THE SIGN CLUSTERS OF THE MASSLESS GAUSSIAN FREE FIELD PERCOLATE ON $\mathbb{Z}^d$, $d \geqslant 3$ (AND MORE)

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Abstract

We investigate the percolation phase transition for level sets of the Gaussian free field on $\mathbb{Z}^d$, with $d \geqslant 3$, and prove that the corresponding critical parameter $h_*(d)$ is strictly positive for all $d \geqslant 3$, thus settling an open question from [20]. In particular, this implies that the sign clusters of the Gaussian free field percolate on $\mathbb{Z}^d$, for all $d \geqslant 3$. Among other things, our construction of an infinite cluster above small, but positive level h involves random interlacements at level $u > 0$, a random subset of $\mathbb{Z}^d$ with desirable percolative properties, introduced in [21] in a rather different context, a certain Dynkin-type isomorphism theorem relating random interlacements to the Gaussian free field [23], and a recent coupling of these two objects [10], lifted to a continuous metric graph structure over $\mathbb{Z}^d$.

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1   Introduction

The present work studies the percolation phase transition of Gaussian free field level sets on $\mathbb{Z}^d$, $d \geqslant 3$, which provides a canonical example for a percolation model with strong, algebraically decaying correlations. It was first proved in [3] that the corresponding critical level $h_*(d)$, see (1.4) below for its definition, satisfies $h_*(d) \geqslant 0$ for every dimension $d \geqslant 3$ and that $h_*(3) < \infty$. It was later shown in [20] that $h_*(d)$ is finite in every dimension $d \geqslant 3$, and strictly positive when d is large, with leading asymptotics as $d \to \infty$ derived in [7]. We prove here that this parameter is actually strictly positive in all dimensions $d \geqslant 3$. This answers a question from [3], see also Remark 3.6 in [20], and fits with numerical evidence from [12], see Section 4.1.2 and Figure 4.1 therein. A corresponding classical result for Bernoulli site percolation, $p_c^{\mathrm{site}}(\mathbb{Z}^d) < \frac{1}{2}$ for $d \geqslant 3$, has been known to hold for several decades already [4].

Our construction of infinite clusters (by which, adopting the usual terminology, we mean unbounded connected components) of excursion sets for the Gaussian free field crucially relies on another object, random interlacements. The model of random interlacements has originally been introduced in [21] to study certain geometric properties of random walk trajectories on large, asymptotically transient, finite graphs. The Dynkin-type isomorphism theorem relating interlacements and the Gaussian free field, see [23], has repeatedly proved a useful tool in their study, see [23], [25], [18], [10], [27] and [1]. In particular, the cable system method introduced in [10] provides a continuous version of this isomorphism theorem, from which some links between the level sets of the Gaussian free field and the vacant sets $\mathcal{V}^u$, $u > 0$, of random interlacements can be derived. This method was used in [27] and [1] to find a suitable coupling between those two sets, and was applied in the case of transient trees. It was also proved in these papers that, under certain conditions on the geometry of the tree $\mathbb{T}$, the critical parameter $h_*(\mathbb{T})$ for level set percolation of the Gaussian free field on $\mathbb{T}$ is strictly positive. As will become apparent below, the isomorphism theorem on the cable system can be paired with renormalization techniques from random interlacements, and in particular from [16], which imply a certain robustness property of $\mathcal{I}^u = \mathbb{Z}^d \setminus \mathcal{V}^u$ with respect to small noise, to yield similar findings on $\mathbb{Z}^d$, for all $d \geqslant 3$.

Let us now describe the results in more details. For $d \geqslant 3$, we consider $\mathbb{Z}^d$ as a graph, with undirected edge set E, and take uniform weights equal to 1 on all edges in E, so that the sum of all weights around a vertex $x \in \mathbb{Z}^d$ is $2d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if and only if $\{x, y\} \in E$. Noting that $\mathbb{Z}^d$, $d \geqslant 3$, is transient for discrete time simple random walk, we define the symmetric Green function by

$$(1.1) \qquad g(x, y) = \frac{1}{2d} E_x \Big[\sum_{k=0}^{\infty} 1_{\{X_k = y\}} \Big], \ x, y \in \mathbb{Z}^d,$$

where $(X_k)_{k \geq 0}$ denotes the simple random walk on $\mathbb{Z}^d$ starting at x under P_x. We also set $g(x) = g(0, x)$, for $x \in \mathbb{Z}^d$. We define $\mathbb{P}^G$, a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$ endowed with its

canonical σ-algebra generated by the coordinate maps Φ_x, $x \in \mathbb{Z}^d$, such that, under $\mathbb{P}^G$,

$$(1.2) \qquad \begin{aligned} &(\Phi_x)_{x \in \mathbb{Z}^d} \text{ is a centered Gaussian field with} \\ &\text{covariance function } \mathbb{E}^G[\Phi_x \Phi_y] = g(x,y) \text{ for all } x, y \in \mathbb{Z}^d. \end{aligned}$$

(Any random field $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ with law $\mathbb{P}^G$ on $\mathbb{R}^{\mathbb{Z}^d}$ will henceforth be called a Gaussian free field on $\mathbb{Z}^d$). We are interested in level sets of Φ, and for every $h \in \mathbb{R}$, denote by $\{x \xleftrightarrow{\geq h} \infty\}$ the event that $x \in \mathbb{Z}^d$ lies in an infinite connected component of

$$(1.3) \qquad E^{\geq h} \equiv E^{\geq h}(\Phi) = \{y \in \mathbb{Z}^d; \Phi_y \geq h\},$$

and by $\eta(h)$ its probability, which does not depend on the choice of x. The function $\eta(\cdot)$ is decreasing, and it is natural to ask whether it is strictly positive or not. This leads to the definition of the critical point

$$(1.4) \qquad h_*(d) \overset{\text{def.}}{=} \inf \{h \in \mathbb{R}; \eta(h) = 0\}.$$

By ergodicity, this definition corresponds to the phase transition for the existence of an infinite connected component in $E^{\geq h}$, see Lemma 1.5 in [20]. It is not a priori clear whether $|h_*| < \infty$ or not, and a summary of the *status quo* was given in the first paragraph. In summary, it is known that $h_*(d) \in [0, \infty)$ for all $d \geq 3$, and that $h_*(d) \sim (2g(0) \log d)^{1/2}$ as $d \to \infty$. Our main result is the following lower bound in all dimensions.

Theorem 1.1.

$$(1.5) \qquad h_*(d) > 0, \quad \textit{for all } d \geq 3.$$

Moreover, there exists $h_1 > 0$ such that for all $h \leq h_1$, there exists $L_0 > 0$ such that $E^{\geq h}$ contains an infinite cluster in the thick slab $\mathbb{Z}^2 \times [0, 2L_0)^{d-2}$.

In fact, one can replace $E^{\geq h}$ by $\{y \in \mathbb{Z}^d; K(h) \geq \Phi_y \geq h\} (\subset E^{\geq h})$ for sufficiently large $K(h)$ in the previous statement, see Remark 5.3,2) below. As an immediate corollary of Theorem 1.1, we note that there exists an open interval $I \subset \mathbb{R}$ containing the origin and such that, for all $h \in I$, the level set $E^{\geq h}$ and its complement $E^{<h} = \mathbb{Z}^d \setminus E^{\geq h}$ *both* percolate (with probability one). This follows readily from (1.5) and the fact that $E^{<h} \overset{\text{law}}{=} E^{\geq -h}$ for all $h \in \mathbb{R}$, by symmetry of Φ. In particular, choosing $h = 0$, this implies that

$$(1.6) \qquad \Phi \text{ almost surely contains two infinite sign clusters (one for each sign).}$$

Put differently, Theorem 1.1 asserts that the critical density $p_c^G(d) = \mathbb{P}^G[\Phi_0 \geq h_*(d)]$ satisfies $p_c^G(d) < \frac{1}{2}$, for all $d \geq 3$, thus mirroring the result $p_c^{\text{site}}(\mathbb{Z}^d) < \frac{1}{2}$, see [4], for independent Bernoulli site percolation on $\mathbb{Z}^d$, $d \geq 3$. However, the elegant geometric arguments developed therein to "interpolate" between two- and three-dimensional structures do not seem to transfer to the current situation: the correlations present a serious impediment. Moreover, there is no obvious monotonicity of $p_c^G(d)$ (or $h_*(d)$) as a function of d. One may

also conjecture that $p_c^G(d) < p_c^{\text{site}}(\mathbb{Z}^d)$, the critical density for independent site percolation on the lattice, based on the reasonable intuition that positive correlations "help" in forming clusters of $E^{\geqslant h}$. We do not currently know a proof of this (nor of the more modest conjecture $p_c^G(d) \leq p_c^{\text{site}}(\mathbb{Z}^d)$).

A key tool in the proof of Theorem 1.1 is a certain isomorphism, see Theorem 2.2 below, which gives a link between random interlacements and the Gaussian free field. We now explain its benefits in some detail, and refer to Section 2 for precise definitions. Suppose that ω denotes the interlacement point process defined in [21], with law $\mathbb{P}^I$, and let ω^u be the process consisting of the trajectories in the support of ω with label at most u. Somewhat informally, ω^u is a Poisson cloud of bi-infinite nearest neighbor trajectories modulo time-shift whose forward and backward parts escape all finite sets in finite time. One naturally associates to ω^u, see for instance (1.8) in [23], a field of occupation times $(\ell_{x,u})_{x \in \mathbb{Z}^d}$, where $\ell_{x,u} = \ell_{x,u}(\omega^u)$ collects the total amount of time spent at x by any of the trajectories in the support of ω^u. The interlacement set at level u is then defined as

$$(1.7) \qquad \mathcal{I}^u = \{x \in \mathbb{Z}^d; \ell_{x,u} > 0\}.$$

It corresponds to the set of vertices visited by at least one trajectory in the support of ω^u. For any $u > 0$, the set $\mathcal{I}^u$ is almost surely unbounded and connected [21]. The following isomorphism was proved in Theorem 0.1 of [23], and has the same spirit as the generalized second Ray-Knight theorem, see for example [8], [11] or [25]:

$$(1.8)$$
$$\left(\ell_{x,u} + \frac{1}{2}\Phi_x^2\right)_{x \in \mathbb{Z}^d} \text{ under } \mathbb{P}^I \otimes \mathbb{P}^G \text{ has the same law as } \left(\frac{1}{2}(\Phi_x + \sqrt{2u})^2\right)_{x \in \mathbb{Z}^d} \text{ under } \mathbb{P}^G.$$

If one attaches to each edge e of $\mathbb{Z}^d$ a line segment I_e of length $\frac{1}{2}$, the resulting "graph" $\widetilde{\mathbb{Z}}^d$ is continuous and called the cable system, see Section 2. On this cable system, one then defines probabilities $\widetilde{\mathbb{P}}^G$ and $\widetilde{\mathbb{P}}^I$ under which the fields $(\Phi_x)_{x \in \mathbb{Z}^d}$ and $(\ell_{x,u})_{x \in \mathbb{Z}^d}$ admit *continuous* extensions $\widetilde{\Phi} = (\widetilde{\Phi}_x)_{x \in \widetilde{\mathbb{Z}}^d}$ and $\widetilde{\ell} = (\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$. It was proved in [10] that for each $u > 0$, a continuous version of the isomorphism (1.8) also holds on $\widetilde{\mathbb{Z}}^d$, see (2.14) below, and in particular (somewhat inaccurately, but see (2.14), (2.15) below for precise statements) the sign of $\widetilde{\Phi}_x + \sqrt{2u}$ is constant as long as $\widetilde{\ell}_{x,u} > 0$, and thus (using (1.10) below) for all $h > 0$, $\widetilde{\mathbb{P}}$-a.s. the set

$$(1.9) \qquad \{x \in \widetilde{\mathbb{Z}}^d; \widetilde{\Phi}_x \geqslant -h\} \text{ contains an unbounded cluster in the cable system } \widetilde{\mathbb{Z}}^d.$$

This result was already known to hold on $\mathbb{Z}^d$ without the isomorphism theorem [3], where it had been derived using a neat contour argument. It is interesting to note that, on the cable system, (1.9) is actually sharp, because $\widetilde{\mathbb{P}}^G$-a.s. the set

$$(1.10) \quad \{x \in \widetilde{\mathbb{Z}}^d; \widetilde{\Phi}_x \geqslant 0\} \text{ does not contain unbounded clusters in the cable system } \widetilde{\mathbb{Z}}^d,$$

see Proposition 5.5 in [10], which sharply contrasts with (1.6). All in all, the infinite cluster in $E^{\geqslant 0}$ (part of $\mathbb{Z}^d$), which exists by Theorem 1.1, "scatters" into finite pieces upon adding the field on the edges, but the infinite cluster of $E^{\geqslant -h}$ does not, for ever so small $h > 0$.

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On our way towards proving Theorem 1.1, we will also show that a truncated version of the level sets in (1.9) contains an unbounded cluster on $\widetilde{\mathbb{Z}}^d$. Indeed, it was proved in [16] that the intersection of the random interlacement set $\mathcal{I}^u$ with a Bernoulli percolation having large success parameter still contains an infinite cluster in $\mathbb{Z}^d$. By showing a similar stability result on the cable system, see Proposition 4.1, and using the isomorphism theorem on the cable system, we will obtain, cf. Theorem 3.1 and Corollary 4.9 below, that the truncated (continuous) level set

$$(1.11) \qquad \widetilde{A}_h \stackrel{\text{def.}}{=} \{x \in \widetilde{\mathbb{Z}}^d; \, -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$$

contains an unbounded cluster on $\widetilde{\mathbb{Z}}^d$ for all $h > 0$ and large enough, but finite $K(h)$ (with $K(h) \to \infty$ slowly as $h \searrow 0$). Let us now briefly describe in how far this helps in proving Theorem 1.1. By construction, one can view $\widetilde{\Phi}$, the Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$, as a Gaussian free field on $\mathbb{Z}^d$ with Brownian bridges of length $\frac{1}{2}$ attached on the edges, see (2.6). On an edge in the set $\widetilde{A}_h$, those Brownian bridges never go below $-h$, which happens with low probability for small h. We are going to use this low probability on the edges to go from $-h \leqslant \Phi \leqslant K(h)$ to $h \leqslant \Phi \leqslant K(h)$ on the *endpoints* of theses edges and for small enough h, see in particular Lemmas 5.1 and 5.2, which will then imply that the set $\{x \in \mathbb{Z}^d; \, \Phi_x \geqslant h\}$ has an infinite cluster on $\mathbb{Z}^d$, as asserted.

We now explain the organization of this article, and highlight its main contributions. In Section 2, we recall the definitions of the Gaussian free field and random interlacements on the cable system, and the link between the two via the aforementioned isomorphism theorem. In Section 3, we collect a few preparatory tools by showing some strong connectivity properties, a large deviation inequality as well as a version of the decoupling inequalities for random interlacements on the cable system. Most of these are well-known, but existing results do not entirely fit our needs.

The construction of the infinite cluster comes essentially in three steps, Proposition 4.1, Corollary 4.9, and Section 5, which are the main reference points of this paper. Proposition 4.1 is a fairly generic result, which, roughly speaking, for *any* coupling of a continuous interlacement and a free field, see (4.1), yields a percolating interlacement cluster, with good control on the free field part, and some room to play with along the edges. Its proof follows a standard static renormalization scheme from [16], [18], assembling the results of Section 3. In Corollary 4.9, we "translate" Proposition 4.1, for a certain choice of the coupling, to show that suitably truncated level sets of the Gaussian free field on the cable system contain an unbounded connected component. The reference level for the excursion sets of Corollary 4.9 is $-h$, for (small) positive h. Section 5 contains the device to "flip the sign" and pass from $-h$ to h on the vertices, as indicated above. Together with Corollary 4.9, this then yields a proof of Theorem 1.1.

In the rest of this article, we denote by c and C positive constants that may change from place to place. Numbered constants such as $C_0, c_0, C_1, C_1', \ldots$ are fixed until the end of the article. All constants are allowed to implicitly depend on the dimension d and a parameter $u_0 > 0$, which will first appear in Lemma 3.2 and throughout the remaining sections.

2 Notation and useful facts about the cable system

In this section, we give a definition of the Gaussian free field and random interlacements on the cable system that will be useful later. We also discuss some aspects of the Markov property for the Gaussian free field and its consequences, and recall the isomorphism theorem which links random interlacements and the Gaussian free field.

For later convenience, we endow the graph $\mathbb{Z}^d$ with a distance function $d(\cdot, \cdot)$ which is half of the usual graph distance, i.e., half of the l^1-distance on $\mathbb{Z}^d$. We define $V_0 = \{2x, \ x \sim 0\}$, so that, for all $x, y \in \mathbb{Z}^d$ with $x \sim y$, we can write $y = x + \frac{1}{2}v_{(x,y)}$ for a unique $v_{(x,y)} \in V_0$. Note that $d(x, x + \frac{1}{2}v) = \frac{1}{2}$, for all $x \in \mathbb{Z}^d$ and $v \in V_0$. We attach to each edge $e = \{x, y\}$ the following interval of length $\frac{1}{2}$:

$$(2.1) \qquad I_e \overset{\text{def.}}{=} \left\{ x + tv_{(x,y)}; \ t \in \left(0, \frac{1}{2}\right) \right\} = \left\{ y + tv_{(y,x)}; \ t \in \left(0, \frac{1}{2}\right) \right\},$$

which is homeomorphic to an open interval of $\mathbb{R}$ of length $\frac{1}{2}$, and we write $\overline{I_e} = I_e \cup \{x, y\}$. The cable system $\widetilde{\mathbb{Z}}^d$ is then defined by glueing these intervals through their endpoints. $\widetilde{\mathbb{Z}}^d \setminus \mathbb{Z}^d$ is now the union of such I_e, one for every edge $e \in E$, and one can view $\widetilde{\mathbb{Z}}^d$ as a subset of $\mathbb{R}^d$ with distance $d(x, y) = \frac{1}{2}\|x - y\|_1$, where $\|\cdot\|_p$ denotes the usual ℓ^p-distance on $\mathbb{R}^d$ for $p \in [1, \infty)$. For all $e \in E$ and $z_1, z_2 \in \overline{I_e}$ we define $(z_1, z_2) \subset \widetilde{\mathbb{Z}}^d$ as the open interval in I_e between z_1 and z_2. We also define the distance between two subsets A_1 and A_2 of $\widetilde{\mathbb{Z}}^d$ by $d(A_1, A_2) = \inf_{x \in A_1, y \in A_2} d(x, y)$, and the diameter of a subset A of $\widetilde{\mathbb{Z}}^d$ by $\delta(A) = \sup_{x, y \in A} d(x, y)$. For $0 \leqslant R_1 < R_2$, we introduce the boxes $[R_1, R_2)^d = \{z \in \mathbb{R}^d; \ z_i \in [R_1, R_2) \text{ for all } i = 1, \ldots, d\}$. The set $\mathbb{Z}^d$ will henceforth be considered as a subset of $\widetilde{\mathbb{Z}}^d$ and we will call vertices the elements of $\mathbb{Z}^d$.

