ARM EXPONENT FOR THE GAUSSIAN FREE FIELD ON METRIC GRAPHS IN INTERMEDIATE DIMENSIONS

Alexander Drewitz[1], Alexis Prévost[2] and Pierre-François Rodriguez[3]

Abstract

We investigate the bond percolation model on transient weighted graphs G induced by the excursion sets of the Gaussian free field on the corresponding metric graph. We assume that balls in G have polynomial volume growth with growth exponent α and that the Green's function for the random walk on G exhibits a power law decay with exponent ν, in the regime $1 \leqslant \nu \leqslant \frac{\alpha}{2}$. In particular, this includes the cases of $G = \mathbb{Z}^3$, for which $\nu = 1$, and $G = \mathbb{Z}^4$, for which $\nu = \frac{\alpha}{2} = 2$. For all such graphs, we determine the leading-order asymptotic behavior for the critical one-arm probability, which we prove decays with distance R like $R^{-\frac{\nu}{2}+o(1)}$. Our results are in fact more precise and yield logarithmic corrections when $\nu > 1$ as well as corrections of order $\log\log R$ when $\nu = 1$. We further obtain very sharp upper bounds on truncated two-point functions close to criticality, which are new when $\nu > 1$ and essentially optimal when $\nu = 1$. This extends previous results from [16].

December 2023

[1]Universität zu Köln
Department Mathematik/Informatik
Weyertal 86–90
50931 Köln, Germany.
adrewitz@uni-koeln.de

[3]Imperial College London
Department of Mathematics
London SW7 2AZ
United Kingdom
p.rodriguez@imperial.ac.uk

[2]University of Geneva
Section of Mathematics
24, rue du Général Dufour
1211 Genève 4, Suisse.
alexis.prevost@unige.ch

1 Introduction

Percolation models exhibit intriguing behavior at and near their critical point, which is notoriously difficult to describe rigorously. Among many quantities of interest, one which plays a central role in this article is the 'one-arm' probability to connect a point to distance $R \geqslant 1$. For Bernoulli site percolation on the two-dimensional triangular lattice, this observable was famously shown [33, 44] to decay polynomially in R at criticality as $R^{-\frac{5}{48}+o(1)}$. On $\mathbb{Z}^d$ for sufficiently high dimension d (namely, $d \geqslant 11$) the decay is known to be of order R^{-2} independently of d [31, 24, 28, 6], a manifestation of mean-field behavior; see also [29] concerning spread-out models, for which this regime has been proven to extend to all $d > 6$. In intermediate dimensions $d = 3, 4, \dots$ however, the mere decay of the critical one-arm probability in itself is an outstanding open problem.

Recently, a bond percolation model with long-range correlations involving the Gaussian free field, which belongs to a different universality class than Bernoulli percolation, has led to significant advances on questions of the above type, notably in the challenging intermediate dimensions [11, 16, 15, 37, 51, 8]. We summarize these below; see also [39, 30, 2, 38, 12] for related results, including in dimension two. One focus of the present article is the one-arm decay of this model at criticality in intermediate dimensions. Another central quantity of interest is the truncated two-point function near criticality, for which we derive very sharp upper bounds. Together, these results have important ramifications concerning the behavior of critical and near-critical cluster volumes. Moreover, in the special case of $\mathbb{Z}^3$, they refine rather drastically recent results of [26], which concern a different (but related) model. Indeed, the quantitative two-point estimates we derive witness rotational invariance *at* the correlation length scale.

