Homework 2

Read the handout on the construction of continuous-time Markov chains distributed in class. It is also available on the lecturer's homepage, and will be needed below.

1. Do Exercise 2.32 in Liggett, *Continuous-time Markov processes*, AMS, 2010.

2. Consider a Poisson process $(N_t)_{t\geq 0}$ of rate $\lambda > 0$ as defined in class (i.e. a continuous-time Markov chain (CTMC) on $\mathbb{N}$ with $p_{i,i+1} = 1$ and $c(i) = \lambda$, $i \in \mathbb{N}$). Assume throughout that $N_0 = 0$.

a) Compute (inductively) the density of Y_k, $k \geq 1$, the time of the k-th jump.

b) Show that N_t is a Poisson random variable with parameter λt, for all $t \geq 0$.

3. Consider a CTMC on a countable state space S. Show the following: If one of the conditions **a)**, **b)** or **c)** holds, then there is no explosion in finite time (the non-explosion condition is given by Eqn. (5) on the handout).

a) There exists a $c > 0$, such that $\lambda(x) < c$ for all $x \in S$.

b) S is finite.

c) Let T be the set of all transient states. Suppose that $P^Z_x[\cap_{n\geq 0}\{Z_n \in T\}] = 0$ for all $x \in S$ and that $\lambda(x) < \infty$ for all $x \in S$.

4. Let $X_\cdot$ be a CTMC (as in Def. 2.1) on a countable S, with transition probabilities $p_t(x, y) = P_x[X_t = y]$, $t \geq 0$, $x, y \in S$. The goal of this exercise is to give a certain derivation of the *forward integral equation*

$$p_t(x, y) = \delta_{x,y} e^{-\lambda(x)t} + \int_0^t \mathrm{d}s \sum_{z\neq y} p_s(x, z) q(z, y) e^{-\lambda(y)(t-s)} \tag{1}$$

Intuitively, and as will become clear below, this amounts to conditioning on the last jump before time t (whenever such a jump exists, i.e. if $T = T_1 \leq t$).

a) Using the notation from around (14) in the handout, show that $\widetilde{P}_x$-a.s. for $n \geq 1$,

$$\begin{aligned}&\widetilde{E}_x[1\{Z_n = y,\, T_n \leq t < T_{n+1}\}|\sigma(Z_0, \dots, Z_{n-1}, T_1, \dots, T_{n-1})]\\&\quad = p_{Z_{n-1},y} 1\{T_{n-1} \leq t\} \int_{T_{n-1}}^{t} \mathrm{d}s\, c(Z_{n-1}) e^{-c(Z_{n-1})(s-T_{n-1})-\lambda(y)(t-s)}.\end{aligned}$$

Hint: first include T_n (the n-th jumping time) in the conditioning.

b) Use **a)** to show (with $T = T_1$)

$$P_x[X_t = y, T \leq t] = \int_0^t \mathrm{d}s \sum_{z \neq y} p_s(x,z)\lambda(z)p_{z,y}e^{-\lambda(y)(t-s)},$$

c) As for the backward equation (cf. the derivation in class), use **b)** to conclude that (1) holds.

5. (M/M/1 queue with finite capacity.) At a hairdresser's shop only one customer can be served at time and there is only one chair for the next customer to wait on. A potential customer leaves if he arrives at the shop when already two customers are inside. The interarrival times and service times of the customers are independent identically distributed random variables T_λ and T_μ having exponential distribution with parameter λ and μ. Let X_t denote the number of customers being in the shop at time t. Compute the jump rate function and the jump transition probability, and the Q-matrix.

Due: Friday April 28th at the beginning of class.