One can define a continuous diffusion X on the cable system $\widetilde{\mathbb{Z}}^d$, via probabilities $\widetilde{P}_z$, $z \in \widetilde{\mathbb{Z}}^d$, with continuous local times with respect to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$. We now describe this construction somewhat informally, and refer to Section 2 of [10] and Section 2 of [9] for precise definitions. One simple way of defining X is to use the Brownian excursions: starting from some vertex $x_0 \in \mathbb{Z}^d$, one chooses randomly and uniformly an edge leaving x_0, and runs a Brownian excursion on it. One repeats this process until one of these Brownian excursions reaches height $\frac{1}{2}$, i.e., until another vertex y is reached on $\widetilde{\mathbb{Z}}^d$. The local time of this process in x_0 is a.s. finite and has an exponential law with parameter $2d$. One then repeats this process starting from y until another vertex is reached, and so forth, thus obtaining a diffusion X starting from $x_0 \in \mathbb{Z}^d$. The projection of its trajectory on $\mathbb{Z}^d$ has the same law as a simple random walk on $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, $v \in V_0$ and $t \in (0, \frac{1}{2})$, one can also define the diffusion X starting from $z := x + tv$ by running a Brownian excursion on either $\{z - sv; \ s \in [0, t]\}$ or $\{z + sv; \ s \in [0, \frac{1}{2} - t]\}$ with probability $\frac{1}{2}$ each, and repeating this process until either x or $x + \frac{1}{2}v$ is reached. Once one of these two vertices is reached, X behaves like the diffusion starting from this vertex, as previously described. We call $g(x, y)$ for $x, y \in \widetilde{\mathbb{Z}}^d$ the Green function of this diffusion with respect to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$, and its restriction to $\mathbb{Z}^d$ is the same as the Green function on $\mathbb{Z}^d$ defined in (1.1), so the identical notation does not bear any risk of confusion.

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We endow the canonical space Ω_0 of continuous real-valued functions on $\widetilde{\mathbb{Z}}^d$ with the canonical σ-algebra generated by the coordinate functions $\widetilde{\Phi}_x$, $x \in \widetilde{\mathbb{Z}}^d$, and let $\widetilde{\mathbb{P}}^G$ be the probability on Ω_0 such that, under $\widetilde{\mathbb{P}}^G$,

(2.2)
$$(\widetilde{\Phi}_x)_{x \in \widetilde{\mathbb{Z}}^d} \text{ is a centered Gaussian field with}$$
$$\text{covariance function } \widetilde{\mathbb{E}}^G[\widetilde{\Phi}_x \widetilde{\Phi}_y] = g(x, y) \text{ for all } x, y \in \widetilde{\mathbb{Z}}^d.$$

With a slight abuse of notation, any random variable $\widetilde{\varphi} = (\widetilde{\varphi}_x)_{x \in \widetilde{\mathbb{Z}}^d}$ on the space of continuous functions $C(\widetilde{\mathbb{Z}}^d, \mathbb{R})$ with law $\widetilde{\mathbb{P}}^G$ under $\widetilde{\mathbb{P}}$ will be called a Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$, and it is plain that the restriction of a Gaussian free field on the cable system to $\mathbb{Z}^d$ is a Gaussian free field on $\mathbb{Z}^d$.

Let us recall the simple Markov property for $\widetilde{\varphi}$. Let $K \subset \widetilde{\mathbb{Z}}^d$ be a compact subset with finitely many components, and let $U = \widetilde{\mathbb{Z}}^d \setminus K$. For all $x \in \widetilde{\mathbb{Z}}^d$, we define

(2.3)
$$\widetilde{\beta}_x^U = \widetilde{E}_x \left[\widetilde{\varphi}_{X_{T_U}} 1_{\{T_U < \infty\}} \right] \quad \text{and} \quad \widetilde{\varphi}_x^U = \widetilde{\varphi}_x - \widetilde{\beta}_x^U,$$

where $T_U := \inf\{t \geqslant 0; X_t \notin U\}$, with the convention $\inf \emptyset = \infty$, is the exit time from U of the diffusion X on $\widetilde{\mathbb{Z}}^d$. Moreover, for all $x, y \in \widetilde{\mathbb{Z}}^d$, we define $g_U(x, y)$ the Green function with respect to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$ in y of the diffusion X under $\widetilde{P}_x$ killed when exiting U. Then,

(2.4)
$$(\widetilde{\varphi}_x^U)_{x \in \widetilde{\mathbb{Z}}^d} \text{ is a centered Gaussian field with}$$
$$\text{covariance function } g_U(x, y) \text{ for all } x, y \in \widetilde{\mathbb{Z}}^d.$$

Furthermore, this field is continuous, vanishes on K and is independent of $\sigma(\widetilde{\varphi}_z, \ z \in K)$. A strong Markov property is also known to hold, but we will not need it here, see Section 1 of [27] for more details.

Following standard notation, we say that $(B_t)_{t \in [0,l]}$ is a *Brownian bridge of length $l > 0$ between x and y of a Brownian motion with variance σ^2 at time 1* under a probability $\mathbb{P}^B$ if the process

(2.5)
$$W_t := B_t - \frac{t}{l} y - \left(1 - \frac{t}{l}\right) x, \quad t \in [0, l],$$

is a centered Gaussian field with covariance function

(2.6)
$$\mathbb{E}^B[W_{s_1} W_{s_2}] = \frac{\sigma^2 s_1 (l - s_2)}{l} \text{ for all } s_1, s_2 \in [0, l] \text{ with } s_1 \leqslant s_2$$

(the process $(W_{lt}/\sqrt{l\sigma^2})_{t \in [0,1]}$ is a standard Brownian bridge). Let $e \in E$, $z_1 \neq z_2 \in \overline{I_e}$, $v \in V_0$ and $t \in (0, \frac{1}{2}]$ such that $z_2 = z_1 + tv$, and let $s_1, s_2 \in [0, t]$ such that $s_1 \leqslant s_2$. Under $\widetilde{P}_{z_1 + s_1 v}$, until time $T_{(z_1, z_2)^c}$, the diffusion X behaves like a Brownian motion on I_e beginning in $z_1 + s_1 v$ until the hitting time of $(z_1, z_2)^c$. Since the local time at 0 of a Brownian motion

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starting in 0 has the same law as the absolute value of this Brownian motion, see for instance Chapter VI, Theorem 2.3 in [17], the local time at $z_1 + s_2 v$ of X under $\widetilde{P}_{z_1+s_1v}$ until time $T_{(z_1,z_2)^c}$ is

$$g_{(z_1,z_2)}(z_1 + s_1v, z_1 + s_2v) = \frac{2s_1(t - s_2)}{t}.$$

The Markov property for the Gaussian free field implies that, under $\widetilde{\mathbb{P}}$ (under which $\widetilde{\varphi}$ is a Gaussian free field),

$$(2.7) \qquad \left(\widetilde{\varphi}_{z_1+sv} - \frac{t - s}{t}\widetilde{\varphi}_{z_1} - \frac{s}{t}\widetilde{\varphi}_{z_2}\right)_{s\in[0,t]}$$

is a centered Gaussian field with covariance function $(g_{(z_1,z_2)}(z_1 + s_1v, z_2 + s_2v))_{s_1,s_2\in[0,t]}$, and is independent of $\sigma(\widetilde{\varphi}_z, \ z \in \widetilde{\mathbb{Z}}^d \setminus (z_1, z_2))$, and thus it is a Brownian bridge of length t between 0 and 0 of a Brownian motion with variance 2 at time 1. In particular, knowing $\widetilde{\varphi} \restriction \mathbb{Z}^d$, the Gaussian free field on the edges $((\widetilde{\varphi}_z)_{z\in I_e})_{e\in E}$ is an independent family of random processes such that, for each $x \sim y \in \mathbb{Z}^d$, the process $(\widetilde{\varphi}_z)_{z\in I_{\{x,y\}}}$ has the same law as a Brownian bridge of length $\frac{1}{2}$ between $\widetilde{\varphi}_x$ and $\widetilde{\varphi}_y$ of a Brownian motion with variance 2 at time 1, as mentioned in Section 2 of [10]. More precisely, let

$$(2.8) \qquad B_t^e = \widetilde{\varphi}_{x_e+tv_{(x_e,y_e)}} - 2t\widetilde{\varphi}_{y_e} - (1 - 2t)\widetilde{\varphi}_{x_e}, \ \text{for all } t \in [0, 1/2] \text{ and } e \in E,$$

where we have given an (arbitrary) orientation (x_e, y_e) for each edge $e = \{x_e, y_e\} \in E$. Then, under $\widetilde{\mathbb{P}}$, $(B^e)_{e\in E}$ is a family of independent Brownian bridges of length $\frac{1}{2}$ between 0 and 0 of a Brownian motion with variance 2 at time 1. Note that this provides an explicit (and simple) construction of a Gaussian free field on the cable system starting from the Gaussian free field $(\varphi_x)_{x\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$: if one links independently each $x \sim y \in \mathbb{Z}^d$ via a Brownian bridge on $I_{\{x,y\}}$ of length $\frac{1}{2}$ between φ_x and φ_y of a Brownian motion with variance 2 at time 1, then the resulting process is a Gaussian free field on the cable system. In view of this construction, we will later need the following result on the probability that the maximum of a Brownian bridge exceeds some value M (see e.g. [2], Chapter IV.26).

Lemma 2.1. *Let x, y be two real numbers, $M \geqslant \max(x, y)$ and, under $\mathbb{P}^B$, $(B_t)_{t\in[0,l]}$ a Brownian bridge of length l between x and y of a Brownian motion with variance σ^2 at time 1. One has*

$$(2.9) \qquad \mathbb{P}^B\left(\sup_{t\in[0,l]} B_t > M\right) = \exp\left(-\frac{2(M - x)(M - y)}{l\sigma^2}\right).$$

Let us now turn to the definition of random interlacements on $\widetilde{\mathbb{Z}}^d$, as in [10] or [27]. The usual definition of random interlacements on $\mathbb{Z}^d$, see, for example, [21] or the monograph [5], can be adapted to define a Poisson point process $\widetilde{\omega}$ on $\widetilde{W}^* \times [0, \infty)$, where $\widetilde{W}^*$ is the space of doubly infinite trajectories on $\widetilde{\mathbb{Z}}^d$ modulo time-shift, endowed with its canonical σ-algebra. Recall the law $(\widetilde{P}_z)_{z\in\widetilde{\mathbb{Z}}^d}$ of the diffusion X on the cable system, started at $z \in \widetilde{\mathbb{Z}}^d$. The intensity measure of $\widetilde{\omega}$ is characterized as follows: for some $R_1, R_2 \in \mathbb{Z}$ with $R_1 \leqslant R_2$,

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let $\widetilde{K} := [R_1, R_2]^d \cap \widetilde{\mathbb{Z}}^d$, let $K := \widetilde{K} \cap \mathbb{Z}^d$, let $\widetilde{\omega}^u$ be the point process which consists of the trajectories in $\widetilde{\omega}$ with label at most $u > 0$, and let $\widetilde{\omega}^u_{\widetilde{K}}$ be the point process comprising the forward trajectories of $\widetilde{\omega}^u$ hitting $\widetilde{K}$ and beginning at the first time $\widetilde{K}$ is reached. Then $\widetilde{\omega}^u_{\widetilde{K}}$ is a Poisson point process with intensity measure $u\widetilde{P}_{e_K} = u \sum_{x \in K} e_K(x)\widetilde{P}_x$, where e_K is the usual equilibrium measure of K.

One then defines

(2.10) $(\widetilde{\ell}_{z,u})_{z \in \widetilde{\mathbb{Z}}^d}$ the field of local times of random interlacements, for $u > 0$,

as the sum of the local times of each of the trajectories in the support of $\widetilde{\omega}^u$. The restriction of these local times to $\mathbb{Z}^d$ has the same law as the occupation times $(\ell_{x,u})_{x \in \mathbb{Z}^d}$ for random interlacements on $\mathbb{Z}^d$ alluded to in the introduction, cf. above (1.7). The random interlacement set is defined as

(2.11) $\widetilde{\mathcal{I}}^u = \{x \in \widetilde{\mathbb{Z}}^d; \widetilde{\ell}_{x,u} > 0\},$

which is an open connected subset of $\widetilde{\mathbb{Z}}^d$. Note that $\{x \in \mathbb{Z}^d; x \in \widetilde{\mathcal{I}}^u\}$ has the same law as $\mathcal{I}^u$, cf. (1.7).

We also recall the following formula for the Laplace transform of $(\ell_{x,u})_{x \in \mathbb{Z}^d}$, see for instance [23], (1.9)–(1.11) or Remark 2.4.4 in [24]: for all $V : \mathbb{Z}^d \to \mathbb{R}$ with finite support $K \subset \mathbb{Z}^d$ and satisfying

(2.12) $\|GV\|_\infty < 1, \text{ where } GV(x) = \sum_{y \in \mathbb{Z}^d} g(x,y)V(y),$

with $g(\cdot, \cdot)$ as in (1.1), one has

(2.13)
$$\widetilde{\mathbb{E}}^I \Big[\exp \Big\{ \sum_{x \in \mathbb{Z}^d} V(x)\widetilde{\ell}_{x,u} \Big\} \Big] = \exp\{u\langle V, (I - GV)^{-1}1\rangle_{\ell^2(\mathbb{Z}^d)}\}$$
$$\Big(= \exp \Big\{ u \sum_{x \in \mathbb{Z}^d} V(x) \sum_{n \geqslant 0} (GV)^n(x) \Big\} \Big).$$

We now describe in words how to construct the random interlacement process $\widetilde{\omega}^u$ at level $u > 0$ on the cable system from the corresponding random interlacement process ω^u on $\mathbb{Z}^d$. Let us assume that one of the trajectories in the support of ω^u on $\mathbb{Z}^d$ is jumping from x to y with $x \sim y \in \mathbb{Z}^d$ (at some point). One adds Brownian excursions on the edges leaving x in a similar fashion as for the construction of the diffusion X: as long as the height of the Brownian excursion is strictly smaller than $\frac{1}{2}$, one chooses uniformly an edge leaving x and runs this excursion on it, and repeats this process until one of these Brownian excursions has height larger than $\frac{1}{2}$, and one runs this excursion on $I_{\{x,y\}}$ until hitting y for the first time. One then repeats this process starting from y and one iterates over all jumps of the trajectory. Note that the process thereby obtained has the same law as X. Doing this independently for all the trajectories in the support of the interlacement process ω^u yields a Poisson point process which has the same law as $\widetilde{\omega}^u$.

Random interlacements are useful in the study of the Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$ because of the existence of a Ray-Knight-type isomorphism theorem proved in Proposition 6.3 of [10], see also (1.30) in [27].

Theorem 2.2. *For each $u > 0$, there exists a coupling $\widetilde{\mathbb{P}}^u$ between two Gaussian free fields $\widetilde{\varphi}$ and $\widetilde{\gamma}$ and a random interlacement process $\widetilde{\omega}$ on the cable system $\widetilde{\mathbb{Z}}^d$ (i.e., under $\widetilde{\mathbb{P}}^u$, the law of $\widetilde{\varphi}$ and $\widetilde{\gamma}$ is $\widetilde{\mathbb{P}}^G$ each, and the law of $\widetilde{\omega}$ is the same as under $\widetilde{\mathbb{P}}^I$) such that $\widetilde{\gamma}$ and $\widetilde{\omega}$ are independent, and $\widetilde{\mathbb{P}}^u$-a.s.,*

$$(2.14) \qquad \frac{1}{2}\left(\widetilde{\varphi}_x + \sqrt{2u}\right)^2 = \widetilde{\ell}_{x,u} + \frac{1}{2}\widetilde{\gamma}_x^2, \quad \text{for all } x \in \widetilde{\mathbb{Z}}^d,$$

where $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$ is the field of local times of the random interlacements process $\widetilde{\omega}$ at level u, cf. (2.10).

This coupling will be essential for the proof of Theorem 1.1. In particular, we are going to use results from the theory of random interlacements, along with the coupling (2.14), to deduce certain properties of the level sets of the Gaussian free field. For now, let us note that $\widetilde{\mathbb{P}}^u$-a.s. on $\widetilde{\mathcal{I}}^u$, cf. (2.11), one has $|\widetilde{\varphi} + \sqrt{2u}| > 0$, where $\widetilde{\varphi}$ refers to the Gaussian free field from the coupling in Theorem 2.2. Since $\widetilde{\mathcal{I}}^u$ is connected (and unbounded, by construction) and since $x \in \widetilde{\mathbb{Z}}^d \mapsto \widetilde{\varphi}_x$ is continuous, either $\widetilde{\varphi}_x > -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$, or $\widetilde{\varphi}_x < -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$. But Proposition 5.5 in [10], cf. also (1.10) above, implies that the set $\{x \in \widetilde{\mathbb{Z}}^d; \widetilde{\varphi}_x < 0\}$, which contains $\{x \in \widetilde{\mathbb{Z}}^d; \widetilde{\varphi}_x < -\sqrt{2u}\}$, only has bounded components, hence

$$(2.15) \qquad \widetilde{\mathbb{P}}^u - \text{a.s.}, \forall x \in \widetilde{\mathcal{I}}^u, \ \widetilde{\varphi}_x > -\sqrt{2u}.$$

In particular, this means that the negative (upper) level sets percolate on $\widetilde{\mathbb{Z}}^d$, see (1.9).

3 Connectivity and a large deviation inequality for $\widetilde{\mathcal{I}}^u$

The following result, which is proved over the next two sections, is essentially a refinement of (2.15), which allows us to truncate $\widetilde{\Phi}$, cf. (2.2), at sufficiently large heights. This important technical step will be helpful in dealing with the fact that $\widetilde{\Phi}$ is a priori unbounded on sets of interest.