Our results hold under certain mild conditions on the base graph, which originate in [27], see also [13] in the context of percolation for the Gaussian free field, and are not specific to Euclidean lattices. We consider $\mathcal{G} = (G, \lambda)$ a transient weighted graph, connected and locally finite. We assume controlled weights, that is $\lambda_{x,y}/\lambda_x \geqslant c$ for some constant $c > 0$, which entails uniform ellipticity for the random walk on $\mathcal{G}$, see for instance [4, Definition 1.2, p.3]. We further impose two natural conditions on the growth of balls and the decay of the Green's function for the random walk on $\mathcal{G}$. Namely, there exist a positive exponent α and $c, C \in (0, \infty)$ such that the volume growth condition

$$(V_\alpha) \qquad cr^\alpha \leqslant \lambda(B(x,r)) \leqslant Cr^\alpha \quad \text{for all } x \in G \text{ and } r \geqslant 1,$$

is satisfied, where $B(x,r)$ refers to the discrete ball of radius r around $x \in G$ in a given metric d on G, and λ denotes the measure induced via the point masses $\lambda_x \stackrel{\text{def.}}{=} \sum_{y\sim x} \lambda_{x,y}$ for $x \in G$. Moreover, there exist an exponent $\nu > 0$ and constants $c, C \in (0, \infty)$ such that

$$(G_\nu) \qquad c \leqslant g(x,x) \leqslant C \text{ and } cd(x,y)^{-\nu} \leqslant g(x,y) \leqslant Cd(x,y)^{-\nu} \quad \text{for all } x \neq y \in G.$$

As explained around [16, (1.18)], in case $d = d_{\text{gr}}$, where d_{gr} is the graph distance on $\mathcal{G}$, these conditions imply that $0 < \nu \leqslant \alpha - 2$. We assume from now on that these bounds on ν are satisfied. An emblematic example of graphs satisfying these conditions are the Euclidean lattices $G = \mathbb{Z}^\alpha$ for integer $\alpha \geqslant 3$ endowed with unit weights $\lambda_{x,y} = 1\{|x-y| = 1\}$, $x, y \in G$; here, $|\cdot|$ denotes the Euclidean distance, and this setting fits the above setup with $\nu = \alpha - 2$. In this case ν ranges over the integer values $\{1, 2, \dots\}$, but intermediate values are also attainable, see [3], and also [13] for concrete examples, including for instance fractal graphs of intermediate dimension. Although the examples mentioned above often entail additional symmetry, no assumptions beyond (V_α) and (G_ν) (along with controlled weights) are required, in particular no transitivity assumption needs to be imposed.

Let φ denote the mean zero Gaussian free field on $\mathcal{G}$ with canonical law $\mathbb{P}$. The excursion sets $\{x \in G : \varphi_x \geqslant a\}$ for varying parameter $a \in \mathbb{R}$, lead to a natural *site* percolation model on G, and have been extensively studied, see e.g. [7, 43, 18, 48, 14, 26, 19, 40]. A variation with improved integrability properties is obtained by considering instead the *bond* percolation model which is obtained by retaining each edge $\{x, y\}$ in G independently with probability

$$1 - \exp\{-2\lambda_{x,y}(\varphi_x - a)_+(\varphi_y - a)_+\}, \quad a \in \mathbb{R}, \tag{1.1}$$

conditionally on φ, where $t_+ = \max\{t, 0\}$ for $t \in \mathbb{R}$. We assume $\mathbb{P}$ to be suitably extended to carry this additional randomness. An edge is called open if it has been retained. Let $E^{\geqslant a}$ denote the set of open edges of $\mathcal{G}$ obtained in this way. In view of (1.1), by a straightforward coupling involving uniform random variables, one sees that $E^{\geqslant a}$ is decreasing in a, thus rendering the phase transition for the percolation problem $(E^{\geqslant a})_{a\in\mathbb{R}}$ well-defined. What lurks behind the choice of the bond disorder (1.1) is an extension of φ to the metric graph $\widetilde{\mathcal{G}} \supset G$, comprising G along with one-dimensional cables joining neighboring vertices in G. The percolation problem for $(E^{\geqslant a})_{a\in\mathbb{R}}$ is equivalent to the percolation problem for excursion sets of the continuous extension of φ to $\widetilde{\mathcal{G}}$; see Section 2 for details.

