The Discrete Gaussian model, II.
Infinite-volume scaling limit at high temperature

Roland Bauerschmidt* Jiwoon Park† Pierre-François Rodriguez‡

Abstract

The Discrete Gaussian model is the lattice Gaussian free field conditioned to be integer-valued. In two dimensions, at sufficiently high temperature, we show that the scaling limit of the infinite-volume gradient Gibbs state with zero mean is a multiple of the Gaussian free field.

This article is the second in a series on the Discrete Gaussian model, extending the methods of the first paper by the analysis of general external fields (rather than macroscopic test functions on the torus). As a byproduct, we also obtain a scaling limit for mesoscopic test functions on the torus.

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1 Introduction and main results

This is the second article in a series on the Discrete Gaussian model, which builds on the foundation provided by the first paper [8]. The Discrete Gaussian model is the Gaussian free field conditioned to be integer-valued. Its two-dimensional version is a model for a crystal interface (in 2+1 dimensions) undergoing a roughening transition, see [16, Section 6] for a textbook treatment. We refer to our first paper [8] for a more extensive introduction and discussion of the literature.

1.1. Discrete Gaussian model in infinite volume. In our first paper [8], we studied the scaling limit of the Discrete Gaussian model for macroscopic test functions on the torus. In the present article, we derive the scaling limit of its infinite-volume gradient Gibbs state, as well as the scaling limit for mesoscopic test functions on the torus, which is a byproduct of the proof of the infinite-volume result. These scaling limit results are the objects of Theorems 1.1 and 1.2 below.

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*University of Cambridge, Statistical Laboratory, DPMMS. E-mail: rb812@cam.ac.uk.

†University of Cambridge, Statistical Laboratory, DPMMS. E-mail: jp711@cam.ac.uk.

‡Imperial College London, Department of Mathematics. E-mail: p.rodriguez@imperial.ac.uk.

The infinite-volume limit of the two-dimensional Discrete Gaussian model will be taken through weak limits with periodic boundary conditions, cf. (1.5), and we permit a general finite-range interaction J in the definition of the model. To be precise, let $J \subset \mathbb{Z}^d \setminus \{0\}$ be finite and symmetric under reflections and lattice rotations, and define the associated normalised range-J Laplacian Δ_J by

$$(\Delta_J f)(x) = \frac{1}{|J|} \sum_{y \in J} (f(x+y) - f(x)), \tag{1.1}$$

for $f : \mathbb{Z}^d \to \mathbb{R}$, where $|J|$ denotes the number of elements of J. Acting on test functions having mean zero, $(-\Delta_J)^{-1}$ has kernel

$$(-\Delta_J)^{-1}(x,y) \sim -\frac{1}{2\pi v_J^2} \log|x-y|, \text{ as } |x-y| \to \infty, \qquad \text{where} \qquad v_J^2 = \frac{1}{2|J|} \sum_{x \in J} x_1^2. \tag{1.2}$$

We now introduce the relevant finite-volume states. Let Λ_N be a two-dimensional discrete torus of side length L^N for integers $L > 1, N \geqslant 1$, and fix an origin $0 \in \Lambda_N$. Given the above step distribution J, the *Discrete Gaussian model* on Λ_N at temperature $\beta \in (0,\infty)$ has expectation, for any $F : (2\pi\mathbb{Z})^{\Lambda_N} \to \mathbb{R}$ with $F(\sigma) = F(\sigma + c)$ for any constant $c \in 2\pi\mathbb{Z}$ and such that the following series converges, defined by

$$\langle F \rangle_{J,\beta}^{\Lambda_N} \propto \sum_{\sigma \in \Omega^{\Lambda_N}} e^{-\frac{1}{2\beta}(\sigma, -\Delta_J \sigma)} F(\sigma) = \sum_{\sigma \in \Omega^{\Lambda_N}} e^{-\frac{1}{4\beta|J|} \sum_{x-y \in J} (\sigma_x - \sigma_y)^2} F(\sigma) \tag{1.3}$$

where the sum over $x - y \in J$ counts every undirected edge $\{x,y\}$ twice and

$$\Omega^{\Lambda_N} = \{\sigma \in (2\pi\mathbb{Z})^{\Lambda_N} : \sigma_{x=0} = 0\}. \tag{1.4}$$

Note that, as in our first paper [8], the factors of 2π in the spacing of the integers in (1.4) are convenient (but could be absorbed by rescaling β), and, to relate better to the Coulomb gas literature (cf. references below), we use $\frac{1}{\beta}$ rather than β to denote the inverse temperature of the Discrete Gaussian model. Equivalent to considering σ modulo constants, one can consider the gradient field $\eta = (\eta_e)_{e \in E}$ where E are the directed nearest-neighbour edges of $\mathbb{Z}^2$ and $\eta_e = \sigma_x - \sigma_y$ when $e = (x,y)$. Known correlation inequalities imply that, for any integer $L > 1$ and any finite-range distribution J, the weak limit of $\langle \cdot \rangle_{J,\beta}^{\Lambda_N}$ as $N \to \infty$ exists (modulo constants or as a gradient field), see Appendix A. For concreteness, we define the infinite-volume limit in terms tori of side lengths 2^N, i.e., when Λ_N has side length 2^N,

$$\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2} := \lim_{N \to \infty} \langle \cdot \rangle_{J,\beta}^{\Lambda_N}. \tag{1.5}$$

This limit $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ is a translation-invariant gradient Gibbs measure and every ergodic measure $\langle \cdot \rangle$ in its extremal decomposition has zero mean, i.e., $\langle \eta_e \rangle = 0$ for all $e \in E$, also see Appendix A. For $J = J_{\text{nn}}$ the usual nearest-neighbour interaction, $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ is *the* unique ergodic gradient Gibbs measure with zero mean on account of Theorem 9.1.1 in [49]. For general J, such a characterisation has not been proved.

As is well-known (see refs. below for an overview over the existing literature on the subject), in the Discrete Gaussian model, the discreteness of the spins is responsible for a phase transition between a rough (or delocalised) high-temperature phase and an ordered (or localised) low-temperature phase. Our results apply to large temperatures β. In contrast, in the regime of small β, a Peierls expansion yields that the Discrete Gaussian field is localised (or 'smooth'), e.g., there actually exists an (ordinary nongradient) Gibbs measure $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ satisfying

$$\langle \sigma_x \sigma_y \rangle_{J,\beta}^{\mathbb{Z}^2} - \langle \sigma_x \rangle_{J,\beta}^{\mathbb{Z}^2} \langle \sigma_y \rangle_{J,\beta}^{\mathbb{Z}^2} \leqslant C e^{-c|x-y|}, \text{ for all } x, y \text{ and } \beta < c; \tag{1.6}$$

see also [12, 46] for very precise results on the extremal behaviour in this regime.

1.2. Main results. Our main result is that the scaling limit of the Discrete Gaussian model $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ defined above is a multiple of the Gaussian free field on $\mathbb{R}^2$ when β is large. To state this precisely, given $f \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} f(x)\,dx = 0$, let $f_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ satisfy $\sum_{x\in\mathbb{Z}^2} f_\varepsilon(x) = 0$ and, with $d = 2$,

$$\begin{gathered}\max_{0\leqslant k\leqslant 2}\max_{x\in\mathbb{Z}^d}|(\varepsilon^{-1}\nabla)^k f_\varepsilon(x)| \leqslant C_f\varepsilon^{1+d/2}, \qquad \operatorname{supp} f_\varepsilon \subset [-C_f\varepsilon^{-1}, C_f\varepsilon^{-1}]^d,\\ \max_{x\in\mathbb{Z}^d}\left|\varepsilon^{-1-d/2}f_\varepsilon(x) - f(\varepsilon x)\right| \to 0,\end{gathered} \tag{1.7}$$

where C_f is a constant and ∇ is the vector of discrete gradients on $\mathbb{Z}^2$, see Section 1.4. For example, if $f = \nabla_i g$ for some $g \in C_c^\infty(\mathbb{R}^2)$ and $i \in \{1,2\}$ then one can take $f_\varepsilon(x) = \varepsilon^{d/2}(g(\varepsilon x + \varepsilon e_i) - g(\varepsilon x))$. Thus the following scaling limit in particular implies that of the gradient field $\nabla\sigma$.

We use the notation $(u,v)_{\mathbb{Z}^2} = \sum_{x\in\mathbb{Z}^2} u(x)v(x)$ for $u, v : \mathbb{Z}^2 \to \mathbb{R}$ square summable, $(f,g)_{\mathbb{R}^2} = \int_{\mathbb{R}^2} f(x)g(x)\,dx$ for $f, g : \mathbb{R}^2 \to \mathbb{R}$ square integrable, and $\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian on $\mathbb{R}^2$.

Theorem 1.1. *Let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour edges. Then there exists $\beta_0(J) \in (0,\infty)$ such that for the infinite-volume Discrete Gaussian Model $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ at temperature $\beta \geqslant \beta_0(J)$, there is $\beta_{\mathrm{eff}}(J,\beta) = \beta + O_J(e^{-c\beta}) \in (0,\infty)$ with a universal constant $c > 0$ (independent of J) such that for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$ and f_ε as in (1.7), as $\varepsilon \to 0$,*

$$\log\bigl\langle e^{(f_\varepsilon,\sigma)_{\mathbb{Z}^2}}\bigr\rangle_{J,\beta}^{\mathbb{Z}^2} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f, (-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}. \tag{1.8}$$

Theorem 1.1 superficially resembles [8, Theorem 1.1], but we emphasise that we are now considering the infinite-volume state; correspondingly the covariance on the right-hand side is now $(-\Delta_{\mathbb{R}^2})^{-1}$ instead of $(-\Delta_{\mathbb{T}^2})^{-1}$. The comparison below [8, Theorem 1.1] with previous results for the Discrete Gaussian model however also applies to the infinite-volume version, i.e., to Theorem 1.1 of this paper.

Theorem 1.1 can be seen as an analogue for the Discrete Gaussian model (with $\beta \geqslant \beta_0(J)$) of the Naddaf–Spencer theorem [48] which applies to strictly convex smooth gradient models. In our first paper [8] we discuss many further references concerning such models and concerning discrete height functions, and we refer to [8] for a more detailed discussion and only list here the most relevant references. For the Discrete Gaussian and XY models, we of course mention the fundamental work of Fröhlich–Spencer [31, 32] as well as the more recent articles [2, 35, 36, 43–45, 50, 51]. For smooth gradient models, there is a very comprehensive picture including stochastic dynamics [33, 34, 38] and recent developments include [3–6, 9, 20–22, 47, 49, 52]. For the smooth but nonconvex gradient models we refer to [10, 11, 18, 19] and in particular [13] and [1] which use the renormalisation group approach. For other discrete height functions, recent works include [17, 26, 27, 39–42]. Our first paper (and therefore this paper as well) relies in important ways on ideas developed in [14, 23, 25, 28, 29].

As a byproduct of the proof of Theorem 1.1 we also obtain the following mesoscopic scaling limit for the Discrete Gaussian model on the torus. (Effective error bounds also follow from the proof.)

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, there exists $L = L(J)$ such that for the Discrete Gaussian model on the torus Λ_N of side length L^N, for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$, f_ε as in (1.7), and any sequence $\varepsilon_N > 0$ such that $\varepsilon_N \to 0$ as $N \to \infty$ while $\varepsilon_N L^N \to \infty$,*

$$\log\bigl\langle e^{(f_{\varepsilon_N},\sigma)}\bigr\rangle_{J,\beta}^{\Lambda_N} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f, (-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \qquad \text{as } N \to \infty. \tag{1.9}$$

Note that the assumption $\varepsilon_N L^N \to \infty$ is necessary. Indeed, if $\varepsilon_N \ll L^{-N}$ then the support of f_ε is not a subset of Λ_N. Moreover, if $\varepsilon_N = L^{-N}$ the limit would correspond to the macroscopic scaling limit considered in [8, Theorem 1.1] which is different from the right-hand side above (given in terms of $(-\Delta_{\mathbb{T}^2})^{-1}$ rather than $(-\Delta_{\mathbb{R}^2})^{-1}$).

For some of the related open questions, we refer to our discussion in [8, Section 1.3], but mention in addition that a characterisation of the gradient Gibbs measures with finite range J as in [49, Theorem 9.1.1] for the nearest-neighbour case would be interesting.

1.3. Outline of the paper. This paper relies heavily on our first article on the Discrete Gaussian model [8], and in particular we use the set-up and notation from Section 2 and Sections 4–6 of that paper. Even though we included some reminders below, we will often refer to [8] to avoid repetitiveness.

The proofs of Theorems 1.1 and 1.2 proceed by decomposing the external field from the moment-generating function into contributions from all scales, with each contribution smooth at the respective scale. This is set up in Section 2. Then, the main technical contribution of the present paper compared to [8] is an extension of the renormalisation group map, originally defined in [8, Section 7], to allow for a scale-dependent external field. This is carried out in Section 5, after technical preparation in the preceding sections.

Different methods to extend a renormalisation group flow by observables for pointwise correlation functions in similar setups to ours were considered in [7, 15, 24, 29]. These approaches do not allow to derive the infinite-volume scaling limit as in our main result, and we expect that the approach we develop here could have applications to other models.

1.4. Notation. We use the notation $|a| \leqslant O(|b|)$ or $a = O(b)$ to denote $|a| \leqslant C|b|$ for an absolute constant $C > 0$ and $a \sim b$ to denote that $\lim a/b = 1$ (where the limit is clear from the context). We stress that all constants appearing below are uniform in β unless explicitly stated.

Throughout the paper, the dimension will be $d = 2$, but we sometimes write d to emphasise the source of the constant 2. Let $e_1, \dots, e_d$ be the basis of unit vectors with nonnegative components spanning $\mathbb{Z}^d$ or the local coordinates of Λ, and set $\hat{e} = \{\pm e_1, \cdots, \pm e_d\}$. For a function $f : \mathbb{Z}^d \to \mathbb{C}$ or $f : \Lambda_N \to \mathbb{C}$, we write $\nabla^\mu f(x) = f(x + \mu) - f(x)$ for $\mu \in \hat{e}$. For any multi-index $\alpha \in \{\pm 1, \cdots, \pm d\}^n$ with $n = |\alpha| \geqslant 1$, we write $\nabla^\alpha f = \nabla^{e_{\alpha_1}} \cdots \nabla^{e_{\alpha_n}} f$. The vector of n-th order discrete partial derivatives are denoted by

$$\nabla^n f(x) = (\nabla^{\mu_1} \cdots \nabla^{\mu_n} f(x) : \mu_k \in \hat{e} \text{ for all } k), \tag{1.10}$$

and we write $|\nabla^n f(x)|$ for the maximum over all of its components. Δ without subscript denotes the *unnormalised* nearest-neighbour Laplacian,

$$\Delta f(x) = \sum_{\mu \in \hat{e}} (f(x + \mu) - f(x)) = \sum_{\mu \in \hat{e}} \nabla^\mu f(x) = \frac{1}{2} \sum_{\mu \in \hat{e}} \nabla^\mu \nabla^{-\mu} f(x), \tag{1.11}$$

whereas Δ_J denotes the *normalised* Laplacian (1.1) with finite-range step distribution J.

2 Scale-dependent external fields

In this section, after briefly reviewing some aspects from the setup of our first paper [8], we proceed to describe how the proofs of the above theorems follow by amending the renormalisation group flow constructed in [8] by suitable external fields $u = (u_j)$, which start to appear at a characteristic scale $j = j_f$ in the renormalisation. We then proceed, assuming these fields u to have a negligible overall effect, as expressed in Theorem 2.3 below, to conclude the proofs of Theorems 1.1 and 1.2. The remaining sections will be geared towards the proof of Theorem 2.3, which appears in Section 6.

2.1. Multiscale decomposition of the field. We first briefly review a few key aspects from the setup of our previous paper [8], which will prevail here. As in Section 1, we denote by Λ_N the discrete torus of side length L^N and we will later impose that L is sufficiently large, see the discussion at the end of Section 3.4 for details; the infinite volume limit will then correspond to the limit $N \to \infty$. As explained in [8, Section 2], it is convenient to work with the mass-regularised Discrete Gaussian model $\langle\cdot\rangle_{\beta,m^2}$ and take $m^2 \downarrow 0$ in the end. This is the probability measure $\langle\cdot\rangle_{\beta,m^2} \equiv \langle\cdot\rangle^{\Lambda_N}_{\beta,m^2}$ obtained by replacing $-\Delta_J$ by $-\Delta_J + m^2$ in (1.3) and Ω^{Λ_N} by $\mathbb{Z}_\beta^{\Lambda_N}$ where $\mathbb{Z}_\beta = 2\pi\beta^{-1/2}\mathbb{Z}$, i.e., dropping the constraint $\sigma_0 = 0$. By [8, Lemma 2.1], then

$$\langle F(\sigma)\rangle_\beta = \lim_{m^2\downarrow 0} \langle F(\sigma)\rangle_{\beta,m^2}, \tag{2.1}$$

for any F as appearing above (1.3) (and in particular for the choice $F(\sigma) = e^{(f,\sigma)}$ for any $f : \Lambda_N \to \mathbb{R}$).

The renormalisation group analysis will involve a decomposition of the covariance

$$C(s,m^2) \overset{\text{def.}}{=} (C(m^2)^{-1} - s\Delta)^{-1}, \qquad \text{with} \quad C(m^2) = (-\Delta_J + m^2)^{-1} - \gamma\,\mathrm{id}, \tag{2.2}$$

where the inverses are interpreted on $\mathbb{R}^{\Lambda_N}$ and Δ is the (unnormalised) nearest-neighbour Laplacian on Λ_N, and γ and s are parameters with $\gamma \in (0, \frac{1}{3})$ and $|s|$ tacitly assumed sufficiently small so that $C(m^2)^{-1} - s\Delta$ is positive definite. As in [8, (4.1)], and without loss of generality, we work from here on under the standing assumptions that $|s| \leqslant \varepsilon_s\theta_J$ (by which (2.2) is well-defined) and that, for an arbitrary constant $C > 0$, we have $\theta_J \geqslant C^{-1}$ and $v_J \geqslant C^{-1}\rho_J$, where θ_J and ρ_J refer to the range and spectral characteristics of J, defined in [8, (3.3), (3.5)], and ε_s is the numerical constant appearing in [8, Proposition 3.4]. The last two conditions hold for any fixed J as in the theorems. (The use of the constant C will yield uniform estimates over families of J as above, see [8, Remark 1.2]. We do not state these in our main theorems above, but still introduce C to follow the same setup as in [8]).

Under these assumptions, it follows that for suitable choice of $\gamma \in (0, \frac{1}{3})$, which we henceforth regard as fixed, one can decompose $C(s,m^2)$ from (2.2) as in [8, Section 4] (see in particular (4.4) therein) to obtain, for all $m^2 > 0$ (and $|s| \leqslant \varepsilon_s\theta_J$),

$$C(s,m^2) = \Gamma_1(s,m^2) + \cdots + \Gamma_{N-1}(s,m^2) + \Gamma_N^{\Lambda_N}(s,m^2) + t_N(s,m^2)Q_N. \tag{2.3}$$

The right-hand side is a sum over positive (semi-)definite (covariance) matrices indexed by Λ_N. The matrix Q_N has all entries equal to $1/|\Lambda_N| = L^{-dN}$ and $t_N(s,m^2)$ is a scalar satisfying [8, (3.16)], in particular, diverging like m^{-2} as $m^2 \downarrow 0$. The covariances Γ_{j+1} and $\Gamma_N^{\Lambda_N}$ in (2.3) refer to those defined in [8, (4.2), (4.3)]. They correspond to a decomposition over scales L^j of the covariance $C(s,m^2)$. By construction, the matrices Γ_j have range $\frac{1}{4}L^j$ and their key analytical features are summarised in [8, Lemma 4.1]. We will frequently use the following notation. For $f : \Lambda_N \to \mathbb{R}$, we define (with a slight abuse of notation) $\Gamma_j(f) = \Gamma_j * f$ where $(\Gamma_j * f)(x) = \sum_y \Gamma_j(x-y)f(y)$ with $\Gamma_j(x) = \Gamma_j(0,x)$, cf. [8, below (3.8)].

This completes the introduction of our setup. We observe that in fact, the parameter s in (2.2), which implements the renormalisation of the temperature of the model, can be fixed from the start in the present paper as $s = s_0^c(J,\beta)$ with the latter as defined in [8, Proposition 8.1]; we will return to this later.

In what follows, we write $\mathbb{E}_\Gamma$ denotes the expectation of a Gaussian field ζ with covariance Γ. We will frequently write $\mathbb{E}$ for $\mathbb{E}_{\Gamma_{j+1}}$ when $j = 1, \ldots, N-2$ and $\mathbb{E}$ for $\mathbb{E}_{\Gamma_N^{\Lambda_N}}$ when $j = N-1$, whenever the scale j is clear from the context. Since $\Gamma_N^{\Lambda_N}$ satisfies exactly the same upper bounds as Γ_j with $j = N$, we will usually not distinguish between the cases $j+1 < N$ and $j+1 = N$. Generally, j without further specification is allowed to take values $j = 1, \ldots, N-1$.

2.2. Strategy. Contrary to the macroscopic torus scaling limit in [8], in which all the scales $j < N$ appearing in (2.3) were treated equally, we will have to distinguish in what follows a characteristic scale j_f at which a given test function f starts to induce a 'perturbation,' cf. (2.9) below, which manifests itself as a shift (or translation) of the corresponding Gaussian field (at the same scale). This is because the infinite volume limit $N \to \infty$ in Theorem 1.1 is decoupled from the characteristic scale j_f, whereas [8] simply takes $j_f = N$. The induced perturbation influences the renormalisation group flow in all the larger scales $j \geqslant j_f$. The technical difficulties arising in this paper are due to these changes. Fortunately, it will turn out that the infinite chain of perturbations will only impact the analysis on a bounded region by the compact support condition on the external field (see Lemma 2.2, for example).

Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a finitely supported test function with $\sum_x f(x) = 0$. Let j_f be the smallest integer ($\geqslant 1$) such that the support of f and Δf is contained in $[0, \frac{1}{4}L^{j_f})^2$ up to a spatial translation. If $f : \Lambda_N \to \mathbb{R}$ then j_f is defined similarly by identifying Λ_N with $([0, L^N) \cap \mathbb{Z})^2 \subset \mathbb{Z}^2$, whence $j_f \leqslant N$. We call j_f the smoothness scale of f and will frequently assume that

$$\|f\|_{\ell^\infty(\mathbb{Z}^2)} \leqslant cL^{-2j_f}, \tag{2.4}$$

where c will be an L-dependent small constant fixed below Lemma 2.2. The interpretation of j_f as a smoothness scale becomes clear when we focus on lattice functions scaled like f_ε given by (1.7). Indeed, each $\varepsilon^{-2} f_\varepsilon(\varepsilon^{-1}x)$ is an approximation of a smooth function, thus j_{f_ε} is the scale where f_ε becomes smooth: $L^{-j_{f_\varepsilon}} \simeq \varepsilon^{-1}$.

The macroscopic scaling limit considered in [8] corresponds to $j_f = N$, but now we are interested in $j_f \ll N$. The analysis of the macroscopic scaling limit proceeded through a translation of the field by $\gamma f + C(s, m^2)(f + s\gamma\Delta f)$ at scale N, with $C(s, m^2)$ as given by (2.2). The term γf and the difference between f and $f + s\gamma\Delta f$ will be insignificant and result from the preliminary renormalisation group step in [8, Section 2.3], which integrates out the i.i.d. field with variance γ, cf. (2.2), thus transforming the original discrete field into a smooth periodic potential (integrated with respect to a Gaussian measure). In view of (2.3), we now rewrite $C(s, m^2)$ as

$$C(s, m^2) = \Gamma_{\leqslant j_f}(s, m^2) + \sum_{j=j_f+1}^{N-1} \Gamma_j(s, m^2) + \Gamma_N^{\Lambda_N}(s, m^2) + t_N(s, m^2) Q_N, \tag{2.5}$$

where, with hopefully obvious notation, $\Gamma_{\leqslant j} = \sum_{1 \leqslant k \leqslant j} \Gamma_k$. Our starting point in this paper for the proofs of Theorems 1.1 and 1.2 is also a translation, but at the smoothness scale j_f rather than the macroscopic scale N, and by $\gamma f + \Gamma_{\leqslant j_f}(f + s\gamma\Delta f)$, see Lemma 2.1 below. An observation (made precise by Lemma 2.2 below) is that $\gamma f + \Gamma_{\leqslant j_f}(f + s\gamma\Delta f)$ is smooth at scale j_f because f is, while on the other hand, $\Gamma_k(f + s\gamma\Delta f)$ is smooth for $k > j_f$ because of the smoothing properties of the covariance Γ_k. We will show that this allows to implement translations iteratively for all scales $k \geqslant j_f$, with small errors accumulating from each scale k starting from $k = j_f$ and that as $j_f \to \infty$ the sum of these errors is governed by the contribution from the scale j_f and tends to 0 as $j_f \to \infty$.

2.3. Scale-dependent external fields. To formulate the above strategy more precisely, first recall (as mentioned above) that the parameter s is fixed as $s = s_0^c(J, \beta)$ from the start of this paper. Further let $s_0 = s = s_0^c(J, \beta)$, and define (as in [8, (2.25)])

$$Z_0(\varphi) = e^{\frac{s_0}{2}(\varphi, -\Delta\varphi) + \sum_{x \in \Lambda_N} \tilde{U}(\varphi_x)}, \quad \varphi \in \mathbb{R}^{\Lambda_N}, \tag{2.6}$$

with the function $\tilde{U}$ given by [8, (2.15)], which is a $2\pi\beta^{-1/2}$ periodic function of a single real variable. The next lemma is a slight reformulation of [8, Lemma 2.3]. For its statement let $\tilde{C}(s, m^2)$ be given as in [8, (2.26)], i.e.,

$$\tilde{C}(s, m^2) = \gamma(1 + s\gamma\Delta) + (1 + s\gamma\Delta) C(s, m^2)(1 + s\gamma\Delta), \tag{2.7}$$

and recall the covariance decomposition (2.5).

Lemma 2.1. *For all $\beta > 0$, $\gamma \in (0, \frac{1}{3})$, $m^2 \in (0, 1]$, $|s| = |s_0|$ small, one has for any $f \in \mathbb{R}^{\Lambda_N}$ such that $\sum_x f(x) = 0$,*

$$\langle e^{(f,\sigma)} \rangle^{\Lambda_N}_{\beta,m^2} \propto e^{\frac{1}{2}(f,\tilde{C}(s,m^2)f)} \mathbb{E}_{C(s,m^2)}\big[Z_0(\varphi + \textstyle\sum_{j=j_f}^N u_j)\big], \tag{2.8}$$

where the expectation acts on φ and

$$u_j = \begin{cases} 0 & (j < j_f) \\ \gamma f + \Gamma_{\leqslant j_f}(f + s\gamma\Delta f) & (j = j_f) \\ \Gamma_j(f + s\gamma\Delta f) & (N > j > j_f) \\ \Gamma_N^{\Lambda_N}(f + s\gamma\Delta f) & (j = N). \end{cases} \tag{2.9}$$

Proof. By [8, Lemma 2.3],

$$\sum_{\sigma \in \mathbb{Z}_\beta^{\Lambda_N}} e^{-\frac{1}{2}(\sigma,(-\Delta_J + m^2)\sigma)} e^{(f,\sigma)} \propto e^{\frac{1}{2}(f,\tilde{C}(s,m^2)f)} \mathbb{E}_{C(s,m^2)}\big[Z_0(\varphi + Af)\big], \tag{2.10}$$

with

$$A = (1 + s\gamma\Delta)^{-1}\tilde{C}(s,m^2) = \gamma + C(s,m^2)(1 + s\gamma\Delta). \tag{2.11}$$

The statement follows by applying the decomposition (2.5) of $C(s,m^2)$ which gives

$$Af = \sum_{j \leqslant N} u_j + t_N Q_N(f + s\gamma\Delta f) = \sum_{j \leqslant N} u_j, \tag{2.12}$$

where the last equality follows because $\sum_x f(x) = 0$, and hence $Q_N f = Q_N \Delta f = 0$. □

The renormalisation group flow constructed in [8], which we now sometimes refer to as the bulk renormalisation group flow, is in terms of the recursion (cf. [8, (7.3)])

$$Z_{j+1}(\varphi') = \mathbb{E}_{\Gamma_{j+1}} Z_j(\varphi' + \zeta), \quad \varphi' \in \mathbb{R}^{\Lambda_N}, \tag{2.13}$$

where here and below, $\mathbb{E}_{\Gamma_{j+1}}$ is the Gaussian expectation with covariance Γ_{j+1} which always acts on the field ζ. To incorporate the scale-dependent external fields $u = (u_j)$ we now define $Z_0(u, \varphi) = Z_0(\varphi)$ and

$$Z_{j+1}(u, \varphi') = \mathbb{E}_{\Gamma_{j+1}} Z_j(u, \varphi' + \zeta + u_j), \quad \varphi' \in \mathbb{R}^{\Lambda_N}, \tag{2.14}$$

with $\Gamma_N^{\Lambda_N}$ instead of Γ_{j+1} when $j + 1 = N$. Finally set

$$\tilde{Z}_N(u, \varphi') = \mathbb{E}_{t_N Q_N} Z_N(u, \varphi' + \zeta + u_N), \quad \varphi' \in \mathbb{R}^{\Lambda_N}. \tag{2.15}$$

Together, (2.14), (2.15) and (2.3) imply in particular that the expectation appearing on the right-hand side of (2.8) can be recast as (with $\mathbb{E}_{C(s,m^2)}$ acting on φ)

$$\mathbb{E}_{C(s,m^2)}\big[Z_0(\varphi + \textstyle\sum_{j=j_f}^N u_j)\big] = \tilde{Z}_N(u, 0). \tag{2.16}$$

Our analysis of the $Z_j(u, \varphi')$ relies on the property that the external fields u_j are smooth on scale j for all j, as demonstrated by the next lemma. Here assume that j_f in (2.9) is the smoothness scale of f, i.e., the smallest integer such that $\operatorname{supp} f$ is contained in a block of side length $\frac{1}{4}L^{j_f}$. By definition, a block of size L is any set of the form $x + ([0, L) \cap \mathbb{Z})^2$ for some $x \in L\mathbb{Z}^2$. Let $\|u_j\|_{C_j^2} = \|u_j\|_{C_j^2(\mathbb{Z}^2)} = \max_{n=0,1,2} \|\nabla_j^n u_j\|_{\ell^\infty(\mathbb{Z}^2)}$, cf. [8, (5.10)]. In the sequel we often tacitly view a function f with domain Λ_N (such as u_j) as defined on $\mathbb{Z}^2$ by identifying Λ_N with $[0, L^N)^2$ and extending f to have value 0 outside this set.

Lemma 2.2. *There exists an L-independent constant $C > 0$ such that the following holds: for all $f : \mathbb{Z}^d \to \mathbb{R}$ satisfying $\sum f = 0$ and such that f and Δf have support in a block of side length $\frac{1}{4}L^{j_f}$, the functions u_j defined by (2.9) have support in blocks of side lengths $\frac{3}{4}L^j$ for $j \leqslant N-1$ and*

$$\|u_j\|_{C_j^2} \leqslant CL^{2j_f+2}\|f\|_{\ell^\infty(\mathbb{Z}^2)}, \qquad j \leqslant N. \tag{2.17}$$

In particular, if (2.4) holds with $c \leqslant (CL^2)^{-1}$, then $\sup_j \|u_j\|_{C_j^2} \leqslant 1$. From here on, we fix (any) such value of c; this choice is implicit when referring to (2.4) in the sequel.

Proof. Let $g = f + s\gamma\Delta f$ and note that by assumption g has support in a block of side length $\frac{1}{4}L^{j_f}$. Also, $\|g\|_{\ell^\infty} \leqslant (1 + 2|s|\gamma)\|f\|_{\ell^\infty} \leqslant C\|f\|_{\ell^\infty}$ since $\|\Delta f\|_{\ell^\infty} \leqslant 8\|f\|_{\ell^\infty}$ for any j. We may identify Γ_j with its convolution kernel, i.e., $\Gamma_j g = \Gamma_j * g$. Then Γ_j is supported in a block of side length $\frac{1}{4}L^j$ and satisfies $\|\nabla_j^\alpha \Gamma_j\|_{\ell^\infty} \leqslant CL^2$ for $|\alpha| \leqslant 2$ where $\nabla_j^\alpha = L^{j|\alpha|}\nabla^\alpha$, see [8, Corollary 4.1], thus

$$\|\nabla_j^\alpha \Gamma_j g\|_{\ell^\infty} \leqslant L^{2j_f}\|\nabla_j^\alpha \Gamma_j\|_{\ell^\infty}\|g\|_{\ell^\infty} \leqslant CL^{2j_f+2}\|f\|_{\ell^\infty}. \tag{2.18}$$

Thus the desired statement holds if $j < N$.

The same estimates hold when $j = N$, i.e., with Γ_j is replaced by $\Gamma_N^{\Lambda_N}$ which satisfies analogous bounds, see [8, Corollary 4.1]. This completes the proof of the bound (2.17).

The statement about the support of the u_j follows immediately from the assumption that the support of f and g have diameter $\frac{1}{4}L^{j_f} \leqslant \frac{1}{4}L^j$ for all $j \geqslant j_f$ and that Γ_j has range $\frac{1}{4}L^j$. □

2.4. Conclusion of the argument. In Section 6 we will show the following theorem from which the proof of Theorem 1.1 can be completed similarly as the torus result in [8, Section 9]. The theorem is stated under somewhat more general condition on the sequence $(u_j)_j = (u_j \in \mathbb{R}^\Lambda)_j$ of given external fields that are uniformly bounded and supported on a single block in the sense that:

($\mathbf{A}_u$) There exists j_u such that $u_j = 0$ for $j < j_u$, $\|u_j\|_{C_j^2} \leqslant 1$ for each $j \leqslant N$, and u_j is supported on the unique $B_0 \in \mathcal{B}_j$ such that $0 \in B_0$ and $d(\partial B_0, \mathrm{supp}(u_j)) > 4$.

For the same reason that j_f was called a smoothness scale of f, we call j_u the smoothness scale (of $u = (u_j)_j$). Note that, by translation invariance of the Discrete Gaussian model on the torus Λ_N, we may assume that f is centred with respect to the block decomposition; that is, $\mathrm{supp}(f)$ and $\mathrm{supp}(\Delta f)$ are contained in the box $m + [0, \frac{1}{4}L_{j_f})^2$, where m is one of the lattice points closest to the center of some block $B \in \mathcal{B}_j$ for all $j_f \leqslant j \leqslant N$. In particular, then, by Lemma 2.2, for all scales $j \leqslant N$, there is a block $B \in \mathcal{B}_j$ such that whenever $L \geqslant C$, $N_5(\mathrm{supp}(u_j)) \subset B$ where $N_k(X)$ denotes the set of points with ℓ^1-distance at most k from the set X. Thus the condition on the support of u_j is not stronger than the condition on the support of f.

Theorem 2.3. *Let J be a finite-range step distribution as in the statements of Theorems 1.1 and 1.2. There are $\beta_0(J) \in (0, \infty)$, a (large) integer $L = L(J)$ (which can be chosen dyadic), and a constant $\alpha > 0$ such that if $u = (u_j)$ satisfies ($\mathbf{A}_u$) there is $C > 0$ such that for $\beta \geqslant \beta_0(J)$ and $N > j_u$,*

$$\left| \frac{\tilde{Z}_N(u,0)}{\tilde{Z}_N(0,0)} - 1 \right| \leqslant CL^{-\alpha j_u}. \tag{2.19}$$

Assuming Theorem 2.3 to hold, and in view of Lemma 2.1, the proofs of Theorems 1.1 and 1.2 are readily completed by means of the following elementary lemma, as explained below. This lemma is the infinite-volume analogue of [8, Lemma 9.2]; we postpone its proof to the end of this section and first give the details for the proof of Theorems 1.1 and 1.2. In what follows, for $N > j_{f_\varepsilon}$, we tacitly identify f_ε with the corresponding function having domain on the torus Λ_N by identifying $\mathrm{supp}(f_\varepsilon)$ with a suitable subset of the torus Λ_N. We write $\tilde{C}^{\Lambda_N} \equiv \tilde{C}$ for the covariance matrix defined in (2.7) to stress the dependence on the underlying torus Λ_N.

Lemma 2.4. *Let $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$ and f_ε be as in (1.7). Then*

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\lim_{m^2\downarrow 0}(f_\varepsilon, \tilde{C}^{\Lambda_N}(s,m^2)f_\varepsilon) = \frac{1}{v_J^2+s}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \tag{2.20}$$

and the statement also holds if the two leftmost limits are replaced by $N\to\infty$ with $\varepsilon = \varepsilon_N \to 0$ while $\varepsilon_N L^N \to \infty$.

Proof of Theorems 1.1 and 1.2. Our proof proceeds as the following. We will first prove our main limit results with f_ε replaced by τf_ε for $\tau > 0$ sufficiently small (depending on C_f of (1.7) and c of (2.4)). The convergence can then be extended to all $\tau \in \mathbb{C}$ by a standard argument which we include for completeness.

Given $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$ and f_ε as in (1.7), set $j_f = \lceil \log_L(8C_f\varepsilon^{-1})\rceil$. Then using the first two conditions in (1.7), it readily follows that f_ε satisfies (2.4) for all $\varepsilon \in (0,1)$ (including that f_ε is supported on a block of side length $L^{j_f}/4$). Now define $u_j = u_j[\varepsilon]$ according to (2.9) with f_ε in place of f. Then by Lemma 2.2, $(\tau u_j)_j$ satisfies $(\mathbf{A}_u)$ with $j_u = j_f$ whenever $\tau > 0$ is small enough depending on C_f and L. Now by Lemma 2.1 and (2.16),

$$\langle e^{\tau(f_\varepsilon,\sigma)}\rangle_{\beta,m^2}^{\Lambda_N} = e^{\frac{\tau^2}{2}(f_\varepsilon,\tilde{C}(s,m^2)f_\varepsilon)}\frac{\tilde{Z}_N(\tau u[\varepsilon],0)}{\tilde{Z}_N(0,0)}. \tag{2.21}$$

Since $(\mathbf{A}_u)$ holds for $\tau u[\varepsilon]$, the assumption of Theorem 2.3 is satisfied uniformly in ε. Therefore

$$\frac{\tilde{Z}_N(\tau u[\varepsilon],0)}{\tilde{Z}_N(0,0)} = 1 + O_f(e^{-\alpha\log(8C_f\varepsilon^{-1})}) = 1 + O_f(\varepsilon^\alpha) \tag{2.22}$$

uniformly in m^2, ε and $j_f < N$. In the context of Theorem 1.1, the last condition $j_f < N$ is immediate as soon as $N \geqslant C(\varepsilon)$ since $\varepsilon > 0$ is fixed while $N \to \infty$; in the context of Theorem 1.2, it follows from our assumption $\varepsilon_N L^N \to \infty$. Finally, by Lemma 2.4, if either first $N \to \infty$ and then $\varepsilon \to 0$, or if $\varepsilon = \varepsilon_N \to 0$ such that $\varepsilon_N L^N \to \infty$, we have

$$\lim_{m^2\downarrow 0}\log\langle e^{\tau(f_\varepsilon,\sigma)}\rangle_{\beta,m^2}^{\Lambda_N} \to \tau^2\frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}. \tag{2.23}$$

By (2.1), using that $\sum f_\varepsilon = 0$, the left-hand side equals $\log\langle e^{\tau(f_\varepsilon,\sigma)}\rangle_{J,\beta}^{\Lambda_N}$. Thus with f_ε replaced by τf_ε with sufficiently small $\tau > 0$, the proof of Theorem 1.1 follows on account of Proposition A.1, and Theorem 1.2 follows directly from the above, i.e.,

$$\log\big\langle e^{\tau(f_\varepsilon,\sigma)_{\mathbb{Z}^2}}\big\rangle_{J,\beta}^{\mathbb{Z}^2} \to \tau^2\frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \qquad \text{as } \varepsilon\to 0, \tag{2.24}$$

$$\log\big\langle e^{\tau(f_{\varepsilon_N},\sigma)}\big\rangle_{J,\beta}^{\Lambda_N} \to \tau^2\frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \qquad \text{as } N\to\infty. \tag{2.25}$$

Now, we show that the domain of τ can be extended to $\mathbb{C}$ using the Gaussian domination inequality. Indeed, by (2.24), we see

$$\begin{aligned}\big\langle (f_\varepsilon,\sigma)_{\mathbb{Z}^2}^{2n}\big\rangle_{J,\beta}^{\mathbb{Z}^2} &\to \frac{(2n)!}{2^n n!}\frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}^n\\ \big\langle (f_\varepsilon,\sigma)_{\mathbb{Z}^2}^{2n+1}\big\rangle_{J,\beta}^{\mathbb{Z}^2} &= 0\end{aligned} \tag{2.26}$$

for each $n \in \mathbb{N}$. Also, for any $T \geqslant 0$, by the Taylor's theorem (for the second equality), there exists $\theta \in [0,1]$ such that

$$\sum_{n>k}\frac{T^n}{n!}|(f_\varepsilon,\sigma)|^n = e^{T|(f_\varepsilon,\sigma)|} - \sum_{n=0}^k\frac{T^n}{n!}|(f_\varepsilon,\sigma)|^n = \frac{e^{\theta T|(f_\varepsilon,\sigma)|}}{(k+1)!} \leqslant \frac{e^{T(f_\varepsilon,\sigma)} + e^{-T(f_\varepsilon,\sigma)}}{(k+1)!}. \tag{2.27}$$

But by [30] (see also [43, Proposition 1.2]), we have the Gaussian domination

$$\langle e^{(g,\sigma)}\rangle_{J,\beta}^{\Lambda_N} \leqslant e^{\frac{\beta}{2}(g,(-\Delta_J)_\Lambda^{-1}g)} \tag{2.28}$$

for any $g : \Lambda_N \to \mathbb{R}$ with $\sum g = 0$, so we obtain

$$\Big|\Big\langle \sum_{n>k} \frac{\tau^n}{n!}(f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N}\Big| \leqslant \frac{2}{(k+1)!} e^{\frac{\beta}{2}T^2(f_\varepsilon,(-\Delta)^{-1}f_\varepsilon)} \tag{2.29}$$

upon letting $T = |\operatorname{Re}(\tau)|$. In other words, $\langle \sum_{n=0}^k \frac{\tau^n}{n!}(f_\varepsilon, \sigma)^n\rangle_{J,\beta}^{\Lambda_N} \to \langle e^{\tau(f_\varepsilon,\sigma)}\rangle_{J,\beta}^{\Lambda_N}$ as $k \to \infty$, uniformly in ε and N, proving

$$\lim_{\varepsilon\to 0, N\to\infty} \lim_{k\to\infty} \Big\langle \sum_{n=0}^{k} \frac{\tau^n}{n!}(f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N} = \lim_{k\to\infty} \lim_{\varepsilon\to 0, N\to\infty} \Big\langle \sum_{n=0}^{k} \frac{\tau^n}{n!}(f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N}. \tag{2.30}$$

But by (2.26), the latter is $\tau^2 \frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2}(f, (-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}$, completing the proof of Theorem 1.1. The extension for Theorem 1.2 is done analogously. □

