov property that

$$\mathbb{E}^I[V_1] = \bar{g}(z) \cdot \frac{\mathrm{cap}_{B_3}(B_2)}{\mathrm{cap}(B_2)}.$$

Finally, for all $u_1 < u_2$ we have by the definitions of $\mathcal{N}_{\mathrm{RI}}$ in (5.12) and $G_\cdot^{\mathrm{RI}}$ in (5.22) (note that similarly as before the clotheslines in these definition are shifted by 1 since the walk therein is started outside of B_3 instead of inside B_2)

$$\mathcal{N}_{\mathrm{RI}}(\omega, B_2, B_3, u_1) \leqslant m \leqslant \mathcal{N}_{\mathrm{RI}}(\omega, B_2, B_3, u_2) \quad \Longrightarrow \quad u_1 \overline{V}_{u_1} \leqslant \frac{G_m^{\mathrm{RI}}(z)}{\mathrm{cap}(B_2)} \leqslant u_2 \overline{V}_{u_2}.$$

Taking $u_1 = m/(M(1 + \varepsilon/3))$ and $u_2 = m/(M(1 - \varepsilon/3))$ we conclude by combining (A.40) for $u = u_1, u_2$ and $\eta = \varepsilon/3$, with the concentration of $\mathcal{N}_{\mathrm{RI}}$ supplied by Lemma 5.6, for $u = u_1, u_2$ and $\varepsilon/3$ in place of ε, as well as the inequality $\mathrm{cap}_{B_3}(B_2) \leqslant C\mathrm{cap}(B_2)$. $\qquad \square$

Remark A.11 (Extensions). As with all results of Section 5, both Lemmas 5.5 and 5.6 involve square boxes. In particular, this means for instance that within the setup of Theorem 5.1 or Corollary 5.2, no ℓ^2-smoothing of boxes as used e.g. in [8] is necessary. This degree of flexibility is relevant for applications to more general classes of graphs (for which a meaningful notion of smoothing is often not even clear) under suitable hypotheses (e.g. polynomial volume growth, polynomial decay of the Green function and the validity of an elliptic Harnack inequality, cf. §A.1), to which the above arguments can likely be extended.

B   Appendix: admissible sets

As explained below Theorem 1.3, an important question is to determine which sets are admissible, i.e. belong to $\mathcal{A}_{\mathbf{T}}$ in (1.24), for these are precisely the 'patterns' that can be seen as part of $\mathcal{L}^\alpha$ for some $\alpha > \frac{1}{2}$. We classify these sets in Theorem B.1 below. Once this is established, one readily deduces Corollary 1.4 from Theorem 1.3; the short proof appears at the end of this appendix.

Let

(B.1) $$\mathcal{A}_{\mathbb{Z}^d} = \{K \subset \mathbb{Z}^d : K \neq \varnothing, \mathrm{cap}(K) \leqslant \tfrac{2}{g(0)}\}$$

so that, in view of (1.19), (1.24), (2.10) and our definition of the capacity for subsets of $\mathbf{T}$, see below (2.3), the family $\mathcal{A}_{\mathbf{T}}$ corresponds precisely to projections onto $\mathbf{T}$ of sets belonging to $\mathcal{A}_{\mathbb{Z}^d}$ when $N \geqslant C$.

Theorem B.1 (Admissible sets)**.**

$$\mathcal{A}_{\mathbb{Z}^d} = \{K \subset \mathbb{Z}^d : |K| \leqslant 2\}, \text{ for all } d \geqslant 4, \text{ and}$$
$$\mathcal{A}_{\mathbb{Z}^3} = \{K \subset \mathbb{Z}^d : |K| \leqslant 2 \text{ or } |K| = 3 \text{ and } K \text{ is connected}\}.$$

The following lemma will be used to reduce to the cases $d \in \{3, 4\}$ when proving Theorem B.1. In the sequel we add subscripts $\mathbb{Z}^d$ to various quantities such as $\mathrm{cap}(\cdot)$ or $g(\cdot)$ to highlight their dependence on the underlying graph.