One knows as a consequence of [15, Theorem 1.1(1)] and Lemma 3.4(2) therein (which implies in view of (G_ν) that the condition (Cap) appearing in [15] holds on $\mathcal{G}$) that the percolation model $(E^{\geqslant a})_{a\in\mathbb{R}}$ has a non-trivial phase transition on any graph $\mathcal{G}$ satisfying the above requirements, that the critical height is $a = 0$ regardless of the particular choice of $\mathcal{G}$, and furthermore that the order parameter of the transition (i.e. the probability of a point to belong to an infinite cluster in $E^{\geqslant a}$) is a continuous function of a.

Our first result concerns the critical 'one-arm' events $\{x \leftrightarrow B(x, R)^{\mathsf{c}}\}$, which refer to the existence of a path in $E^{\geqslant 0}$ connecting 0 and $B(x, R)^{\mathsf{c}} = G \backslash B(x, R)$. To date, the best available bounds on the critical one-arm probability $\mathbb{P}(x \leftrightarrow B(x, R)^{\mathsf{c}})$ are as follows. By [16, (1.22)–(1.23)], one knows that for all $\nu > 0$, $x \in G$ and $R \geqslant 1$, and some $c, C \in (0, \infty)$ (see the end of this introduction regarding our convention with constants),

$$\mathbb{P}\big(x \leftrightarrow B(x, R)^{\mathsf{c}}\big) \geqslant cR^{-\frac{\nu}{2}}, \tag{1.2}$$

and

$$\mathbb{P}\big(x \leftrightarrow B(x, R)^{\mathsf{c}}\big) \leqslant C \begin{cases} R^{-\frac{\nu}{2}}, & \text{if } \nu < 1 \\ \big(\frac{R}{\log R}\big)^{-\frac{1}{2}}, & \text{if } \nu = 1 \\ R^{-\frac{1}{2}}, & \text{if } \nu > 1 \end{cases}. \tag{1.3}$$

The bounds (1.2)–(1.3) were first derived in [11] for $G = \mathbb{Z}^\alpha$, with $\alpha = \nu + 2 (\geqslant 3)$, via different methods. Later on, in the general graph setting of [16], they were seen to follow immediately from a comparison between the radius and capacity observables for the cluster of a point, using that the latter is integrable, see [16, Cor. 1.3]; in fact, the proof of (1.2)–(1.3) does not even require the assumption (V_α). The bounds (1.2)–(1.3) thus express little more than the fact that the cluster of x reaching distance R can be anything from a line to the full box. When $\nu < 1$, the two are indistinguishable to leading order in N when measured in terms of their capacity, but discrepancies start to arise when $\nu \geqslant 1$, which corresponds to dimension three and higher on Euclidean lattices. In the mean-field regime of $\mathbb{Z}^\alpha$ for any integer $\alpha > 6$, it was recently proved in [8] that $\mathbb{P}\big(x \leftrightarrow B(x, R)^{\mathsf{c}}\big) \asymp R^{-2}$; see also [51] for more results valid in high dimensions.

The mismatch between (1.2) and (1.3) for $\nu \geqslant 1$ warrants further investigation, in particular in the regime of 'intermediate' dimensions, e.g. for integer α with $3 \leqslant \alpha \leqslant 6$ on $\mathbb{Z}^\alpha$, which is the object of our first main result. From here on, we impose the standing assumption

$$1 \leqslant \nu \leqslant \frac{\alpha}{2}. \tag{1.4}$$

In particular, (1.4) includes the cases $G = \mathbb{Z}^\alpha$, $\alpha \in \{3,4\}$, with unit weights, for which $\nu = \alpha - 2 (\leqslant \frac{\alpha}{2})$, but also e.g. certain fractal graphs as will be mentioned below. Our first result is an improved upper bound on the one-arm probability at criticality.