Proof of Lemma 2.4. In what follows, given $f_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$, we denote by $\hat{f}_\varepsilon$ its Fourier transform, defined as in [8, (3.19)]. By definition of $\tilde{C}(s, m^2)$ and since $\hat{f}_\varepsilon(0) = 0$, one has

$$\lim_{N\to\infty} \lim_{m^2\to 0} (f_\varepsilon, \tilde{C}(s, m^2) f_\varepsilon) = \frac{\varepsilon^2}{4\pi^2} \int_{[-\pi/\varepsilon,\pi/\varepsilon]^2} \frac{\lambda_J(\varepsilon p)^{-1}(1 - s\gamma\lambda(\varepsilon p))}{1 + s\lambda(\varepsilon p)(\lambda_J(\varepsilon p)^{-1} - \gamma)} |\hat{f}_\varepsilon(\varepsilon p)|^2 \, dp, \tag{2.31}$$

where $\lambda(p)$ is the Fourier multiplier of the (unnormalised) discrete Laplacian $-\Delta$ and $\lambda_J(p)$ that of the (normalised) range-J Laplacian $-\Delta_J$, see [8, Section 3.2]. By [8, Lemma 3.6],

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\lambda(\varepsilon p) = |p|^2, \qquad \lim_{\varepsilon\to 0} \varepsilon^{-2}\lambda_J(\varepsilon p) = v_J^2|p|^2, \tag{2.32}$$

and the fraction in the integrand in (2.31) is bounded by $C|p|^{-2}$ uniformly in ε and $p \in [-\pi/\varepsilon, \pi/\varepsilon]^2$. Moreover, as we now argue, (1.7) implies that $\hat{f}_\varepsilon(\varepsilon p) \to \hat{f}(p)$ as $\varepsilon \to 0$ for each $p \in \mathbb{R}^2$ and that $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|(1 + |p|)^{-3}$. To see this in detail, we start from

$$\hat{f}_\varepsilon(\varepsilon p) = \sum_{y\in\varepsilon\mathbb{Z}^2} f_\varepsilon(y/\varepsilon)e^{-iy\cdot p}. \tag{2.33}$$

For $|\hat{f}(p) - \hat{f}_\varepsilon(\varepsilon p)| \to 0$ pointwise, use $f \in C_c^\infty(\mathbb{R}^2)$ and the last condition in (1.7) to see that, with $[\cdot]$ denoting the integer part,

$$|\hat{f}(p) - \hat{f}_\varepsilon(\varepsilon p)| \leqslant \int_{\mathbb{R}^2} |f(y)(e^{-iy\cdot p} - e^{-i\varepsilon[y/\varepsilon]\cdot p})| \, dy + \int_{\mathbb{R}^2} |f(y) - \varepsilon^{-2} f_\varepsilon([y/\varepsilon])| \, dy \to 0. \tag{2.34}$$

To see the bound on $\hat{f}_\varepsilon(\varepsilon p)$, use summation by parts to write

$$\lambda(p)|\hat{f}_\varepsilon(p)| = |\widehat{\Delta f}_\varepsilon(p)| = |\sum_{x\in\mathbb{Z}^2} e^{-ip\cdot x}\Delta f_\varepsilon(x)| \leqslant \|\Delta f_\varepsilon\|_{\ell^1(\mathbb{Z}^2)}. \tag{2.35}$$

By (1.7),

$$\|\Delta f_\varepsilon\|_{\ell^1(\mathbb{Z}^2)} \leqslant C_f^2(\varepsilon^{-1} + 1)^2 \|\Delta f_\varepsilon\|_{\ell^\infty(\mathbb{Z}^2)} \leqslant 2C_f^2 \|(\varepsilon^{-1}\nabla)^2 f_\varepsilon\|_{\ell^\infty(\mathbb{Z}^2)} \leqslant 2C_f^3\varepsilon^2, \tag{2.36}$$

and by [8, Lemma 3.6], we have that $\frac{1}{\varepsilon^2|p|^2}\lambda(\varepsilon p) \geqslant \frac{4}{\pi^2}$. Thus it follows that $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|^{-2}$. On the other hand, since $\sum f_\varepsilon = 0$ and $\|f_\varepsilon\|_{\ell^\infty} \leqslant C_f\varepsilon^2$, also

$$|\hat{f}_\varepsilon(\varepsilon p)| = \Big| \sum_{y\in\varepsilon\mathbb{Z}^2} f_\varepsilon(y/\varepsilon)(e^{-iy\cdot p} - 1) \Big| \leqslant \|f_\varepsilon\|_{\ell^\infty} \sum_{y\in\varepsilon\mathbb{Z}^2 : |y|\leqslant C_f} |y \cdot p| \leqslant CC_f^4|p|, \tag{2.37}$$

and therefore $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|(1+|p|)^{-3}$ when combined with $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|^{-2}$.

Finally, using the convergence in Fourier space and that the integrand is dominated by $C|p|^{-2} \times (|p|(1+|p|)^{-3})^2 \leqslant C(1+|p|)^{-6}$ which is integrable over $\mathbb{R}^2$, the Dominated convergence theorem implies

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\lim_{m^2\to 0}(f_\varepsilon, \tilde{C}(s,m^2)f_\varepsilon) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{1}{v_J^2+s}|p|^{-2}|\hat{f}(p)|^2\,dp = \frac{1}{v_J^2+s}(f,(-\Delta_{\mathbb{R}^2})^{-1}f) \quad (2.38)$$

as claimed. □

3 Norms and contraction estimates

We now prepare the ground for the proof of Theorem 2.3, which will essentially follow by suitably extending the RG flow developed in [8]. This extension is designed to accomodate the external field u. In the present section, we discuss the necessary amendments to the norms introduced in [8, Section 5] required to carry this out, as well as the resulting contraction estimates, cf. [8, Section 6].

3.1. Norms and regulators without external field. We recall some essential elements of [8]. Given Λ_N, the discrete two-dimensional torus of side lengths L^N and a distinguished point $0 \in \Lambda_N$, let $\pi_N : \mathbb{Z}^2 \to \Lambda_N$ be the canonical projection with $\pi_N(0) = 0$. Then for each $j = 0, \cdots, N$, $\mathcal{B}_j$ (j-blocks) will be the sets of the form $\pi_N(([0, L^j) \cap \mathbb{Z})^2 + nL^j)$ for $n \in \mathbb{Z}^2$, $\mathcal{P}_j$ (j-scale polymers) are any subsets (not necessarily connected) of Λ_N that can be obtained as the union of j-blocks. For various notions related to $\mathcal{P}_j$, see [8, Section 4]. Functions $F(X,\varphi)$ smooth in φ that only depend on $\varphi|_{X^*}$ for each $X \in \mathcal{P}_j$ are called polymer activities at scale j, see [8, Section 5]; here X^* refers to the small-set neighborhood of X, see [8, Section 4.1].

In (2.13), Z_j will always be parametrised as

$$Z_j(\varphi) = e^{-E_j|\Lambda_N|}\sum_{X\in\mathcal{P}_j(\Lambda_N)} e^{U_j(\Lambda_N\setminus X,\varphi)}K_j(X,\varphi) \quad (3.1)$$

$$U_j(X,\varphi) = \frac{1}{2}s_j|\nabla\varphi|_X^2 + \sum_{q\geqslant 1} L^{-2j}z_j^{(q)}\sum_{x\in X}\cos(\sqrt{\beta}q\varphi(x)) = \frac{1}{2}s_j|\nabla\varphi|_X^2 + W_j(X,\varphi), \quad (3.2)$$

with $U_j(\emptyset) = 0$ and initial conditions $s_0 \in \mathbb{R}$ given, $E_0 = 0$, $K_0(X,\varphi) = 1_{X=\emptyset}$ and $z_0 = (z_0^{(q)})_{q\geqslant 0}$ given, where the latter refer to the Fourier coefficients of the periodic potential $\tilde{U}$ in (2.6), see [8, (2.18)]. The coordinates U_j and K_j are polymer activities, and in [8, Sections 5 and 7], they are controlled using the norms $\|\cdot\|_{\Omega_j^U}$ and $\|\cdot\|_{\Omega_j^K}$. The latter norm needs an extension in the current work, so it will be reviewed in some detail here. It is defined in terms of positive parameters r, A, L, κ_L, c_2, c_4, c_w, h, which will essentially be fixed as in [8] in Section 3.4 below. The definition of the norms involves the regulator G_j, which is a weight defined for $X \in \mathcal{P}_j$ and $\varphi \in \mathbb{R}^{\Lambda_N}$ by

$$G_j(X,\varphi) = \exp\left(\kappa_L\left(\|\nabla_j\varphi\|^2_{L_j^2(X)} + c_2\|\nabla_j\varphi\|^2_{L_j^2(\partial X)} + \sum_{B\in\mathcal{B}_j(X)}\|\nabla_j^2\varphi\|^2_{L^\infty(B^*)}\right)\right), \quad (3.3)$$

where $\mathcal{B}_j(X)$ is the set of j-blocks constituting X, ∂X denotes the inner ℓ^1-vertex boundary of X, and with the relevant L^p-norms as introduced in [8, Definition 5.2]. The semi-norms and norms on polymer activities are then given by (cf. [8, Definition 5.4])

$$\|D^nF(X,\varphi)\|_{n,T_j(X,\varphi)} = \sup\left\{D^nF(X,\varphi)(f_1,\cdots,f_n) : \|f_k\|_{C_j^2(X^*)} \leqslant 1\ \forall k\right\} \quad (3.4)$$

$$\|F(X,\cdot)\|_{h,T_j(X)} = \sup_{\varphi\in\mathbb{R}^{\Lambda_N}} G_j(X,\varphi)^{-1}\sum_{n=0}^{\infty}\frac{h^n}{n!}\|D^nF(X,\varphi)\|_{n,T_j(X,\varphi)} \quad (3.5)$$

$$\|F\|_{\Omega_j^K} = \|F\|_{h,T_j} = \sup_{X\in\mathcal{P}_j^c} A^{|X|_j}\|F(X,\cdot)\|_{h,T_j(X)}. \quad (3.6)$$

We will also need the following somewhat more technical properties of the norms and regulators. For $X \in \mathcal{P}_j$ and $\varphi \in \mathbb{R}^{\Lambda_N}$, recall the definition $w_j(X,\varphi)^2 = \sum_{B \in \mathcal{B}_j(X)} \max_{n=1,2} \|\nabla_j^n \varphi\|^2_{L^\infty(B^*)}$ and then that of the strong regulators

$$\exp\Big(c_w \kappa_L w_j(X,\varphi)^2\Big), \quad g_j(X,\varphi) = \exp\Big(c_4 \kappa_L \sum_{a=0,1,2} W_j(X, \nabla_j^a \varphi)^2\Big), \tag{3.7}$$

where $W_j(X, \nabla_j^a \varphi)^2 = \sum_{B \in \mathcal{B}_j(X)} \|\nabla_j^a \varphi\|^2_{L^\infty(B^*)}$. For sharp integrability estimates, we subdecomposed in [8, Section 4.3] each scale j into M fractional scales $j+s$ with $s = 0, \dots, 1-1/M$ when $L = \ell^M$ with ℓ an integer. Each covariance Γ_{j+1} from the finite-range decomposition (2.5) has the corresponding subdecomposition

$$\Gamma_{j+1} = \Gamma_{j,j+1/M} + \cdots + \Gamma_{j+(M-1)/M, j+1}. \tag{3.8}$$

The regulators G_{j+s} and the strong regulators g_{j+s} are also defined on these fine scales, see [8, (5.15)] and analogously for G_{j+s}. The crucial property of G_{j+s} and g_{j+s} is stated in the next lemma, which is an extension of [8, Lemma 5.13] and proved in Appendix B. The fields ξ_o, ξ_B appearing in the next lemma will correspond in practice to shifts induced by the external fields.

Lemma 3.1. *For $X \in \mathcal{P}_{j+s}$ and φ, ξ_o, $\xi_B \in \mathbb{R}^{\Lambda_N}$ for each $B \in \mathcal{B}_{j+s}(X)$, define*

$$\begin{aligned} &\log G_{j+s}(X, \varphi, \xi_o, (\xi)_{B \in \mathcal{B}_{j+s}(X)}) \\ &= \kappa_L \|\nabla_{j+s}(\varphi + \xi_o)\|^2_{L^2_{j+s}(X)} + \kappa_L c_2 \|\nabla_{j+s}(\varphi+\xi_o)\|^2_{L^2_{j+s}(\partial X)} + \kappa_L \sum_{B \in \mathcal{B}_{j+s}(X)} \|\nabla^2_{j+s}(\varphi + \xi_B)\|^2_{L^\infty(B^*)}. \end{aligned} \tag{3.9}$$

Assume $0 \leqslant j < N$, $L = \ell^M$. For any choice of c_2 small enough compared to 1, there exist $c_4 = c_4(c_2)$ and an integer $\ell_0 = \ell_0(c_1, c_2)$ (both large), such that for all $\ell \geqslant \ell_0$, $M \geqslant 1$, $s \in \{0, \frac{1}{M}, \dots, 1-\frac{1}{M}\}$ and $\kappa_L > 0$, for $X \in \mathcal{P}^c_{j+s}$,

$$G_{j+s}(X, \varphi, \xi_o, (\xi_B)_{B \in \mathcal{B}_{j+s}(X)}) \leqslant \max_{\mathfrak{a} \in \{o\} \cup \mathcal{B}_{j+s}(X)} g_{j+s}(X_{s+M^{-1}}, \xi_{\mathfrak{a}}) G_{j+s+M^{-1}}(X_{s+M^{-1}}, \varphi). \tag{3.10}$$

and $X_{s+M^{-1}}$ is the smallest $(j+s+M^{-1})$-polymer containing X (see [8, Section 4.3]).

3.2. Norms and regulators with external field. To incorporate the effect of the scale-dependent external fields, we need an extension of the norms and regulators that take the external field into account. The following definition introduces modified regulators that effectively control the polymer activities perturbed by the external fields (u_j).

Definition 3.2. *Given $(u_j)_j$ satisfying $(\mathbf{A}_u)$, define the Ψ-regulators (cf. (3.3))*

$$\begin{aligned} &G_j^\Psi(X, \varphi; u_j) \\ &= \sup_{t \in [0,1]} \exp\Big(\kappa_L \|\nabla_j(\varphi + t u_j)\|^2_{L^2_j(X)} + c_2 \kappa_L \|\nabla_j(\varphi + t u_j)\|^2_{L^2_j(\partial X)} + \kappa_L W_j^\Psi(X, \varphi; u_j)^2\Big) \end{aligned} \tag{3.11}$$

where

$$W_j^\Psi(X, \nabla_j^2 \varphi; u_j)^2 = \sum_{B \in \mathcal{B}_j(X)} \sup_{t_B \in [0,1]} \|\nabla_j^2(\varphi + t_B u_j)\|^2_{L^\infty(B^*)}. \tag{3.12}$$

The dependence on u_j will often be hidden.

Remark 3.3. The main motivation for G_j^Ψ is to have $\sup_{t \in [0,1]} G_j(X, \varphi + t u_j) \leqslant G_j^\Psi(X, \varphi)$ and hence

$$\|K(X, \varphi + t u_j)\|_{h, T_j(X,\varphi)} \leqslant \|K(X)\|_{h, T_j(X)} G_j^\Psi(X, \varphi), \quad t \in [0,1], . \tag{3.13}$$

Note that we could not use $\sup_{t\in[0,1]} G_j(X,\varphi+tu_j)$ for G_j^Ψ because this definition does not factorise into connected components, i.e.,

$$\sup_{t\in[0,1]} G_j(X\cup Y,\varphi+tu_j) \neq \sup_{t_1,t_2\in[0,1]} G_j(X,\varphi+t_1u_j)G_j(Y,\varphi+t_2u_j) \tag{3.14}$$

if $X \not\sim Y = \emptyset$ but $X^*\cap Y^* \neq \emptyset$. This is why we introduced the W_j^Ψ.

Also note that since $\|\nabla_j u_j\|^2_{L^2_j(X)}$, $\|\nabla_j u_j\|^2_{L^2_j(\partial X)}$, $W_j(X,\nabla_j^2 u_j)$ are each bounded by some multiple of $\|u_j\|^2_{C_j^2}$, in particular, there exists finite $C>0$, independent of X, such that, under $(\mathbf{A}_u)$,

$$G_j^\Psi(X,0;u_j) \leqslant C. \tag{3.15}$$

The following are the key properties of G_j^Ψ (cf. the properties of G_j in [8, Section 5]).

Proposition 3.4. *Let $(u_j)_j$ satisfy $(\mathbf{A}_u)$. Then there exists $C_\Psi>0$ such that for L as in the assumption of Lemma 3.1 and sufficiently small c_2, $c_w>0$, $(G_j^\Psi)_{j\geqslant 0} \equiv (G_j^\Psi(\cdot;u_j))_{j\geqslant 0}$ satisfies for each $(X,\varphi)\in\mathcal{P}_j\times\mathbb{R}^{\Lambda_N}$, $j\geqslant 0$,*

1. $G_j^\Psi(X,\varphi) \geqslant G_j(X,\varphi)$,
2. $G_j^\Psi(X,\varphi) = \prod_{Y\in\mathrm{Comp}_j(X)} G_j^\Psi(Y,\varphi)$,
3. $e^{c_w\kappa_L w_j(X,\varphi+tu_j)^2} G_j^\Psi(Y,\varphi) \leqslant C_\Psi G_j^\Psi(X\cup Y,\varphi)$ *if $X\cap Y=\emptyset$ and $t\in[0,1]$,*
4. $\mathbb{E}[G_j^\Psi(X,\varphi'+\zeta)] \leqslant C_\Psi 2^{|X|_j} G_{j+1}(X,\varphi')$ *for all $\varphi'\in\mathbb{R}^{\Lambda_N}$.*

Proof. By definition of G_j^Ψ, properties (1) and (2) are clear. For (3), first observe from the definition of $w_j(X,\varphi)$ (see above (3.7)) that for each $t'\in[0,1]$ and some geometric constant $C>0$,

$$\begin{aligned} w_j(X,\varphi+tu_j)^2 &\leqslant 2\sum_{B\in\mathcal{B}_j(X)} \Big(\|\nabla_j(\varphi+t'u_j)\|^2_{L^\infty(B^*)} + (t-t')^2\|\nabla_j u_j\|^2_{L^\infty(B^*)} \qquad (3.16)\\ &\qquad + \|\nabla_j^2(\varphi+t'u_j)\|^2_{L^\infty(B^*)} + (t-t')^2\|\nabla_j^2 u_j\|_{L^\infty(B^*)}\Big) \\ &\leqslant 2\sum_{B\in\mathcal{B}_j(X)} \big(\|\nabla_j(\varphi+t'u_j)\|^2_{L^\infty(B^*)} + \|\nabla_j^2(\varphi+t'u_j)\|^2_{L^\infty(B^*)}\big) + C\|u_j\|^2_{C_j^2}. \qquad (3.17)\end{aligned}$$

We then note that for any $B\in\mathcal{B}_j(X)$, $x_0\in B$ and $x\in B^*$, there is another constant $C>0$ such that

$$|\nabla_j^\mu\varphi(x)| \leqslant |\nabla_j^\mu\varphi(x_0)| + C\|\nabla_j^2\varphi\|_{L^\infty(B^*)}, \tag{3.18}$$

for all $\mu\in\hat{e}$ (for example, cf. [8, (A.37)], applied to $f=\nabla_j^\mu\varphi$ and recall that x_0 and x belong to some small set X, whence $|X|_j\leqslant C$) and hence $\|\nabla_j\varphi\|^2_{L^\infty(B^*)} \leqslant 2\max_{\mu\in\hat{e}}|\nabla^\mu\varphi(x_0)|^2 + 2C^2\|\nabla_j^2\varphi\|^2_{L^\infty(B^*)}$. Summing over all $x_0\in B$, this implies

$$\|\nabla_j\varphi\|^2_{L^\infty(B^*)} \leqslant 2\|\nabla_j\varphi\|^2_{L^2_j(B)} + 2C^2\|\nabla_j^2\varphi\|^2_{L^\infty(B^*)}. \tag{3.19}$$

Plugging this into (3.17), we get

$$c_w w_j(X,\varphi+tu_j)^2 \leqslant Cc_w\|u_j\|^2_{C_j^2} + \frac{1}{2}\|\nabla_j(\varphi+t'u_j)\|^2_{L^2_j(X)} + \frac{1}{2}W_j^\Psi(X,\nabla_j^2\varphi)^2 \tag{3.20}$$

for c_w sufficiently small. The discrepancy between the left- and right-hand sides of item (3) of the statement of the proposition due to the boundary term of G_j can be treated by the discrete Sobolev trace theorem [8, Corollary A.2], which shows that there is $C > 0$ such that

$$c_2 \|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(\partial Y \setminus \partial(X \cup Y))} \leqslant C c_2 \Big(\|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(X)} + \|\nabla^2_j(\varphi + t' u_j)\|^2_{L^\infty(X)} \Big), \qquad (3.21)$$

so if c_2 is sufficiently small so that $Cc_2 \leqslant \frac{1}{2}$, then this together with (3.20) gives

$$\begin{aligned} &c_w w_j(X, \varphi + t u_j)^2 + \|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(Y)} + c_2 \|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(\partial Y)} + W^\Psi_j(Y, \nabla^2_j \varphi)^2 \qquad (3.22)\\ &\leqslant C c_w \|u_j\|^2_{C^2_j} + \|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(X \cup Y)} + c_2 \|\nabla_j(\varphi + t' u_j)\|^2_{L^2_j(\partial(X \cup Y))} + W^\Psi_j(X \cup Y, \nabla^2_j \varphi)^2. \end{aligned}$$

After taking supremum over t', it follows that (3) holds for any $C_\Psi \geqslant \exp(C c_w \kappa_L \|u_j\|^2_{C^2_j})$, and C_Ψ can be chosen independent of j because of $(\mathbf{A}_u)$.