Lemma B.2. *For all $d \geqslant 3$ and finite $K \subset \mathbb{Z}^d$, the function*

$$d' \in \{d, d+1, \dots\} \mapsto \frac{\mathrm{cap}_{\mathbb{Z}^{d'}}(K \times \{0\}^{d'-d})}{\mathrm{cap}_{\mathbb{Z}^{d'}}(\{0\})}$$

is non-decreasing.

Proof. To stress its dependence on dimension, we write $P_x^{\mathbb{Z}^d}$ for the canonical law of simple random walk on $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. For all $x, y \in K$ writing $x' = (x, 0, \dots, 0)$ and $y' = (y, 0, \dots, 0) \in \mathbb{Z}^{d'}$ one has

$$\frac{g_{\mathbb{Z}^d}(x, y)}{g_{\mathbb{Z}^d}(0,0)} = P_x^{\mathbb{Z}^d}(H_y < \infty) = P_{x'}^{\mathbb{Z}^{d'}}(H_{\{y\} \times \mathbb{Z}^{d'-d}} < \infty) \geqslant P_{x'}^{\mathbb{Z}^{d'}}(H_{\{y\} \times \{0\}^{d'-d}} < \infty) = \frac{g_{\mathbb{Z}^{d'}}(x', y')}{g_{\mathbb{Z}^{d'}}(0,0)}.$$

The claim follows using a well-known variational characterisation of the capacity, see [31, (1.61)], whose proof easily extend to infinite transient graphs. $\square$

Our next result will be used as a further reduction step, by which fully determining $\mathcal{A}_{\mathbb{Z}^d}$ in (1.24) will effectively boil down to computing the capacities of a small number (at most ten) of sets. To simplify notation we will from now for each $d' \leqslant d$ identify $\mathbb{Z}^{d'}$ with $\mathbb{Z}^{d'} \times \{0\}^{d-d'} \subset \mathbb{Z}^d$. Recall the sets $K_1 = \{(0,0), (0,1), (0,2)\}$ and $K_2 = \{(0,0), (0,1), (1,0)\}$ from (1.25), viewed as subsets of $\mathbb{Z}^2 \times \{0\}^{d-2}$ for $d \geqslant 3$ according to our convention, and which correspond to all connected sets with cardinality three. Further, let

$$
\begin{aligned}
&A_1 = \{(0,0,0), (0,2,0), (0,1,1)\}, \quad A_2 = \{(0,0,0), (0,2,0), (0,3,0)\}, \\
&A_3 = \{(0,0,0), (1,1,0), (0,3,0)\}, \quad A_4 = \{(0,0,0), (0,2,0), (1,2,0)\}, \\
&A_5 = \{(0,0,0), (1,1,0), (1,2,0)\}, \quad A_6 = \{(0,0,0), (0,2,0), (1,1,1)\}, \\
&A_7 = \{(0,0,0), (1,1,0), (1,1,1)\}, \quad A_8 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\}.
\end{aligned}
$$

(B.2)

The following result mirrors Theorem B.1. Its first part (B.3) will be enough to treat the cases $d \geqslant 4$; the more refined (B.4) will be used to deal with the case $d = 3$. In what follows, K and K' are called isomorphic if K can be obtained from K' by lattice symmetries.

Proposition B.3 $(d \geqslant 3, K \subseteq \mathbb{Z}^d, |K| \geqslant 3)$.

$$(B.3) \qquad\qquad \inf_K \mathrm{cap}(K) \geqslant \min_{i=1,2} \mathrm{cap}(K_i),$$

$$(B.4) \qquad\qquad \widetilde{\inf_K} \mathrm{cap}(K) \geqslant \min_{1 \leqslant i \leqslant 8} \mathrm{cap}(A_i),$$

where $\widetilde{\inf}_K$ refers to a restricted infimum over sets K not isomorphic to K_1 or K_2.