Theorem 1.1. *There exists $C \in (0,\infty)$ such that for all $x \in G$ and $R \geqslant 3$,*

$$\mathbb{P}\big(x \leftrightarrow B(x,R)^{\mathsf{c}}\big) \leqslant C q(R) R^{-\frac{\nu}{2}}, \tag{1.5}$$

where

$$q(R) = \begin{cases} \log\log R, & \text{if } \nu = 1, \\ (\log R)^{\frac{1}{2}(\nu-1)}(\log\log R)^{\nu}, & \text{if } 1 < \nu < \frac{\alpha}{2}, \\ (\log R)^{\nu}(\log\log R)^{\nu}, & \text{if } \nu = \frac{\alpha}{2}. \end{cases} \tag{1.6}$$

In particular, for the critical one-arm exponent one has that

$$\frac{1}{\rho} \overset{\text{def.}}{=} -\lim_{R\to\infty} \frac{\log \mathbb{P}(x \leftrightarrow B(x,R)^{\mathsf{c}})}{\log R} = \frac{\nu}{2}. \tag{1.7}$$

Thus, in comparison with (1.2)–(1.3), Theorem 1.1 yields an improvement in the discrepancy between upper and lower bounds: first for $\nu = 1$, we obtain an improvement of the upper bound from a factor of $\log(R)^{\frac{1}{2}}$ to $\log\log R$; this applies in particular to the lattice $\mathbb{Z}^3$; see Remark 4.4 for more details. Even more distinctively, in the regime $1 < \nu \leqslant \frac{\alpha}{2}$, Theorem 1.1 implies that the polynomial lower bound (1.2) is in fact sharp up to logarithmic factors. For instance $\mathbb{Z}^4$ corresponds to the case $\nu = \frac{\alpha}{2} = 2$; an example with $1 < \nu < \frac{\alpha}{2}$ is the graphical Sierpinski carpet in $\alpha = 4$ dimensions; see [5] and [13, Remark 3.10,2)]. This sharpness up to logarithmic factors implies that the limit in (1.7) exists and defines a critical one-arm exponent. There is however no specific reason to believe that the upper bounds we derive are sharp up to constants. In fact we prove slightly more than (1.6). For instance, in case $\nu = 1$, the proof yields (1.5) with $q(R) = (\log\log R)^{\frac{2}{3}+\varepsilon}$ for any $\varepsilon > 0$; see (4.19) for best available bounds in all cases.

The improvement from (1.3) to (1.5)–(1.6) is a matter of showing that the cluster of x is sufficiently 'thick' when measured in terms of capacity. This 'capacity thickening' is due to the underlying presence of random walk-like objects within this cluster, whose capacity is particularly large under the assumption (1.4). We will return to this below. As mentioned above, on $\mathbb{Z}^\alpha$ for (integer) $\alpha > 6$, it was proved in [8] that $\rho = 1/2$, and noting that $2/\nu = 1/2$ exactly when $\alpha = 6$ (and hence $\nu = 4$), we conjecture that (1.7) remains true on $\mathbb{Z}^\alpha$ for $\alpha \in \{5,6\}$.

Theorem 1.1 has further important consequences, which we now discuss. Indeed, this is because our improved bounds for $q(\cdot)$ from (1.5) and (1.6) feed into the expressions for the lower bound on the near-critical one-arm probability and two-point function obtained in [16, Proposition 6.1 and (8.3)]. We detail this in the latter case. For each $a \in \mathbb{R}$ and $x, y \in G$, let

$$\tau_a^{\mathrm{tr}}(x,y) = \tau_a^{\mathrm{tr}}(y,x) \overset{\text{def.}}{=} \mathbb{P}\big(x \leftrightarrow y \text{ in } E^{\geqslant a}, x \nleftrightarrow \infty \text{ in } E^{\geqslant a}\big) \tag{1.8}$$

denote the truncated two-point function, where '$x \leftrightarrow y$ in A' means that there is a path of edges in A from x to y. Then, under the additional hypothesis that $d = d_{\mathrm{gr}}$ one obtains the following: there exist constants $c, C_1 \in (0,\infty)$ such that for all $a \in \mathbb{R}$ with $|a| \leqslant c$ and all $x, y \in G$ with $d(x,y) \geqslant C_1 |a|^{-2} q(a^{-1}) \log(q(a^{-1}))$ when $\nu = 1$, and with $d(x,y) \geqslant C_1 |a|^{-\frac{2}{\nu}} q(a^{-1})$ when $1 < \nu < \frac{\alpha}{2}$, one has