Theorem 3.1. *For each $h_0 > 0$, there exist positive constants C_0 and c_0 only depending on d and h_0, with $C_0 h_0^{-c_0} \geqslant 1$, such that for all $0 < h \leqslant h_0$, with*

$$(3.1) \qquad K(h) = \sqrt{\log\left(\frac{C_0}{h^{c_0}}\right)},$$

$\widetilde{\mathbb{P}}^G$*-a.s. the set*

$$(3.2) \qquad \widetilde{A}_h \overset{\text{def.}}{=} \left\{x \in \widetilde{\mathbb{Z}}^d; -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\right\}$$

contains an unbounded connected component in $\widetilde{\mathbb{Z}}^d$.

The proof of Theorem 3.1 will involve an application of the isomorphism (2.14), and therefore hinges on a corresponding statement "in the world of random interlacements," see Proposition 4.1 at the beginning of the next section. The proof of the latter requires some preliminary results on the geometry of $\widetilde{\mathcal{I}}^u$, which we gather now. In the remainder of this section, we consider, under $\widetilde{\mathbb{P}}^I$, and for each $u > 0$, the random interlacement set $\widetilde{\mathcal{I}}^u$ at level u on the cable system, see (2.11), and $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$ the field of local times of the underlying interlacement process $\widetilde{\omega}^u$, see (2.10). The following lemma asserts that $\widetilde{\mathcal{I}}^u$ is typically well-connected.

Lemma 3.2. *Let $d \geqslant 3$, $\varepsilon \in (0,1)$ and $u_0 > 0$. There exist constants $c = c(d, \varepsilon, u_0)$ and $C = C(d, \varepsilon, u_0)$ such that for all $u \in (0, u_0]$ and $R \geqslant 1$,*

$$(3.3) \quad \widetilde{\mathbb{P}}^I\left(\bigcap_{x,y \in \widetilde{\mathcal{I}}^u \cap [0,R)^d} \{x \leftrightarrow y \text{ in } \widetilde{\mathcal{I}}^u \cap [-\varepsilon R, (1+\varepsilon)R)^d\} \right) \geqslant 1 - C \exp\left(-cR^{1/7}u \right),$$

where, for measurable $A \subset \widetilde{\mathbb{Z}}^d$, the event $\{x \leftrightarrow y \text{ in } \widetilde{\mathcal{I}}^u \cap A\}$ refers to the existence of a continuous path in the subset $\widetilde{\mathcal{I}}^u \cap A$ of the cable system connecting x and y.

This property is essentially known, see for instance Proposition 1 of [15] or Lemma 3.1 in [16]. However, we need to keep careful track of the dependence of error terms on the intensity u. For the reader's convenience, we have included a proof of Lemma 3.2 in the Appendix.

Next, we will need to know how much time the trajectories of random interlacements typically spend in a large box with sufficiently high precision. This can be conveniently formulated in terms of a large deviation inequality for the local times.

Lemma 3.3. *Let $d \geqslant 3$ and $\varepsilon \in (0,1)$. There exist constants $c = c(d, \varepsilon)$ and $C = C(d, \varepsilon)$ such that for all $u > 0$ and $R \geqslant 1$,*

$$(3.4) \quad \widetilde{\mathbb{P}}^I\left(\left| \frac{1}{R^d} \sum_{x \in [0,R)^d \cap \mathbb{Z}^d} \widetilde{\ell}_{x,u} - u \right| > \varepsilon \cdot u \right) \leqslant C \exp\left(-cR^{d-2}u \right).$$

Proof. Abbreviate $B_R = [0,R)^d \cap \mathbb{Z}^d$ and, for $\lambda > 0$, let $V(x) = (|B_R|)^{-1}\lambda 1_{\{x \in B_R\}}$. It follows, cf. (2.12) for notation, that uniformly in $x \in \mathbb{Z}^d$,

$$(3.5) \quad |GV(x)| = \lambda \sum_{y \in B_R} \frac{g(x,y)}{|B_R|} \leq K_1 \lambda R^{2-d},$$

using that $g(x,y) \leq c'|x-y|^{2-d}$, for $x, y \in \mathbb{Z}^d$. Hence, for $\lambda = \lambda_0 R^{d-2}$, with $\lambda_0 < K_1^{-1}$, one obtains that $\|GV\|_\infty < 1$, for all $R \geq 1$. In view of (3.5), applying Markov's inequality and using (2.13) then yields, for all $\lambda_0 < K_1^{-1}$ and $R \geq 1$,

$$\widetilde{\mathbb{P}}^I\left(\frac{1}{R^d} \sum_{x \in [0,R)^d \cap \mathbb{Z}^d} \widetilde{\ell}_{x,u} > (1+\varepsilon)u \right) \leqslant \exp\left\{ -\lambda_0 R^{d-2}u\left((1+\varepsilon) - \left(1 + \sum_{n \geqslant 1}(K_1\lambda_0)^n\right) \right) \right\}.$$

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The right-hand side is bounded from above by $C \exp\left(-cR^{d-2}u\right)$ upon choosing $\lambda_0(\varepsilon) < K_1^{-1}$ small enough such that $\sum_{n \geqslant 1}(K_1\lambda_0)^n \leq \varepsilon/2$. In a similar fashion one bounds for V, λ as above,

$$\widetilde{\mathbb{P}}^I\left(\frac{1}{R^d}\sum_{x\in[0,R)^d\cap\mathbb{Z}^d}\widetilde{\ell}_{x,u} < (1-\varepsilon)u\right) = \widetilde{\mathbb{P}}^I\left(\exp\left\{-\sum_{x\in\mathbb{Z}^d}V(x)\widetilde{\ell}_{x,u}\right\} > e^{-(1-\varepsilon)\lambda u}\right)$$

$$\leqslant \exp\left\{\lambda_0 R^{d-2}u\left((1-\varepsilon) - \left(1 - \sum_{n\geqslant 1}(K_1\lambda_0)^n\right)\right)\right\},$$

from which (3.4) readily follows. □

As a direct application of Lemmas 3.2 and 3.3, we derive lower bounds for the probabilities of the following events.

Definition 3.4. For all u, $u' > 0$, and integer $R \geq 1$, the events $E_R^{u,u'}$ and $F_R^{u,u'}$ are defined as follows:

(a) $E_R^{u,u'}$ occurs if and only if for each $e \in \{0,1\}^d$, the set $(eR + [0,R)^d) \cap \widetilde{\mathcal{I}}^u$ contains a connected component $\mathcal{A}_e$ such that

$$\sum_{y\in\mathcal{A}_e\cap\mathbb{Z}^d}\widetilde{\ell}_{y,u} > \frac{3}{4}u'R^d,$$

and such that the components $(\mathcal{A}_e)_{e\in\{0,1\}^d}$ are all connected in $\widetilde{\mathcal{I}}^u \cap [0,2R)^d$.

(b) $F_R^{u,u'}$ occurs if and only if for all $e \in \{0,1\}^d$,

$$\sum_{y\in(eR+[0,R)^d)\cap\mathbb{Z}^d}\widetilde{\ell}_{y,u} < \frac{5}{4}u'R^d.$$

Note that for fixed $u' > 0$, the events $(E_R^{u,u'})_{u>0}$ are increasing in u (since $u \mapsto \widetilde{\ell}_{y,u}$ is) and that the events $(F_R^{u,u'})_{u>0}$ are decreasing in u. The following consequence of Lemmas 3.2 and 3.3 is tailored to our purposes in the next section.

Corollary 3.5. *Let $d \geqslant 3$ and $u_0 > 0$. There exist $\delta \in (0,1)$, positive and finite constants $C = C(d,u_0)$ and $c = c(d,u_0)$ such that for all $u \in (0,u_0]$ and $R \geqslant 1$,*

(3.6) $$\widetilde{\mathbb{P}}^I\left(E_R^{u(1-\delta),u}\right) \geqslant 1 - C\exp\left(-cR^{1/7}u\right)$$

and

(3.7) $$\widetilde{\mathbb{P}}^I\left(F_R^{u(1+\delta),u}\right) \geqslant 1 - C\exp\left(-cR^{d-2}u\right).$$

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Proof. Let $\delta = \frac{1}{6}$. We begin with (3.7). In view of Definition 3.4, it follows from Lemma 3.3 applied with $u(1+\delta)$ instead of u and translation invariance that for all $e \in \{0,1\}^d$ and $u > 0$,

$$\widetilde{\mathbb{P}}^I \left(\sum_{x \in (eR + [0,R)^d) \cap \mathbb{Z}^d} \widetilde{\ell}_{x, u(1+\delta)} < \frac{5}{4} R^d u \right)$$

$$\geqslant \widetilde{\mathbb{P}}^I \left(\frac{1}{R^d} \sum_{x \in (eR + [0,R)^d) \cap \mathbb{Z}^d} \widetilde{\ell}_{x, u(1+\delta)} < \frac{15}{14} u(1+\delta) \right) \overset{(3.4)}{\geqslant} 1 - C\exp(-cR^{d-2}u),$$

which is (3.7).

In order to obtain (3.6), fix any two constants $\varepsilon = \varepsilon(d) \in (0,1)$ and $\mu = \mu(d) \in (0,1)$ in such a way that $(1 - 8\varepsilon)^d (1-\mu)(1-\delta) = \frac{3}{4}$. For all $e \in \{0,1\}^d$, we define the inner boxes $\mathsf{B}_e(\varepsilon) = eR + [2\lfloor \varepsilon R \rfloor, R - 2\lfloor \varepsilon R \rfloor)^d$. It is sufficient to prove (3.6) for R satisfying $\varepsilon R \geqslant 1$, which we now tacitly assume. We then have $|\mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d| \cdot (1-\mu)(1-\delta) \geqslant \frac{3}{4} R^d$, where $|A|$ denotes the cardinality of $A \subset \mathbb{Z}^d$. According to Lemma 3.3,

$$\widetilde{\mathbb{P}}^I \left(\sum_{x \in \mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d} \widetilde{\ell}_{x, u(1-\delta)} > \frac{3}{4} R^d u \right) \geqslant \widetilde{\mathbb{P}}^I \left(\frac{1}{|\mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d|} \sum_{x \in \mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d} \widetilde{\ell}_{x, u(1-\delta)} > (1-\mu)u(1-\delta) \right)$$

$$\geqslant 1 - C\exp\left(-cR^{d-2}u\right).$$

We now define $\mathcal{A}_e^1 = \mathsf{B}_e(\varepsilon) \cap \widetilde{\mathcal{I}}^{u(1-\delta)}$. According to Lemma 3.2, for every $e \in \{0,1\}^d$, all the vertices of $\mathcal{A}_e^1$ are connected in $\widetilde{\mathcal{I}}^{u(1-\delta)} \cap (eR + [\lfloor \varepsilon R \rfloor, R - \lfloor \varepsilon R \rfloor)^d)$ with probability at least $1 - C\exp(-cR^{1/7}u)$, and on the corresponding event we define $\mathcal{A}_e \subset \widetilde{\mathcal{I}}^{u(1-\delta)} \cap (eR + [\lfloor \varepsilon R \rfloor, R - \lfloor \varepsilon R \rfloor)^d)$ such that $\mathcal{A}_e^1 \subset \mathcal{A}_e$ and $\mathcal{A}_e$ is connected.

Still according to Lemma 3.2, all the $\mathcal{A}_e$ for $e \in \{0,1\}^d$ are connected with each other in $\widetilde{\mathcal{I}}^{u(1-\delta)} \cap [0, 2R)^d$ with probability at least $1 - C\exp(-cR^{1/7}u)$, which gives (3.6). $\qquad\square$

Since the events $E_R^{u,u'}$ and $F_R^{u,u'}$ are defined in terms of local times and not in terms of the occupation field $\left(1_{\{x \in \widetilde{\mathcal{I}}^u\}}\right)_{x \in \mathbb{Z}^d}$, we now give a slightly different version of the decoupling inequality presented in [14] valid for the local times on the cable system. This inequality will later enable us to use $E_R^{u,u'}$ and $F_R^{u,u'}$ as seed events of a suitable multi-scale argument. In what follows, let $\widetilde{\mathbf{Q}}^u$, $u > 0$, be the law on $\overline{\Omega} = [0,\infty)^{\widetilde{\mathbb{Z}}^d}$ of the local times $\left(\widetilde{\ell}_{x,u}\right)_{x \in \widetilde{\mathbb{Z}}^d}$ of random interlacements on the cable system $\widetilde{\mathbb{Z}}^d$, and let $(\tau_x)_{x \in \widetilde{\mathbb{Z}}^d}$ denote the canonical coordinate functions on $\overline{\Omega}$, i.e., for all $f \in \overline{\Omega}$ and $x \in \widetilde{\mathbb{Z}}^d$, $\tau_x(f) = f(x)$.

Theorem 3.6. *Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two measurable non-intersecting subsets of $\widetilde{\mathbb{Z}}^d$. Assume that $s := d(\widetilde{A}_1, \widetilde{A}_2) \geqslant 1$, and that the minimum r of the diameters of $\widetilde{A}_1$ and $\widetilde{A}_2$ is finite. Then there exist $\kappa_0(d)$ and $\kappa_1(d)$ such that for all $u > 0$ and $\varepsilon \in (0,1)$, for any functions $f_i : \overline{\Omega} \to [0,1]$ which are $\sigma(\tau_x, \, x \in \widetilde{A}_i)$ measurable for each $i \in \{1,2\}$, and which are both increasing or both decreasing,*

$$(3.8) \qquad \widetilde{\mathbf{Q}}^u[f_1 f_2] \leqslant \widetilde{\mathbf{Q}}^{u(1\pm\varepsilon)}[f_1] \widetilde{\mathbf{Q}}^{u(1\pm\varepsilon)}[f_2] + \kappa_0(r+s)^d \exp(-\kappa_1 \varepsilon^2 u s^{d-2}),$$

where the plus sign corresponds to the case where the f_i's are increasing and the minus sign to the case where the f_i's are decreasing.

Proof. Let A_1 and A_2 be the smallest subsets of $\mathbb{Z}^d$ such that for all $i \in \{1, 2\}$, and all $x \in \widetilde{A}_i$, there exist $y, z \in A_i$ such that $x \in \overline{I}_{\{y,z\}}$. Note that $\widetilde{A}_i \cap \mathbb{Z}^d \subset A_i$. Since $d(\widetilde{A}_1, \widetilde{A}_2) \geqslant 1$, the sets A_1 and A_2 are not intersecting (recall that the distance between two neighbors of $\mathbb{Z}^d$ is $\frac{1}{2}$). For two measures μ_1 and μ_2, denote $\mu_1 \ll \mu_2$ the absolute continuity of μ_1 with respect to μ_2. The proof of the main decoupling result, Theorem 2.1 in [14], see in particular Section 5 therein, implies that, for each $u > 0$, there exists a coupling $\mathbb{Q}_u^I$ between the random interlacement process ω on $\mathbb{Z}^d$ and two independent Poisson point processes ω_1 and ω_2 having the same law as ω, such that, for $B \subset \mathbb{Z}^d$, denoting by $(\omega^u)_{|B}$ the point process consisting of the restriction to B of the trajectories in ω^u which hit B,

$$(3.9) \quad \mathbb{Q}_u^I\big[(\omega_i^{u(1-\varepsilon)})_{|A_i} \ll (\omega^u)_{|A_i} \ll (\omega_i^{u(1+\varepsilon)})_{|A_i}, \ i = 1, 2\big] \geqslant 1 - \kappa_0(r+s)^d \exp(-\kappa_1 \varepsilon^2 u s^{d-2}).$$

For each $u > 0$ and $i \in \{1, 2\}$, under an extended probability $\widetilde{\mathbb{Q}}_u^I$, one then constructs an interlacement process $\widetilde{\omega}_i^{u(1-\varepsilon)}$ at level $u(1-\varepsilon)$ on the cable system by adding independent Brownian excursions to the edges visited by the trajectories of the random interlacement process $\omega_i^{u(1-\varepsilon)}$ on $\mathbb{Z}^d$, see Section 2 below (2.11). We now construct a random interlacement process $\widetilde{\omega}^u$ at level u using $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and ω^u. Its trajectories are the trajectories of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ which have a projection on $\mathbb{Z}^d$ already contained in ω^u (i.e. all the trajectories of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ on the event in (3.9)) and the trajectories of ω^u which are not already in $\omega_i^{u(1-\varepsilon)}$, lifted to $\widetilde{\mathbb{Z}}^d$ using additional independent (of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and ω^u) Brownian excursions on the edges. We repeat this construction to obtain a random interlacement process $\widetilde{\omega}_i^{u(1+\varepsilon)}$ at level $u(1+\varepsilon)$ in a similar way from $\widetilde{\omega}^u$ and $\omega_i^{u(1+\varepsilon)}$. Then, an analogue of (3.9) holds for these processes $\widetilde{\omega}^u$, $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and $\widetilde{\omega}_i^{u(1+\varepsilon)}$ under $\widetilde{\mathbb{Q}}_u^I$. In particular, denoting by $\widetilde{\ell}_{x,u}$, $\widetilde{\ell}_{x,u(1-\varepsilon)}^i$ and $\widetilde{\ell}_{x,u(1+\varepsilon)}^i$ their respective local time fields on the cable system, see (2.10), it follows that

$$(3.10) \quad \widetilde{\mathbb{Q}}_u^I\big[\widetilde{\ell}_{x,u(1-\varepsilon)}^i \leqslant \widetilde{\ell}_{x,u} \leqslant \widetilde{\ell}_{x,u(1+\varepsilon)}^i, \ x \in \widetilde{A}_i, \ i = 1, 2\big] \geqslant 1 - \kappa_0(r+s)^d \exp(-\kappa_1 \varepsilon^2 u s^{d-2})$$

The inequalities in (3.8) are a direct consequence of (3.10). $\qquad\square$

4 Percolation for the truncated level set

In this section, we prove Theorem 3.1: for each $h > 0$, there exists a finite constant $K(h)$ such that the level set of the Gaussian free field on the cable system truncated below level $-h$ and above level $K(h)$ contains an unbounded connected component. We will actually show a similar statement for random interlacements and use the coupling from Theorem 2.2 to obtain a slightly refined version of Theorem 3.1, stated below in Corollary 4.9. The corresponding statement for random interlacement, see Proposition 4.1, essentially asserts that one can intersect the continuous interlacement set $\widetilde{\mathcal{I}}^u$, the set $\{x \in \mathbb{Z}^d; |\varphi_x| < K\}$ and

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a Bernoulli family on the edges with parameter p and still retain an unbounded connected component in $\widetilde{\mathbb{Z}}^d$ for sufficiently large K and p close enough to 1. The proof of this statement bears similarities to the proof of Theorem 2.1 in [16], where it is shown that the intersection of $\mathcal{I}^u$ and a Bernoulli family with parameter p on $\mathbb{Z}^d$, not necessarily independent from $\mathcal{I}^u$, contains an infinite connected component in $\mathbb{Z}^d$ for large enough p.