Proof. We start by making the following observation, which will be used extensively throughout the proof. Using again the variational characterization [31, (1.61)] of $\mathrm{cap}(\cdot)$, one sees that,

(B.5)   if $K, K' \subset \mathbb{Z}^d$ are such that there exists a bijection $\varphi : K \to K'$ with $g(x,y) \leqslant g(\varphi(x), \varphi(y))$ for all $x, y \in K$, then $\mathrm{cap}(K) \geqslant \mathrm{cap}(K')$.

We proceed to show (B.3). By monotonicity of $K \mapsto \mathrm{cap}(K)$, it is sufficient to prove the claim for a set K with $|K| = 3$. Write $K = \{x, y, z\}$ and without loss of generality suppose that $|x - z|_1 \geqslant 2$. Then by Lemma 2.1 and translational and rotational invariance we have

$$g(x, z) \leqslant \sup_{v:|v|_1=2} g(0, v) = (g((0,0), (0,2)) \vee g((0,0), (1,1))).$$

Similarly,

$$(g(x, y) \vee g(y, z)) \leqslant \sup_{v:|v|_1=1} g(0, v) = g((0,0), (0,1)) = g((0,1), (0,2)) = g((0,1), (1,1)).$$

So if $g((0,0), (0,2)) \geqslant g((0,0), (1,1))$, then (B.5) applies with $K' = \{(0,0), (0,1), (0,2)\} = K_1$, $\varphi(x) = (0,0)$, $\varphi(y) = (0,1)$ and $\varphi(z) = (0,2)$ and yields that $\text{cap}(K) \geqslant \text{cap}(K_1)$. If instead $g((0,0), (0,2)) \leqslant g((0,0), (1,1))$, then (B.5) applies similarly with $K' = \{(0,0), (0,1), (1,1)\} = K_2$ to give $\text{cap}(K) \geqslant \text{cap}(K_2)$. Overall, (B.3) follows.

We now show (B.4), and first assume to this effect that $|K| = 3$ and K is not isomorphic to K_1 or K_2. Writing $K = \{x, y, z\}$, one notices since K is not isomorphic to K_1 or K_2, then unless K is isomorphic to A_1, in which case $\text{cap}(K)$ is evidently bounded from below by the right-hand side of (B.4), the set K must contain two points at ℓ^1-distance at least three, say x and z. Without loss of generality suppose that $|x - y|_1 \geqslant 2$ (otherwise $|y - z|_1 \geqslant 2$ by the triangle inequality). Then by Lemma 2.1,

$$g(x, z) \leqslant \sup_{v:|v|_1=3} g(0, v), \qquad g(x, y) \leqslant \sup_{v:|v|_1=2} g(0, v), \qquad g(y, z) \leqslant \sup_{v:|v|_1=1} g(0, v).$$

We then consider three different cases depending on which $v \in \{(0,3,0), (1,2,0), (1,1,1)\}$ achieves $\sup_{v:|v|_1=3} g(0, v)$. Then for each possible v we consider two different cases depending on which $v' \in \{(0,2,0), (1,1,0)\}$ achieves $\sup_{v':|v'|_1=2} g(0, v')$. Note also that $\sup_{v'':|v''|_1=1} g(0, v'')$ must be achieved at $g((0,0,0), (0,0,1))$ by symmetry. If for instance the previous suprema are achieved at $v = (0,3,0)$ and $v' = (0,2,0)$, (B.5) implies $\text{cap}(K) \geqslant \text{cap}(A_2)$. Considering all five other possible cases gives us that if K is not isomorphic to K_1 or K_2, then

$$\text{cap}(K) \geqslant \min_{1 \leqslant i \leqslant 7} \text{cap}(A_i),$$

which completes the proof in the case $|K| = 3$. Suppose next that $|K| \geqslant 4$. It suffices to prove that assuming $\text{cap}(K) < \min_{i \leqslant 7} \text{cap}(A_i)$, then $\text{cap}(K) = \text{cap}(A_8)$. The assumption that $\text{cap}(K) < \min_{i \leqslant 7} \text{cap}(A_i)$ implies that all $K' \subseteq K$ with $|K'| = 3$ must satisfy $\text{cap}(K') < \min_{i \leqslant 7} \text{cap}(A_i)$. From the proof above for $|K| = 3$ we can now deduce that any such K' must be isomorphic to either K_1 or K_2. By an elementary geometric argument, it then follows that the only possible shape for K for which this is possible is $K = A_8$ modulo isomophisms. Hence $\text{cap}(K) = \text{cap}(A_8)$, which finishes the proof. $\qquad \square$