For (4), we may assume that $j \leqslant N-2$, since $\Gamma^{\Lambda_N}_N$ satisfies the same estimates as Γ_N. We use the regulator decomposition: by Lemma 3.1,

$$G^\Psi_j(X, \varphi' + \zeta; u_j) \leqslant \prod_{k=1}^M \sup_{t \in [0,1]} g_{j + \frac{k-1}{M}}(X_{k/M}, \xi_k + 1_{k=1} t u_j) G_{j+1}(\overline{X}, \varphi') \qquad (3.23)$$

whenever $\zeta = \sum_k \xi_k$ and $X_{k/M}$ is the smallest polymer in $\mathcal{P}_{j+k/M}$ containing $X \in \mathcal{P}_j$ and $\overline{X} = X_1$. Using the covariance subdecomposition (3.8), we may decompose $\zeta \sim \mathcal{N}(0, \Gamma_{j+1})$ as the sum of independent $\xi_k \sim \mathcal{N}(0, \Gamma_{j+k/M, j+(k+1)/M})$. Then each $\mathbb{E}^{\xi_k}[g_{j+(k-1)/M}(X_{k/M}, \xi_k + 1_{k=1} t u_j)]$ are bounded using [8, Lemma 5.12]. For $k = 1$, we have from the definition of g_j that

$$g_j(X_{M^{-1}}, \xi_1 + t u_j) \leqslant g_j(X_{M^{-1}}, \xi_1)^2 g_j(X_{M^{-1}}, u_j)^2 \leqslant g_j(X_{M^{-1}}, \xi_1)^2 e^{c\kappa_L \|u_j\|_{C^2_j}} \qquad (3.24)$$

for some $c > 0$. Also for any $k \in \{1, \dots, M\}$, [8, Lemma 5.12] gives

$$\mathbb{E}^{\xi_k}[g_{j+(k-1)/M}(X_{k/M}, \xi_k)] \leqslant \mathbb{E}^{\xi_k}[g_{j+(k-1)/M}(X_{k/M}, \xi_k)^2] \leqslant 2^{|X|_j/M} \qquad (3.25)$$

with the choice of L and ℓ as in Lemma 3.1 (cf. [8, Appendix A.2]). Therefore

$$\mathbb{E}[G^\Psi_j(X, \varphi' + \zeta)] \leqslant e^{c\kappa_L \|u_j\|_{C^2_j}} 2^{|X|_j} G_{j+1}(\overline{X}, \varphi') \qquad (3.26)$$

which implies the claim with the same choice of C_Ψ as in (3). □

Next we define a norm corresponding to the Ψ-regulators. This norm is defined in the same way as the $\|\cdot\|_{h,T_j}$-norm in (3.6) except that there is, apart from the use of G^Ψ_j instead of G_j, also a change of the parameter A (large-set regulator) from A to $A/2$. This is to compensate a combinatorial factor coming from reblocking in the next section, which will not significantly affect the resulting estimates.

Definition 3.5. *Define, for $\Psi_j : \mathcal{P}_j \times \mathbb{R}^{\Lambda_N} \to \mathbb{R}$ such that $\Psi_j(X) = \prod_{Y \in \mathrm{Comp}_j(X)} \Psi_j(Y)$,*

$$\|\Psi_j(X)\|_{h, T^\Psi_j(X)} = \sup_\varphi G^\Psi_j(X, \varphi)^{-1} \|\Psi_j(X, \varphi)\|_{h, T_j(X, \varphi)} \qquad (3.27)$$

$$\|\Psi_j\|_{h, T^\Psi_j} = \sup_{X \in \mathcal{P}^c_j} (A/2)^{|X|_j} \|\Psi_j(X)\|_{h, T^\Psi_j(X)}. \qquad (3.28)$$

3.3. Contraction estimates. This short section can be regarded as an extension of [8, Section 6], but some results are now generalized to apply to the norm $\|\cdot\|_{h,T_j^\Psi(X)}$. In the following we write

$$G_j^*(X,\varphi) = \begin{cases} G_j(X,\varphi) & \text{if } * = 0 \\ G_j^\Psi(X,\varphi) & \text{if } * = \Psi. \end{cases} \tag{3.29}$$

Note that by applying Proposition 3.4 to both u_j and $u_j' \equiv 0$ (which also satisfies $(\mathbf{A}_u)$), one obtains that

$$\mathbb{E}[G_j^*(X,\varphi'+\zeta)] \leqslant C_\Psi 2^{|X|_j} G_{j+1}(X,\varphi'), \qquad \text{for both } * \in \{0,\Psi\}. \tag{3.30}$$

We also use the notation $\|\cdot\|_{h,T_j^*(X)}$ for either $\|\cdot\|_{h,T_j(X)}$ or $\|\cdot\|_{h,T_j^\Psi(X)}$ and $\|\cdot\|_{h,T_j^*}$ for either $\|\cdot\|_{h,T_j}$ or $\|\cdot\|_{h,T_j^\Psi}$ when $*=0$ or Ψ, respectively.

Below, we refer to $2\pi/\sqrt{\beta}$-periodic polymer activities to be the functions $F(X,\varphi)$ such that $t \mapsto F(X,\varphi+t)$ is $2\pi/\sqrt{\beta}$-periodic, see [8, Definition 6.1]. Then its charge-q part is defined by the Fourier expansion

$$F(X,\varphi+t) = \sum_{q\in\mathbb{Z}} e^{i\sqrt{\beta}qt}\hat{F}_q(X,\varphi), \quad t\in\mathbb{R}, \tag{3.31}$$

and F is called neutral if $F = \hat{F}_0$. Recall that the norm $\|\cdot\|_{h,T_j^*(X)}$ in (3.27) depends implicitly on a choice of $u=(u_j)_j$ and the notion of small sets $\mathcal{S}_j$ at scale j from [8, Section 4.1].

Proposition 3.6. *Let $X\in\mathcal{S}_j$, and let F be a $2\pi/\sqrt{\beta}$-periodic polymer activity such that $\|F\|_{h,T_j^*(X)} < \infty$ where $*\in\{0,\Psi\}$. Let $(u_j)_j$ satisfy $(\mathbf{A}_u)$. Then for some $C>0$ and $L\geqslant L_0$, the following hold.*

- *If F has charge q with $|q|\geqslant 1$, then for all $\varphi'\in\mathbb{R}^{\Lambda_N}$,*
$$\|\mathbb{E}F(X,\varphi'+\zeta)\|_{h,T_{j+1}(X,\varphi')} \leqslant Ce^{\sqrt{\beta}|q|h}e^{-(|q|-1/2)r\Gamma_{j+1}(0)}\|F(X)\|_{h,T_j^*(X)}G_{j+1}(\overline{X},\varphi'). \tag{3.32}$$
- *If F is neutral, then for all $\varphi'\in\mathbb{R}^{\Lambda_N}$,*
$$\|\mathbb{E}[F(X,\varphi'+\zeta)-F(X,\zeta)]\|_{h,T_{j+1}(X,\varphi')} \leqslant CL^{-1}(\log L)^{1/2}\|F(X)\|_{h,T_j^*(X)}G_{j+1}(\overline{X},\varphi'). \tag{3.33}$$

Proof. The first item, (3.32) for $*=0$ is just [8, Lemma 6.13].

For $*=\Psi$, it suffices to argue that the conclusion of [8, Lemma 6.12] continues to hold under the modified assumption that $\|F\|_{h,T_j^\Psi(X)}<\infty$. Indeed with this at hand, the proof of (3.32) proceeds exactly as that of [8, Lemma 6.13], except that one invokes (3.30) above rather than [8, Proposition 5.9] towards the end of that proof. As to why the identity [8, (6.43)] still holds, one simply observes upon inspecting its proof that an analogue of the argument in [8, (6.50)–(6.52)] involving $\|F(X)\|_{h,T_j^*(X)}$ still applies when combining (3.23) (which generalises [8, Lemma 5.13]) with [8, (5.36)].

To see the second point, we proceed similarly as in [8, Lemma 6.17]: writing $(\mathrm{Rem}_0\,\mathbb{E}F)(X,\varphi') = \mathbb{E}[F(X,\varphi'+\zeta)-F(X,\zeta)]$, Taylor's theorem and neutrality of F give

$$(\mathrm{Rem}_0\,\mathbb{E}F)(X,\varphi') = \int_0^1 dt\,(1-t)D\,\mathrm{Rem}_0\,\mathbb{E}F(X,\zeta+t\varphi')(\delta\varphi'), \tag{3.34}$$

where $\delta\varphi'(x) = \varphi'(x)-\varphi'(x_0)$ for a fixed point $x_0\in X$. But since $D\,\mathrm{Rem}_0\,\mathbb{E}F(X,\varphi') = \mathbb{E}DF(X,\cdot+\varphi')$, the left-hand side of (3.34) is bounded in absolute value by

$$\begin{aligned} h^{-1}&\int_0^1 dt\,(1-t)\|\mathbb{E}DF(X,\cdot+t\varphi')\|_{h,T_{j+1}^*(X,t\varphi')}\|\delta\varphi'\|_{C_{j+1}^2(X^*)} \\ &\leqslant h^{-1}C_dL^{-1}\int_0^1 dt\,(1-t)\|F(X)\|_{h,T_j^*(X)}\mathbb{E}[G_j^*(X,t\varphi'+\zeta)]\|\delta\varphi'\|_{C_{j+1}^2(X^*)}, \end{aligned} \tag{3.35}$$

applying [8, (6.31)] in the second line. Moreover, $\mathbb{E}[G_j^*(X, t\varphi' + \zeta)] \leqslant C_\Psi 2^{|X|_j} G_{j+1}(\overline{X}, t\varphi')$ for both $* \in \{0, \Psi\}$, as follows readily from (3.30), and by [8, (6.100)] (applied with $n = 2$),

$$\mathbb{E}[G_j^*(X, t\varphi' + \zeta)] \|\delta\varphi'\|_{C^2_{j+1}(X^*)} \leqslant C(\log L)^{1/2} G_{j+1}(\overline{X}, \varphi'). \tag{3.36}$$

On the other hand, for $n \geqslant 1$, $D^n \operatorname{Rem}_0 = D^n$ and thus by [8, (6.31)], we immediately get

$$|D^n(\operatorname{Rem}_0 \mathbb{E}F)(X, \varphi)(f_1, \cdots, f_n)| \leqslant (C_g L)^{-n} \|D^n \mathbb{E}F(X, \varphi' + \zeta)\|_{h, T_j(X, \varphi')} \prod_{k=1}^n \|f_k\|_{C^2_{j+1}(X^*)} \tag{3.37}$$

for some constant $C_g > 0$. We obtain (3.33) from (3.35), (3.37) by summing $\frac{h^n}{n!} \|D^n \operatorname{Rem}_0 \mathbb{E}F(X, \varphi')\|_{h, T_j(X, \varphi')}$ over $n \geqslant 0$. □

Finally, we recall the definition of the reblocking operator from [8, Definition 6.19], defined for a j-scale polymer activity F by

$$\mathbb{S}F(X) = \sum_{Y \in \mathcal{P}_j^c}^{\overline{Y} = X} F(Y), \quad X \in \mathcal{P}_{j+1}^c \tag{3.38}$$

and extended to disconnected $Z \in \mathcal{P}_{j+1}$ by $\mathbb{S}F(Z) = \prod_{X \in \operatorname{Comp}_{j+1}(Z)} \mathbb{S}F(X)$. The following lemma extends the reblocking estimate from [8, Proposition 6.20]. The only difference is that the bound on the right-hand side also holds for the weaker norm $\|\cdot\|_{h, T_j^\Psi}$.

Proposition 3.7. *There exists a geometric constant $\eta > 0$ and $\varepsilon_{rb} := A^{-8}$ such that the following holds. Let F be a polymer activity supported on large sets and satisfy $\|F\|_{h, T_j^*} \leqslant \varepsilon_{rb}$. Then for any $L \geqslant 5$, $(A/2)^\eta \geqslant L(2eL^2)^{1+\eta}$, $X \in \mathcal{P}_{j+1}$ and $* \in \{0, \Psi\}$,*

$$\|\mathbb{S}\mathbb{E}[F(X, \cdot + \zeta)]\|_{h, T_{j+1}(X)} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}} \|F\|_{h, T_j^*}. \tag{3.39}$$

Proof. The case $* = 0$ is exactly [8, Proposition 6.20]. The case $* = \Psi$ is obtained by following the same proof, but A is replaced by $A/2$ in view of the definition of $\|\cdot\|_{h, T_j^\Psi}$, see (3.28). □

3.4. Choice of parameters. Finally, we explain how the parameters in the norms above are chosen.

First of all, the parameters κ_L, c_2, c_4, c_w are chosen as in [8, Section 5] (see the end of Section 5.1 and Remark 5.11 therein), except that we impose the extra conditions resulting from the assumptions of Lemma 3.1 and Proposition 3.4. These do not contradict the conditions from [8, Section 5] as they only impose further smallness conditions on c_w, c_2, c_4.

Next, given a finite-range step distribution J, we fix an additional parameter $r \in (0, 1]$ such that (with $C = \sqrt{2} c_h c_f^{-1}$, an absolute constant from [8, Lemma 7.4], cf. also [8, (7.6) and Lemma 6.11] regarding the choices of c_f and c_h, respectively)

$$Cr \leqslant \rho_J^2, \tag{3.40}$$

and we always impose the condition (with $C = 2\max\{c_f^{-2}, c_f^{-1}\}$, also an absolute constant from [8, Lemma 7.4])

$$\beta \geqslant C. \tag{3.41}$$

The parameter h is then chosen as in [8, Definition 7.2] as $h = \max\{c_f^{1/2}, r c_h \rho_J^{-2} \sqrt{\beta}, \rho_J^{-1}\}$.

Finally, we will assume that $L \geqslant L_0$ and $A \geqslant A_0(L)$ with L_0 and $A_0(L)$ chosen to satisfy the assumptions of [8, Theorem 7.7] as well as of those of Lemma 3.1, Proposition 3.6, and Proposition 3.7 above. Moreover, we will always tacitly assume from here on that L is ℓ-adic, i.e., of the form $L = \ell^M$ for some integer $M \geqslant 1$, where $\ell := \min\{2^n : 2^n \geqslant \ell_0\}$ is the smallest dyadic integer larger than ℓ_0 (with ℓ_0 as supplied by Lemma 3.1, now fixed since c_2 is). This ensures that i) Lemma 3.1 is always in force and ii) eventually, (1.5) can be used (since L is automatically dyadic). Later in Sections 5 and 6, further lower bound conditions on L and A will be imposed, which are consistent with our standing assumptions $L \geqslant L_0$ and $A \geqslant A_0(L)$.

4 Reblocking the external field

We will use a renormalisation group analysis in Section 5 to study the flow of the partition functions defined by (2.14). Ideally, we would like to write the renormalisation group maps in identical form as those of [8, Section 7], but the introduction of the external field u_j breaks the algebraic form of U_j (see (3.2)) and the symmetry of the system that we used to define the localisation operators Loc in [8]. Thus, we will first reduce the problem caused by the external field to a setting where the form of U_j stays the same as in the original renormalisation group steps and then bound the perturbation created by this operation. This is achieved by the following proposition and lemma. For $X \in \mathcal{P}_j$, recall that $\mathcal{P}_j(X)$ denotes the set of all j-polymers Y such that $Y \subset X$.

Definition 4.1. *Given $u_j \in \mathbb{R}^{\Lambda_N}$ and scale-j polymer activities K_j and U_j, define for $X \in \mathcal{P}_j^c$,*

$$\mathcal{F}_\Psi[u_j, U_j, K_j\,;\,j](X,\varphi) = -K_j(X,\varphi) + \sum_{Y \in \mathcal{P}_j(X)} (e^{U_j(\cdot,\varphi+u_j)} - e^{U_j(\cdot,\varphi)})^{X\backslash Y} K_j(Y, \varphi + u_j) \tag{4.1}$$

where

$$(e^{U_j(\cdot,\varphi+u_j)} - e^{U_j(\cdot,\varphi)})^Z \overset{\text{def.}}{=} \prod_{B \in \mathcal{B}_j(Z)} (e^{U_j(B,\varphi+u_j)} - e^{U_j(B,\varphi)}), \quad \textit{for } Z \in \mathcal{P}_j, \tag{4.2}$$

and $\mathcal{F}_\Psi[u_j, U_j, K_j\,;\,j](Z,\varphi) = \prod_{X \in \mathrm{Comp}_j(Z)} \mathcal{F}_\Psi[u_j, U_j, K_j\,;\,j](X,\varphi)$ for general $Z \in \mathcal{P}_j$.

The dependence of $\mathcal{F}_\Psi$ on the scale j will often be omitted when it is clear from the context. The following is a purely algebraic statement. Note in particular that the assumptions on U_j, K_j appearing below will be satisfied by the choices in (3.1), (3.2).

Proposition 4.2. *Assume that for some scale-j polymer activities K_j and U_j,*

$$Z_j(\varphi) = e^{-E_j|\Lambda_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(\Lambda\backslash X,\varphi)} K_j(X,\varphi), \tag{4.3}$$

and that U_j is additive over blocks, i.e., $U_j(X \cup Y) = U_j(X) + U_j(Y)$ for all $X \cap Y = \emptyset$, $X, Y \in \mathcal{P}_j$. Let $\Psi_j = \mathcal{F}_\Psi[u_j, U_j, K_j; j]$. Then

$$Z_j(\varphi + u_j) = e^{-E_j|\Lambda_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(\Lambda\backslash X,\varphi)} \prod_{Z \in \mathrm{Comp}_j(X)} (K_j + \Psi_j)(Z,\varphi). \tag{4.4}$$

If K_j, U_j are $2\pi/\sqrt{\beta}$-periodic, then so is Ψ_j. If u_j satisfies $(\mathbf{A}_u)$, then $\Psi_j(X) = 0$ whenever $B_0^* \cap X = \emptyset$.

Proof. This is a result of a simple reblocking argument. Using the assumption

$$Z_j(\varphi + u_j) = e^{-E_j|\Lambda_N|} \sum_{X \in \mathcal{P}_j} e^{U_j(X,\varphi+u_j)} K_j(\Lambda\backslash X, \varphi + u_j), \tag{4.5}$$

by making the substitution

$$e^{U_j(X,\varphi+u_j)} = \prod_{B \in \mathcal{B}_j(X)} e^{U_j(B,\varphi+u_j)} = \sum_{Y \in \mathcal{P}_j(X)} (e^{U_j(\cdot,\varphi+u_j)} - e^{U_j(\cdot,\varphi)})^Y e^{U_j(X\backslash Y,\varphi)} \tag{4.6}$$

we immediately obtain that

$$Z_j(\varphi + u_j) = e^{-E_j|\Lambda_N|} \sum_{Y \in \mathcal{P}_j} e^{U_j(Y,\varphi)} \sum_{X' \in \mathcal{P}_j(\Lambda\backslash Y)} (e^{U_j(\cdot,\varphi+u_j)} - e^{U_j(\cdot,\varphi)})^{X'} K_j(\Lambda\backslash(Y \cup X'), \varphi + u_j). \tag{4.7}$$

Then we arrive at (4.4) after factoring the above expression into connected components of $\Lambda\backslash Y$.

The asserted periodicity of Ψ_j is plainly inherited from K_j, U_j and the last remark is a consequence of the fact that $K_j(X, \varphi + u_j) = K_j(X, \varphi)$ for $0 \notin X^*$ and u_j satisfying $(\mathbf{A}_u)$. $\square$

For the next estimates, recall the definition of the space Ω_j^U from [8, Definition 7.1] and of Ω_j^K from [8, Definition 7.2]. In particular, the parameters these spaces and their norms depend on are always assumed to satisfy the conditions specified in Section 3.4.

Lemma 4.3. *Suppose $(u_j)_j$ satisfies $(\mathbf{A}_u)$. Given U_j in form (3.2) and K_j a $2\pi/\sqrt{\beta}$-periodic polymer activity, let $\Psi_j = \mathcal{F}_\Psi[u_j, U_j, K_j]$. Then there exist $C > 0$ and $\varepsilon_\Psi > 0$ such that, whenever $\|\omega_j\|_{\Omega_j} := \max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\Omega_j^K}\} \leqslant \varepsilon_\Psi$,*

(1) $\|\Psi_j\|_{h,T_j^\Psi} \leqslant C\|\omega_j\|_{\Omega_j}$;

(2) *for $X \in \mathcal{S}_j$, $\|\mathbb{E}[\Psi_j(X, \varphi' + \zeta) - \hat{\Psi}_{j,0}(X, \zeta)]\|_{h,T_{j+1}(X,\varphi')} \leqslant A^{-|X|_j} \alpha_{\text{Loc}}^\Psi \|\Psi_j\|_{h,T_j^\Psi} G_{j+1}(\overline{X}, \varphi')$*

where $\alpha_{\text{Loc}}^\Psi = CL^{-1}(\log L)^{1/2} + C\min\left\{1, \sum_{q\geqslant 1} e^{\sqrt{\beta}qh} e^{-(q-1/2)r\beta\Gamma_{j+1}(0)}\right\}$ and $\hat{\Psi}_{j,0}$ is the charge-0 term of Ψ_j.