$$\tau_a^{\mathrm{tr}}(x,y) \geqslant \tau_0^{\mathrm{tr}}(x,y) \begin{cases} \exp\Big\{-\dfrac{C_1|a|^2 d(x,y)}{\log(|a|^2 d(x,y))}\Big\}, & \text{if } \nu = 1, \\ \exp\Big\{-C_1|a|^{\frac{2}{\nu}} d(x,y) \log\big(|a|^{\frac{2}{\nu}} d(x,y)\big)^{\nu-1}\Big\}, & \text{if } 1 < \nu < \frac{\alpha}{2}. \end{cases} \tag{1.9}$$

We refer to the end of Section 5 for the short derivation of these bounds, which follow from Theorem 1.1 in combination with [16, (8.3)]. Note, however, that when $\nu = \frac{\alpha}{2}$, one cannot improve on the bound of [16, Theorem 1.7], for reasons explained in [16, Remark 8.1,1)]. Regarding the one-arm probability, [16, Proposition 6.1] combined with Theorem 1.1 yields the exact same lower bounds as in (1.9), but with $\tau_a^{\mathrm{tr}}(x,y)$ and $\tau_0^{\mathrm{tr}}(x,y)$ replaced by the near-critical and critical one-arm probabilities, as well as $d(x,y)$ replaced by the radius r.

Our second main result Theorem 1.2 yields an upper bound on the truncated two-point function $\tau_a^{\mathrm{tr}}(x,y)$ from (1.8), thereby assessing the sharpness of the lower bound (1.9) (up to log corrections when $\nu > 1$), as well as the resulting lower bound on the correlation length, cf. the discussion around (1.12) below. In order to put this into context, we recall that the to date best upper bound was proved in [16, Theorem 1.4]: there exists $c_1 > 0$ such that for all $a \in \mathbb{R}$ and $x, y \in G$,

$$\tau_a^{\mathrm{tr}}(x,y) \leqslant \tau_0^{\mathrm{tr}}(x,y) \begin{cases} \exp\Big\{ -\dfrac{c_1|a|^2 d(x,y)}{\log(d(x,y)) \vee 1}\Big\}, & \text{if } \nu = 1, \\ \exp\big\{ -c_1|a|^2 d(x,y)\big\}, & \text{if } \nu > 1; \end{cases} \tag{1.10}$$

see also [11, Theorem 4] for a similar result in the case $G = \mathbb{Z}^\alpha$, albeit without prefactor $\tau_0^{\mathrm{tr}}(x,y)$. To be precise, (1.10) only requires $\mathcal{G}$ to have controlled weights and to satisfy (G_ν). Let us also remark here that in the case $\nu < 1$, one already has matching upper and lower bounds, see [16, Theorem 1.4], by which $\log(\tau_a^{\mathrm{tr}}(x,y)/\tau_0^{\mathrm{tr}}(x,y))$ behaves like $-(|a|^{\frac{2}{\nu}} d(x,y))^\nu$. Our second main result improves the bound (1.10) in the regime of parameters $1 \leqslant \nu \leqslant \frac{\alpha}{2}$ from (1.4).

Theorem 1.2. *There exist $c_2, C_2 \in (0,\infty)$ such that for all $a \in \mathbb{R}$ and $x, y \in G$,*

$$\tau_a^{\mathrm{tr}}(x,y) \leqslant C_2 \tau_0^{\mathrm{tr}}(x,y) \begin{cases} \exp\Big\{ -\dfrac{c_2|a|^2 d(x,y)}{\log(|a|^2 d(x,y)) \vee 1}\Big\}, & \text{if } \nu = 1, \\ \exp\big\{ -c_2|a|^{\frac{2}{\nu}} d(x,y)\big\}, & \text{if } 1 < \nu < \frac{\alpha}{2}, \\ \exp\Big\{ -\dfrac{c_2|a|^{\frac{2}{\nu}} d(x,y)}{\log(|a|^{-1}) \vee 1}\Big\}, & \text{if } \nu = \frac{\alpha}{2}. \end{cases} \tag{1.11}$$