Henceforth, for a given $p \in (0,1)$ (and $d \geqslant 3$), let $\widetilde{\mathbb{Q}}^p$ be any coupling between a Gaussian free field $\widetilde{\varphi}$, a random interlacement process $\widetilde{\omega}$ and a family of independent Bernoulli random variables on the edges $\mathcal{B}^p = (\theta_e^p)_{e \in E}$ with parameter p, i.e.,

(4.1)
$$\text{under } \widetilde{\mathbb{Q}}^p, \text{ the law of } (\widetilde{\varphi}_x)_{x \in \widetilde{\mathbb{Z}}^d} \text{ is } \widetilde{\mathbb{P}}^G, \text{ the process } \widetilde{\omega} \text{ has the same law as under } \widetilde{\mathbb{P}}^I$$
$$\text{and } (\theta_e^p)_{e \in E} \text{ is an i.i.d. family of } \{0,1\}\text{-valued random variables with}$$
$$\widetilde{\mathbb{Q}}^p(\theta_e^p = 1) = p \text{ for each } e \in E.$$

In particular, $\widetilde{\varphi}$, $\widetilde{\omega}$ and $\mathcal{B}^p$ need not be independent, and in fact, we will later use a coupling such that (2.15) holds. For any level $u > 0$, we define the random interlacement set $\widetilde{\mathcal{I}}^u$ as in (2.11) and the local times $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$ as in (2.10) in terms of $\widetilde{\omega}$. We further denote by φ the restriction of $\widetilde{\varphi}$ to $\mathbb{Z}^d$ and by $\mathcal{I}^u$ the restriction of $\widetilde{\mathcal{I}}^u$ to $\mathbb{Z}^d$.

Proposition 4.1. $(d \geqslant 3, u_0 > 0, (4.1))$

There exist positive constants C_1, c_1, C_1' and c_1', only depending on d and u_0, satisfying $C_1 u_0^{-c_1} \geqslant 1$ and $C_1' u_0^{c_1'} < 1$, such that for all $u \in (0, u_0]$, with

(4.2)
$$\widetilde{K}(u) \overset{\text{def.}}{=} \sqrt{\log\left(\frac{C_1}{u^{c_1}}\right)} \quad \text{and} \quad p(u) \overset{\text{def.}}{=} 1 - C_1' u^{c_1'},$$

there exists $L_0(u) > 0$ such that, if $p \in [p(u), 1]$, then $\widetilde{\mathbb{Q}}^p$-a.s. the set

(4.3) $\widetilde{A}_{u,p}' = \left(\widetilde{\mathcal{I}}^u \setminus \mathcal{I}^u\right) \cup \left\{x \in \mathcal{I}^u; |\varphi_x| \leqslant \widetilde{K}(u) \text{ and } \forall y \sim x, |\varphi_y| \leqslant \widetilde{K}(u) \text{ and } \theta_{\{x,y\}}^p = 1\right\}$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \cap [0, 2L_0(u))^{d-2}$.

We now comment on (4.3). First, note that $\mathcal{I}^u \subset \mathbb{Z}^d$, so saying that $\widetilde{A}_{u,p}'$ contains an unbounded connected component implies that $\widetilde{A}_{u,p}' \cap \mathbb{Z}^d$ contains an infinite path such that all the edges of this path are in $\widetilde{\mathcal{I}}^u$, and for all vertices x on this path and all $y \sim x$, $|\varphi_x| \leqslant \widetilde{K}(u)$ and $\theta_{\{x,y\}}^p = 1$. Proposition 4.1 is true for any choice of coupling probability $\widetilde{\mathbb{Q}}^p$ satisfying (4.1). Once its proof is completed, we will choose the coupling introduced in (2.14). This will automatically enforce the lower bound $-h$ required for the proof of Theorem 3.1, since $\widetilde{A}_{u,p}' \subset \mathcal{I}^u$. A good choice of $\theta_{\{x,y\}}^p$, cf. Lemma 4.10 below, will then allow to control the field along the edges. To this effect, (4.3) essentially guarantees that on $\widetilde{A}_{u,p}'$ we are dealing with Brownian bridges whose boundary values are uniformly bounded.

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The proof of Proposition 4.1 follows a strategy very similar to the proof of Theorem 2.1 in [16], but we need to pay diligent attention to the dependence on u in order to obtain the explicit bounds (4.2). We use a renormalisation scheme akin to the one introduced in Section 4 of [16], which uses a sprinkling technique developed in [21] and later improved in [22] and [14]. For $n \geqslant 0$ and $L_0 \geq 1$, we define the geometrically increasing sequence

$$(4.4) \qquad L_n = l_0^n L_0, \quad \text{where } l_0 = 4l(d) \text{ and } l(d) = 4(5 \cdot 4^d + 1)$$

and the coarse-grained lattice model

$$\mathbb{G}_0^{L_0} = L_0 \mathbb{Z}^d \text{ and } \mathbb{G}_n^{L_0} = L_n \mathbb{Z}^d \subset \mathbb{G}_{n-1}^{L_0} \text{ for } n \geqslant 1.$$

Note that, albeit only implicitly, the sequence L_n depends on the choice of L_0, which is the only parameter in this scheme. For $x \in \mathbb{G}_n^{L_0}$, we further introduce the boxes

$$(4.5) \qquad \Lambda_{x,n}^{L_0} = \mathbb{G}_{n-1}^{L_0} \cap (x + [0, L_n)^d),$$

and note that $\{\Lambda_{x,n}^{L_0}; \ x \in \mathbb{G}_n^{L_0}\}$ forms a partition of $\mathbb{G}_{n-1}^{L_0}$. For a given collection of events indexed by $\mathbb{G}_0^{L_0}$, that we denote by $A = (A_x)_{x \in \mathbb{G}_0^{L_0}}$, we define recursively the events $G_{x,n}^{L_0}(A)$ such that $G_{x,0}^{L_0} = A_x$ for all $x \in \mathbb{G}_0^{L_0}$, and for all $n \geqslant 1$ and $x \in \mathbb{G}_n^{L_0}$,

$$(4.6) \qquad G_{x,n}^{L_0}(A) = \bigcup_{\substack{x_1, x_2 \in \Lambda_{x,n}^{L_0} \\ |x_1 - x_2|_\infty \geqslant \frac{L_n}{l(d)}}} G_{x_1, n-1}^{L_0}(A) \cap G_{x_2, n-1}^{L_0}(A),$$

where $|\cdot|_\infty$ stands for the ℓ^∞-distance on $\mathbb{Z}^d$. For each $x \in \mathbb{Z}^d$, let T_x be the translation operator on the space of point measures on W^*, the space of doubly infinite trajectories on $\widetilde{\mathbb{Z}}^d$ modulo time-shift such that, if μ is such a measure, then $T_x(\mu)$ is the point measure where each trajectory in the support of μ has been translated by x. Moreover, in a slight abuse of notation, let τ_x be defined by

$$(4.7) \qquad \widetilde{\omega} \circ \tau_x = T_x(\widetilde{\omega}).$$

We introduce a family of events on the space Ω_{coup} on which $\widetilde{\mathbb{Q}}^p$, cf. (4.1), is defined. Recall the events $E_{L_0}^{u,u'}$ and $F_{L_0}^{u,u'}$ from Definition 3.4 (henceforth tacitly viewed as subsets of Ω_{coup}).

Definition 4.2. For each $u > 0$, integer $L_0 \geq 1$, $K > 0$ and $p \in [0, 1]$ let

(a) $(\mathbf{E}_x^{L_0,u})_{x \in \mathbb{G}_0^{L_0}}$ be the family of increasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{E}_x^{L_0,u} = \tau_x^{-1}(E_{L_0}^{u,u})$ occurs,

(b) $(\mathbf{F}_x^{L_0,u})_{x \in \mathbb{G}_0^{L_0}}$ be the family of decreasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{F}_x^{L_0,u} = \tau_x^{-1}(F_{L_0}^{u,u})$ occurs,

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(c) $(\mathbf{C}_x^{L_0,K})_{x\in\mathbb{G}_0^{L_0}}$ be the family of decreasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{C}_x^{L_0,K}$ occurs if and only if for all $y \in (x + [-1, 2L_0 + 1)^d) \cap \mathbb{Z}^d$, we have $\varphi_y \leqslant K$,

(d) $(\widehat{\mathbf{C}}_x^{L_0,K})_{x\in\mathbb{G}_0^{L_0}}$ be the family of increasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\widehat{\mathbf{C}}_x^{L_0,K}$ occurs if and only if for all $y \in (x + [-1, 2L_0 + 1)^d) \cap \mathbb{Z}^d$, we have $\varphi_y \geqslant -K$,

(e) $(\mathbf{D}_x^{L_0,p})_{x\in\mathbb{G}_0^{L_0}}$ be the family of events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{D}_x^{L_0,p}$ occurs if and only if for all $e \in (x + [-1, 2L_0 + 1)^d) \cap E$, we have $\theta_e^p = 1$.

A vertex $x \in \mathbb{G}_0^{L_0}$ is called a good (L_0, u, K, p) vertex if

(4.8) $$\mathbf{C}_x^{L_0,K} \cap \widehat{\mathbf{C}}_x^{L_0,K} \cap \mathbf{D}_x^{L_0,p} \cap \mathbf{E}_x^{L_0,u} \cap \mathbf{F}_x^{L_0,u}$$

occurs, and otherwise a bad (L_0, u, K, p) vertex.

The reason for the choices in Definition 3.4 and (4.8), with regards to Proposition 4.1, comes in the following.

Lemma 4.3. $(u > 0, L_0 \geqslant 1, p \in (0, 1])$

If $(x_0, x_1, \dots)$ is an unbounded nearest neighbor path of good $(L_0, u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0}$, then the set $\bigcup_{i=0}^{\infty}(x_i + [0, 2L_0)^d)$ contains an unbounded connected path in $\widetilde{A}'_{u,p}$, cf. (4.3).

Proof. Let x, y two neighbors in $\mathbb{G}_0^{L_0}$, and assume that there exists $e \in \{0, 1\}^d$ such that $x + eL_0 = y$. Since $\mathbf{E}_x^{L_0,u}$ holds, there exist two random sets $\mathcal{A}_{x,0} \subset \widetilde{\mathcal{I}}^u \cap (x + [0, L_0)^d)$ and $\mathcal{A}_{x,e} \subset \widetilde{\mathcal{I}}^u \cap (x + eL_0 + [0, L_0)^d)$ which are connected in $\widetilde{\mathcal{I}}^u \cap (x + [0, 2L_0)^d)$, and such that the sum of the local times on the vertices of each of those two sets is larger than $\frac{3}{4}uL_0^d$. Moreover, since $\mathbf{F}_y^{L_0,u}$ occurs, the sum of the local times on the vertices of $\mathcal{A}_{x,e} \cup \mathcal{A}_{y,0}$ is smaller than $\frac{5}{4}uL_0^d$ because $\mathcal{A}_{x,e} \cup \mathcal{A}_{y,0} \subset \widetilde{\mathcal{I}}^u \cap (x_{i+1} + [0, L_0)^d)$. Hence, $\mathcal{A}_{x,e} \cap \mathcal{A}_{y,0} \neq \emptyset$, and this implies that $\mathcal{A}_{x,0}$ is connected to $\mathcal{A}_{y,0}$ in $\widetilde{\mathcal{I}}^u \cap (x + [0, 2L_0)^d)$.

Applying the above to each of the neighbors in our path $(x_0, x_1, \dots)$, we get that for all $i \in \mathbb{N}_0$, $\mathcal{A}_{x_i,0}$ is connected to $\mathcal{A}_{x_{i+1},0}$ in $\widetilde{\mathcal{I}}^u \cap ((x_i + [0, 2L_0)^d) \cup (x_{i+1} + [0, 2L_0)^d))$. Thus, one can find an unbounded connected path in $\widetilde{\mathcal{I}}^u \cap \bigcup_{i=0}^{\infty}(x_i + [0, 2L_0)^d)$, and this path is actually in $\widetilde{A}'_{u,p}$ since $|\varphi_x| \leqslant \widetilde{K}(u)$ and $\theta_e^p = 1$ for all $x, e \in \bigcup_{i=1}^{\infty}(x_i + [-1, 2L_0 + 1)^d)$ by Definition 4.2, (c), (d), (e). $\square$

To prove that an unbounded nearest neighbor path of good $(L_0, u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0}$ exists for a suitable choice of the parameters, we pair our good (seed) events with the renormalisation scheme (4.6) to show that, if being a good seed is typical, i.e., if it occurs with probability sufficiently close to 1, then the probability of being good "at level n" cf. (4.6) and (4.21) below, is overwhelming. The respective bounds for all events of interest, cf. Definition 4.2, can be found in Lemmas 4.4, 4.6 and 4.7 below. We first consider the events $(\mathbf{E}_x^{L_0,u})_{x\in\mathbb{Z}^d}$ and $(\mathbf{F}_x^{L_0,u})_{x\in\mathbb{Z}^d}$, and take advantage of Corollary 3.5 and Theorem 3.6 to show the following.

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Lemma 4.4. $(u_0 > 0)$

There exist $C_2 = C_2(d, u_0)$ and $C_2' = C_2'(d, u_0)$ such that for all $u \in (0, u_0]$ and $L_0 \geq 1$ with $L_0^{1/7} u \geqslant C_2$,

$$(4.9) \qquad\qquad \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0} \left((\mathbf{E}^{L_0,u})^c \right) \right] \leqslant 2^{-2^n},$$

and for all $u \in (0, u_0]$ and $L_0 \geq 1$ with $L_0^{1/7} u \geqslant C_2'$,

$$(4.10) \qquad\qquad \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0} \left((\mathbf{F}^{L_0,u})^c \right) \right] \leqslant 2^{-2^n}.$$

Proof. We only prove (4.9). The proof of (4.10) is similar. Fix $\delta \in (0, 1)$ as in Corollary 3.5, and let $\delta' \in (0, 1)$ be small enough such that

$$(4.11) \qquad\qquad u_n := \frac{u(1 - \delta)}{\prod_{k=1}^{n} \left(1 - \frac{\delta'}{2^k} \right)} < u, \ \forall n \geqslant 0$$

(with $u_0 = u(1-\delta)$). For all $x \in \mathbb{G}_0^{L_0}$, let $\mathbf{E}_x^{L_0,u,u'} = \tau_x^{-1}(E_{L_0}^{u,u'})$, cf. (4.7) and Definition 3.4, and note that $\mathbf{E}_x^{L_0,u,u} = \mathbf{E}_x^{L_0,u}$. Let $n \geqslant 0$. For all $L_0 \geqslant 1$, $n \in \mathbb{N}_0$, $i \in \{1,2\}$ and $x_i \in \mathbb{G}_n^{L_0}$ such that $|x_1 - x_2|_\infty \geqslant \frac{L_{n+1}}{l(d)} = 4L_n$, the events $G_{x_i,n}^{L_0} \left((\mathbf{E}^{L_0,u_{n+1},u})^c \right)$ are $\sigma(\widetilde{l}_{z,u}, \ z \in x_i + [0, L_n + L_0)^d)$ measurable, and

$$s_n := d \left(x_1 + [0, L_n + L_0)^d, x_2 + [0, L_n + L_0)^d] \right) \geqslant d(x_1, x_2) - \frac{1}{2}(L_n + L_0) \geqslant L_n.$$

By Theorem 3.6 applied with $\varepsilon = 1 - \frac{u_n}{u_{n+1}} = \delta' 2^{-n-1}$, and since the events $(\mathbf{E}^{L_0,u,u'})^c$ are decreasing in u, there exist two constants C and c independent of u, n and L_0 such that

(4.12)
$$\widetilde{\mathbb{Q}}^p \left[G_{x_1,n}^{L_0} \left((\mathbf{E}^{L_0,u_{n+1},u})^c \right) \cap G_{x_2,n}^{L_0} \left((\mathbf{E}^{L_0,u_{n+1},u})^c \right) \right]$$
$$\leqslant \widetilde{\mathbb{Q}}^p \left[G_{x_1,n}^{L_0} \left((\mathbf{E}^{L_0,u_n,u})^c \right) \right] \widetilde{\mathbb{Q}}^p \left[G_{x_2,n}^{L_0} \left((\mathbf{E}^{L_0,u_n,u})^c \right) \right] + C(s_n + L_n + L_0)^d \exp \left(-cu s_n^{d-2} 4^{-n} \right).$$

We have chosen $l_0^{d-2} > 8$, see (4.4), whence for $L_0^{1/7} u \geqslant c(d, u_0)$,

$$(4.13) \quad l_0^{2d} C(s_n + L_n + L_0)^d \exp \left(-cu s_n^{d-2} 4^{-n} \right) \leqslant C' \exp \left(-cu L_0^{1/7} 2^n \right) \leqslant \frac{1}{(4 l_0^{2d}) 2^{2^{n+1}}}.$$

We now prove by induction over n that for all $x \in \mathbb{G}_n^{L_0}$, and all $u \in (0, u_0]$,

$$(4.14) \qquad\qquad \widetilde{\mathbb{Q}}^p \left[G_{x,n}^{L_0} \left((\mathbf{E}^{L_0,u_n,u})^c \right) \right] \leqslant \frac{1}{(2 l_0^{2d}) 2^{2^n}}, \quad \text{if } L_0^{1/7} u \geq c(d, u_0).$$

For $n = 0$, the bound on the right-hand side of (4.14) is purely numerical. Thus, it is clear from Corollary 3.5, and since $G_{x,0}^{L_0} \left((\mathbf{E}^{L_0,u_0,u})^c \right) = \mathbf{E}_x^{L_0,u_0,u}$, see above (4.6), that if one takes

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$L_0^{1/7}u$ large enough (only depending on u_0 and d), then (4.14) holds for $n = 0$ on account of (3.6). Suppose now it holds for $n - 1 \geq 0$. Then, according to (4.6)

$$\widetilde{\mathbb{Q}}^p \left[G_{x,n}^{L_0} \left((\mathbf{E}^{L_0, u_n, u})^c \right) \right] \leq l_0^{2d} \sup_{\substack{x_1, x_2 \in \Lambda_{x,n}^{L_0} \\ |x_1 - x_2|_\infty \geq \frac{L_n}{l(d)}}} \widetilde{\mathbb{Q}}^p \left[G_{x_1, n-1}^{L_0} \left((\mathbf{E}^{L_0, u_n, u})^c \right) \cap G_{x_2, n-1}^{L_0} \left((\mathbf{E}^{L_0, u_n, u})^c \right) \right]$$

$$\leq \frac{1}{(2l_0^{2d})2^{2^n}},$$

where the last equality follows from (4.12), (4.13) and the induction hypothesis, and (4.14) follows. The claim (4.9) then follows from (4.11), (4.14) and the fact that the $\left(\mathbf{E}^{L_0, u, u'} \right)^c$ are decreasing events in u. $\qquad \square$

We now turn to the Gaussian free field part $\mathbf{C}^{L_0, K}$ and $\widehat{\mathbf{C}}^{L_0, K}$, see (c) and (d) in Definition 4.2, of the good events in (4.8). Sprinkling techniques have been used successfully in investigating level set percolation of the Gaussian free field, see for example [20], [18] or [7], and also [19] with regard to non-Gaussian measures. These techniques imply similar results as for random interlacements, and this is mainly due to the fact that decoupling inequalities as (3.8) also hold for the Gaussian free field. With hopefully obvious notation, in writing $\Phi + c$ below, with Φ as in (1.2), we mean the field whose value is shifted by $c \in \mathbb{R}$ everywhere.