Proof. To prove (1), we first notice that by [8, Lemma 7.4] (whose assumptions are satisfied by the assumptions of this lemma) and (3.13), for $t \in \{0, 1\}$,

$$\|U_j(B, \varphi + tu_j)\|_{h,T_j(B,\varphi)} \leqslant CA^{-1}\|U_j\|_{\Omega_j^U} w_j(B, \varphi + tu_j)^2, \quad B \in \mathcal{B}_j \tag{4.8}$$

$$\|K_j(X, \varphi + tu_j)\|_{h,T_j(X,\varphi)} \leqslant A^{-|X|_j}\|K_j\|_{\Omega_j^K} G_j^\Psi(X, \varphi), \quad X \in \mathcal{P}_j. \tag{4.9}$$

Also, using $\|e^F - 1\|_{h,T_j(B,\varphi)} \leqslant \|F\|_{h,T_j(B,\varphi)} e^{\|F\|_{h,T_j(B,\varphi)}}$,

$$\begin{aligned} &\|e^{U_j(B,\varphi+u_j)} - e^{U_j(B,\varphi)}\|_{h,T_j(B,\varphi)} \\ &\quad \leqslant CA^{-1}\|U_j\|_{\Omega_j^U} \max_{t\in\{0,1\}} w_j(B, \varphi + tu_j) e^{CA^{-1}\|U_j\|_{\Omega_j^U} \max_{t\in\{0,1\}} w_j(B,\varphi+tu_j)} \end{aligned} \tag{4.10}$$

Using the submultiplicativity of the $\|\cdot\|_{h,T_j(B,\varphi)}$-norm to bound the powers of $e^{U_j(B,\varphi+u_j)} - e^{U_j(B,\varphi)}$ and Proposition 3.4 (3), it follows that

$$\frac{\|\Psi_j(X, \varphi)\|_{h,T_j(X,\varphi)}}{G_j^\Psi(X, \varphi)} \leqslant \sum_{Y\in\mathcal{P}_j(X)} (C\|\omega_j\|_{\Omega_j})^{|X\setminus Y|_j + |\text{Comp}_j(Y)|} A^{-|X|_j} \leqslant C'\|\omega_j\|_{\Omega_j}(A/2)^{-|X|_j} \tag{4.11}$$

whenever $\|\omega\|_{\Omega_j}$ is sufficiently small. This proves (1). To show (2), take $X \in \mathcal{S}_j$ and recall that Ψ_j is $2\pi/\sqrt{\beta}$-periodic to decompose

$$\Psi_j(X, \varphi) = \sum_{q\in\mathbb{Z}} \hat{\Psi}_{j,q}(X, \varphi) \tag{4.12}$$

where $\hat{\Psi}_{j,q}$ is the charge-q term of Ψ_j. Then apply (3.32) to bound $\mathbb{E}[\hat{\Psi}_{j,q}(X, \varphi + \zeta)]$ for $|q| \geqslant 1$ and (3.33) to bound $\mathbb{E}[\hat{\Psi}_{j,0}(X, \varphi + \zeta) - \hat{\Psi}_{j,0}(X, \varphi)]$. □

5 The renormalisation group map with external field

To prove the infinite-volume scaling limit we need an extended version of the renormalisation group maps that admits an external field at every scale. In this section we extend the (bulk) renormalisation group map from [8, Section 7] to allow for such an external field. The starting point is the generalisation of the parametrisation of the partition function from [8, (7.4)] to take into account a local perturbation. In accordance with (4.4), partition functions will now be parametrised as

$$Z_j(\varphi, \Psi_j; (\Psi_k)_{k<j}|\Lambda_N) = e^{-E_j|\Lambda_N|+e_j} \sum_{X\in\mathcal{P}_j(\Lambda_N)} e^{U_j(\Lambda\setminus X,\varphi)} \prod_{Y\in\text{Comp}_j(X)} \big(K_j(Y, \varphi; (\Psi_k)_{k<j}) + \Psi_j(Y, \varphi)\big), \tag{5.1}$$

and where e_j is a scalar coupling constant (like E_j), but originating from a bounded number of blocks near the origin. Then the renormalisation group flow corresponding to

$$Z_{j+1}(\varphi', 0; (\Psi_k)_{k\leqslant j}|\Lambda_N) = \mathbb{E}Z_j(\varphi' + \zeta, \Psi_j; (\Psi_k)_{k<j}|\Lambda_N), \qquad (j < N-1), \tag{5.2}$$

will be considered. Here recall that $\mathbb{E} = \mathbb{E}_{\Gamma_{j+1}}$ for $j \leqslant N-1$ and $\mathbb{E} = \mathbb{E}_{\Gamma_N^{\Lambda_N}}$ for the last step $j = N-1$.

5.1. Renormalisation group flow without external field. When $\Psi_k \equiv 0$ for each $k < j$, then we will just denote $K_j(\cdot; (\Psi_k)_{k<j})$ by $K_j(\cdot; 0)$; this corresponds to the setting of [8]. Here we briefly recall the main estimates for the renormalisation group map in this setting from [8, Sections 7 and 8]. This maps acts on the coupling constants $E_j \in \mathbb{R}$, U_j of the form (3.2), and $K_j(\cdot; 0)$ from [8, Sections 7 and 8]. In particular, U_j can be identified with its coupling constants s_j and $z_j = (z_j^{(q)})_{q\geqslant 1}$.

Also, we use the abbreviations $\omega_j = (U_j, K_j)$ and $\|\omega_j\|_{\Omega_j} = \max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\Omega_j^K}\}$, where norms are still as in [8, Definitions 7.1–7.2] with the parameters they depend on always assumed to satisfy the conditions of Section 3.4.

The following theorem puts together [8, Theorems 7.6 and 7.7] for $j+1 \leqslant N$ with its analogue [8, Proposition 9.1] for the last step $j+1 = N$.

Theorem 5.1. *Fix a finite-range step distribution J as in Theorem 1.1. There exist $\varepsilon_{nl}(\beta, A, L)$ such that the following holds for $\|\omega_j\|_{\Omega_j} \leqslant \varepsilon_{nl}$. For all N and $0 \leqslant j \leqslant N-1$, there is a map*

$$\Phi_{j+1}^{\Lambda_N} = (\mathcal{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathcal{K}_{j+1}) : (E_j, s_j, z_j, K_j(\cdot; 0)) \mapsto (E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}(\cdot; 0)), \tag{5.3}$$

such that (5.1), (5.2) *hold with $e_j \equiv 0$ and $\Psi_k \equiv 0$, and Z_0 given by* (2.6)*. The maps $\mathcal{E}_{j+1} - E_j$, $\mathfrak{s}_{j+1}$, $\mathfrak{z}_{j+1}$, and $\mathcal{K}_{j+1}$ are functions of ω_j satisfying*

$$|\mathfrak{s}_{j+1}(\omega_j) - s_j| \leqslant CA^{-1}\|\omega_j\|_{\Omega_j}, \tag{5.4}$$

$$|(\mathcal{E}_{j+1} - E_j)(\omega_j) + s_j \nabla^{(e_1, -e_1)}\Gamma_{j+1}(0)| \leqslant CA^{-1}L^{-2j}\|\omega_j\|_{\Omega_j}, \tag{5.5}$$

$$\mathfrak{z}_{j+1}(\omega_j) = L^2 e^{-\frac{1}{2}\beta q^2 \Gamma_{j+1}(0)} z_j, \tag{5.6}$$

for some $C > 0$ and there exists $\varepsilon_{nl} > 0$ such that whenever $\|\omega_j\|_{\Omega_j} \leqslant \varepsilon_{nl}$, $\mathcal{K}_{j+1}$ is continuously (Fréchet-)differentiable and admits a decomposition $\mathcal{K}_{j+1} = \mathcal{L}_{j+1} + \mathcal{M}_{j+1}$ satisfying the estimates

$$\|\mathcal{L}_{j+1}(\omega_j)\|_{\Omega_{j+1}^K} \leqslant C_1 L^2 \alpha_{\mathrm{Loc}} \|\omega_j\|_{\Omega_j} \tag{5.7}$$

$$\|\mathcal{M}_{j+1}(\omega_j)\|_{\Omega_{j+1}^K} \leqslant C_2(\beta, A, L)\|\omega_j\|_{\Omega_j}^2 \tag{5.8}$$

$$\|D\mathcal{M}_{j+1}(\omega_j)\|_{\Omega_{j+1}^K} \leqslant C_2(\beta, A, L)\|\omega_j\|_{\Omega_j} \tag{5.9}$$

for some $C_1, C_2(\beta, A, L) > 0$, where $\mathcal{L}_{j+1}$ is linear in ω_j and

$$\alpha_{\mathrm{Loc}} = CL^{-3}(\log L)^{3/2} + C\min\Big\{1, \sum_{q\geqslant 1} e^{\sqrt{\beta}qh} e^{-(q-1/2)r\beta\Gamma_{j+1}(0)}\Big\}. \tag{5.10}$$

The next theorem concerns the existence of initial conditions independent of N such that the renormalisation group flow exists for all N, i.e., that for all $N \geqslant 1$ and all $j \leqslant N-1$,

$$(E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}) = \Phi_{j+1}^{\Lambda_N}(E_j, s_j, z_j, K_j) \tag{5.11}$$

such that $\|(U_j, K_j)\|_{\Omega_j} < \varepsilon_{nl}$ for each $j \leqslant N$. With $j \leqslant N-1$ instead of $j \leqslant N$, the theorem is exactly [8, Proposition 8.1], and the bounds (5.12) for $j = N$ follows from the bounds with $j = N-1$ by a single application of Theorem 5.1.

Theorem 5.2. *For any finite-range step distribution J as in Theorem 1.1, there exist $\beta_0(J) \in (0,\infty)$, $s_0^c(J,\beta) = O(e^{-\frac{1}{4}\gamma\beta})$, and $\alpha = \alpha(J,\beta) > 0$, such that for $\beta \geqslant \beta_0(J)$ the solution to the flow equation* (5.11) *with parameter $s = s_0^c(J,\beta)$ and initial conditions $s_0 = s_0^c(J,\beta)$, $z_0 = \tilde{z}(\beta)$ as in [8, Lemma 2.2], and $K_0(X) = 1_{X=\emptyset}$, satisfies for all $j \leqslant N$ and $N > 1$,*

$$\|U_j\|_{\Omega_j^U} \leqslant O(e^{-\frac{1}{4}\gamma\beta}L^{-\alpha j}), \qquad \|K_j\|_{\Omega_j^K} \leqslant O(e^{-\frac{1}{4}\gamma\beta}L^{-\alpha j}). \tag{5.12}$$

In fact, one can take any $\alpha > 0$ such that $CL^2\alpha_{\mathrm{Loc}} \leqslant L^{-\alpha}$ for sufficiently large C.

5.2. Extended coordinates. We next define the extended renormalisation group coordinates that incorporate a perturbation Ψ. First, recall the definition of polymer activities from [8, Definition 5.1] and the definition of (bulk) renormalisation group coordinates K_j from [8, Definition 7.2]. The extended version of the K-coordinate is then defined as follows.

Definition 5.3. *The coordinate $\vec{K}_j = (K_j(\cdot;0), K_j(\cdot;(\Psi_k)_{k<j}))$ is a pair of $2\pi/\sqrt{\beta}$-periodic polymer activities such that $K_j(X,\varphi;0)$ is even and invariant under the lattice symmetries. For pairs of such polymer activities, define*

$$\|\vec{K}_j\|_{\Omega_j^{\vec{K}}} = \max\{\|K_j(\cdot;0)\|_{h,T_j}, \|K_j(\cdot;(\Psi_k)_{k<j})\|_{h,T_j}\}. \tag{5.13}$$

Let $\Omega_j^{\vec{K}}$ be the Banach space (cf. [8, Appendix B]) of such pairs where the maximum is finite.

We also need a new definition of the product space of $(U_j, \vec{K}_j, \Psi_j)$ as follows.

Definition 5.4 (Extended coordinates)**.** *Define the normed space of polymer activity perturbations based at the origin by*

$$\Omega_j^\Psi = \{\Psi_j \textit{ is } 2\pi/\sqrt{\beta}\textit{-periodic} : \|\Psi_j\|_{h,T_j^\Psi} < \infty,\ \Psi_j(X) = 0 \text{ if } 0 \notin X^*\} \tag{5.14}$$

equipped with the norm $\|\cdot\|_{\Omega_j^\Psi} = \|\cdot\|_{h,T_j^\Psi}$. Also let $\overline{\Omega}_j = \Omega_j^U \times \Omega_j^{\vec{K}} \times \Omega_j^\Psi$, i.e.,

$$\overline{\Omega}_j = \{\omega_j = (U_j, \vec{K}_j, \Psi_j) : \|\omega_j\|_{\overline{\Omega}_j} < +\infty\}, \quad \|\omega_j\|_{\overline{\Omega}_j} = \max\{\|U_j\|_{\Omega_j^U}, \|\vec{K}_j\|_{\Omega_j^{\vec{K}}}, \|\Psi_j\|_{\Omega_j^\Psi}\}. \tag{5.15}$$

Given $\varepsilon_\Psi, C_\Psi > 0$ also define $\mathcal{Y}_j \equiv \mathcal{Y}_j(\varepsilon_\Psi, C_\Psi) \subset \overline{\Omega}_j$ be the closed subset defined by the conditions

(1) $K_j(X,\varphi;0) = K_j(X,\varphi;(\Psi_k)_{k<j})$ if $0 \notin X^*$;

(2) $\|\Psi_j\|_{\Omega_j^\Psi} \leqslant C_\Psi \max\{\|U_j\|_{\Omega_j^U}, \|\vec{K}_j\|_{\Omega_j^{\vec{K}}}\}$ and $\|\omega_j\|_{\overline{\Omega}_j} \leqslant \varepsilon_\Psi$;

(3) For $X \in \mathcal{S}_j$,

$$\|\mathbb{E}[\Psi_j(X, \cdot + \zeta) - \hat{\Psi}_{j,0}(X,\zeta)]\|_{h,T_{j+1}(\overline{X})} \leqslant C_\Psi A^{-|X|_j}\alpha_{\mathrm{Loc}}^\Psi \|\omega_j\|_{\overline{\Omega}_j} \tag{5.16}$$

with $\alpha_{\mathrm{Loc}}^\Psi$ as defined below Lemma 4.3 (2).

In particular, if we define $\Psi_j = \mathcal{F}_\Psi[u_j, K_j(\cdot;(\Psi_k)_{k<j}), U_j; j]$ for given U_j, $\vec{K}_j$, then, if their assumptions are satisfied, Proposition 4.2 and Lemma 4.3 imply $(U_j, K_j(\cdot;0), K_j(\cdot;(\Psi_k)_{k<j}), \Psi_j) \in \mathcal{Y}_j(\varepsilon_\Psi, C_\Psi)$ for some $\varepsilon_\Psi, C_\Psi > 0$ whenever $\|U_j\|_{\Omega_j^U}, \|\vec{K}_j\|_{\Omega_j^{\vec{K}}} \leqslant \varepsilon_\Psi$.

5.3. Definition of the extended renormalisation group map. We will now introduce the extended renormalisation group map with the extra coordinates $K_j(\cdot;(\Psi_k)_{k<j})$ and Ψ_j, which we denote by

$$\overline{\Phi}_{j+1}:(E_j,e_j,s_j,z_j,\vec{K}_j,\Psi_j)\mapsto(E_{j+1},e_{j+1},s_{j+1},z_{j+1},\vec{K}_{j+1},0) \tag{5.17}$$

(cf. [8, (7.12)] for Φ_{j+1} which we now call the bulk part of the renormalisation group map); here the e_j are scalar coupling constants taking the role for the perturbation due to the external field that the E_j have for the bulk part of the renormalisation group map. In analogy with Φ_{j+1}, we will also denote the components of the map $\overline{\Phi}_{j+1}$ by $(\mathcal{E}_{j+1},\mathfrak{e}_{j+1},\mathcal{U}_{j+1},\mathcal{K}^0_{j+1},\mathcal{K}^\Psi_{j+1})$ and require that

$$(\mathfrak{e}_{j+1}-\mathcal{E}_{j+1}|\Lambda|)(E_j,e_j,\cdot)=(\mathfrak{e}_{j+1}-\mathcal{E}_{j+1}|\Lambda|)(0,0,\cdot)+e_j-E_j|\Lambda|. \tag{5.18}$$

The last condition can be imposed because the scalar prefactor $e^{-E_j|\Lambda|+e_j}$ appearing in Z_j (see (5.1)) is mapped to the corresponding quantity at scale $j+1$ and hence does not contribute to the dynamics, see the discussion below [8, (7.12)] for the bulk case. Moreover, when we write $\mathfrak{e}_{j+1}$, $\mathcal{E}_{j+1}$ without e_j,E_j as their arguments, they are just $\mathfrak{e}_{j+1}(0,0,\cdot)$ and $\mathcal{E}_{j+1}(0,0,\cdot)$ respectively.

We are thinking of $\overline{\Phi}_{j+1}$ as Φ_{j+1} with a perturbation, which entails that the $\mathcal{E}_{j+1}$, $\mathcal{U}_{j+1}$ and $\mathcal{K}^0_{j+1}$ will be given as in [8, Section 7], i.e., by Definitions 7.8 and 7.9 in that paper respectively. The other coordinates $\mathfrak{e}_{j+1}$ and $\mathcal{K}^\Psi_{j+1}$ are defined explicitly as follows. The definition of $\mathcal{K}^\Psi_{j+1}$ is almost the same as that of $\mathcal{K}_{j+1}$ except for the perturbed activity Ψ_j and the one-point energy $\mathfrak{e}_{j+1}$ arising from it.

Definition 5.5. *For $0\leqslant j\leqslant N-1$, let ζ be the centred Gaussian random variable with covariance Γ_{j+1} if $j\leqslant N-2$ and $\Gamma_N^{\Lambda_N}$ if $j=N-1$. Then for each $Y\in\mathcal{P}_j$, define the map $(\vec{K}_j,\Psi_j)\mapsto\mathfrak{e}_{j+1}(\vec{K}_j,\Psi_j)$ by*

$$\mathfrak{e}'_{j+1}(Y,\vec{K}_j,\Psi_j)=\sum_{B\in\mathcal{B}_j(B_0^*\cap Y)}\sum_{Z\in\mathcal{S}_j}^{B\subset Z}\frac{1}{|Z\cap B_0^*|_j}\mathbb{E}[\hat{\Psi}_{j,0}(Z,\zeta)+\hat{K}_{j,0}(Z,\zeta;(\Psi_k)_{k<j})-\hat{K}_{j,0}(Z,\zeta;0)] \tag{5.19}$$

where we recall that B_0 is the unique j-block such that $0\in B_0$ and let

$$\mathfrak{e}_{j+1}(\vec{K}_j,\Psi_j)=\mathfrak{e}'_{j+1}(\Lambda_N,\vec{K}_j,\Psi_j). \tag{5.20}$$

The map $(U_j,\vec{K}_j,\Psi_j)\mapsto K^\Psi_{j+1}$ is defined by

$$\begin{aligned}\mathcal{K}^\Psi_{j+1}(U_j,\vec{K}_j,\Psi_j,X)=&\sum^{*}_{X_0,X_1,Z,(B_{Z''})}e^{\mathcal{E}_{j+1}|T|-\mathfrak{e}_{j+1}(T)}e^{\mathcal{U}_{j+1}(X\setminus T)}\\&\times\mathbb{E}\Big[(e^{U_j}-e^{-\mathcal{E}_{j+1}|B|+\mathfrak{e}'_{j+1}(B)+\mathcal{U}_{j+1}})^{X_0}(\overline{K}^\Psi_j-\mathcal{E}^\Psi K_j)^{[X_1]}\Big]\prod_{Z''\in\mathrm{Comp}_{j+1}(Z)}J^\Psi_j(B_{Z''},Z''),\end{aligned} \tag{5.21}$$

where the polymer powers follow the convention [8, (7.23), (7.24)], the summation $*$ is running over disjoint $(j+1)$-polymers X_0,X_1,Z such that $X_1\not\sim Z$, $B_{Z''}\in\mathcal{B}_{j+1}(Z'')$ for each $Z''\in$

$\mathrm{Comp}_{j+1}(Z)$, $T = X_0 \cup X_1 \cup Z$ *and* $X = \cup_{Z''} B^*_{Z''} \cup X_0 \cup X_1$, *and*

$$\mathcal{E}^\Psi K_j(X,\varphi') = \sum_{B \in \mathcal{B}_{j+1}(X)} J_j^\Psi(B,X,\varphi') \tag{5.22}$$

$$Q_j^\Psi(D,Y,\varphi') = 1_{Y\in\mathcal{S}_j}\Big(\mathrm{Loc}_{Y,D}\,\mathbb{E}[K_j(Y,\varphi'+\zeta;0)] + \frac{1_{D\subset\mathcal{B}_j(B_0^*\cap Y)}}{|Y\cap B_0^*|_j}\mathbb{E}[\hat{\Psi}_{j,0}(Y,\zeta) + \hat{K}_{j,0}(Y,\zeta;(\Psi_k)_{k<j}) - \hat{K}_{j,0}(Y,\zeta;0)]\Big) \tag{5.23}$$

$$J_j^\Psi(B,X,\varphi') = 1_{B\in\mathcal{B}_{j+1}(X)} \sum_{D\in\mathcal{B}_j(B)} \sum_{Y\in\mathcal{S}_j}^{D\in\mathcal{B}_j(Y)} Q_j^\Psi(D,Y,\varphi')(1_{\overline{Y}=X} - 1_{B=X}). \tag{5.24}$$

$$\overline{K}_j^\Psi(X,\varphi'+\zeta) = \sum_{Y\in\mathcal{P}_j}^{\overline{Y}=X} e^{U_j(X\setminus Y,\varphi'+\zeta)}\Big((K_j(Y,\varphi'+\zeta;(\Psi_k)_{k<j}) + \Psi_j(Y,\varphi'+\zeta))\Big) \tag{5.25}$$

for $D\in\mathcal{B}_j$, $B\in\mathcal{B}_{j+1}$, $Y\in\mathcal{P}_j$ *and* $X\in\mathcal{P}_{j+1}$.

Note that each $(j+1)$-block $B_{Z''}$ appearing in the summation defining $\mathcal{K}_{j+1}^\Psi$ is such that $Z''\in B^*_{Z''}$ since $J_j^\Psi(B_{Z''},Z'',\varphi')$ vanishes whenever $Z''\notin\mathcal{S}_{j+1}$.

In the remainder of the argument, we will focus on the case $j\leqslant N-2$, and hence $\zeta\sim\mathcal{N}(0,\Gamma_{j+1})$. The argument is identical for the case $j\leqslant N-1$ because $\Gamma_N^{\Lambda_N}$ satisfies the same estimates as Γ_N.