In view of Theorem 1.2, the bounds (1.9) and (1.11) now exactly match up to constants when $\nu = 1$, and match up to logarithmic correction in $|a|^{\frac{2}{\nu}} d(x,y)$ when $1 < \nu < \frac{\alpha}{2}$. We believe that our upper bound (1.11) is sharp, that is one can remove the term $\log(|a|^{\frac{2}{\nu}} d(x,y))^{\nu-1}$ in (1.9); we hope to return to this elsewhere. More importantly, as opposed to (1.10), the upper bounds in (1.11) are functions of $|a|^{\frac{2}{\nu}} d(x,y)$ for all $1 \leqslant \nu < \frac{\alpha}{2}$. Thus, combining (1.9), (1.11), and [16, Theorem 1.7] for the lower bound when $\nu = \frac{\alpha}{2}$, and defining

$$\xi \equiv \xi(a) \stackrel{\text{def.}}{=} |a|^{-\frac{2}{\nu}} \quad (\text{with } \xi(0) \stackrel{\text{def.}}{=} \infty), \tag{1.12}$$

this entails that ξ is the length scale after which the ratio $\tau_a^{\mathrm{tr}}/\tau_0^{\mathrm{tr}}$ starts to decay to zero rapidly (with corrections of order $\log(\xi)$ when $\nu = \frac{\alpha}{2}$). In particular, it follows that the critical exponent ν_c associated to the correlation length is equal to $\frac{2}{\nu}$ for all $1 \leqslant \nu \leqslant \frac{\alpha}{2}$, which extends the results from [16, (1.28)]. Let us also emphasize that, contrary to (1.10) or the bounds in case $\nu < 1$ mentioned below (1.10) (cf. also the disconnection results of [48]), the upper bounds in (1.11) have a functional dependence on the parameter a which is *not* Gaussian (not even when $\nu = 1$).

When $G = \mathbb{Z}^3$ (with unit weights), following ideas from [26], see also [41] for related results, one can actually strengthen this even more and (almost) match the constants C_1 from (1.9) and c_2 from (1.11). More precisely, denoting by S^2 the two-dimensional unit sphere, and abbreviating for each $x \in \mathbb{R}^3$ by $[x]$ the vertex closest to x in $\mathbb{Z}^3$ (with some arbitrary choice if x is equidistant to two or more vertices of $\mathbb{Z}^3$), the following result is proved at the end of Section 5.

Corollary 1.3. *Let $G = \mathbb{Z}^3$. There exists $c \in (0,\infty)$ and for all $\eta \in (0,1)$, there exists $C = C(\eta) \in (0,\infty)$ such that for all $e \in S^2$, $a \in \mathbb{R}$ with $|a| \leqslant c$, and all $\lambda \geqslant C$,*

$$-\frac{\pi}{6}(1+\eta)\frac{\lambda}{\log(\lambda)} - C\log\log\xi \leqslant \log\left(\frac{\tau_a^{\mathrm{tr}}(0,[\lambda\xi e])}{\tau_0^{\mathrm{tr}}(0,[\lambda\xi e])}\right) \leqslant -\frac{\pi}{6}(1-\eta)\frac{\lambda}{\log(\lambda)}. \tag{1.13}$$

The expected rotational invariance at criticality is manifested in (1.13) by the fact that the bounds obtained are functions of λ alone (with $\log\log$ corrections) and do not depend on the choice of $e \in S^2$. For the related (but harder) model where one considers excursion sets of the discrete free field [7, 43], bounds witnessing a degree of rotational invariance similar to (1.13) were derived in [26], but only asymptotically in the limit $\lambda \to \infty$ for fixed parameter a. We refer to [44, 23] and references therein for rotational invariance results at criticality when $\alpha = 2$.