Combining Lemma B.2 and Proposition B.3 with the next result, we will soon see that in order to identify the set $\mathcal{A}_{\mathbb{Z}^d}$ for all $d \geqslant 3$, it will be enough to compute the Green's function $g(0)$ and the capacities of the sets K_1 and K_2 from (1.25) in dimensions three and four, as well as the capacities of the sets A_i, $i \in \{1, \ldots, 8\}$ from (B.2) in dimension three. The following lemma gathers these numerical computations, and, in doing so, also isolates the parts of the argument which rely on computer-assisted methods. Note that we express our numerical results with an absolute error of 10^{-30} as this might be useful in the future, but we will actually only need a precision 10^{-3}.

Lemma B.4. *With an absolute error of at most 10^{-30}, one has when $d = 3$*

$$g_{\mathbb{Z}^3}(0) = 1.516386059151978018156012159681$$
$$\text{cap}_{\mathbb{Z}^3}(K_1) \vee \text{cap}_{\mathbb{Z}^3}(K_2) = 1.271113197748638670916474203095$$
$$\min_{1 \leqslant i \leqslant 8} \text{cap}_{\mathbb{Z}^3}(A_i) = 1.335471948363948449723770501931$$

and when $d = 4$

$$g_{\mathbb{Z}^4}(0) = 1.239467121848481712678697664859$$
$$\text{cap}_{\mathbb{Z}^4}(K_1) \wedge \text{cap}_{\mathbb{Z}^4}(K_2) = 1.849398784221098051683201012328.$$

Before explaining how the values in Lemma B.4 are obtained, let us conclude the proof of Theorem B.1.

Proof of Theorem B.1. We first observe that the inclusion

(B.6)
$$\mathcal{A}_{\mathbb{Z}^d} \supset \{K \subset \mathbb{Z}^d : 0 < |K| \leqslant 2\}$$

holds true for all $d \geqslant 3$ on account of (2.6). We now proceed in increasing order of difficulty, and start with the case $d = 4$. By Lemma B.4 we have that $\mathrm{cap}_{\mathbb{Z}^4}(K_1) \wedge \mathrm{cap}_{\mathbb{Z}^4}(K_2) \geqslant 1.84 > 1.62 \geqslant 2/g_{\mathbb{Z}^4}(0)$. Using (B.3), we deduce that there are no sets $K \in \mathcal{A}_{\mathbb{Z}^4}$ with $|K| \geqslant 3$. It follows that the inclusion in (B.6) is in fact an equality.

Next when $d \geqslant 5$, one deduces from Lemma B.2 and the previous case that $g_{\mathbb{Z}^d}(0)(\mathrm{cap}_{\mathbb{Z}^d}(K_1) \wedge \mathrm{cap}_{\mathbb{Z}^d}(K_2)) \geqslant g_{\mathbb{Z}^4}(0)(\mathrm{cap}_{\mathbb{Z}^4}(K_1) \wedge \mathrm{cap}_{\mathbb{Z}^4}(K_2)) > 2$. Using (B.3), we can conclude similarly as before.