The next theorem is the extension of [8, Theorem 7.5] with essentially the same proof; see Appendix C for the proof. It shows that Z_{j+1} defined by the map $\overline{\Phi}_{j+1}$ is indeed the desired partition function of scale $j+1$.

Theorem 5.6. *Let* $Z_j(\varphi,\Psi_j;(\Psi_k)_{k<j}|\Lambda)$ *and* $Z_{j+1}(\varphi',0;(\Psi_k)_{k\leqslant j}|\Lambda)$ *be defined by* (5.1) *with coordinates* $(E_j,e_j,U_j,\vec{K}_j,\Psi_j)$ *and* $(E_{j+1},e_{j+1},U_{j+1}\vec{K}_{j+1},0) = \overline{\Phi}_{j+1}(E_j,e_j,U_j,\vec{K}_j,\Psi_j)$ *respectively. Then they satisfy* (5.2) *(and* (5.18) *holds).*

5.4. Estimates for the extended renormalisation group map. Since we have already established estimates on the bulk components $\mathcal{E}_{j+1}$, $\mathcal{U}_{j+1}$, and $\mathcal{K}_j^0\equiv\mathcal{K}_j(\cdot;0)$ of the renormalisation group map in [8, Theorems 7.6 and 7.7], we only need additional estimates for $\mathfrak{e}_{j+1}$ and $\mathcal{K}_{j+1}^\Psi$. Since we will not need a stable manifold theorem to tune parameters, a cruder control of these suffices.

Theorem 5.7. *Let* $(u_j)_j$ *satisfy* $(\mathbf{A}_u)$ *and the parameters be as in Section 3.4. If* $(U_j,\vec{K}_j,\Psi_j)\in\mathcal{Y}_j(\varepsilon,C_\Psi)$, *for some* $\varepsilon>0$ *and* C_Ψ *as given by Proposition 3.4,*

$$|\mathfrak{e}'_{j+1}(B,\Psi_j,\vec{K}_j)| \leqslant CC_\Psi A^{-1}\|\omega_j\|_{\overline{\Omega}_j}, \quad B\in\mathcal{B}_j. \tag{5.26}$$

Proof. Let $X\in\mathcal{S}_j$ be such that $0\in X^*$ and $B\in\mathcal{B}_j(X)$. By (4.12) and the definition of $\|\cdot\|_{\Omega_j^\Psi}(=\|\cdot\|_{h,T_j^\Psi})$, see (3.27)-(3.28),

$$\big|\mathbb{E}[\hat{\Psi}_{j,0}(X,\zeta)]\big| \leqslant (A/2)^{-|X|_j}\|\Psi_j\|_{\Omega_j^\Psi}\mathbb{E}[G_j^\Psi(X,\zeta)] \tag{5.27}$$

and by the assumption $(U_j,\vec{K}_j,\Psi_j)\in\mathcal{Y}_j$, we also have $\|\Psi_j\|_{\Omega_j^\Psi}\leqslant C_\Psi\|\omega_j\|_{\overline{\Omega}_j}$. Similarly,

$$\big|\mathbb{E}[\hat{K}_{j,0}(X,\zeta;(\Psi_k)_{k<j}) - \hat{K}_{j,0}(X,\zeta;0)]\big| \leqslant 2A^{-|X|_j}\|\vec{K}_j\|_{\Omega_j^{\vec{K}}}\mathbb{E}[G_j(X,\zeta)] \tag{5.28}$$

and note that $\|\vec{K}_j\|_{\Omega_j^{\vec{K}}}\leqslant\|\omega_j\|_{\overline{\Omega}_j}$ by definition, see (5.15). By Proposition 3.4 and since $|X|_j\leqslant 4$ for $X\in\mathcal{S}_j$, we have that

$$\mathbb{E}[G_j(X,\zeta)] \leqslant \mathbb{E}[G_j^\Psi(X,\zeta)] \leqslant C_\Psi 2^{|X|_j} \leqslant 16C_\Psi. \tag{5.29}$$

Hence, by definition of $\mathfrak{e}'_{j+1}$ in (5.19), we obtain (5.26). □

Theorem 5.8 (Estimate for remainder coordinate)**.** *Let $0 \leqslant j \leqslant N-1$ and the parameters be as in Section 3.4. Further assume $(\mathbf{A}_u)$ to hold and let C_Ψ be given by Proposition 3.4. Then the map $\mathcal{K}^\Psi_{j+1}(U_j, \vec{K}_j, \Psi_j)$ admits a decomposition*

$$\mathcal{K}^\Psi_{j+1}(U_j, \vec{K}_j, \Psi_j) = \mathcal{L}^\Psi_{j+1}(\vec{K}_j, \Psi_j) + \mathcal{M}^\Psi_{j+1}(U_j, \vec{K}_j, \Psi_j) \tag{5.30}$$

such that the following estimates hold: the map $\mathcal{L}^\Psi_{j+1}$ is linear in $(\vec{K}_j, \Psi_j)$ and there exist L_0', $A_0'(L)$, $\tilde{\varepsilon}_{nl} \equiv \tilde{\varepsilon}_{nl}(\beta, A, L, C_\Psi) > 0$ (only polynomially small in its arguments), $C_1 > 0$ independent of A and L and $C_2 = C_2(\beta, A, L, C_\Psi) > 0$ (only polynomially large in its arguments) such that for $L \geqslant L_0'$, $A \geqslant A_0'(L)$, $\omega_j = (U_j, \vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$,

$$\|\mathcal{L}^\Psi_{j+1}(\vec{K}_j, \Psi_j)\|_{\Omega^\Psi_{j+1}} \leqslant C_1 C_\Psi \big(L^2 \alpha_{\mathrm{Loc}} \|K_j(\cdot;0)\|_{\Omega^K_j} + \alpha^\Psi_{\mathrm{Loc}} \|\omega_j\|_{\overline{\Omega}_j}\big), \tag{5.31}$$

with $\alpha^\Psi_{\mathrm{Loc}}$ from Lemma 4.3, and $\mathcal{M}^\Psi_{j+1}(U_j, \vec{K}_j, \Psi_j)$ is continuously Fréchet-differentiable with

$$\|\mathcal{M}^\Psi_{j+1}(\omega_j)\|_{\Omega^\Psi_{j+1}} \leqslant C_2(\beta, A, L, C_\Psi) \|\omega_j\|^2_{\overline{\Omega}_j} \tag{5.32}$$

$$\|D\mathcal{M}^\Psi_{j+1}(\omega_j)\|_{\Omega^\Psi_{j+1}} \leqslant C_2(\beta, A, L, C_\Psi) \|\omega_j\|_{\overline{\Omega}_j}. \tag{5.33}$$

5.5. Proof of Theorem 5.8: bound of linear part. We first introduce $\mathcal{L}^\Psi_{j+1}$. Proceeding as in [8, Section 7.4], we may write the terms linear in U_j, $\vec{K}_j$ from (5.21) by keeping only the terms in (5.21) with

$$\#(X_0, X_1, Z) := |X_0|_{j+1} + |\operatorname{Comp}_{j+1}(X_1)| + |\operatorname{Comp}_{j+1}(Z)| \leqslant 1 \tag{5.34}$$

and replacing exponentials by their linear approximations. This linearisation process is identical to that of [8, Section 7.4]. For $X \in \mathcal{P}^c_{j+1}$, this gives

$$\begin{aligned}
&\mathcal{L}^\Psi_{j+1}(X, \varphi') \\
&= \sum_{Y : \overline{Y} = X} 1_{Y \in \mathcal{P}^c_j} \Big(\mathbb{E}[K_j(Y, \zeta + \varphi'; (\Psi_k)_{k \leqslant j-1}) + \Psi_j(Y, \zeta + \varphi')] - 1_{Y \in \mathcal{S}_j} \sum_{D \in \mathcal{B}_j(Y)} Q^\Psi_j(D, Y, \varphi') \Big) \\
&\quad + \sum_{\substack{\overline{D} = X \\ D \in \mathcal{B}_j}} \Big(\mathbb{E}[U_j(D, \zeta + \varphi')] + \mathcal{E}_{j+1}|D| - \mathfrak{e}'_{j+1}(D) - \mathcal{U}_{j+1}(D, \varphi') + \sum_{\substack{D \in \mathcal{B}_j(Y) \\ Y \in \mathcal{S}_j}} Q^\Psi_j(D, Y, \varphi') \Big) \\
&=: \mathcal{L}^{(1)}_{j+1}(\vec{K}_j)(X, \varphi') + \mathcal{L}^{(2)}_{j+1}(\Psi_j)(X, \varphi') + \mathcal{L}^{(3)}_{j+1}(\vec{K}_j)(X, \varphi')
\end{aligned} \tag{5.35}$$

where, using the choice of $\mathcal{U}_{j+1}$ and $\mathfrak{e}'_{j+1}$, see [8, (7.21)] and (5.19), respectively, we set

$$\mathcal{L}^{(1)}_{j+1}(\vec{K}_j)(X, \varphi') = \sum_{Y : \overline{Y} = X} 1_{Y \in \mathcal{P}^c_j} \mathbb{E} K_j(Y, \zeta + \varphi'; 0) - 1_{Y \in \mathcal{S}_j} \mathbb{E}[\operatorname{Loc}_Y \hat{K}_{j, q=0}(Y, \varphi' + \zeta; 0)], \tag{5.36}$$

$$\mathcal{L}^{(2)}_{j+1}(\Psi_j)(X, \varphi') = \sum_{Y : \overline{Y} = X} 1_{Y \in \mathcal{P}^c_j} \mathbb{E}[\Psi_j(Y, \varphi' + \zeta) - 1_{Y \in \mathcal{S}_j} \hat{\Psi}_{j,0}(Y, \zeta)], \tag{5.37}$$

$$\mathcal{L}^{(3)}_{j+1}(\vec{K}_j)(X, \varphi') = \sum_{Y : \overline{Y} = X} 1_{Y \in \mathcal{P}^c_j} \mathbb{E}[D_j(Y, \varphi' + \zeta) - 1_{Y \in \mathcal{S}_j} \hat{D}_{j,0}(Y, \zeta)] \tag{5.38}$$

and

$$D_j(Y, \varphi) := K_j(Y, \varphi; (\Psi_k)_{k<j}) - K_j(Y, \varphi; 0). \tag{5.39}$$

In fact, the $\mathcal{L}_{j+1}$ in Theorem 5.1 (see [8, Section 7.4]) is identical to $\mathcal{L}^{(1)}_{j+1}$, i.e.,

$$\mathcal{L}_{j+1}(K_j(\cdot;0)) = \mathcal{L}^{(1)}_{j+1}(\vec{K}_j) \tag{5.40}$$

and also $\mathcal{L}^\Psi_{j+1}$ is a function of $(\vec{K}_j, \Psi_j)$, not depending on U_j.

Proof of (5.31) *of Theorem 5.8.* We will show that the bound (5.31) holds for any choice of $\tilde{\varepsilon}_{nl} \leqslant \varepsilon_{nl}$, where the latter refers to the (bulk) value supplied by Theorem 5.1, see above (5.7). Thus, let $\omega_j = (U_j, \vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$. By (5.7) and (5.40), we already know that

$$\|\mathcal{L}^{(1)}_{j+1}(\vec{K}_j)\|_{\Omega^K_{j+1}} \leqslant C_1 L^2 \alpha_{\mathrm{Loc}} \|K_j(\cdot;0)\|_{\vec{\Omega}^K_j}. \tag{5.41}$$

The estimate on $\mathcal{L}^{(2)}_{j+1}$ follows from the decomposition

$$\mathcal{L}^{(2)}_{j+1}(X,\varphi') = \sum_{Y\in\mathcal{S}_j,\, 0\in Y^*}^{\overline{Y}=X} \mathbb{E}[\Psi_j(Y,\varphi'+\zeta) - \hat{\Psi}_{j,0}(Y,\zeta)] + \mathbb{S}(1_{Y\notin\mathcal{S}_j}\mathbb{E}[\Psi_j(\cdot,\cdot+\zeta)])(X,\varphi'). \tag{5.42}$$

The summation is running over $Y^* \ni 0$ now because of the assumption that $\Psi_j(Y,\varphi) = 0$ if $0 \notin Y^*$ (which is a part of the assumption $(\vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$). Then the first term is bounded by $CA^{-|X|_{j+1}}\alpha^\Psi_{\mathrm{Loc}}\|\omega_j\|_{\overline{\Omega}_j}$ because of the assumption $(\vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon_\Psi, C_\Psi)$ and (5.16) implied by it (here we also used that $|\overline{Y}|_{j+1} \leqslant |Y|_j$). The second term is bounded using Proposition 3.7 with $* = \Psi$ with L and $A = A(L)$ sufficiently large:

$$\|\mathbb{S}\big[1_{Y\in\mathcal{P}^c_j\setminus\mathcal{S}_j}\mathbb{E}[\Psi_j(\cdot,\cdot+\zeta)]\big]\|_{h,T^\Psi_{j+1}(X)} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}}\|\Psi_j\|_{\Omega^\Psi_j} \leqslant C\alpha^\Psi_{\mathrm{Loc}}A^{-|X|_{j+1}}\|\Psi_j\|_{\Omega^\Psi_j} \tag{5.43}$$

Finally, we bound

$$\mathcal{L}^{(3)}_{j+1}(X,\varphi') = \sum_{Y\in\mathcal{S}_j,\, 0\in Y^*}^{\overline{Y}=X} \mathbb{E}[D_j(Y,\varphi'+\zeta) - \hat{D}_{j,0}(Y,\zeta)] + \mathbb{S}(1_{Y\notin\mathcal{S}_j}\mathbb{E}[D_j(\cdot,\cdot+\zeta)])(X,\varphi'). \tag{5.44}$$

Again, the assumption $D_j(Y,\zeta+\varphi') = 0$ for $Y^* \not\ni 0$ (which, as above, is a part of the assumption $(U_j, \vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$) effectively restricts the sum in the first term to $Y^* \ni 0$, then Proposition 3.6 with case $* = 0$ applies to give the bound $CA^{-|X|_{j+1}}\alpha^\Psi_{\mathrm{Loc}}\|\vec{K}_j\|_{\Omega^{\vec{K}}_j}$. For the second term, Proposition 3.7 with $* = 0$ gives the bound same bound with the same choice of L and A as above. □

5.6. Proof of Theorem 5.8: bound of non-linear part. Analogously as in [8, Section 7.5], the non-linear part $\mathcal{M}^\Psi_{j+1} := \mathcal{K}^\Psi_{j+1} - \mathcal{L}^\Psi_{j+1}$ (with $\mathcal{L}^\Psi_{j+1}$ as defined by the first line of (5.35)) can be decomposed into four parts,

$$\mathcal{M}^\Psi_{j+1}(U_j, \vec{K}_j, X, \varphi') = \sum_{k=1}^{4} \mathfrak{M}^{\Psi,(k)}_{j+1}(\mathfrak{K}^\Psi_j(\omega_j), X, \varphi') \tag{5.45}$$

with

$$\mathfrak{M}^{\Psi,(1)}_{j+1}(\mathfrak{K}^{\Psi}_j(\omega_j),X) = \sum^{*}_{X_0,X_1,Z,(B_{Z''})} 1_{\#(X_0,X_1,Z)\geqslant 2} e^{\mathcal{E}_{j+1}|X|-\mathfrak{e}'_{j+1}(X)} e^{\overline{U}^{\Psi}_{j+1}(X\backslash T)} \times \mathbb{E}\Big[(e^{U_j}-e^{\overline{U}^{\Psi}_{j+1}})^{X_0}(\overline{K}^{\Psi}_j-\mathcal{E}^{\Psi}K_j)^{[X_1]}\Big] \prod_{Z''\in\mathrm{Comp}_{j+1}(Z)} J^{\Psi}_j(B_{Z''},Z'') \quad (5.46)$$

$$\mathfrak{M}^{\Psi,(2)}_{j+1}(\mathfrak{K}^{\Psi}_j(\omega_j),X) = \sum^{*}_{X_0,X_1,Z,(B_{Z''})} 1_{\#(X_0,X_1,Z)\leqslant 1}(e^{\mathcal{E}_{j+1}|X|-\mathfrak{e}'_{j+1}(X)} e^{\overline{U}^{\Psi}_{j+1}(X\backslash T)}-1) \times \mathbb{E}\Big[(e^{U_j}-e^{\overline{U}^{\Psi}_{j+1}})^{X_0}(\overline{K}^{\Psi}_j-\mathcal{E}^{\Psi}K_j)^{[X_1]}\Big] \prod_{Z''\in\mathrm{Comp}_{j+1}(Z)} J^{\Psi}_j(B_{Z''},Z'') \quad (5.47)$$

$$\mathfrak{M}^{\Psi,(3)}_{j+1}(\mathfrak{K}^{\Psi}_j(\omega_j),X) = \sum^{X_0=X}_{|X_0|_{j+1}=1} \mathbb{E}\Big[\Big(e^{U_j}-e^{\overline{U}^{\Psi}_{j+1}}-U_j+\overline{U}^{\Psi}_{j+1}\Big)^{X_0}\Big] \quad (5.48)$$

$$\mathfrak{M}^{\Psi,(4)}_{j+1}(\mathfrak{K}^{\Psi}_j(\omega_j),X) = \mathbb{E}\Big[\sum^{\overline{Y}=X}_{Y\in\mathcal{P}_j} e^{U_j(Y)}(K_j(\cdot;(\Psi_k)_{k<j})+\Psi_j)(X\backslash Y)-\mathbb{S}(K_j+\Psi_j)(X)\Big] \quad (5.49)$$

where $\mathfrak{K}^{\Psi}_j \equiv \mathfrak{K}^{\Psi}_j(\omega_j)$ is short for the collection

$$(\mathcal{E}_{j+1}|X|-\mathfrak{e}'_{j+1}(X),\, U_j,\, \overline{U}^{\Psi}_{j+1},\, K_j(\cdot;(\Psi_k)_{k<j})+\Psi_j,\, \overline{K}^{\Psi}_j,\, \mathcal{E}^{\Psi}K_j,\, J^{\Psi}_j)(\omega_j), \quad (5.50)$$

we consider $X\mapsto \mathcal{E}_{j+1}|X|-\mathfrak{e}_{j+1}1_{0\in X}$ as a polymer activity,

$$\overline{U}^{\Psi}_{j+1}(X,\varphi') := -\mathcal{E}_{j+1}|X|+\mathfrak{e}'_{j+1}(X)+U_{j+1}(X,\varphi'), \quad (5.51)$$

and the rest of the notations are those of Definition 5.5. Also notice that U_{j+1} is used in place of $\mathcal{U}_{j+1}$ to simplify notations. These look somewhat complicated, but in view of [8, Lemma 7.12], it is actually sufficient to check some regularity properties of terms appearing in each $\mathfrak{M}^{\Psi,(k)}_{j+1}$ to show the differentiability of $\mathcal{M}_{j+1}$ along with the desired estimates (5.32) and (5.33). We now proceed to supply the necessary details. Our discussion follows closely the line of arguments yielding [8, Lemmas 7.11 and 7.12]. We first gather the estimates that will lead to a suitable analogue of [8, Lemma 7.11]. This is the object of the next lemma.

Lemma 5.9. *Under the assumptions of Theorem 5.8, for any $\delta>0$, there exists $\varepsilon=\varepsilon(\delta,\beta,L,C_\Psi)>0$ such that for $\omega_j\in\mathcal{Y}_j(\varepsilon,C_\Psi)$, $B\in\mathcal{B}_{j+1}$, $k\in\{0,1,2\}$,*

$$\|\mathfrak{U}(B,\varphi)\|_{h,T_j(\varphi,B)} \leqslant C(\delta,\beta,L,C_\Psi)(1+\delta c_w\kappa_L w_j(B,\varphi)^2)\|\omega_j\|_{\overline{\Omega}_j}, \quad (5.52)$$

$$\|e^{\mathfrak{U}(B,\varphi)}-\sum_{m=0}^{k}\frac{1}{m!}(\mathfrak{U}(B,\varphi))^m\|_{h,T_j(\varphi,B)} \leqslant C(\delta,\beta,L,C_\Psi)e^{\delta c_w\kappa_L w_j(B,\varphi)^2}\|\omega_j\|^{k+1}_{\overline{\Omega}_j}, \quad (5.53)$$

where $\mathfrak{U}$ is either U_j or $\overline{U}^{\Psi}_{j+1}$. The same inequalities hold with $\mathfrak{U}(B)$ and $C(\delta,\beta,L,C_\Psi)$ replaced by $\mathcal{E}_{j+1}|B|-\mathfrak{e}'_{j+1}(B)$ and $C(\beta,L,C_\Psi)$, respectively, and δ set to 0.

Proof. For $\mathfrak{U}=U_j$ or $\mathcal{E}_{j+1}|B|$, the asserted bounds are then an immediate consequence of [8, Lemma 7.14]. For the remaining choices of $\mathfrak{U}$, recall the definition and the bound on $\overline{U}_{j+1}$ provided by [8, (7.49)] and [8, (7.71)] that for $B\in\mathcal{B}_{j+1}$,

$$\|\overline{U}_{j+1}(B,\varphi)\|_{h,T_j(B,\varphi)} \leqslant C(\delta,\beta,L)\big(1+\delta c_w\kappa_L w_j(B,\varphi)^2\big)\|\omega_j\|_{\overline{\Omega}_j}. \quad (5.54)$$

Also by Theorem 5.7, we have

$$|\mathfrak{e}'_{j+1}(B,\omega_j)| \leqslant CC_\Psi A^{-1}\|\omega_j\|_{\overline{\Omega}_j}, \quad (5.55)$$

and since $\overline{U}_j^\Psi(X,\varphi) = \mathfrak{e}'_{j+1}(X) + \overline{U}_{j+1}(X,\varphi')$ by (5.51), we have

$$\|\overline{U}_{j+1}^\Psi(B,\varphi)\|_{h,T_j(B,\varphi)} \leqslant C(\delta,\beta,L,C_\Psi)\big(1+\delta c_w \kappa_L w_j(B,\varphi)^2\big)\|\omega_j\|_{\Omega_j}, \tag{5.56}$$

showing (5.52). For the second inequality, assume $\varepsilon \leqslant 1/C(\delta,\beta,L,C_\Psi)$ and $\|\omega_j\|_{\overline{\Omega}_j} \leqslant \varepsilon$, then the submultiplicativity of norm and (5.56) shows

$$\|e^{\overline{U}_{j+1}^\Psi}\|_{h,T_j(B,\varphi)} \leqslant e^{\|\overline{U}_{j+1}^\Psi\|_{h,T_j(B,\varphi)}} \leqslant C(\delta,\beta,L,C_\Psi) e^{\delta c_w \kappa_L w_j(B,\varphi)^2}. \tag{5.57}$$

Then (5.56) and (5.57) shows (5.53).