Theorem 4.5 ([13, Corollary 1.3]). *Let A_1 and A_2 be two non intersecting subsets of $\mathbb{Z}^d$, define $s = d(A_1, A_2)$ and assume that the minimum r of their diameters is finite. Then, there exist positive constants $\kappa_0'(d)$ and $\kappa_1'(d)$ such that, for all $\varepsilon \in (0, 1)$, and any two functions $f_i : \mathbb{R}^{\mathbb{Z}^d} \to [0, 1]$ which are $\sigma(\Phi_x, x \in A_i)$ measurable for each $i \in \{1, 2\}$, and either both increasing or both decreasing,*

$$(4.15) \qquad \mathbb{E}^G[f_1(\Phi) f_2(\Phi)] \leq \mathbb{E}^G[f_1(\Phi \pm \varepsilon)] \mathbb{E}^G[f_2(\Phi \pm \varepsilon)] + \kappa_0'(r + s)^d \exp(-\kappa_1' \varepsilon^2 s^{d-2}),$$

where the plus sign corresponds to the case where the f_i's are increasing and the minus sign to the case where the f_i's are decreasing.

Theorem 1.2 in [13] gives a slightly better inequality, but (4.15) will be sufficient for our purposes, and readily yields the following analogue of Lemma 4.4 for the events pertaining to the free field.

Lemma 4.6. *There exist constants $C_3(d) > 1$ and $C_3'(d) > 0$ such that for all $L_0 \geq C_3$ and $K > 0$ with*

$$(4.16) \qquad\qquad\qquad K \geq C_3' \sqrt{\log(L_0)},$$

one has

$$(4.17) \qquad \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0} \left((\mathbf{C}^{L_0, K})^c \right) \right] \leq 2^{-2^n} \quad and \quad \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0} \left((\widehat{\mathbf{C}}^{L_0, K})^c \right) \right] \leq 2^{-2^n}.$$

Proof. One knows from (2.35) and (2.38) in [20] that if $K > C\sqrt{\log(L_0)}$ for some constant C large enough,

$$(4.18) \quad \widetilde{\mathbb{Q}}^p\left[\left(\widehat{\mathbf{C}}_0^{L_0,K}\right)^c\right] = \widetilde{\mathbb{Q}}^p\left[\left(\mathbf{C}_0^{L_0,K}\right)^c\right] = \widetilde{\mathbb{Q}}^p\left(\sup_{x\in[-1,2L_0+1)^d} \varphi_x > K\right) \leqslant e^{-\frac{\left(K-C\sqrt{\log(L_0)}\right)^2}{2g(0)}}.$$

The claim (4.17) now follows by induction over n from (4.18) and Theorem 4.5 in exactly the same way as Lemma 4.4 was obtained from Corollary 3.5 and Theorem 3.6. $\qquad\square$

Finally, we collect a simple estimate for the Bernoulli part of our good events $\mathbf{D}^{L_0,p}$, see part (e) in Definition 4.2.

Lemma 4.7 ([16, Lemma 4]). *There exists $C_4 = C_4(d)$ such that for all $L_0 \geqslant 1$ and $p \in (0,1)$ satisfying*

$$(4.19) \quad\quad\quad\quad\quad p \geqslant \exp\left(-\frac{C_4}{L_0^d}\right),$$

one has

$$(4.20) \quad\quad\quad\quad\quad \widetilde{\mathbb{Q}}^p\left[G_{0,n}^{L_0}\left((\mathbf{D}^{L_0,p})^c\right)\right] \leqslant 2^{-2^n}.$$

The bounds of Lemmas 4.4, 4.6 and 4.7 allow for a proof of Proposition 4.1 by means of a standard duality argument. On account of Lemma 4.3, this requires an estimate on the probability to see certain long (dual) paths. The relevant events, see (4.22), can be suitably expressed in terms of bad vertices at level n, as Lemma 4.8 asserts.

Recall the definition of good (L_0, u, K, p) vertices in (4.8). For $n \geq 0$, we call $x \in \mathbb{G}_n^{L_0}$ a bad $n - (L_0, u, K, p)$ vertex if the event

$$(4.21) \quad G_{x,n}^{L_0}\left((\mathbf{C}^{L_0,K})^c\right) \cup G_{x,n}^{L_0}\left((\widehat{\mathbf{C}}^{L_0,K})^c\right) \cup G_{x,n}^{L_0}\left((\mathbf{D}^{L_0,p})^c\right) \cup G_{x,n}^{L_0}\left((\mathbf{E}^{L_0,u})^c\right) \cup G_{x,n}^{L_0}\left((\mathbf{F}^{L_0,u})^c\right)$$

occurs, and a good $n - (L_0, u, K, p)$ vertex otherwise. Note that a good $0 - (L_0, u, K, p)$ vertex is simply a good (L_0, u, K, p) vertex. We say that $(x_0, x_1, \ldots, x_n, \ldots)$ is a $*$-path in $\mathbb{G}_0^{L_0}$ if for all $i \in \{0, 1, \ldots\}$, $x_i \in \mathbb{G}_0^{L_0}$ and $\|x_i - x_{i+1}\|_\infty = L_0$. For each $u, K > 0$, integer $L_0 \geq 1$, $p \in (0,1)$, $0 < M < N$ with M, N multiples of L_0, and $x \in \mathbb{G}_0^{L_0}$, let

$$(4.22) \quad\begin{aligned} H_M^N(x; L_0, u, K, p) = \big\{ &(x + [-M, M]^d) \text{ is connected to } (x + \partial[-N, N]^d) \text{ by} \\ &\text{a } *\text{-path of bad } (L_0, u, K, p) \text{ vertices in } \mathbb{G}_0^{L_0} \big\}. \end{aligned}$$

Here, $\partial[-N, N]^d$ denotes the boundary of the set $[-N, N]^d$. The following lemma asserts that $H_{L_n}^{2L_n}(x; L_0, u, K, p)$ can only happen if there is a bad $n - (L_0, u, K, p)$ vertex in the box of radius $2L_n$ around x.

Lemma 4.8. *For all integers $n \geq 0$ and $L_0 \geq 1$, $u, K > 0$, $p \in (0,1)$, and $x \in \mathbb{G}_n^{L_0}$,*

$$(4.23) \quad\quad H_{L_n}^{2L_n}(x; L_0, u, K, p) \subset \bigcup_{y\in\mathbb{G}_n^{L_0}\cap(x+[-2L_n,2L_n)^d)} \{y \text{ is } n - (L_0, u, K, p) \text{ bad}\}.$$

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Proof. This is a consequence of Lemma 4.4 of [18] (with $N \equiv 5$, $r \equiv l(d)$ and L_0, l_0 as in (4.4) above). We include the proof for the reader's convenience. We proceed by induction over n: it is clear that (4.23) is true for $n = 0$, and we assume that it holds for any choice of x up to level $n - 1$. If $H_{L_n}^{2L_n}(x; L_0, u, K, p)$ occurs, there exists a $*$-path π of bad (L_0, u, K, p) vertices in $\mathbb{G}_0^{L_0}$ from $(x + [-L_n, L_n]^d)$ to $(x + \partial[-2L_n, 2L_n]^d)$. This path intersects the concentric ℓ^∞-spheres $(x + \partial[-L_n - 16iL_{n-1}, L_n + 16iL_{n-1}]^d)$ for all $i \in \{0, \ldots, m - 1\}$, where $m = 5 \cdot 4^d + 1$ (recall that $l_0 = 16m$). In view of (4.4), for all $i \in \{0, \ldots, m - 1\}$, one can thus find $y_i \in \mathbb{G}_{n-1}^{L_0} \cap (x + \partial[-L_n - 16iL_{n-1}, L_n + 16iL_{n-1}]^d)$ such that $\pi \cap (y_i + [-L_{n-1}, L_{n-1}]^d) \neq \emptyset$.

For each $i \in \{0, \ldots, m - 1\}$, since $(y_i + [-2L_{n-1}, 2L_{n-1}]^d) \subset (x + [-2L_n, 2L_n]^d)$, the connected $*$-path π in $\mathbb{G}_0^{L_0}$ connects $(y_i + [-L_{n-1}, L_{n-1}]^d)$ to $(y_i + \partial[-2L_{n-1}, 2L_{n-1}]^d)$, and thus the induction hypothesis implies that there exists $z_i \in (y_i + [-2L_{n-1}, 2L_{n-1})^d)$ which is $(n-1) - (L_0, u, K, p)$ bad, and in particular $z_i \in \mathbb{G}_{n-1}^{L_0}$. There are $m = 5 \cdot 4^d + 1$ such z_i, and since there are only 4^d vectors in $\mathbb{G}_n^{L_0} \cap (x + [-2L_n, 2L_n)^d)$, one can find x_0 in this set such that $\Lambda_{x_0,n}^{L_0}$, cf. (4.5), contains at least 6 different z_i. By (4.21), one can thus find $k \neq j$ in $\{0, \ldots, m - 1\}$ and $A_0 \in \{(\mathbf{C}^{L_0,K})^c, (\widehat{\mathbf{C}}^{L_0,K})^c, (\mathbf{D}^{L_0,p})^c, (\mathbf{E}^{L_0,u})^c, (\mathbf{F}^{L_0,u})^c\}$ such that z_k and z_j are in $\Lambda_{x_0,n}^{L_0}$, and $G_{z_k,n-1}^{L_0}(A_0)$ and $G_{z_j,n-1}^{L_0}(A_0)$ both occur. Moreover,

$$\|z_k - z_j\|_\infty \geqslant \|y_k - y_j\|_\infty - 4L_{n-1} \geqslant 12L_{n-1} \geqslant L_n/l(d),$$

which, in view of (4.6), implies that $G_{x_0,n}^{L_0}(A_0)$ occurs, and thus x_0 is $n - (L_0, u, K, p)$ bad. $\square$

By Lemmas 4.4, 4.6 and 4.7, we know that for all $u \in (0, u_0]$, and for a suitable choice of the parameters L_0, K and p, the probability that a vertex is $n - (L_0, u, K, p)$ bad is very small. Lemma 4.8 then yields that a $*$-path of (L_0, u, K, p) bad vertices in $\mathbb{G}_0^{L_0}$ exists with very small probability only, and on account of Lemma 4.3, we can prove Proposition 4.1 using a Peierls argument.

Proof of Proposition 4.1. Choose a constant $C_5 = C_5(d, u_0)$ large enough such that, upon defining

$$L_0(u) = \lceil C_5/u^7 \rceil,$$

one has $L_0(u)^{1/7}u \geqslant \max(C_2, C_2')$, cf. Lemma 4.4, and $L_0(u) \geqslant C_3$, cf. Lemma 4.6, for all $u \in (0, u_0]$. One can now find constants c_1, C_1, c_1' and C_1' such that if (4.2) holds, then $\widetilde{K}(u) \geqslant C_3'\sqrt{\log(L_0(u))}$, cf. (4.16), and $p(u) \geqslant \exp(-\frac{C_4}{L_0(u)^d})$, cf. (4.19), for all $u \in (0, u_0]$. Let us now fix arbitrarily some $u \in (0, u_0]$ and $p \in [p(u), 1]$. Lemma 4.8, Lemmas 4.4 and

4.6 and Lemma 4.7 can now be applied with $L_0 = L_0(u)$, $K = \widetilde{K}(u)$ and p, to yield

$$\widetilde{\mathbb{Q}}^p \left(H^{2L_n}_{L_n}(0; L_0(u), u, \widetilde{K}(u), p) \right)$$

$$\overset{(4.23)}{\leqslant} 4^d \widetilde{\mathbb{Q}}^p \big(0 \text{ is } n - (L_0(u), u, \widetilde{K}(u), p) \text{ bad} \big)$$

$$\overset{(4.21)}{\leqslant} 4^d \Big\{ \widetilde{\mathbb{Q}}^p \Big[G^{L_0(u)}_{0,n} \Big((\mathbf{C}^{L_0(u), \widetilde{K}(u)})^c \Big) \Big] + \widetilde{\mathbb{Q}}^p \Big[G^{L_0(u)}_{0,n} \Big((\widehat{\mathbf{C}}^{L_0(u), \widetilde{K}(u)})^c \Big) \Big]$$

$$+ \widetilde{\mathbb{Q}}^p \Big[G^{L_0(u)}_{0,n} \Big((\mathbf{D}^{L_0(u), p})^c \Big) \Big] + \widetilde{\mathbb{Q}}^p \Big[G^{L_0(u)}_{0,n} \Big((\mathbf{E}^{L_0(u), u})^c \Big) \Big] + \widetilde{\mathbb{Q}}^p \Big[G^{L_0(u)}_{0,n} \Big((\mathbf{F}^{L_0(u), u})^c \Big) \Big] \Big\}$$

$$\leqslant 5 \cdot 4^d \cdot 2^{-2^n},$$

using (4.9), (4.10), (4.17) and (4.20) in the last step. Since this bound holds for all $n \geqslant 0$, and, in view of (4.4), $H^N_0(0; L_0(u), u, \widetilde{K}(u), p) \subset H^{2L_n}_{L_n}(0; L_0(u), u, \widetilde{K}(u), p)$ for any $n \in \mathbb{N}$ such that $2L_n \leqslant N$, one can find constants $c, C > 0$ depending only on d, u and u_0 such that, for all integers N,

$$(4.24) \qquad \widetilde{\mathbb{Q}}^p \left(H^N_0 \big(0; L_0(u), u, \widetilde{K}(u), p \big) \right) \leqslant C \exp\left(-cN^c \right).$$

Given (4.24), the argument proceeds as follows. For any set $A \subset \mathbb{Z}^2 \times \{0\}^{d-2}$, define $(x_0, \ldots, x_n, \ldots)$ to be a nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \cap \{0\}^{d-2})$ that connects A to ∞ if the $x_i \in \mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \cap \{0\}^{d-2})$ are good $(L_0(u), u, \widetilde{K}(u), p)$ vertices, $\|x_i - x_{i+1}\|_1 = L_0(u)$ for all $i \in \{0, 1, \ldots\}$, $x_0 \in A$ and $\|x_i\|_\infty \to \infty$, as $i \to \infty$. Now, assume that there exists no unbounded nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$, and in particular that for all $M \in L_0(u) \cdot \mathbb{N} \equiv \mathbb{N}_u$ there is no nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices that connects $[M, M]^2 \times \{0\}^{d-2}$ to ∞. Then by planar duality, for all $M \in \mathbb{N}_u$, there exists a $*$-path π around $[-M, M]^2 \times \{0\}^{d-2}$ in $\mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ of bad $(L_0(u), u, \widetilde{K}(u), p)$ vertices. If $N \geqslant M$ denotes the smallest multiple of $L_0(u)$ such that $x_N \equiv (N, 0) \in \mathbb{N}_u \times \{0\}^{d-1}$ is in π, then $H^N_0(x_N; L_0(u), u, \widetilde{K}(u), p)$ occurs. Thus, the probability that there is no infinite nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ that connect $[M, M]^2 \times \{0\}^{d-2}$ to ∞ is bounded by

$$\sum_{N \in \mathbb{N}_u : N \geqslant M} \widetilde{\mathbb{Q}}^p \left(H^N_0(0; L_0(u), u, \widetilde{K}(u), p) \right) \leqslant \sum_{N \in \mathbb{N}_u : N \geqslant M} C \exp\left(-cN^c \right).$$

This is true for all $M \in \mathbb{N}_u$, hence the probability of having no unbounded nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}^{L_0(u)}_0 \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ is 0. Lemma 4.3 then implies that the set $\widetilde{A}'_{u,p}$ percolates (almost surely), for any $u \in (0, u_0]$ and $p \in [p(u), 1]$, and the claim of Proposition 4.1 follows. $\qquad \square$

With Proposition 4.1 at hand, it is possible to deduce Theorem 3.1 for a good choice of coupling $\widetilde{\mathbb{Q}}^p$ in (4.1). The idea is to use (2.14), and to suitably couple the Bernoulli percolation $\mathcal{B}^p$ with $\{|\widetilde{\varphi}| \leqslant K(h)\}$ on the edges. We prove the following slightly stronger version of Theorem 3.1.

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Corollary 4.9. *For each $h_0 > 0$, there exist positive constants C_0 and c_0, only depending on d and h_0, with $C_0 h_0^{-c_0} \geqslant 1$ such that, for all $0 < h \leqslant h_0$, with $K(h)$ as in (3.1), there exists $L_0 = L_0(h) > 0$ such that $\widetilde{\mathbb{P}}^G$-a.s. the set*
(4.25)
$$\widetilde{A}_h''(\widetilde{\Phi}) \stackrel{def.}{=} \left\{ x \in \widetilde{\mathbb{Z}}^d \setminus \mathbb{Z}^d; \, \widetilde{\Phi}_x \geqslant -h \right\} \cup \left\{ x \in \mathbb{Z}^d; \, \forall v \in V_0, \, \forall t \in \left[0, \frac{1}{2}\right], \, |\widetilde{\Phi}_{x+tv}| \leqslant K(h) \right\}$$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^d$.