Summing the bounds (1.9) and (1.11) over all $y \in G$, one also obtains bounds on the average volume of a (bounded) cluster at level $a \neq 0$, and one can furthermore deduce from Theorem 1.1 bounds on the tail of the volume of the critical cluster. This generalizes results from [16, Corollary 1.5 and 1.6] for $\nu \leqslant 1$ to the regime $1 < \nu < \frac{\alpha}{2}$, with improved logarithmic corrections when $\nu = 1$. Let $|K|$ denote the cardinality of a set $K \subset G$.

Corollary 1.4. *There exist $c, C \in (0,\infty)$ such that for all $x \in G$ and $n \geqslant 1$, denoting by $\mathcal{K}^a$ the open cluster of x at level a, under the assumption (1.4) on ν and α and with $q(\cdot)$ as in (1.6),*

$$\mathbb{P}(|\mathcal{K}^0| \geqslant n) \leqslant Cn^{-\frac{\nu}{2\alpha-\nu}}q(n). \tag{1.14}$$

Moreover, for all $a \neq 0$, if $\nu < \frac{\alpha}{2}$ and $d = d_{\mathrm{gr}}$, one has

$$c|a|^{-\frac{2\alpha}{\nu}+2}\exp\{-Cq(\xi)\} \leqslant \mathbb{E}[|\mathcal{K}^a|1\{|\mathcal{K}^a| < \infty\}] \leqslant C|a|^{-\frac{2\alpha}{\nu}+2}, \tag{1.15}$$

and if $\nu < 3$ is additionally fulfilled, then

$$\gamma \overset{\text{def.}}{=} -\lim_{a\to 0}\frac{\log\mathbb{E}[|\mathcal{K}^a|1\{|\mathcal{K}^a| < \infty\}]}{\log|a|} = \frac{2\alpha}{\nu} - 2. \tag{1.16}$$

For instance, (1.16) applies when $\mathcal{G}$ is the four-dimensional Sierpinski carpet, see [13, Remark 3.10,1)] as to why. The bound (1.14) can be deduced from Theorem 1.1 similarly as [16, Corollary 1.6], and it in particular implies that the critical exponent δ (defined in such a way that $n^{-1/\delta}$ controls the tail in (1.14)) is smaller than or equal to $\frac{2\alpha}{\nu} - 1$ when it exists. The bound (1.15) can be deduced from [16, (8.3)] and (1.11) similarly as [16, Corollary 1.5], and (1.16) follows readily noting that $q(\xi)/\log(\xi) \to 0$ as $\xi \to \infty$ when $\nu < 3 \wedge \frac{\alpha}{2}$, see (1.6). If $\nu = \frac{\alpha}{2}$, one can still bound the average near-critical volume in (1.15) from above by $C|a|^{-\frac{2\alpha}{\nu}+2}\log(|a|^{-1})^{\alpha-\nu}$, and hence $\gamma \geqslant \frac{2\alpha}{\nu} - 2$ when it exists, but we cannot prove a matching lower bound anymore since there is no bound similar to (1.9) currently available in this regime.

We now comment on the proofs. A key role in deriving the upper bounds in Theorems 1.1 and 1.2 is played by a certain obstacle set $\mathcal{O}$ with 'good' properties, introduced in its general form in the paragraph leading to Lemma 2.1 below. The relevant obstacle set $\mathcal{O}$, which is a carefully chosen part of $E^{\geqslant a}$, is different in each proof and described below. Incidentally, obstacle sets with similar visibility requirements on the random walk (cf. Lemma 2.1 below and [21, (1.9)]) were key in deriving the 'near-critical' couplings for finite range interlacements, which were instrumental in the recent proof of sharpness for the vacant set of random interlacements [20, 21, 22], another model with a similar correlation structure as $(E^{\geqslant a})_{a\in\mathbb{R}}$. The interlacement set will also play a role in this article, as explained below.