□

We now state the analogue of [8, Lemma 7.11] in the present context.

Lemma 5.10. *Under assumptions of Theorem 5.8, there exist* ~~$c_w > 0$~~ $\varepsilon \equiv \varepsilon(\beta,L) > 0$, $\eta > 0$, $C \equiv C(c_w,\beta,L,C_\Psi)$ *and* $C_A \equiv C_A(c_w,L,A,C_\Psi)$ *such that*

$$\|De^{\mathfrak{U}(B,\varphi)}\|_{h,T_j(B,\varphi)} \leqslant Ce^{c_w\kappa_L w_j(B,\varphi)^2} \tag{5.58}$$

$$\|D^2e^{\mathfrak{U}(B,\varphi)}\|_{h,T_j(B,\varphi)} \leqslant Ce^{c_w\kappa_L w_j(B,\varphi)^2} \tag{5.59}$$

$$\|DJ_j^\Psi(B,Z,\varphi')\|_{h,T_j(B,\varphi')} \leqslant CA^{-1}e^{c_w\kappa_L w_j(B,\varphi')^2} \tag{5.60}$$

$$\|D\overline{K}_j^\Psi(Z,\varphi)\|_{h,T_j(Z,\varphi)} \leqslant C_A A^{-(1+\eta)|Z|_{j+1}} G_j^\Psi(Z,\varphi) \tag{5.61}$$

$$\|D\mathcal{E}^\Psi K_j(Z,\varphi')\|_{h,T_j(Z,\varphi')} \leqslant C_A A^{-(1+\eta)|Z|_{j+1}} e^{c_w\kappa_L w_j(Z,\varphi')^2} \tag{5.62}$$

for $B \in \mathcal{B}_{j+1}$, $Z \in \mathcal{P}_{j+1}$ *whenever* $\omega_j \in \mathcal{Y}_j(\varepsilon(L),C_\Psi)$ *and* $\mathfrak{U}$ *is either* U_j *or* $\overline{U}_{j+1}^\Psi$ *or* $\mathcal{E}_{j+1}|B| - \mathfrak{e}'_{j+1}(B)$. *In the final case,* $e^{c_w\kappa_L w_j(B,\varphi)^2}$ *can be omitted.*

Proof. The proof is mostly the same as that of [8, Lemma 7.15–7.16]. The bounds (5.58) and (5.59) are consequences Lemma 5.9, cf. the discussion around [8, (7.76)–(7.77)]. The bound (5.62) follows directly from (5.60) (cf. [8, Lemma 7.15]), which in turn follows from a bound on $\|DQ_j^\Psi\|_{h,T_j(Y,\varphi')}$ (namely, (5.66) below). To obtain this bound, notice that for $D \in \mathcal{B}_j$, $Y \in \mathcal{S}_j$,

$$Q_j^\Psi(D,Y,\varphi') = Q_j(D,Y,\varphi') + 1_{Y\in\mathcal{S}_j}\frac{1_{D\subset Y\cap B_0^*}}{|Y\cap B_0^*|_j}\mathbb{E}^\zeta[\hat{\Psi}_{j,0}(Y,\zeta) + \hat{D}_{j,0}(Y,\zeta)] \tag{5.63}$$

where $D_j(Y,\zeta) = K_j(Y,\zeta;(\Psi_k)_{k<j}) - K_j(Y,\zeta;0)$ and Q_j is defined by [8, (7.26)]. But [8, (7.75)] already bounds $Q_j(D,Y,\varphi)$, so we actually only have to bound $\mathbb{E}[\hat{\Psi}_{j,0}(Y,\zeta) + \hat{D}_{j,0}(Y,\zeta)]$. But

$$\|\mathbb{E}[\hat{\Psi}_{j,0}(Y,\zeta)]\|_{h,T_j(Y,\varphi')} \leqslant C(A/2)^{-|Y|_j}\|\Psi_j\|_{\Omega_j^\Psi}\mathbb{E}[G_j^\Psi(Y,\zeta)] \leqslant C_\Psi C(A/4)^{-|Y|_j}\|\Psi_j\|_{\Omega_j^\Psi}, \tag{5.64}$$

$$\|\mathbb{E}[\hat{D}_{j,0}(Y,\zeta)]\|_{h,T_j(Y,\varphi')} \leqslant CA^{-|Y|_j}\|\vec{K}_j\|_{\Omega_j^{\vec{K}}}\mathbb{E}[G_j(Y,\zeta)] \leqslant C(A/2)^{-|Y|_j}\|\vec{K}_j\|_{\Omega_j^{\vec{K}}} \tag{5.65}$$

so it follows that Q_j^Ψ is differentiable with

$$\|DQ_j^\Psi(D,Y,\varphi')\|_{h,T_j(Y,\varphi')} \leqslant CA^{-|Y|_j}e^{c_w\kappa_L w_j(D,\varphi')}. \tag{5.66}$$

For (5.61), notice that if we write $\overline{\mathcal{F}}$ for the function

$$\overline{\mathcal{F}}(U_j,K_j) = \overline{K}_j := \sum_{Y\in\mathcal{P}_j}^{\overline{Y}=X} e^{U_j(X\setminus Y)}K_j(Y), \tag{5.67}$$

it follows that $\overline{K}_j^\Psi = \overline{\mathcal{F}}(U_j, K_j(\cdot;(\Psi_k)_{k<j}) + \Psi_j)$. So by inspecting the proof of [8, Lemma 7.16], one sees that $D\overline{K}_j^\Psi$ satisfies exactly the same bound as $D\overline{K}_j$ (see [8, (7.28), (7.61)] for its definition and bound), only with A replaced by $A/2$, i.e.,

$$G_j^\Psi(Z,\varphi)^{-1}\|D\overline{K}_j^\Psi(Z,\varphi)\|_{h,T_j(Z,\varphi)} \leqslant C_A(A/2)^{-(1+\eta)|Z|_{j+1}}. \tag{5.68}$$

But for A large enough, this is less than $C_A A^{-(1+\eta')|Z|_{j+1}}$ for some $\eta' \in (0,\eta)$ as needed. □

Proof of continuous differentiability of $\mathcal{M}^{\Psi}_{j+1}$ and (5.32), (5.33). For $j \leqslant N-2$, [8, Lemma 7.12] implies that the bounds on $\mathfrak{K}^{\Psi}_j(\omega_j) = (\mathcal{E}_{j+1}, U_j, \overline{U}^{\Psi}_{j+1}, K_j(\cdot;(\Psi_k)_{k<j}) + \Psi_j, \overline{K}^{\Psi}_j, \mathcal{E}^{\Psi} K_j, J^{\Psi}_j)$ provided by Lemma 5.9 and Lemma 5.10 are sufficient to prove the differentiability and bounds on $\mathfrak{M}^{\Psi,(k)}_{j+1}(\mathfrak{K}_j(\omega_j))$, $k \in \{1,2,3,4\}$. In fact, (5.61) now imposes bound in terms of G^{Ψ}_j instead of G_j but this does not affect the proof because [8, Lemma 7.12] uses the properties of G_j that (1) $e^{c_w \kappa_L w_j(X)^2} G_j(Y) \leqslant G_j(X \cup Y)$ if $X \cap Y = \emptyset$, (2) $G_j(X) = \prod_{\mathrm{Comp}_j(X)} G_j(X)$ and (3) $\mathbb{E}[G_j(X, \varphi' + \zeta)] \leqslant 2^{|X|_j} G_{j+1}(X, \varphi')$. But the same properties are verified on account of Proposition 3.4, while the constant C_Ψ only contributes as a multiplicative factor in each estimate.

For $j = N-1$, all of the arguments of Sections-5.5–5.6 continue to apply as $\Gamma^{\Lambda_N}_N$ satisfies exactly the same bounds as required for Γ_j when $j = N$. □

6 Proof of Theorem 2.3

In Section 5, we defined the extended renormalisation map $\overline{\Phi}_{j+1}$ corresponding to the finite torus Λ_N. In this section, we analyse the limit (as $N \to \infty$) of the final renormalisation group coordinates $(E_N, e_N, U_N, \vec{K}_N, \Psi_N)_{N \geqslant 0}$ obtained by the iteration of the renormalisation group map up to scale N, with initial conditions provided by Theorem 5.2. This limit is not exactly as the same as the limit $j \to \infty$ of the local infinite volume limit; in the former limit the size of the torus Λ_N is also varying as $N \to \infty$. For this reason, we temporarily write the dependence on Λ_N of the coordinates explicitly in the following theorem and the corollary, e.g., the coordinates will be denoted $(E^{\Lambda_N}_j, e^{\Lambda_N}_j, U^{\Lambda_N}_j, \vec{K}^{\Lambda_N}_j, \Psi^{\Lambda_N}_j)$ and the renormalisation group map will be denoted $\Phi^{\Lambda_N}_{j+1}$ and $\overline{\Phi}^{\Lambda_N}_{j+1}$ for the bulk and the extended flows, respectively.

Theorem 6.1. *Let J be any finite-range step distribution as in Theorem 1.1, choose the parameters as in Section 3.4, assume that $\beta \geqslant \beta_0(J)$ as in Theorem 5.2, and let $(E^{\Lambda_N}_j, U^{\Lambda_N}_j, K^{\Lambda_N}_j)$ be the (bulk) renormalisation group map on Λ_N as in Theorem 5.2, i.e.,*

$$(E^{\Lambda_N}_{j+1}, U^{\Lambda_N}_{j+1}, K^{\Lambda_N}_{j+1}(\cdot;0)) = \Phi^{\Lambda_N}_{j+1}(E^{\Lambda_N}_j, U^{\Lambda_N}_j, K^{\Lambda_N}_j(\cdot;0)), \quad 0 \leqslant j \leqslant N-1. \tag{6.1}$$

Assume that $(u_j)_{j \geqslant 0}$ satisfies $(\mathbf{A}_u)$, and define $(e_j)_{0 \leqslant j \leqslant N}$, $(\Psi^{\Lambda_N}_j)_{0 \leqslant j \leqslant N}$, $(K^{\Lambda_N}_j(\cdot;(\Psi^{\Lambda_N}_k)_{k<j}))_{0 \leqslant j \leqslant N}$ inductively by

$$\Psi^{\Lambda_N}_j = \mathcal{F}_\Psi[u_j, U^{\Lambda_N}_j, K^{\Lambda_N}_j(\cdot;(\Psi_k)_{k<j}); j] \tag{6.2}$$

$$K^{\Lambda_N}_{j+1}(\cdot;(\Psi^{\Lambda_N}_k)_{k \leqslant j}) = \mathcal{K}^{\Psi,\Lambda_N}_{j+1}(U^{\Lambda_N}_j, \vec{K}^{\Lambda_N}_j, \Psi^{\Lambda_N}_j) \tag{6.3}$$

$$e^{\Lambda_N}_{j+1} = e^{\Lambda_N}_j + \mathfrak{e}^{\Lambda_N}_{j+1}(\vec{K}^{\Lambda_N}_j, \Psi^{\Lambda_N}_j) \tag{6.4}$$

with initial conditions $K^{\Lambda_N}_0(X) = 1_{X=\emptyset}$ and $e^{\Lambda_N}_0 = 0$. Then there exists $C > 0$ such that for all $N \geqslant 1$ and $0 \leqslant j \leqslant N$, if L and j_u are large enough, then

$$\max\{\|\vec{K}^{\Lambda_N}_j\|_{\Omega^{\vec{K}}_j}, \|\Psi^{\Lambda_N}_j\|_{\Omega^{\Psi}_j}\} \leqslant CL^{-\alpha j}, \tag{6.5}$$

with decay factor $\alpha \equiv \alpha(\beta, J) > 0$ as in Theorem 5.2.

Proof. The asserted exponential decay in j (uniform in N) is almost immediate from Theorems 5.1, 5.2, and 5.8, as we now explain. Throughout the remainder of the proof, we drop the superscripts N and Λ_N. All the following estimates hold uniformly in N. By Theorem 5.2, it has already been shown that $\omega_j \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$ and $\|(U_j, K_j(\cdot;0))\|_{\Omega_j} \leqslant CL^{-\alpha j}$ for all $j \leqslant N$. We will now argue that there is $C' > 0$ such that, for all j, both

$$\|K_j(\cdot;(\Psi_k)_{k<j}) - K_j(\cdot;0)\|_{\Omega^K_j} \leqslant C'L^{-\alpha j}, \tag{6.6}$$

$$\|\Psi_j\|_{\Omega^{\Psi}_j} \leqslant C_\Psi(C+C')L^{-\alpha j} \tag{6.7}$$

hold, where C refers to the constant in the bound $\|(U_j, K_j(\cdot;0))\|_{\Omega_j} \leqslant CL^{-\alpha j}$. The claim then immediately follows by combining these two estimates with (5.12). We now show these two bounds by induction. For $j \leqslant j_u$ there is nothing to prove, as $\Psi_j \equiv 0$ and $K_j(\cdot;(\Psi_k)_{k<j}) \equiv K_j(\cdot;0)$. Now assume (6.6) and (6.7) hold for some $j \in [j_u, N)$. If j_u is sufficiently large, then these bounds and Lemma 4.3 imply that ω_j falls into the admissible range of Theorem 5.8, i.e., $(U_j^{\Lambda_N}, \vec{K}_j^{\Lambda_N}, \Psi_j^{\Lambda_N}) \in \mathcal{Y}_j(\tilde{\varepsilon}_{nl}, C_\Psi)$. Then (5.31) and linearity of $\mathcal{L}_{j+1}^\Psi$ give for $\omega_j = (U_j, \vec{K}_j, \Psi_j) \in \mathcal{Y}_j(\varepsilon, C_\Psi)$ (with $\varepsilon \leqslant \tilde{\varepsilon}_{nl}$)

$$\|\mathcal{L}_{j+1}^\Psi(\omega_j)(\cdot;0) - \mathcal{L}_{j+1}^\Psi(\omega_j)(\cdot;(\Psi_k)_{k<j},0)\|_{\Omega_{j+1}^K} \leqslant C_1 C_\Psi \alpha_{\mathrm{Loc}}^\Psi \|K_j^{\Lambda_N}(\cdot;(\Psi_k)_{k<j}) - K_j^{\Lambda_N}(\cdot;0)\|_{\Omega_j^K} \tag{6.8}$$

and (5.33) gives

$$\|\mathcal{M}_{j+1}^\Psi(\omega_j)(\cdot;0) - \mathcal{M}_{j+1}^\Psi(\omega_j)(\cdot;(\Psi_k)_{k<j},0)\|_{\Omega_{j+1}^K} \leqslant C_2 \|K_j^{\Lambda_N}(\cdot;(\Psi_k)_{k<j}) - K_j^{\Lambda_N}(\cdot;0)\|_{\Omega_j^K}\varepsilon. \tag{6.9}$$

Here $((\Psi_k)_{k<j},0)$ refers to $(\Psi'_k)_{k\leqslant j}$ with $\Psi'_k = \Psi_k$ for $k<j$ and $\Psi'_j = 0$. For ε sufficiently small in $\alpha_{\mathrm{Loc}}^\Psi$ and $(C_2(\beta,A,L))^{-1}$, (6.6), (6.8) and (6.9) imply

$$\|K_{j+1}^{\Lambda_N}(\cdot;0) - K_{j+1}^{\Lambda_N}(\cdot;(\Psi_k)_{k<j},0)\|_{\Omega_{j+1}^K} \leqslant 2C_1 C_\Psi C' \alpha_{\mathrm{Loc}}^\Psi L^{-\alpha j} \tag{6.10}$$

Similar arguments gives

$$\|K_{j+1}^{\Lambda_N}(\cdot;(\Psi_k)_{k\leqslant j}) - K_{j+1}^{\Lambda_N}(\cdot;(\Psi_k)_{k<j},0)\|_{\Omega_{j+1}^K} \leqslant 2C_1 C_\Psi \alpha_{\mathrm{Loc}}^\Psi \|\Psi_j\|_{\Omega_j^\Psi} \tag{6.11}$$

Together with (6.7), these inequalities imply

$$\|K_{j+1}^{\Lambda_N}(\cdot;(\Psi_k)_{k\leqslant j}) - K_{j+1}^{\Lambda_N}(\cdot;0)\|_{\Omega_{j+1}^K} \leqslant C'' L^\alpha \alpha_{\mathrm{Loc}}^\Psi L^{-\alpha(j+1)}. \tag{6.12}$$

To proceed, we need the fact that $L^\alpha \leqslant C(L^2\alpha_{\mathrm{Loc}})^{-1}$ for some $C>0$, see the last remark of Theorem 5.2. Also since $\alpha_{\mathrm{Loc}}^\Psi = (\log L)^{-1} O(L^2\alpha_{\mathrm{Loc}})$, we now have $L^\alpha \alpha_{\mathrm{Loc}}^\Psi \leqslant C/\log L$ and therefore

$$\|K_{j+1}^{\Lambda_N}(\cdot;(\Psi_k)_{k\leqslant j}) - K_{j+1}^{\Lambda_N}(\cdot;0)\|_{\Omega_{j+1}^K} \leqslant \frac{C'''}{\log L} L^{-\alpha(j+1)} \tag{6.13}$$

which completes the induction step for (6.6) after choosing $C' \log L \geqslant C'''$. To obtain (6.7) at scale $j+1$, one now uses that $\|(U_{j+1}, K_{j+1}(\cdot;0))\|_{\Omega_j} \leqslant CL^{-\alpha(j+1)}$ by Theorem 5.2, and the fact that $\|K_{j+1}(\cdot;(\Psi_k)_{k\leqslant j})\|_{\Omega_{j+1}^K} \leqslant (C+C')L^{-\alpha j+1}$ which follows by combining with the newly proved (6.6) at scale $j+1$, along with the fact that $\|\Psi_{j+1}\|_{\Omega_{j+1}^\Psi} \leqslant C_\Psi \|\vec{K}_{j+1}(\cdot;(\Psi_k)_{k\leqslant j}))\|_{\Omega_{j+1}^{\vec{K}}}$ by Lemma 4.3.

□

Corollary 6.2. *Under the assumptions of Thereom 6.1,*

$$|e_N^{\Lambda_N}| \leqslant O\Big(\sum_{j\geqslant j_u} \|\vec{K}_j^{\Lambda_N}\|_{\Omega_j^{\vec{K}}}\Big) \leqslant O(L^{-\alpha j_u}) \tag{6.14}$$

for j_u from $(\mathbf{A}_u)$, uniformly in N.

Proof. We start from the the explicit expression $e_n^{\Lambda_N} = \sum_{j\leqslant n-1} \mathfrak{e}_{j+1}(\vec{K}_j^{\Lambda_N}, \Psi_j^{\Lambda_N})$ and use (5.26). To see that the sum actually only starts from $j = j_u$, note that, by construction, $\Psi_k \equiv 0$ for $k < j_u$, and hence $K_j^{\Lambda_N}(\cdot;(\Psi_k)_{k<j}) = K_j^{\Lambda_N}(\cdot;0)$ for $j \leqslant j_u$ which implies that $\mathfrak{e}_{j+1} = 0$ by its definition, (5.19). Hence $|e_N^{\Lambda_N}| \leqslant C\sum_{j_u\leqslant j\leqslant N-1} \|\vec{K}_j^{\Lambda_N}\|_{\Omega_j^{\vec{K}}}$ and the sum is uniformly bounded in N because $\|\vec{K}_j^{\Lambda_N}\|_{\Omega_j^{\vec{K}}} = O(L^{-\alpha j})$ uniformly in N. □

Theorem 2.3 is almost direct from the above two results.

Proof of Theorem 2.3. We first note that Lemma 2.2 implies that $(u_j)_{j\geqslant 0}$ defined by (2.9) satisfies $(\mathbf{A}_u)$ with some $j_u = j_f$, and so Theorem 6.1 and Corollary 6.2 may be used. We then assume that $\beta \geqslant \beta_0(J)$ with $\beta_0(J)$ as supplied by Theorem 5.2, pick $L = L(J)$ large enough (and of the form specified in Section 3.4) such that the conclusions Theorem 6.1 hold and set $A(J) = A_0'(L)$ for this choice of L.