We first comment on why Corollary 4.9 implies Theorem 3.1. Since $\widetilde{\Phi}$ is continuous, asserting that $\widetilde{A}_h''$ has an unbounded component is tantamount to saying that there exists an infinite path in the set $\{x \in \widetilde{\mathbb{Z}}^d; \, \widetilde{\Phi}_x \geqslant -h\}$, and that in addition, for every $y \in \widetilde{\mathbb{Z}}^d$ at distance less than $\frac{1}{2}$ from a vertex on this path, $|\widetilde{\Phi}_y| \leqslant K(h)$ holds. In particular this infinite path is also in $\widetilde{A}_h$, see (3.2), and Theorem 3.1 follows.

The key to the proof of Corollary 4.9 is the following lemma, by which one can essentially couple a Bernoulli percolation $\{\mathcal{B}^p = 1\}$ on the edges with sufficiently large success parameter $p \geqslant 1 - C_1' u^{c_1'}$, cf. (4.2), with $\{|\widetilde{\varphi}| \leqslant K'\}$ on the edges for K' large enough.

Lemma 4.10. *Let $\widetilde{\varphi}$ be a Gaussian free field on the cable system under $\widetilde{\mathbb{P}}$. For all $u_0 > 0$, there exist positive constants C_0 and c_0 such that, for all $u \in (0, u_0]$, with $\widetilde{K}(u)$ and $p(u)$ as defined in (4.2), $h = \sqrt{2u}$ and $K(h)$ as defined in (3.1), the following holds: under $\widetilde{\mathbb{P}}$, there exists a family of independent Bernoulli variables $\mathcal{B}^{\widetilde{p}(u)} = (\theta_e^{\widetilde{p}(u)})_{e \in E}$ with parameter $\widetilde{p}(u) \geqslant p(u)$, and the property that*

(4.26)
$$\begin{aligned} &\text{for all } e = \{x, y\} \in E, \text{ if } |\varphi_x| \leqslant \widetilde{K}(u) \text{ and } |\varphi_y| \leqslant \widetilde{K}(u), \\ &\text{then } \left\{ \theta_e^{\widetilde{p}(u)} = 1 \Rightarrow \forall z \in \overline{I_e}, \, |\widetilde{\varphi}_z| \leqslant K(h) \right\}. \end{aligned}$$

Proof. Let $u \in (0, u_0]$ and $h = \sqrt{2u}$. With C_1, c_1, C_1' and c_1' as given by Proposition 4.1, fix constants C_0 and c_0 depending only on u_0 and d such that

(4.27)
$$\begin{aligned} K(h) &\stackrel{(3.1)}{=} \sqrt{\log\left(\frac{C_0}{(2u)^{c_0/2}}\right)} \geqslant \sqrt{\log\left(\frac{C_1}{u^{c_1}}\right)} + \sqrt{-\frac{1}{2}\log\left(\frac{C_1' u^{c_1'}}{2}\right)} \\ &\stackrel{(4.2)}{=} \widetilde{K}(u) + \sqrt{-\frac{1}{2}\log\left(\frac{1 - p(u)}{2}\right)}. \end{aligned}$$

Let $(B^e)_{e \in E}$ be as defined in (2.8), and recall that $(B^e)_{e \in E}$ is an i.i.d. family of Brownian bridges with length $\frac{1}{2}$ of a Brownian motion with variance 2 at time 1. For all $e \in E$, define

(4.28)
$$\theta_e^{\widetilde{p}(u)} = \begin{cases} 1, & \text{if } |B_t^e| \leqslant K(h) - \widetilde{K}(u) \text{ for all } t \in \left[0, \frac{1}{2}\right], \\ 0, & \text{otherwise.} \end{cases}$$

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Then $\left(\theta_e^{\widetilde{p}(u)}\right)_{e\in E}$ is an i.i.d. family of Bernoulli variables with parameter

$$\widetilde{p}(u) \overset{\text{def.}}{=} \widetilde{\mathbb{P}}\left(\forall t \in [0, 1/2],\ |B_t^e| \leqslant K(h) - \widetilde{K}(u)\right).$$

Moreover, by symmetry (the boundary values of B^e are both 0) and Lemma 2.1,

$$\widetilde{p}(u) \geqslant 1 - 2\widetilde{\mathbb{P}}\left(\sup_{t\in[0,1/2]} B_t^e \geqslant K(h) - \widetilde{K}(u)\right) \overset{(2.9)}{\geqslant} 1 - 2\exp\left(-2(K(h) - \widetilde{K}(u))^2\right) \overset{(4.27)}{\geqslant} p(u),$$

and, using (2.8) and (4.28), for all $e = \{x, y\} \in E$ such that $\varphi_x \leqslant \widetilde{K}(u)$ and $\varphi_y \leqslant \widetilde{K}(u)$,

$$\theta_e^{\widetilde{p}(u)} = 1 \quad \Rightarrow \quad \forall t \in [0, 1/2],\ |\widetilde{\varphi}_{x+tv_{(x,y)}} - (1 - 2t)\varphi_x - 2t\varphi_y| \leqslant K(h) - \widetilde{K}(u)$$
$$\Rightarrow \quad \forall z \in \overline{I_e},\ |\widetilde{\varphi}_z| \leqslant K(h),$$

whence (4.26). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Corollary 4.9. Let $u_0 = \frac{h_0^2}{2}$, and, for any $h \in (0, h_0]$, define $u = \frac{h^2}{2}$. Let $\widetilde{\mathbb{P}}^u$ be the coupling from Theorem 2.2, under which there exist a Gaussian free field $\widetilde{\varphi}$ and a random interlacement process $\widetilde{\omega}$ such that (2.15) holds. For this $\widetilde{\varphi}$, let $\mathcal{B}^{\widetilde{p}(u)} = \mathcal{B}^{\widetilde{p}(u)}(\widetilde{\varphi})$ be the family of independent Bernoulli variables under $\widetilde{\mathbb{P}}^u$ introduced in Lemma 4.10. This yields a coupling $\widetilde{\mathbb{Q}}^{\widetilde{p}(u)}$ satisfying (4.1), with parameter $\widetilde{p}(u) \geqslant p(u)$. One can now apply Proposition 4.1 to obtain that, $\widetilde{\mathbb{P}}^u$-a.s., the set $\widetilde{A}'_{u,\widetilde{p}(u)}$, cf. (4.3), contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, L_0(u))^d$, and thus (4.26) yields that $\widetilde{\mathbb{P}}^u$-a.s. the set

$$(4.29) \qquad \left(\widetilde{\mathcal{I}}^u \cap (\widetilde{\mathbb{Z}}^d \setminus \mathbb{Z}^d)\right) \cup \left\{x \in \mathbb{Z}^d;\ \forall v \in V_0,\ \forall t \in [0, 1/2],\ |\widetilde{\varphi}_{x+tv}| \leqslant K(h)\right\}$$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, L_0(u))^d$. Now (2.15) implies that the set defined in (4.29) is included in $\widetilde{A}''_h(\widetilde{\varphi})$, and Corollary 4.9 follows. $\quad\square$

5 Percolation for positive level set

In this section, we prove our main result, Theorem 1.1, with the help of Theorem 3.1. A key ingredient is the following observation: we have shown (in Theorem 3.1) that the set $\{x \in \widetilde{\mathbb{Z}}^d;\ -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$ contains an unbounded connected component for large enough $K(h)$, cf. (3.1). Suppose that $x \in \mathbb{Z}^d$ is a vertex inside this unbounded component, and that I_e is attached to x (recall that $\Phi = \widetilde{\Phi} \restriction \mathbb{Z}^d$). Then, since $\widetilde{\Phi}$ behaves like a Brownian bridge on $\overline{I_e}$, see (2.8), the probability that $\widetilde{\Phi}_z \geqslant -h$ for all $z \in I_e$ becomes very small as $h \searrow 0$. In fact, if $e = \{x, y\}$, for sufficiently small $h > 0$, it is more costly to keep $\widetilde{\Phi} \geqslant -h$ along the entire cable $\overline{I_e}$, than to require $\Phi_x \geqslant h$ (*at the vertex x only!*), knowing that $-h \leqslant \Phi_x \leqslant K(h)$ and $|\Phi_y| \leqslant K(h)$, see Lemma 5.1 for the corresponding

statement. Accordingly, the probability that the set $\{x \in \widetilde{\mathbb{Z}}^d; -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$ contains an unbounded connected component becomes smaller than the probability that the set $\{x \in \mathbb{Z}^d; h \leqslant \Phi_x \leqslant K(h)\}$ contains an infinite cluster (in $\mathbb{Z}^d$) as h goes to 0, which implies Theorem 1.1. The precise formulation comes through a slightly abstract coupling argument, which is the content of Lemma 5.2.

Comparing the probability that $\Phi_x \geqslant h$ knowing that $-h \leqslant \Phi_x \leqslant K(h)$ with the probability that the Brownian bridge on I_e remains above level $-h$ in a uniform way requires some control on the Gaussian free field $\widetilde{\Phi}$ in the neighborhood of x, and for this purpose we are actually going to use Corollary 4.9 instead of Theorem 3.1. We define, for $x \in \mathbb{Z}^d$ and $v \in V_0$, the open subsets $U^{x,v}$ and U^x of $\widetilde{\mathbb{Z}}^d$ by

$$(5.1) \qquad U^{x,v} = x + \left[0, \frac{1}{4}v\right) \text{ and } U^x = \bigcup_{v \in V_0} U^{x,v} = x + \bigcup_{v \in V_0} \left[0, \frac{1}{4}v\right).$$

We call $\mathcal{K}^x \equiv \partial U^x$ the boundary of U^x, which has exactly $2d$ elements, and define $\mathcal{K} = \bigcup_{x \in \mathbb{Z}^d} \mathcal{K}^x$. Henceforth, we set

$$(5.2) \qquad \qquad \qquad h_0 = 1$$

in all the previous definitions and results, and in particular in Corollary 4.9 (this value is chosen arbitrarily in $(0, \infty)$). For any $h \in (0, 1]$, we define $K(h)$ as in (3.1) (with c_0, C_0 numerical constants depending only on d by the choice (5.2)). We further define two families of events $(E_h^x)_{x \in \mathbb{Z}^d}$ and $(F_h^{x,v})_{x \in \mathbb{Z}^d, v \in V_0}$ (part of Ω_0, cf. above (2.2)) by

$$(5.3)$$
$$E_h^{x,v} = \left\{\widetilde{\Phi}_{x+\frac{1}{4}v} \geqslant -h\right\} \cap \left\{\forall y \in \mathcal{K}^x; |\widetilde{\Phi}_y| \leqslant K(h)\right\} \text{ and } F_h^{x,v} = \left\{\forall z \in U^{x,v}; \widetilde{\Phi}_z \geqslant -h\right\},$$

as well as

$$(5.4) \qquad \qquad E_h^x = \bigcup_{v \in V_0} E_h^{x,v} \text{ and } G_h^x = \bigcup_{v \in V_0} \left(E_h^{x,v} \cap F_h^{x,v}\right),$$

and the (random) subsets of $\mathbb{Z}^d$

$$(5.5) \qquad \qquad E_h = \{x \in \mathbb{Z}^d; E_h^x \text{ occurs}\} \text{ and } G_h = \{x \in \mathbb{Z}^d; G_h^x \text{ occurs}\}.$$

For all $K \subset \widetilde{\mathbb{Z}}^d$, we denote by $\mathcal{A}_K$ the σ-algebra $\sigma\left(\widetilde{\varphi}_z, z \in K\right)$. We note that the sets U^x are disjoint when x varies, cf. (5.1) and (2.1), and that the events $E_h^{x,v}$ are $\mathcal{A}_{\mathcal{K}^x}$-measurable. Corollary 4.9 implies that G_h contains an infinite connected component, and the goal is to go from this to the percolation of $E_h \cap \{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$. The following lemma makes the above observation, see the discussion at the beginning of this section, precise.

Lemma 5.1. *There exists $h_1 \in (0, 1]$ such that for all $h \in (0, h_1]$ and $x \in \mathbb{Z}^d$,*

$$(5.6) \qquad \qquad \widetilde{\mathbb{P}}^G\left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \leqslant \widetilde{\mathbb{P}}^G\left(E_h^x \cap \{\Phi_x \geqslant h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right).$$

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Proof. Let us fix some $x \in \mathbb{Z}^d$. It is sufficient to prove that there exists $h_1 \in (0,1]$ such that for all $h \in (0, h_1]$ and all $v \in V_0$,

$$(5.7) \qquad 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right) \leqslant \frac{1}{2d} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right).$$

Indeed, if (5.7) holds, then

$$\widetilde{\mathbb{P}}^G \left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x} \right) = \widetilde{\mathbb{P}}^G \left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right) + \widetilde{\mathbb{P}}^G \left(G_h^x \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right)$$

$$\overset{(5.4)}{\leqslant} \widetilde{\mathbb{P}}^G \left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right) + \sum_{v \in V_0} \widetilde{\mathbb{P}}^G \left(E_h^{x,v} \cap F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right)$$

$$\overset{(5.7)}{\leqslant} \widetilde{\mathbb{P}}^G \left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right) + \frac{1}{2d} \sum_{v \in V_0} \widetilde{\mathbb{P}}^G \left(E_h^{x,v} \cap \{h \leqslant \Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right)$$

$$\leqslant \widetilde{\mathbb{P}}^G \left(E_h^x \cap \{\Phi_x \geqslant h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right),$$

noting that $G_h^x, E_h^{x,v} \subset E_h^x$ in the last inequality, and (5.6) follows. We now show (5.7). Let us fix some $v \in V_0$. We begin with the study of Φ_x, by decomposing it suitably. It follows from the Markov property, cf. (2.3), that $\widetilde{\Phi}_x^{U^x} = \Phi_x - \widetilde{\beta}_x^{U^x}$ is a centered Gaussian variable with variance $g_{U^x}(x,x)$. The value of the variance $g_{U^x}(x,x) \equiv \sigma_0^2$ does not depend on $x \in \mathbb{Z}^d$ (it actually follows from Section 2 of [10] that $\sigma_0^2 = \frac{1}{4d}$). Moreover, on the event $E_h^{x,v}$, it is clear that $|\widetilde{\beta}_x^{U^x}| \leqslant K(h)$. Thus, on the event $E_h^{x,v}$, since the harmonic average $\widetilde{\beta}_x^{U^x}$ is $\mathcal{A}_{\mathcal{K}^x}$-measurable, we obtain, for all $h > 0$,

$$\widetilde{\mathbb{P}}^G \left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right) = \widetilde{\mathbb{P}}^G \left(-h \leqslant \widetilde{\Phi}_x^{U^x} + \widetilde{\beta}_x^{U^x} \leqslant 2h \,\Big|\, \mathcal{A}_{\mathcal{K}^x} \right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}} \int_{-h}^{2h} \exp\left(-\frac{\left(y - \widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2} \right) \mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2} \right) \int_{-h}^{2h} \exp\left(-\frac{y^2}{2\sigma_0^2} \right) \exp\left(\frac{y \widetilde{\beta}_x^{U^x}}{\sigma_0^2} \right) \mathrm{d}y$$

$$(5.8) \qquad \leqslant \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2} \right) \times 3h \exp\left(\frac{2hK(h)}{\sigma_0^2} \right).$$

A similar calculation shows that on the event $E_h^{x,v}$, for $h > 0$,

$$(5.9) \quad \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right) \geqslant \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2} \right) \times h \exp\left(-\frac{2h(K(h)+h)}{\sigma_0^2} \right).$$

Define
$$C_6(d) = \sup_{h \in (0,1]} \left\{ 3 \exp\left(\frac{2hK(h)}{\sigma_0^2} \right) \times \exp\left(\frac{2h(K(h)+h)}{\sigma_0^2} \right) \right\},$$

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and note that $C_6 < \infty$ since $hK(h) \to 0$ as $h \to 0$, cf. (3.1). Hence, by (5.8) and (5.9),

$$(5.10) \quad 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right) \leqslant C_6 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right), \text{ for } h \in (0,1].$$

Let us now turn to the events $F_h^{x,v}$. It follows again from the Markov property for the Gaussian free field, see in particular the discussion below (2.7), that, knowing $\mathcal{A}_{\mathcal{K}^x \cup \{x\}}$, the process $(\widetilde{\Phi}_z)_{z \in U^{x,v}}$ is a Brownian bridge of length $\frac{1}{4}$ between Φ_x and $\widetilde{\Phi}_{x+\frac{1}{4}v}$ of a Brownian motion with variance 2 at time 1. Using Lemma 2.1, one can then find $h_1 \in (0,1]$ such that, for all $h \in (0, h_1]$, on the event $E_h^{x,v} \cap \{-h \leqslant \Phi_x \leqslant 2h\}$,

$$\widetilde{\mathbb{P}}^G \left(F_h^{x,v} \,|\, \mathcal{A}_{\mathcal{K}^x \cup \{x\}} \right) = \widetilde{\mathbb{P}}^G \left(\min_{z \in U^{x,v}} \widetilde{\Phi}_z \geqslant -h \,\Big|\, \mathcal{A}_{\mathcal{K}^x \cup \{x\}} \right)$$

$$= 1 - \exp\left(-4(h + \Phi_x)(h + \widetilde{\Phi}_{x+\frac{1}{4}v}) \right)$$

$$\leqslant 1 - \exp\left(-12h(K(h) + h) \right)$$

$$(5.11) \qquad\qquad\qquad \leqslant \frac{1}{2dC_6}.$$

We now conclude using (5.10) and (5.11): for all $h \in (0, h_1]$,

$$1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x} \right) \overset{(5.3)}{=} \mathbb{E}\left[1_{E_h^{x,v} \cap \{-h \leqslant \Phi_x \leqslant 2h\}} \widetilde{\mathbb{P}}^G \left(F_h^{x,v} \,|\, \mathcal{A}_{\mathcal{K}^x \cup \{x\}} \right) \,\Big|\, \mathcal{A}_{\mathcal{K}^x} \right]$$

$$\overset{(5.11)}{\leqslant} \frac{1}{2dC_6} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right)$$

$$\overset{(5.10)}{\leqslant} \frac{1}{2d} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x} \right),$$

which is (5.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.1 roughly asserts that it is more likely to have $\{\Phi_x \geqslant h\}$ than to have G_h^x (on E_h^x) and we know by Corollary 4.9 that G_h has an infinite connected component. The proof of Theorem 1.1 now hinges on the following result.