Finally, assume that $d = 3$. Then by Lemma B.4 one has

$$\mathrm{cap}_{\mathbb{Z}^3}(K_1) \vee \mathrm{cap}_{\mathbb{Z}^3}(K_2) \leqslant 1.28 < 1.31 \leqslant \frac{2}{g_{\mathbb{Z}^3}(0)} \text{ and } \min_{1 \leqslant i \leqslant 8} \mathrm{cap}_{\mathbb{Z}^3}(A_i) \geqslant 1.33 > 1.32 \geqslant \frac{2}{g_{\mathbb{Z}^3}(0)}.$$

Using (B.4), we deduce that the only sets $K \in \mathcal{A}_{\mathbb{Z}^3}$ with $|K| \geqslant 3$ are isomorphic to K_1 and K_2. Together with (B.6), the claim follows. □

We now explain in detail how Lemma B.4 is obtained. The first step in our algorithm consists in computing the Green's function $g(x, y)$ for any x, y belonging to the sets for which we want to compute the capacity, which will be enough in view of (B.10) below. We follow the strategy developed in [18, Appendix B]. Let us provide some details for the reader's convenience. The main idea is to use the formula

(B.7)
$$g(x) = \int_{-\infty}^{\infty} F_x(u) \mathrm{d}u, \text{ where } F_x(u) = de^u \prod_{k=1}^{d} \exp(-e^u) I_{|x_k|}(e^u) \text{ for all } x \in \mathbb{Z}^d,$$

and $I_k(t)$ denotes the modified Bessel function of the first kind with parameter k at time t. The formula (B.7) is a simple consequence of [23, (2.10)] and the substitution $u \mapsto de^u$. In [18] the integral in (B.7) is approximated by a finite sum in five steps, which we now summarize.

1) Replace the integral in (B.7) by the Riemann sum $h \sum_{m=-\infty}^{\infty} F_x(mh)$ for some small $h > 0$ to be chosen later. We denote the absolute error made in this step by $\mathrm{Error1}(h, d)$, which corresponds to [18, (B.64)] for the choice $s = \arctan(2\pi/h)$.

2) Remove $h \sum_{m=-\infty}^{-(M+1)} F_x(mh)$ in the previous Riemann sum, for some large M to be chosen later. We denote the absolute error made in this step by $\mathrm{Error2}(h, d, M)$, cf. [18, (B.68)]. Note in particular that Error1 and Error2 are uniform in $x \in \mathbb{Z}^d$.

3) Replace $h \sum_{m=M+1}^{\infty} F_x(mh)$ in the previous Riemann sum by

$$\frac{d}{(2\pi)^{\frac{d}{2}}} \frac{h \exp\left(-(M+1)(\frac{d}{2}-1)h\right)}{1 - \exp\left(-(\frac{d}{2}-1)h\right)}.$$

We denote the absolute error made in this step for any $|x|_\infty \leqslant N$ by $\mathrm{Error3}(h, d, M, N)$, which corresponds to [18, (B.72)]. Note that in [18, (B.72)] it is assumed that $N \leqslant 54$ and $Mh \geqslant 45$, as will be the case for us in (B.9).

4) For some large $T > 0$ and $J \in \mathbb{N}$ with $T \leqslant J/2$ to be chosen later, replace the occurrence of $e^{-t}I_k(t)$ (part of F_x) in the remaining finite sum by $T(t, k, J)$ for each $t \leqslant T$, where

$$T(t, k, J) = e^{-t} \left(\frac{t}{2}\right)^k \sum_{j=0}^{J} \frac{(t^2/4)^j}{j!(j+k)!}.$$

The relative error made in this step is $\mathrm{Error4}(T, J)$, which is uniform in k; it corresponds to [18, (B.40)].

5) For some $\widetilde{J} \in \mathbb{N}$ to be chosen later, replace $e^{-t}I_k(t)$ by $A(t, k, \widetilde{J})$ for each $t > T$ and $k \leqslant N$, where

$$A(t, k, \widetilde{J}) = \frac{1}{\sqrt{2\pi t}} \sum_{j=0}^{\widetilde{J}} \frac{(-1)^j (k, j)}{(2t)^j}, \quad \text{where } (k, j) = \frac{1}{4^j j!} \prod_{i=1}^{j} (4k^2 - (2i-1)^2).$$

We denote the relative error made in this step by $\mathrm{Error5}(T, \widetilde{J}, N)$, which corresponds to [18, (B.7) and (B.46)-(B.47)].