For a constant field ζ, we have $\nabla\zeta = 0$ and $G_N^\Psi(X,\zeta) = G_N^\Psi(X,0)$ so, with W_N denoting the non-gradient term (involving the cosines) in (3.2) with $j = N$,

$$\begin{aligned} e^{E_N|\Lambda_N|-e_N} Z_N(u,\zeta+u_N) &= e^{\frac{1}{2}s_N|\nabla(\zeta+u_N)|^2_{\Lambda_N}+W_N(\Lambda_N,\zeta+u_N)} + K_N(\Lambda_N,\zeta+u_N;(\Psi_k)_{k<N}) \\ &= e^{\frac{1}{2}s_N|\nabla\zeta|^2_{\Lambda_N}+W_N(\Lambda_N,\zeta)} + K_N(\Lambda_N,\zeta;(\Psi_k)_{k<N}) + \Psi_N(\Lambda_N,\zeta) \\ &= 1 + O\big(\|W_N\|_{\Omega_N^U} + \|K_N(\cdot;(\Psi_k)_{k<N})\|_{\Omega_N^K} G_N^\Psi(\Lambda_N,0)\big) \end{aligned} \tag{6.15}$$

whenever $\|W_N\|_{\Omega_N^U} \leqslant 1$ and we have used $\Psi_N = \mathcal{F}_\Psi[u_N, U_N, K_N(\cdot;(\Psi_k)_{k<N}); N]$ and Proposition 4.2 for the second equality. Also Lemma 4.3 bounds Ψ_N in terms of $K_N(\cdot;(\Psi_k)_{k<N})$ in the third equality. Then by (2.15) and (3.15),

$$\begin{aligned} \tilde{Z}_N(u,0) &= \mathbb{E}_{t_N Q_N} Z_N(u,\zeta+u_N) \\ &= e^{-E_N|\Lambda_N|+e_N}\big(1 + O(\|W_N\|_{\Omega_N^U} + \|K_N(\cdot;(\Psi_k)_{k<N})\|_{\Omega_N^K})\big). \end{aligned} \tag{6.16}$$

For $\|W_N\|_{\Omega_N^U} + \|K_N(\cdot;0)\|_{\Omega_N^K}$ sufficiently small, it follows that

$$\frac{\tilde{Z}_N(u,0)}{\tilde{Z}_N(0,0)} = \exp(e_N)\frac{1 + O(\|W_N\|_{\Omega_N^U} + \|K_N(\cdot;(\Psi_k)_{k<N})\|_{\Omega_N^K})}{1 + O(\|W_N\|_{\Omega_N^U} + \|K_N(\cdot;0)\|_{\Omega_N^K})}. \tag{6.17}$$

But $\|W_N\|_{\Omega_N^U} \leqslant CL^{-\alpha N}$ by Theorem 5.2, $\|\vec{K}_N\|_{\Omega_N^{\vec{K}}} \leqslant C_1 L^{-\alpha N}$ by Theorem 6.1, and $|e_N| \leqslant C_2 L^{-\alpha j_f}$ by Corollary 6.2. This implies the desired conclusion. □

A Existence of infinite-volume limit

We recall the Fröhlich–Park–Ginibre inequalities: Let Λ be finite, let C be a positive definite matrix, and let $\langle\cdot\rangle_C$ be the expectation of the associated (generalised) Discrete Gaussian model:

$$\langle F\rangle_C \propto \sum_{\sigma\in\mathbb{Z}^\Lambda} e^{-\frac{1}{2}(\sigma,C^{-1}\sigma)}F(\sigma). \tag{A.1}$$

By taking limits, the definition of $\langle\cdot\rangle_C$ can also be extended to C positive semidefinite. The finite volume states $\langle\cdot\rangle_{J,\beta}^\Lambda$ given by (1.3) then correspond to $C = \beta(-\Delta_J)^{-1}$ when σ is identified up to constants (as we do), see also [8, Lemma 2.1]. The results of [30, Section 3] (see also [43, Proposition 1.2]) then imply that for $f : \Lambda \to \mathbb{R}$ with $\sum f = 0$:

$$\langle e^{(f,\sigma)}\rangle_{J,\beta}^\Lambda \leqslant e^{\frac{1}{2}(f,(-\Delta_J)_\Lambda^{-1}f)}, \tag{A.2}$$

$$\langle (f,\sigma)^2\rangle_{J,\beta}^\Lambda \leqslant (f,(-\Delta_J)_\Lambda^{-1}f). \tag{A.3}$$

Moreover, [30, Corollary 3.2 (1)] implies that

$$\langle e^{i(\varphi,f)}\rangle_{C_1} \leqslant \langle e^{i(\varphi,f)}\rangle_{C_2} \qquad \text{if } C_2 \leqslant C_1. \tag{A.4}$$

Proposition A.1. *Let $L > 1$ be an integer. For any finite-range step distribution J and any sequence of discrete tori Λ_N with side lengths L^N, with $N \in \mathbb{N}$, the measures $\langle\cdot\rangle_{J,\beta}^{\Lambda_N}$ converge weakly as $N \to \infty$ (when the field is identified up to constants). For any $f : \mathbb{Z}^d \to \mathbb{R}$ with compact support and $\sum f = 0$, one also has $\langle e^{(f,\sigma)}\rangle_{J,\beta}^{\Lambda_N} \to \langle e^{(f,\sigma)}\rangle$ where $\langle\cdot\rangle = \lim_{N\to\infty}\langle\cdot\rangle_{J,\beta}^{\Lambda_N}$ is the weak limit.*

Proof. We consider the Laplacian $-\Delta^{\Lambda_N}$ as an operator on $\ell^2(\mathbb{Z}^d)$ with domain

$$D(-\Delta^{\Lambda_N}) = \{f \in \ell^2(\mathbb{Z}^d) : f(0) = 0,\ f(x) = f(x + L^N y) \text{ for any } y \in \mathbb{Z}^d\}. \tag{A.5}$$

Then clearly $D(-\Delta^{\Lambda_N}) \subset D(-\Delta^{\Lambda_{N+1}})$ and $-\Delta^{\Lambda_N} = -\Delta^{\Lambda_{N+1}}$ on $D(-\Delta^{\Lambda_N})$. This implies $-\Delta^{\Lambda_N} \geqslant -\Delta^{\Lambda_{N+1}}$ and hence $(-\Delta^{\Lambda_N})^{-1} \leqslant (-\Delta^{\Lambda_{N+1}})^{-1}$. From (A.4), it follows that for any $f : \mathbb{Z}^d \to \mathbb{R}$ compactly supported and with $\sum f = 0$, $S_N(f) = \langle e^{i(f,\varphi)}\rangle_{J,\beta}^{\Lambda_N}$ is increasing in N. In particular, since also $S_N(f) \leqslant 1$, the limit $S(f) = \lim_{N\to\infty} S_N(f)$ exists. To show $S(f)$ is the characteristic function of a probability measure on $(2\pi\mathbb{Z})^{\mathbb{Z}^2}$/constants to which $\langle\cdot\rangle_{J,\beta}^{\Lambda_N}$ converges weakly, we will apply Minlos' theorem. To this end, we consider $(2\pi\mathbb{Z})^{\mathbb{Z}^2}$/constants as a topological vector space with the topology defined by the condition that $\varphi_k \to \varphi$ in $(2\pi\mathbb{Z})^{\mathbb{Z}^2}$/constants if $(\varphi_k, g) \to (\varphi, g)$ for all compactly supported $g : \mathbb{Z}^d \to \mathbb{R}$ with $\sum g = 0$. In particular, $(2\pi\mathbb{Z})^{\mathbb{Z}^2}$/constants is the dual of a nuclear space. To apply Minlos' theorem we need to check that S is continuous in this topology. But this is immediate from the correlation inequality (A.3) which implies that for any $g : \mathbb{Z}^2 \to \mathbb{R}$ with compact support and $\sum g = 0$,

$$|S(f+g) - S(f)| = \lim_{N\to\infty} |S_N(f+g) - S_N(f)| \leqslant \lim_{N\to\infty} (g, (-\Delta_J^{\Lambda_N})^{-1} g) = (g, (-\Delta_J)^{-1} g), \tag{A.6}$$

from which the continuity is clear.

The final statement about the convergence of $\langle e^{(f,\sigma)}\rangle_{J,\beta}^{\Lambda_N}$ follows from the weak convergence and (A.2) which implies that the random variables $e^{(f,\sigma)}$ are uniformly integrable. □

It is also standard, see [37] and analogous extensions to the gradient Gibbs setting as in [33,34], that any limit as in the previous proposition is translation invariant and satisfies the gradient Gibbs property. Moreover, the limit satisfies the analogous correlation inequalities.

Proposition A.2. *The measure $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ has tilt 0, i.e., for each gradient Gibbs state in the ergodic decomposition of $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ the gradient field has mean 0.*

Proof. The proof is analogous to that of [34, Theorem 3.2]. The correlation decay can be replaced by the following application of the Riemann–Lebesgue lemma. For $g : \mathbb{Z}^2 \to \mathbb{R}^d$ with compact support, where now $\nabla\sigma : \mathbb{Z}^d \to \mathbb{R}^d$ denotes the vector of discrete forward derivatives, (A.3) implies

$$\langle (g, \nabla\sigma)^2 \rangle_{J,\beta}^{\mathbb{Z}^2} \leqslant C \int_{[-\pi,\pi]^2} \frac{|\hat{g}(p)\cdot p|^2}{|p|^2}\, dp. \tag{A.7}$$

Thus the distributional Fourier transform of $\langle \nabla_{e_i}\sigma(0) \nabla_{e_i}\sigma(x)\rangle$ is integrable in the Fourier variable. From this, the Riemann-Lebesgue lemma implies that

$$\langle \nabla_{e_i}\sigma(x) \nabla_{e_i}\sigma(y)\rangle_{J,\beta}^{\mathbb{Z}^2} \to 0 \qquad (|x-y| \to \infty). \tag{A.8}$$

In particular, for every $i = 1, \dots, d$, with $Q_R = [-R, R]^2 \cap \mathbb{Z}^2$,

$$\left\langle \left(\liminf_{R\to\infty} \frac{1}{|Q_R|} \sum_{x\in Q_R} \nabla_{e_i}\sigma(x) \right)^2 \right\rangle_{J,\beta} \leqslant \liminf_{R\to\infty} \frac{1}{|Q_R|^2} \sum_{x,y\in Q_R} |\langle \nabla_{e_i}\sigma(x)\nabla_{e_i}\sigma(y)\rangle_{J,\beta}| = 0. \tag{A.9}$$

This implies that every measure μ in the ergodic decomposition of $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ has mean 0 for $\nabla\sigma$ (see e.g. [34, Theorem 3.2] for a similar argument): indeed, for any such μ, by (A.9) and ergodicity, one deduces that $|Q_R|^{-1} \sum_{x\in Q_R} \nabla_{e_i}\sigma(x)$ converges μ-a.s. and that the limit vanishes, whence $E_\mu[\nabla_{e_i}\sigma(x)] = 0$. □

B Properties of the regulator with external field

Proof of Lemma 3.1. In the proof, the notation

$$W_{j+s}(X,\nabla_j^a\varphi)^2=\sum_{B\in\mathcal{B}_{j+s}(X)}\|\nabla_{j+s}^a\varphi\|_{L^\infty(B^*)}^2 \tag{B.1}$$

will be used. For brevity, $s+M^{-1}$ will be denoted s' and $X_{s'}$ will be denoted X'. We will bound each term appearing in $\log G_{j+s}(X,\varphi+\xi_o)$. First, $\|\nabla\varphi\|_{L^2(X)}^2$ will be isolated from $\|\nabla(\varphi+\xi_o)\|_{L^2(X)}^2$. Let $B\in\mathcal{B}_{j+s}(X)$ and without loss of generality, let B, l_i $(i=1,2,3,4)$ be as above but $B=[1,L^{j+s}]^2$. Then by discrete integration by parts,

$$\sum_{x\in B}\nabla^{e_1}\varphi(x)\nabla^{e_1}\xi_o(x)=-\sum_{x\in l_3}\xi_o(x)\nabla^{-e_1}\varphi(x)-\sum_{x\in l_1}\xi_o(x+e_1)\nabla^{e_1}\varphi(x)+\sum_{x\in B}\xi_o(x)\nabla^{e_1}\nabla^{-e_1}\varphi(x). \tag{B.2}$$

Hence in particular, summing this over each direction $\pm e_1,\pm e_2$, $B\in\mathcal{B}_{j+s}(X)$, and using the AM-GM inequality,

$$\begin{aligned}t(\nabla\varphi,\nabla\xi_o)_X &\leqslant \tau t\|\xi_o\|_{L^2_{j+s}(X)}^2+\tau^{-1}t\|\nabla_{j+s}^2\varphi\|_{L^2_{j+s}(X)}^2+\tau t\|\xi_o\|_{L^2_{j+s}(\partial X)}^2+\tau^{-1}t\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2\\ &\leqslant 2\tau W_{j+s}(X,\xi_o)^2+\tau^{-1}\big(\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2+W_{j+s}(X,\nabla_{j+s}^2\varphi)^2\big)\end{aligned} \tag{B.3}$$

for any $\tau>0$, and hence

$$\begin{aligned}\|\nabla_{j+s}(\varphi+\xi_o)\|_{L^2_{j+s}(X)}^2 &\leqslant \|\nabla_{j+s'}\varphi\|_{L^2_{j+s'}(X)}^2+\|\nabla_{j+s}\xi_o\|_{L^2_{j+s}(X)}^2\\ &\quad+2\tau W_{j+s}(X,\xi_o)^2+\tau^{-1}\big(\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2+W_{j+s}(X,\nabla_{j+s}^2\varphi)^2\big).\end{aligned} \tag{B.4}$$

Next, we will use rather trivial bound on the other two terms of $\log G_{j+s}$:

$$\|\nabla_{j+s}(\varphi+\xi_o)\|_{L^2_{j+s}(\partial X)}^2\leqslant 2\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2+2W_{j+s}(X,\nabla_{j+s}\xi_o)^2 \tag{B.5}$$

$$\|\nabla_j^2(\varphi+\xi_B)\|_{L^\infty(B^*)}^2\leqslant 2\|\nabla_j^2\varphi\|_{L^\infty(B^*)}^2+2\|\nabla_j^2\xi_B\|_{L^\infty(B^*)}^2 \tag{B.6}$$

By (B.4), (B.5), (B.6) and setting $c_4=\max\{2c_1,2\tau c_1,2c_2\}$,

$$\begin{aligned}\frac{1}{\kappa_L}\log G_{j+s}(X,\varphi,\xi_o,(\xi_B)_B) &\leqslant c_1\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(X)}^2+(2c_2+c_1\tau^{-1})\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2\\ &\quad+2c_1(1+\tau^{-1})W_{j+s}(X,\nabla_{j+s}^2\varphi)+\frac{1}{\kappa_L}\log\max_{\mathfrak{a}\in\{o\}\cup\mathcal{B}_{j+s}(X)}g_{j+s}(X,\xi_{\mathfrak{a}}).\end{aligned} \tag{B.7}$$

Now by repeated application of the discrete Sobolev trace theorem [8, (A.4)],

$$\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X)}^2\leqslant\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X')}^2+10\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(X'\backslash X)}^2+10W_{j+s}(\nabla_{j+s}^2\varphi,X'\backslash X) \tag{B.8}$$

hence by choosing $\tau=c_1c_2^{-1}$ and $30c_2\leqslant c_1$,

$$\begin{aligned}&\frac{\log(G_{j+s}(X,\varphi,\xi;t,(t_B))/\max_{\mathfrak{a}}g_{j+s}(X,\xi_{\mathfrak{a}}))}{\kappa_L}\\ &\leqslant c_1\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(X')}^2+3c_2\|\nabla_{j+s}\varphi\|_{L^2_{j+s}(\partial X')}^2+2c_1(1+\tau^{-1})W_{j+s}(\nabla_{j+s}^2\varphi,X')\\ &\leqslant c_1\|\nabla_{j+s'}\varphi\|_{L^2_{j+s'}(X')}^2+3\,\ell^{-1}c_2\|\nabla_{j+s'}\varphi\|_{L^2_{j+s'}(\partial X')}^2+2\,\ell^{-2}c_1(1+\tau^{-1})W_{j+s'}(\nabla_{j+s'}^2\varphi,X').\end{aligned} \tag{B.9}$$

Hence the conclusion follows upon taking ℓ large enough. □

C Reblocking and fluctuation integral

Proof of Theorem 5.6. Throughout the proof, we write

$$\varphi = \varphi' + \zeta \tag{C.1}$$

with $\zeta \sim \Gamma_{j+1}$ and φ', ζ independent, and the fluctuation integral $\mathbb{E}$ acts on the variable ζ. As explained in [8, below (7.12)], we may assume that $E_j = 0$ and $e_j = 0$. The first step is the reblocking

$$Z_j(\varphi, \Psi_j; (\Psi_k)_{k<j}) = \sum_{X\in\mathcal{P}_j} e^{U_j(\Lambda\setminus X)}(K_j(X; (\Psi_k)_{k<j}) + \Psi_j(X)) = \sum_{X\in\mathcal{P}_{j+1}} e^{U_j(\Lambda\setminus X)}\overline{K}_j^{\Psi}(X) \tag{C.2}$$

where $\overline{K}_j^{\Psi}$ is defined in (5.25). In the next step, $e^{-E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}}$ replaces e^{U_j} using the identity

$$\begin{aligned} e^{U_j(\Lambda\setminus X',\varphi)} &= \prod_{B\in\mathcal{B}_{j+1}(\Lambda\setminus X')} \Big(\big(e^{U_j(B,\varphi)} - e^{-E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}(B,\varphi')}\big) + e^{-E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}(B,\varphi')}\Big) \\ &= \sum_{Y\in\mathcal{P}_{j+1}(\Lambda\setminus X')} e^{-E_{j+1}|\Lambda\setminus(X'\cup Y)|+1_{0\in\Lambda\setminus(X'\cup Y)}e_{j+1}+U_{j+1}(\Lambda\setminus(X'\cup Y),\varphi')}\big(e^{U_j(\varphi)} - e^{E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}(\varphi')}\big)^Y \end{aligned} \tag{C.3}$$

and similarly $\overline{K}_j^{\Psi} - \mathcal{E}^{\Psi}K_j$ replaces $\overline{K}_j^{\Psi}$ (recall $\mathcal{E}^{\Psi}K_j$ from (5.22)) using the identity

$$\begin{aligned} \overline{K}_j^{\Psi}(X',\varphi) &= \prod_{Z'\in\mathrm{Comp}_{j+1}(X')} \big(\mathcal{E}^{\Psi}K_j(Z',\varphi') + (\overline{K}_j^{\Psi}(Z',\varphi) - \mathcal{E}^{\Psi}K_j(Z',\varphi'))\big) \\ &= \sum_{\substack{Z\not\sim X'\setminus Z \\ Z\in\mathcal{P}_{j+1}(X')}} \mathcal{E}^{\Psi}K_j(\varphi')^{[Z]}(\overline{K}_j^{\Psi}(\varphi) - \mathcal{E}^{\Psi}K_j(\varphi'))^{[X'\setminus Z]}. \end{aligned} \tag{C.4}$$

Using the specific form of $\mathcal{E}^{\Psi}K_j$ given by (5.22) the last right-hand side can be rewritten as

$$\mathcal{E}^{\Psi}K_j(\varphi')^{[Z]} = \sum_{(B_{Z''})_{Z''}} \prod_{Z''} J_j^{\Psi}(B, Z'') \tag{C.5}$$

where the last sum $(B_{Z''})_{Z''}$ runs over collections of blocks $B_{Z''} \in \mathcal{B}_{j+1}(Z'')$ and $Z'' \in \mathrm{Comp}_{j+1}(Z)$. Rewriting $X'' = X' \cup Y$, the expectation $\mathbb{E}Z_j$ can now be written as

$$\begin{aligned} Z_{j+1}(\varphi', 0; (\Psi_k)_{k\leqslant j}) &= \mathbb{E}Z_j(\varphi'+\zeta, \Psi_j; (\Psi_k)_{k<j}) \\ &= e^{-E_{j+1}|\Lambda|}\mathbb{E}\Bigg[\sum_{X''\in\mathcal{P}_{j+1}} e^{1_{0\in\Lambda\setminus X''}e_{j+1}+U_{j+1}(\Lambda\setminus X'')}e^{E_{j+1}|X''|} \\ &\qquad\times \sum_{X'\subset X''} (e^{U_j} - e^{-E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}})^{X''\setminus X'} \\ &\qquad\times \sum_{Z\subset X'} (\overline{K}_j^{\Psi} - \mathcal{E}^{\Psi}K_j)^{[X'\setminus Z]} \sum_{(B_{Z''})} \prod_{Z''\in\mathrm{Comp}_{j+1}(Z)} J_j^{\Psi}(B, Z'')\Bigg]. \end{aligned} \tag{C.6}$$

The final result is obtained after taking $e^{e_{j+1}}$ out and another resummation: we write $X_0 = X''\setminus X'$, $X_1 = X'\setminus Z$, $T = X_0 \cup X_1 \cup Z = X''$ and define for $X = \cup_{Z''}B^*_{Z''} \cup X_0 \cup X_1$,

$$\begin{aligned} K_{j+1}(X, \varphi'; (\Psi_k)_{k\leqslant j}) &= \sum_{X_0,X_1,Z,(B_{Z''})} e^{E_{j+1}|T|-1_{0\in T}e_{j+1}}e^{U_{j+1}(X\setminus T)} \\ &\times \mathbb{E}\Big[(e^{U_j} - e^{-E_{j+1}|B|+1_{0\in B}e_{j+1}+U_{j+1}})^{X_0}(\overline{K}_j^{\Psi} - \mathcal{E}^{\Psi}K_j)^{[X_1]}\Big] \prod_{Z''\in\mathrm{Comp}_{j+1}(Z)} J_j^{\Psi}(B, Z''). \end{aligned} \tag{C.7}$$

Note that only $T \subset X$ contribute because, by definition of $\mathcal{E}^\Psi K_j$, the whole expression vanishes when $Z \notin \mathcal{S}_{j+1}$. Therefore

$$Z_{j+1}(\varphi', 0; (\Psi_k)_{k \leqslant j}) = e^{-E_{j+1}|\Lambda| + e_{j+1}} \sum_{Z \in \mathcal{P}_{j+1}} e^{U_{j+1}(\Lambda \setminus Z, \varphi')} K_{j+1}(Z, \varphi'; (\Psi_k)_{k \leqslant j}) \tag{C.8}$$

which is the desired form. The factorisation property of $K_{j+1}(\cdot; (\Psi_k)_{k \leqslant j})$ is inherited from that of e^{U_j}, $e^{U_{j+1}}$ and K_j. □

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