Lemma 5.2. *Let $(E_i, \mathcal{E}_i)$ for $i \in \{1, 2, 3\}$ be three measurable spaces, let X, Y and Z be three independent random variables under $\mathbb{P}$ with values in E_1, E_2 and E_3 respectively, and let $A \in \mathcal{E}_1 \otimes \mathcal{E}_3$ and $B \in \mathcal{E}_2 \otimes \mathcal{E}_3$ be such that*

$$(5.12) \qquad\qquad \mathbb{P}\big((X, Z) \in A \,|\, Z \big) = \mathbb{P}\big((Y, Z) \in B \,|\, Z \big), \ \mathbb{P}\text{-}a.s.$$

Then, on an extended probability space, there exists a random variable $\widetilde{Y}$ independent of Z and with the same law as Y under $\mathbb{P}$ such that

$$(5.13) \qquad\qquad (X, Z) \in A \quad \Leftrightarrow \quad (\widetilde{Y}, Z) \in B, \ \mathbb{P}\text{-}a.s.$$

N.B.: with a slight abuse of notation, the probability measure on the extended space is still denoted by $\mathbb{P}$ in (5.13). We will use this notation throughout the proof.

Proof. We may assume that $\mathbb{P}\left((X, Z) \in A\right) \in (0, 1)$, otherwise we simply take $\widetilde{Y} = Y$. By suitably extending the probability space on which $\mathbb{P}$ is defined, we introduce a family of independent E_2-valued random variables $\widetilde{Y}_z^i$, $i = 0, 1$ and $z \in E_3$, independent of X, Y and Z such that for each $z \in E_3$ and $C \in \mathcal{E}_2$,

$$\mathbb{P}\left(\widetilde{Y}_z^0 \in C\right) = \mathbb{P}\left(Y \in C \,\big|\, (Y, z) \notin B\right) \quad \text{and} \quad \mathbb{P}\left(\widetilde{Y}_z^1 \in C\right) = \mathbb{P}\left(Y \in C \,\big|\, (Y, z) \in B\right)$$

(if $\mathbb{P}((Y, z) \notin B) = 0$ we just fix an arbitrary value, say 0, for $\widetilde{Y}_z^0$, and if $\mathbb{P}((Y, z) \in B) = 0$ we do the same for $\widetilde{Y}_z^1$). Let $\mathbb{P}_Z$ be the law of Z, then, for $\mathbb{P}_Z$-almost all $z \in E_3$, $(\widetilde{Y}_z^0, z) \notin B$ and $(\widetilde{Y}_z^1, z) \in B$, $\mathbb{P}$-a.s. Define $p = 1_{\{(X, Z) \in A\}}$, and $\widetilde{Y} = \widetilde{Y}_Z^p$. We have that for all $F_2 \in \mathcal{E}_2$ and $F_3 \in \mathcal{E}_3$,

$$\begin{aligned}
\mathbb{P}\left((\widetilde{Y}, Z) \in F_2 \times F_3\right) &= \int_{F_3} \mathbb{P}\left(\widetilde{Y}_z^p \in F_2\right) \mathrm{d}\mathbb{P}_Z(z) \\
&= \int_{F_3} \mathbb{P}\left(\widetilde{Y}_z^0 \in F_2\right) \mathbb{P}\left((X, z) \notin A\right) + \mathbb{P}\left(\widetilde{Y}_z^1 \in F_2\right) \mathbb{P}\left((X, z) \in A\right) \mathrm{d}\mathbb{P}_Z(z) \\
&= \int_{F_3} \mathbb{P}\left(Y \in F_2, (Y, z) \notin B\right) + \mathbb{P}\left(Y \in F_2, (Y, z) \in B\right) \mathrm{d}\mathbb{P}_Z(z) \\
&= \mathbb{P}\left(Y \in F_2\right) \mathbb{P}\left(Z \in F_3\right),
\end{aligned}$$

where we used (5.12) in the third equality. Thus, $\widetilde{Y}$ is independent from Z and has the same law as Y. Moreover, for $\mathbb{P}_Z$ almost all $z \in E_3$,

$$(X, z) \in A \quad \Leftrightarrow \quad p = 1 \quad \Leftrightarrow \quad (\widetilde{Y}_z^p, z) \in B, \ \mathbb{P}\text{-a.s.}$$

and this gives (5.13). $\qquad\square$

We now proceed to the

Proof of Theorem 1.1. Let $h \in (0, h_1]$, with h_1 as in Lemma 5.1. We will show that $E^{\geqslant h}(\widetilde{\Phi}^+)$, cf. (1.3), where $\widetilde{\Phi}^+$ is a Gaussian free field, see (5.16) below, percolates in a sufficiently thick slab for any such value of h, thus obtaining Theorem 1.1. The Markov property for the Gaussian free field, see (2.4), (5.1) and (2.1), implies that the family $(\widetilde{\Phi}_\cdot^{U^x})_{x \in \mathbb{Z}^d}$ is i.i.d. and independent of $\mathcal{A}_\mathcal{K}$, and that for all $x \in \mathbb{Z}^d$ and $z \in U^x$,

$$\widetilde{\Phi}_z = \widetilde{\beta}_z^{U^x} + \widetilde{\Phi}_z^{U^x}$$

where $(\widetilde{\beta}_\cdot^{U^x})_{x \in \mathbb{Z}^d}$ is $\mathcal{A}_\mathcal{K}$-measurable. Let $(\lambda_x)_{x \in \mathbb{Z}^d}$ be an i.i.d. family of random variables independent from $\widetilde{\Phi}$ under $\widetilde{\mathbb{P}}^G$ (tacitly assumed to be suitably extended), with common law $\mathrm{Unif}([0, 1])$. Note that for all $x \in \mathbb{Z}^d$,

$$\begin{aligned}
\widetilde{\mathbb{P}}^G &\left(\lambda_x \leqslant \widetilde{\mathbb{P}}^G\left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \,\Big|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&= \widetilde{\mathbb{P}}^G\left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \overset{(5.4)}{=} \widetilde{\mathbb{P}}^G\left(\bigcup_{v \in V_0} E_h^{x, v} \cap \left\{\forall z \in U^{x, v}; \ \widetilde{\beta}_z^{U_x} + \widetilde{\Phi}_z^{U^x} \geqslant -h\right\} \,\Big|\, \mathcal{A}_{\mathcal{K}^x}\right),
\end{aligned}$$

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where $E_h^{x,v} \in \mathcal{A}_{\mathcal{K}^x}$ for all $v \in V_0$. For each $x \in \mathbb{Z}^d$, we can now use Lemma 5.2 with $X = \lambda_x$, $Y = \widetilde{\Phi}_{\cdot}^{U^x}$ and $Z = \widetilde{\beta}_{\cdot}^{U^x}$, to find an i.i.d. family of random variables $(\widetilde{\Phi}_{\cdot}^{-,U^x})_{x \in \mathbb{Z}^d}$ independent of $\mathcal{A}_{\mathcal{K}^x}$ and with the same law as $(\widetilde{\Phi}_{\cdot}^{U^x})_{x \in \mathbb{Z}^d}$ (note that the latter are independent, as x varies), such that, for each $x \in \mathbb{Z}^d$,

$$(5.14) \qquad \left\{ \lambda_x \leqslant \widetilde{\mathbb{P}}^G \left(G_h^x \mid \mathcal{A}_{\mathcal{K}^x} \right) \right\} = \bigcup_{v \in V_0} E_h^{x,v} \cap \left\{ \forall z \in U^{x,v}; \, \widetilde{\beta}_z^{U_x} + \widetilde{\Phi}_z^{-,U^x} \geqslant -h \right\}.$$

Similarly, we find an i.i.d. family of random variables $(\widetilde{\Phi}_{\cdot}^{+,U^x})_{x \in \mathbb{Z}^d}$ independent of $\mathcal{A}_{\mathcal{K}^x}$ and with the same law as $(\widetilde{\Phi}_{\cdot}^{U^x})_{x \in \mathbb{Z}^d}$, such that, for each $x \in \mathbb{Z}^d$,

$$(5.15) \qquad \left\{ \lambda_x \leqslant \widetilde{\mathbb{P}}^G \left(E_h^x \cap \{ \Phi_x \geqslant h \} \mid \mathcal{A}_{\mathcal{K}^x} \right) \right\} = E_h^x \cap \left\{ \widetilde{\beta}_x^{U_x} + \widetilde{\Phi}_x^{+,U^x} \geqslant h \right\}.$$

We now define, for all $x \in \mathbb{Z}^d$ and $z \in U^x \cup \mathcal{K}^x$,

$$(5.16) \qquad \widetilde{\Phi}_z^- = \widetilde{\beta}_z^{U^x} + \widetilde{\Phi}_z^{-,U^x} \quad \text{and} \quad \widetilde{\Phi}_z^+ = \widetilde{\beta}_z^{U^x} + \widetilde{\Phi}_z^{+,U^x}.$$

Then $\widetilde{\Phi}^-$ and $\widetilde{\Phi}^+$ have the same law as $\widetilde{\Phi}$ and are thus Gaussian free fields on the cable system. Using Lemma 5.1, (5.14) and (5.15), we obtain that for all $x \in \mathbb{Z}^d$,

$$(5.17) \qquad\qquad G_h^{-,x} \subset E_h^{+,x} \cap \{ \widetilde{\Phi}_x^+ \geqslant h \}$$

and in particular

$$(5.18) \qquad\qquad G_h^- \subset E_h^+ \cap \{ x \in \mathbb{Z}^d; \, \widetilde{\Phi}_x^+ \geqslant h \}$$

where $G_h^{-,x}$ and G_h^- (resp. $E_h^{+,x}$ and E_h^+), are defined as in (5.4) and (5.5), but for the Gaussian free field $\widetilde{\Phi}^-$ (resp. $\widetilde{\Phi}^+$). In deducing (5.17), we have also used that $E_h^{+,x}(\widetilde{\Phi}^+) = E_h^x(\widetilde{\Phi})$, since both events are $\mathcal{A}_{\mathcal{K}^x}$-measurable, see (5.3), and thus independent of $\widetilde{\Phi}_{\cdot}^{U^x}$ and $\widetilde{\Phi}_{\cdot}^{+,U^x}$.

Finally, by Corollary 4.9, there exists $L_0(h) > 0$ such that the set $\widetilde{A}_h''(\widetilde{\Phi}^-)$, cf. (4.25), contains a.s. an unbounded connected component M in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$. By definition, see (5.5), $M \cap \mathbb{Z}^d \subset G_h^-$, and thus, on account of (5.18), $M \cap \mathbb{Z}^d \subset \{ x \in \mathbb{Z}^d; \, \widetilde{\Phi}_x^+ \geqslant h \}$, hence $\{ x \in \mathbb{Z}^d; \, \widetilde{\Phi}_x^+ \geqslant h \}$ contains a.s. an infinite connected component in $\mathbb{Z}^2 \times [0, 2L_0)^{d-2}$. $\qquad \square$

Remark 5.3. 1) The result of [20] is actually slightly better than Theorem 1.1 in high dimensions: if d is large enough, there exist $h_2 = h_2(d) > 0$ and $L_0 = L_0(d) \geqslant 1$ such that the level set $\{ x \in \mathbb{Z}^d; \, \Phi_x \geqslant h_2 \}$ percolates in the slab $\mathbb{Z}^2 \times [0, 2L_0) \times \{0\}^{d-3}$. However, in all dimensions $d \geqslant 3$, the set $\{ x \in \mathbb{Z}^d; \, \Phi_x \geqslant h \}$ never percolates for $h \geqslant 0$ in $\mathbb{Z}^2 \times \{0\}^{d-2}$, as explained in Remark 3.6.1 of [20].

2) It is possible to get a result similar to Theorem 3.1 for the positive level set of the Gaussian free field Φ on $\mathbb{Z}^d$ just constructed, thus obtaining the following strenghtening of

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Theorem 1.1. For all $h \leqslant h_1$, let $K(h)$ be as in (3.1) for $h_0 = 1$, then the set $\{x \in \mathbb{Z}^d; h \leqslant \Phi_x \leqslant K(h)\}$ contains a.s. an infinite connected component. Indeed, using an argument similar to that of Lemma 5.1, one can prove that, conditionally on $\mathcal{A}_{\mathcal{K}^x}$, the probability of $G_h^x \cap \{\Phi_x \leqslant K(h)\}$ is smaller than the probability of $E_h^x \cap \{h \leqslant \Phi_x \leqslant K(h)\}$ and the result follows.

3) Theorem 2 in [16] can also easily be extended to the Gaussian free field: for each $h \leqslant h_1$, the set $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$ contains an almost surely transient component. Indeed, looking at the proof of Theorem 2 in [16], see also Theorem 1 in [15], we can use (4.24) instead of (5.1) in [16] to obtain that the set $\widetilde{A}'_{u,p}$ defined in (4.3) contains an unbounded connected and transient component for $p \in [p(u), 1]$. Using the same coupling as in Lemma 4.10, we get that this is also true for the set $\widetilde{A}''_h$ defined in (4.25), and the coupling of Lemma 5.2 tells us that $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$ also contains an infinite connected and transient component for $h \leqslant h_1$.

4) Another parameter $\overline{h} \leqslant h_*$ has been introduced in [6], and a similar one has been used in [26]. This parameter describes a strong percolative regime for $E^{\geqslant h}$, when $h < \overline{h}$, i.e. all connected components of $E^{\geqslant h}$ in $[-R, R]^d$ with diameter at least $\frac{R}{10}$ are connected in $[-2R, 2R]^d$ with large enough probability when R goes to ∞. It has been proved that $\overline{h} > -\infty$ and it is believed that actually $\overline{h} = h_*$, but it is still unknown whether $\overline{h} \geqslant 0$ or not. Our methods may perhaps help in that regard.

Appendix: Proof of Lemma 3.2

The proof of Lemma 3.2 is very close to the proof of Proposition 1 in [15], but we need to remove the dependence on u of the constants, and make the dependence on u of the error term explicit instead. We will henceforth refer to [15] whenever possible, and in particular, Lemmas 3 to 6 and 11 in [15] do not involve u at all, so we will use them without proof. Recall ω^u, the interlacement process on $\mathbb{Z}^d$, and $\widetilde{\omega}^u$, the interlacement process on the cable system, obtained from ω^u by adding independent Brownian excursions as explained below (2.11). We denote by $\widehat{\mathcal{I}}^u$ the set of edges traversed by at least one of the trajectories in $\text{supp}(\omega^u)$. Now observe that the event that every x and y in $\widetilde{\mathcal{I}}^u \cap [0, R)^d$ be connected in $\widetilde{\mathcal{I}}^u \cap [-\varepsilon R, (1 + \varepsilon)R)^d$, which is the event of interest in (3.3), is more likely than every x and y in $\widehat{\mathcal{I}}^u \cap [-1, R+1)^d$ being connected in $\widehat{\mathcal{I}}^u \cap [-\varepsilon R, (1 + \varepsilon)R)^d$. Thus, we only need to show the respective statement of Lemma 3.2 for $\widehat{\mathcal{I}}^u$ instead of $\widetilde{\mathcal{I}}^u$, cf. Lemma A.5 below.

The idea of the proof is to show that there exists $C \geqslant 1$ such that for every integer $R \geqslant 1$ and every $x, y \in \mathcal{I}^u \cap [-R, R)^d$, the vertices x and y are connected through edges in $\widehat{\mathcal{I}}^u \cap [-CR, CR)^d$ with high enough probability. It is quite hard to directly link x and y, especially if R is large. Therefore, let us define $\omega_{i,3}^{u/3}$ for $i \in \{1, 2, 3\}$, three independent Poisson point process with the same law as $\omega^{u/3}$, such that $\omega^u = \sum_{i=1}^3 \omega_{i,3}^{u/3}$. Let us call $\mathcal{I}_{i,3}^{u/3}$ the set of vertices visited by at least one of the trajectories from $\text{supp}(\omega_{i,3}^{u/3})$, denote by $\widehat{\mathcal{I}}_{i,3}^{u/3}$ the set of edges traversed by at least one of the trajectories from $\text{supp}(\omega_{i,3}^{u/3})$,

and let $C_i^{u/3}(z, R)$ be the set of vertices connected to z by edges in $\widehat{\mathcal{I}}_{i,3}^{u/3} \cap [-R, R)^d$ for $i \in \{1, 2, 3\}$ and $z \in \mathbb{Z}^d$. We are going to prove that, if $x \in \mathcal{I}_{1,3}^{u/3}$ and $y \in \mathcal{I}_{2,3}^{u/3}$, then $C_1^{u/3}(x, R)$ and $C_2^{u/3}(y, R)$ are big enough, and that one can connect these two sets by edges in $\widehat{\mathcal{I}}_{3,3}^{u/3} \cap [-CR, CR)^d$ with high probability. In particular, this will imply that x and y are connected through edges in $\widehat{\mathcal{I}}^u \cap [-CR, CR)^d$ with high probability.

We first recall a property of the Poisson distribution (see for example (2.11) in [15]): let N be a random variable which has Poisson distribution with parameter λ, then there exist constants $c < 1$ and $C > 1$ independent of λ such that

$$\mathbb{P}\left(c\lambda \leqslant N \leqslant C\lambda\right) \geqslant 1 - C \exp\left(-c\lambda\right).$$

For $A \subset \mathbb{Z}^d$ finite let N_A^u be the number of trajectories in $\operatorname{supp}(\omega^u)$ which enter A, and write $X_1, \ldots, X_{N_A^u}$ for the corresponding trajectories, parametrized such that $X_i(0) \in A$ and $X_i(-n) \notin A$ for all $n > 0$. Note that $X_1, \ldots, X_{N_A^u}$ depend on u and A even if this is only implicit in the notation. Then N_A^u is a Poisson variable with parameter $u\mathrm{cap}(A)$ and

(A.1) $\quad\quad\quad \mathbb{P}^I\left(cu\mathrm{cap}(A) \leqslant N_A^u \leqslant Cu\mathrm{cap}(A)\right) \geqslant 1 - C\exp\left(-cu\mathrm{cap}(A)\right).$

Here, $\mathrm{cap}(A)$ is the capacity of the set A, i.e., the total mass of the equilibrium measure of A. The following standard bounds will soon prove to be useful: For any $A \subset [-R, R)^d \cap \mathbb{Z}^d$ and $R \geqslant 1$,

(A.2) $\quad\quad \mathrm{cap}(A) \leqslant \mathrm{cap}([-R, R)^d) \leqslant CR^{d-2}$ and $\mathrm{cap}([-R, R)^d) \geqslant cR^{d-2}.$

The next lemma gives a bound on the probability to connect the two sets $C_1^{u/3}(x, R)$ and $C_2^{u/3}(y, R)$ in $\widehat{\mathcal{I}}_3^{u/3} \cap [-CR, CR)^d$ in terms of capacity.