Combining all these steps one can approximate $g_{\mathbb{Z}^d}(x)$ by

$$\widetilde{g}(x, d, h, M, T, J, \widetilde{J}) \stackrel{\text{def.}}{=} \sum_{m=-M}^{M} dh e^{hm} \prod_{k=1}^{d} \left(T(e^{mh}, |x_k|, J) \mathbb{1}_{e^{mh} \leqslant T} + A(e^{mh}, |x_k|, \widetilde{J}) \mathbb{1}_{e^{mh} > T} \right)$$
$$+ \frac{d}{(2\pi)^{\frac{d}{2}}} \frac{h \exp\left(-(M+1)(\frac{d}{2}-1)h\right)}{1 - \exp\left(-(\frac{d}{2}-1)h\right)}.$$

The function $\widetilde{g}$ consists of finite sums and products of usual functions, and can thus be approximated using a computer with high precision. Such computations were performed in high dimensions in [17, Section 5] using a Mathematica notebook called "SRW.nb" available at [16]. We modified this notebook to include as well the computation of the capacity, and a version called "Cap.nb" is available at [26]. One still needs to choose the parameters h, M, T, J and $\widetilde{J}$. Since $g_{\mathbb{Z}^d}(0) \leqslant 2$, we have for all $x \in \mathbb{Z}^d$ with $|x|_\infty \leqslant N$

$$\begin{aligned}
(\text{B.8}) \qquad |g(x) - \widetilde{g}(x, d, h, M, T, J, \widetilde{J})| \leqslant{} &\mathrm{Error1}(h, d) + \mathrm{Error2}(h, d, M) + \mathrm{Error3}(h, d, M, N) \\
&+ 2\big((1 + \mathrm{Error4}(T, J) \vee \mathrm{Error5}(T, \widetilde{J}, N))^d - 1\big).
\end{aligned}$$

One can find the exact formulas for these errors in the file "Errors.nb", also available at [26], where they are also computed. To make these errors small, one typically needs to choose h small, and hM, T, J/T and $\widetilde{J}$ large. Choosing

$$(\text{B.9}) \qquad N = 3, \quad d \in \{3, 4\}, \quad h = \frac{76}{630}, \quad M = 630, \quad T = 80, \quad J = 139 \text{ and } \widetilde{J} = 30$$

we obtain that the total error in (B.8) is at most 10^{-32}. We then compute the values of $\widetilde{g}$ for these parameters and any x appearing in the sets from Lemma B.4, which are stored in the file "SRWIntegralsData.nb".

Let us now briefly explain how the capacities can be deduced from these Green's functions. For each $K \subset \mathbb{Z}^d$, denoting by G_K the matrix $(g(x, y))_{x,y \in K}$, by e_K the vector $(e_K(x))_{x \in K}$ and by $\mathbf{1}$ the vector of size $|K|$ with all coordinates equal to one, we have by (2.4)

$$(\text{B.10}) \qquad\qquad\qquad\qquad G_K e_K = \mathbf{1}.$$

Once G_K is known, one can thus use Mathematica again to solve the linear system (B.10), which yields the equilibrium measure and the capacity after summation. Note that we do not know the values of G_K, but only their approximate values $\widetilde{G}_K$ with $|\widetilde{G}_K(x,y) - G_K(x,y)| \leqslant 10^{-32}$ for each $x, y \in K$. Denoting by $\widetilde{e}_K$ the computed solution of $\widetilde{G}_K \widetilde{e}_K = 1$, we have $e_K(x) - \widetilde{e}_K(x) = G_K^{-1}(\widetilde{G}_K - G_K)\widetilde{e}_K(x)$. Using (1.38) and Proposition 1.11 in [31], one can moreover easily show that $\sum_{x,y \in K} |G_K^{-1}(x,y)| \leqslant 2|K|$, and for our choices of K one easily deduces that the values we obtain for the capacity have an error of at most $5 \cdot 10^{-31}$. This solution to the system (B.10) is also implemented in our Mathematica notebook "Cap.nb", and the results are stored in the file "SRWCapacityData.nb". Running this program, which should take under fifteen minutes on modern computers, finishes the proof of Lemma B.4.