Lemma A.1. *There exist constants $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $R > 0$ and $u > 0$, for all subsets U and V of $[-R, R)^d$,*

$$\mathbb{P}^I\left(U \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} V\right) \geqslant 1 - C\exp\left(-cR^{2-d}u\mathrm{cap}(U)\mathrm{cap}(V)\right).$$

Proof. If there is a trajectory among $(X_1, \ldots, X_{N_U^u})$, which hits V after 0 and before leaving $[-CR, CR)^d$, then U is connected to V through edges of $\widehat{\mathcal{I}}^u \cap [-CR, CR)^d$. We can use Lemma 11 in [15] to lower bound the probability of a trajectory to behave accordingly by $cR^{2-d}\mathrm{cap}(V)$, and thus we infer

$$\mathbb{P}^I\left(U \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} V\right) \geqslant 1 - \mathbb{P}^I(N_U^u < cu\mathrm{cap}(U)) - \left(1 - cR^{2-d}\mathrm{cap}(V)\right)^{cu\mathrm{cap}(U)}$$

$$\overset{(A.1),\,(A.2)}{\geqslant} 1 - C\exp\left(-cR^{2-d}u\mathrm{cap}(U)\mathrm{cap}(V)\right).$$

$\square$

We are now going to prove that $\mathrm{cap}\big(C_1^{u/3}(x,R)\big)$ and $\mathrm{cap}\big(C_2^{u/3}(y,R)\big)$ are large enough with high probability, and in particular that they grow faster in R than $R^{\frac{d-2}{2}}$. For all $u > 0$, $A \subset \mathbb{Z}^d$ finite and T a positive integer, we define the set $\Psi(u,A,T)$ by

$$\text{(A.3)} \qquad \Psi(u,A,T) = \bigcup_{i=1}^{N_A^u} \{X_i(n),\ 0 \leqslant n \leqslant T\}.$$

Lemma A.2. *For all $\varepsilon \in (0,1)$, $k \geqslant 1$ and $\delta \geqslant \varepsilon$, there exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0, u_0]$, $A \subset \mathbb{Z}^d$ finite and T a positive integer,*

$$\text{(A.4)} \qquad \begin{aligned} \mathbb{P}^I\left(\mathrm{cap}\left(\Psi\left(u,A,T\right)\right) \geqslant c \min\left(u\mathrm{cap}(A)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right) \\ \geqslant 1 - C \exp\left(-c \min\left(T^{\varepsilon/2}, u\mathrm{cap}(A)\right)\right), \end{aligned}$$

and, if $A \subset B = [-kT^\delta, kT^\delta)$,

$$\text{(A.5)} \qquad \mathbb{P}^I\left(\Psi(u,A,T) \subset B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d\right) \geqslant 1 - C \exp\left(-cT^\varepsilon u\right).$$

Proof. (A.4) is a simple consequence of Lemma 6 in [15] and (A.1). In order to prove (A.5), let us first define $h(T,\varepsilon)$, the probability that the simple random walk on $\mathbb{Z}^d$ beginning in 0 leaves $[-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d$ before time T. Hoeffding's inequality yields that $h(T,\varepsilon) \leqslant C \exp(-cT^\varepsilon)$. Moreover, taking $B = [-kT^\delta, kT^\delta)$,

$$\begin{aligned} \mathbb{P}^I\left(\Psi(u,A,T) \subset B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d\right) &\geqslant \mathbb{P}^I\left(\Psi(u,B,T) \subset B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d\right) \\ &\geqslant 1 - \mathbb{P}^I\left(N_B^u \geqslant Cu\mathrm{cap}(B)\right) - Cu\mathrm{cap}(B)h(T,\varepsilon) \\ &\overset{(A.1),(A.2)}{\geqslant} 1 - C \exp\left(-cuT^{(d-2)\delta}\right) - CuT^{(d-2)\delta}\exp\left(-cT^\varepsilon\right) \\ &\geqslant 1 - C \exp\left(-cT^\varepsilon u\right). \end{aligned}$$

$\square$

We now iterate this process to find the desired bound on $\mathrm{cap}\left(C^u(z,R)\right)$. Consider a sequence of independent random interlacement processes $(\omega_k)_{k \geqslant 1}$ at level u which define an independent sequence $(\Psi_k)_{k \geqslant 2}$ such that for all $k \geqslant 2$, Ψ_k has the same law as Ψ (see (A.3) for notation). For each $z \in \mathbb{Z}^d$, let X^z be the trajectory with the smallest label u contained in ω_1 such that $X^z(0) = z$. For all $z \in \mathbb{Z}^d$, $u > 0$ and T positive integer, we recursively define a sequence of subsets $(U_u^{(k)}(z,T))_{k \geqslant 1}$ of $\mathbb{Z}^d$ by

$$U_u^{(1)}(z,T) = \{X^z(n),\ 0 \leqslant n \leqslant T\},$$

and, for all $k \geqslant 2$,

$$U_u^{(k)}(z,T) = \Psi_k\left(u, U_u^{(k-1)}(z,T), T\right).$$

In the next lemma, we iterate the results of Lemma A.2 to find lower bounds on the capacity of $U_u^{(d-2)}(z,T)$ and upper bounds on the diameter of $U_u^{(d-2)}(z,T)$.

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Lemma A.3. *For all $\varepsilon \in (0, \frac{1}{3}]$, there exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0, u_0]$, $z \in \mathbb{Z}^d$ and positive integer T,*

$$(A.6) \qquad \mathbb{P}^I \left(\bigcap_{k=1}^{d-2} \left\{ \mathrm{cap}\left(U_u^{(k)}(z, T) \right) \geqslant u^{k-1} \left(cT^{\frac{1-\varepsilon}{2}} \right)^k \right\} \right) \geqslant 1 - C \exp\left(-cT^{\varepsilon/2} u \right)$$

and

$$(A.7) \qquad \mathbb{P}^I \left(\bigcup_{k=1}^{d-2} \left\{ U_u^{(k)}(z, T) \subseteq z + [-kT^{\frac{1+\varepsilon}{2}}, kT^{\frac{1+\varepsilon}{2}})^d \right\} \right) \geqslant 1 - C \exp\left(-cT^\varepsilon u \right).$$

Proof. Let us introduce the shorthand $U_u^{(k)} = U_u^{(k)}(z, T)$, and note that, albeit only implicitly, $U_u^{(k)}$ depends on z and T. We first prove (A.6). For $u_0 c \leqslant 1$, using recursion, the event in (A.6) is implied by the event

$$\left\{ \mathrm{cap}\left(U_u^{(1)} \right) \geqslant cT^{\frac{1-\varepsilon}{2}} \right\} \cap \bigcap_{k=2}^{d-2} \left\{ \mathrm{cap}\left(U_u^{(k)} \right) \geqslant c \min\left(u\,\mathrm{cap}\left(U_u^{(k-1)} \right) T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}} \right) \right\}.$$

Lemma 6 in [15] tells us that

$$\mathbb{P}^I \left(\mathrm{cap}\left(U_u^{(1)} \right) \geqslant cT^{\frac{1-\varepsilon}{2}} \right) \geqslant 1 - C \exp\left(-cT^{\varepsilon/2} \right) \geqslant 1 - C \exp\left(-cT^{\varepsilon/2} u \right).$$

We only need to prove (A.6) if $cT^{\varepsilon/2} u \geqslant 1$, and then $cT^{\frac{1-\varepsilon}{2}} u \geqslant T^{\varepsilon/2}$ and (A.4) gives that for all $1 \leqslant k \leqslant d-3$,

$$\mathbb{P}^I \left(\mathrm{cap}\left(U_u^{(k+1)} \right) \geqslant c \min\left(u\,\mathrm{cap}\left(U_u^{(k)} \right) T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}} \right) \,\middle|\, \mathrm{cap}\left(U_u^{(k)} \right) \geqslant u^{k-1} \left(cT^{\frac{1-\varepsilon}{2}} \right)^k \right)$$

$$\geqslant 1 - C \exp\left(-c \min\left(T^{\varepsilon/2}, \left(cT^{\frac{1-\varepsilon}{2}} u \right)^k \right) \right) \geqslant 1 - C \exp\left(-cT^{\varepsilon/2} u \right).$$

The proof of (A.7) is simpler: we only need to use the fact that $h(T, \varepsilon) \leqslant C \exp(-cT^\varepsilon)$ for $k = 1$ (see the proof of Lemma A.2 for the definition of $h(T, \varepsilon)$), and then proceed by induction with (A.5) for $k \geqslant 2$. $\qquad\square$

Corollary A.4. *For all $\varepsilon \in (0, \frac{1}{2}]$, there exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0, u_0]$, $z \in \mathbb{Z}^d$ and $R > 0$,*

$$\mathbb{P}^I \left(z \in \mathcal{I}^u, \ \mathrm{cap}(C^u(z, R)) < cR^{(1-\varepsilon)(d-2)} u^{d-3} \right) \leqslant C \exp\left(-cR^{\varepsilon/2} u \right),$$

where $C^u(z, R)$ is the set of vertices connected to z by edges in $\widehat{\mathcal{I}}^u \cap [-R, R)^d$.

Proof. For all $\widetilde{u} \in (0, u_0]$, let $\omega^{(d-2)\widetilde{u}} := \sum_{i=1}^{d-2} \omega_i^{\widetilde{u}}$, where $(\omega_i^{\widetilde{u}})_{i \geqslant 1}$ are the independent random interlacement processes at level $\widetilde{u}$ used in the definition of $\left(U_{\widetilde{u}}^{(k)}(.,.) \right)_{k \geqslant 1}$, see above

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Lemma A.3. Furthermore, let $\mathcal{I}_1^{\widetilde{u}}$ be the random interlacement set associated with $\omega_1^{\widetilde{u}}$ and $\mathcal{I}^{(d-2)\widetilde{u}}$ the random interlacement set associated with $\omega^{(d-2)\widetilde{u}}$. By definition, $\omega^{(d-2)\widetilde{u}}$ has the law of a random interlacement process at level $(d-2)\widetilde{u}$, and if $z \in \mathcal{I}_1^{\widetilde{u}}$ then $U_{\widetilde{u}}^{(k)}(z,T)$ is a connected subset of $\mathcal{I}^{(d-2)\widetilde{u}}$ for all $k \in \{1, \ldots, d-2\}$, $z \in \mathbb{Z}^d$ and positive integer T. In particular, if $z \in \mathcal{I}_1^{\widetilde{u}}$ and if the event in (A.7) occurs with ε in that formula taking the value of some $\delta \in (0, \frac{1}{3})$, then

$$U_{\widetilde{u}}^{(d-2)}(z,T) \subset C^{(d-2)\widetilde{u}}\big(z, (d-2)T^{\frac{1+\delta}{2}}\big).$$

Using Lemma A.3 we obtain, for all $\delta \in (0, \frac{1}{3})$, $\widetilde{u} \in (0, u_0]$, $z \in \mathbb{Z}^d$ and positive integer T,

$$\mathbb{P}^I\left(z \in \mathcal{I}_1^{\widetilde{u}}, \ \mathrm{cap}\left(C^{(d-2)\widetilde{u}}(z, (d-2)T^{\frac{1+\delta}{2}})\right) < cT^{\frac{(1-\delta)(d-2)}{2}}\widetilde{u}^{d-3}\right) \leqslant C\exp\left(-cT^{\delta/2}\widetilde{u}\right).$$

The result follows by taking $\widetilde{u} = \frac{u}{d-2}$, $T = \left\lfloor \left(\frac{R}{d-2}\right)^{2-\varepsilon}\right\rfloor$ and $\delta = \frac{\varepsilon}{2-\varepsilon}$. $\qquad\square$

We now have all the tools required to connect $x, y \in \mathcal{I}^u \cap [-R, R)^d$ through edges in $\widehat{\mathcal{I}}^u \cap [-CR, CR)^d$, as mentioned at the beginning of the Appendix.

Lemma A.5. *There exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0, u_0]$, $R > 0$ and $x, y \in [-R, R)^d$,*

$$\mathbb{P}^I\left(x, y \in \mathcal{I}^u, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} y\right\}^c\right) \leqslant C\exp\left(-cR^{1/6}u\right).$$

Proof. Using the notation introduced at the beginning of the Appendix, we have

$$\mathbb{P}^I\left(x, y \in \mathcal{I}^u, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} y\right\}^c\right) \leqslant \sum_{i,j=1}^3 \mathbb{P}^I\left(x \in \mathcal{I}_{i,3}^{u/3}, y \in \mathcal{I}_{j,3}^{u/3}, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} y\right\}^c\right).$$

Let us now fix $i, j \in \{1, 2, 3\}$ and let $k \in \{1, 2, 3\}$ be different from i and j. We define the events

$$E_1 = \left\{\mathrm{cap}\left(C_i^{u/3}(x, R)\right) \geqslant CR^{\frac{2(d-2)}{3}}u^{d-3}\right\}, \quad E_2 = \left\{\mathrm{cap}\left(C_j^{u/3}(y, R)\right) \geqslant CR^{\frac{2(d-2)}{3}}u^{d-3}\right\},$$

and note that $E_1 \subset \{x \in \mathcal{I}_{i,3}^{u/3}\}$ and $E_2 \subset \{y \in \mathcal{I}_{j,3}^{u/3}\}$. Thus,

(A.8)
$$\mathbb{P}^I\left(x \in \mathcal{I}_{i,3}^{u/3}, y \in \mathcal{I}_{j,3}^{u/3}, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} y\right\}^c\right)$$
$$\leqslant \mathbb{P}^I\left((E_1 \cap E_2) \setminus \left\{C_i^u(x) \overset{\widehat{\mathcal{I}}_{k,3}^{u/3} \cap [-CR,CR)^d}{\longleftrightarrow} C_j^u(y)\right\}\right)$$
$$+ \mathbb{P}^I\left(\{x \in \mathcal{I}_{i,3}^{u/3}\} \setminus E_1\right) + \mathbb{P}^I\left(\{y \in \mathcal{I}_{j,3}^{u/3}\} \setminus E_2\right).$$

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For $uR^{1/6} \geqslant 1$, we can now use Lemma A.1 to bound the first summand of (A.8) as

$$\mathbb{P}^I\left((E_1 \cap E_2) \setminus \left\{ C_i^u(x) \overset{\widehat{\mathcal{I}}_{k,3}^{u/3} \cap [-CR,CR]^d}{\longleftrightarrow} C_j^u(y) \right\}\right) \leqslant C \exp\left(-cR^{2-d}u \times R^{\frac{4(d-2)}{3}} u^{2(d-3)}\right)$$

$$\leqslant C \exp\left(-cR^{1/6}u\right).$$

The second summand of (A.8) can also be bounded using Corollary A.4 with $\varepsilon = \frac{1}{3}$, and the result follows. $\qquad \square$

We now come to the

Proof of Lemma 3.2. Lemma 3.2 is a simple consequence of Lemma A.5. Indeed, let us define $R' = \lfloor \varepsilon R/2C \rfloor$ with C as in Lemma A.5, and we can assume without loss of generality that $\varepsilon R \geqslant 2C$. We define for each $z \in \mathbb{Z}^d$ the events

$$A_z^{(1)} = \left\{\mathcal{I}^u \cap \left(z + [-R', R')^d\right) \neq \emptyset\right\} \text{ and } A_z^{(2)} = \bigcap_{x,y \in \mathcal{I}^u \cap (z+[-2R', 2R')^d)} \left\{ x \overset{\widehat{\mathcal{I}}^u \cap z + [-\varepsilon R, \varepsilon R)^d}{\longleftrightarrow} y \right\}.$$

Note that these events depend on our choice of u and R even if it does not appear in the notation. It follows from the definition of random interlacements, (A.2) and Lemma A.5 that

$$\mathbb{P}^I\left(A_z^{(1)}\right) \geqslant 1 - \exp\left(-cR^{d-2}u\right) \text{ and } \mathbb{P}^I\left(A_z^{(2)}\right) \geqslant 1 - CR^{2d} \exp\left(-cR^{1/6}u\right).$$

In particular we get that

$$\mathbb{P}^I\left(\bigcap_{z \in [0,R)^d \cap \mathbb{Z}^d} A_z^{(1)} \cap A_z^{(2)}\right) \geqslant 1 - CR^{3d} \exp\left(-cR^{1/6}u\right) \geqslant 1 - C \exp\left(-cR^{1/7}u\right).$$

Let us call A the event on the left-hand side in the previous line, and suppose that A occurs. Then, for all $x, y \in \mathcal{I}^u \cap [0, R)^d$, one can find a path of nearest neighbors between x and y in $[0, R)^d$. Moreover, if z and z' are two neighbors in $[0, R)^d$, then $(z + [-R', R')^d) \cup (z' + [-R', R')^d) \subset z + [-2R', 2R')^d$, so every vertex in

(A.9) $\mathcal{I}^u \cap (z + [-R', R')^d)$ is connected to every vertex in $\mathcal{I}^u \cap (z + [-R', R')^d)$

by a path of edges in $\widehat{\mathcal{I}}^u \cap (z + [-\varepsilon R, \varepsilon R)^d) \subset \widehat{\mathcal{I}}^u \cap [-\varepsilon R, (1+\varepsilon)R)^d$, and the sets in (A.9) are not empty. This tells us that if A occurs, then every $x, y \in \mathcal{I}^u \cap [0, R)^d$ can be connected by edges in $\widehat{\mathcal{I}}^u \cap [-\varepsilon R, (1 + \varepsilon)R)^d$, and thus A implies the event on the left-hand side of (3.2). $\qquad \square$

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