Remark B.5. In order to check the consistency of our results, let us also mention that the value of $g_{\mathbb{Z}^3}(0)$ can alternatively also be computed using the formula from [23, (2.11)]. This formula corresponds to three times the integral I_3 from [33], whose approximate value can for instance found at http://oeis.org/A091672. The difference between the value obtained by this method and the value obtained by our Mathematica notebook is $2.8 \cdot 10^{-33}$, which is consistent with our error of 10^{-32}. Note that this error seems to mainly come from the term Error2 in (B.8).

We conclude this appendix with the short

Proof of Corollary 1.4. Let $\eta > 0$ be small enough so that $\frac{1}{2} + \eta < \alpha_*(K_1) \wedge \alpha_*(K_2)$ when $d = 3$ and $\frac{1}{2} + \eta < \alpha_*$ when $d = 4$ (with α_* as in (1.10)). Now fix any two-point set K such that $\frac{1}{2} < \alpha_*(K) < \frac{1+\eta}{2}$ and that there exists $D < \infty$ verifying for any $x, y \in \mathbb{T}$ that $\alpha_*(K) < \alpha_*(\{x,y\})$ if and only if $|x-y|_1 \leqslant D$. Such a set can always be found since $\alpha_*(\{x,y\}) \searrow \frac{1}{2}$ as $|x-y|_1 \to \infty$, and since $(x,y) \mapsto \alpha_*(\{x,y\})$ is decreasing in $|x-y|_1$ by (2.1) and (2.6).

Consider now first the case $d = 3$. Applying Theorem 1.3 (specifically, the first line of (1.22)) to this choice of K and with $\alpha = \frac{1}{2} + \eta (> \alpha_*(K))$ yields a coupling $\mathbf{Q}$ with the property (1.28) upon identifying $\mathcal{D}^{\cdot}$ with $\mathcal{B}_K^{\cdot}$. In view of (1.20), (1.24), the condition on η and by Theorem B.1, the patterns sampled independently as part of $\mathcal{D}^\alpha$ for $\alpha > \frac{1+\eta}{2}$ are precisely of the form *i)-iii)* in Corollary 1.4.

Let now $\mathcal{D}_-^\alpha$ be obtained from $\mathcal{D}^\alpha$ by removing all sets A included in (1.20) corresponding to images of K_i by torus isomorphisms for any $i \in \{1,2\}$. By Lemma 6.4 applied with $K = K_i$ one knows that $\lim_N \mathbf{Q}(D_{K_i}(\mathcal{L}^{\frac{1}{2}+\eta}) \geqslant N^{c(\eta)}) = 1$ for some $c(\eta) > 0$. On the other hand, by a similar calculation as below (7.15) one sees that $\lim_N \mathbf{Q}(D_{K_i}(\mathcal{D}_-^{\frac{1+\eta}{2}}) \geqslant N^{c(\eta)}) = 0$. It follows that the limit in (1.28) vanishes if one replaces $\mathcal{D}^{\cdot}$ by $\mathcal{D}_-^{\cdot}$.

Finally, if $d \geqslant 4$, the above coupling $\mathbf{Q}$ satisfies (1.28) but does not include patterns of type *iii)* on account of Theorem B.1. However, including images A by torus isomorphisms of K_i for $i = 1, 2$ independently with probability $p^\alpha(A)$ for $\alpha > \frac{1}{2}$ makes no difference. Indeed by Markov's inequality, the probability to sample such A in Q_N is bounded by $CN^d \mathbb{P}(K_1 \subset \mathcal{L}^\alpha) \to 0$ as $N \to \infty$ on account of (6.6) since $\frac{\alpha}{\alpha_*(K_1)} > 1$ for $\alpha > \frac{1}{2}$ when $d \geqslant 4$. $\qquad\square$

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