For the purpose of proving Theorem 1.1, the obstacle set $\mathcal{O}$ will be made up of a selection of 'big' loops stemming from a loop soup at critical intensity, see (4.22); these big loops are hard to avoid for the cluster of the origin, and when hit ensure that the latter is sufficiently 'fat' when measured in terms of its capacity. The pertinence of the loops stems from the metric version of Le Jan's isomorphism [36, 37]. A key input, see Proposition 4.1, is a sufficiently strong quantitative control guaranteeing the presence of big loops. This estimate is the main driving force of the proof.

For the purposes of proving Theorem 1.2, the relevant obstacle set $\mathcal{O}$ comprises pieces of interlacement trajectories in $\mathcal{I}^u$, with $u = \frac{a^2}{2}$ and a as appearing in Theorem 1.2. Interlacements enter by means of Sznitman's isomorphism theorem [49]. They appear when $a \neq 0$ and witness the emergence of an infinite cluster. Within the proof, interlacements allow, together with a version of Lupu's formula involving killing on $\mathcal{O}$, to control the two-point function when $\mathcal{O}$ is 'good'; see in particular (5.2) and Corollary 2.2. A key input to control the goodness features of $\mathcal{O}$ that eventually give rise to the precise bounds in (1.2) comes via state-of-the-art coarse-graining techniques, quantitative in both the parameter a and the relevant spatial scales, which were developed and progressively refined in [48, 26, 1, 42], the last of which will be applied here.

We now briefly comment on the relevance of the condition (1.4). In the proofs of both Theorems 1.1 and 1.2, the obstacle set $\mathcal{O}$ is composed of random walk-type objects, either comprising big loops or pieces of interlacement trajectories. Consequently, whenever a cluster of large radius intersects $\mathcal{O}$ at a vertex $x \in G$, its capacity in the vicinity of x is not only larger than the capacity of a line (as needed to satisfy the large radius constraint), but also than the capacity of a random walk. Whenever $\nu \geqslant 1$, the capacity of the latter is typically much larger than the capacity of a line, see e.g. [13, Lemmas 4.4 and A.1]. This boosts the capacity of this cluster upon intersection with $\mathcal{O}$. Furthermore, when $\nu \leqslant \frac{\alpha}{2}$, the capacity of a random walk in a ball is typically close to the capacity of this ball, and hence any cluster with large radius locally resembles a ball if measured in terms of capacity whenever it intersects $\mathcal{O}$. This key random walk input explains our standing assumption (1.4).

We now describe the organization of this article. Section 2 introduces the framework, some notation and the important notion of (good) obstacle set $\mathcal{O}$. Section 3 gathers preliminary estimates on the loop measure for certain sets of interest, at the level of generality needed for the present article. In combination with the notion of obstacle set developed in Section 2, these estimates are used in the proof of Theorem 1.1, which is presented in Section 4. The proofs of Theorem 1.2, Corollary 1.3 and (1.9) are given in Section 5.

Throughout this article we always tacitly assume that (1.4) holds. All constants belong to $(0,\infty)$ and may implicitly depend on ν and α in the sequel (as well as on the constants from (V_α), (G_ν) and the ellipticity condition on controlled weights), but not on the choice of base point 0 in (2.1) below, which is arbitrary. Any further dependence on parameters will be stated explicitly. Numbered constants $c_1, C_1, \dots$ are fixed once they first appear, whereas the constants c and C may change from place to place. It will be convenient to abbreviate

$$b_1 = \begin{cases} 1, & \text{if } \nu = 1, \\ 0, & \text{else}, \end{cases} \quad \text{and} \quad b_2 = \begin{cases} 1, & \text{if } \alpha = 2\nu, \\ 0, & \text{else}. \end{cases} \tag{1.17}$$

2 Preliminaries

In this section we collect a small amount of notation and gather a useful preliminary result on the connectivity function for the free field in the presence of a certain obstacle (set) $\mathcal{O} \subset G$ on which the random walk is killed, see Corollary 2.2. The obstacle set has certain good properties, which make it hard for the random walk to avoid, see Lemma 2.1. The related results will play an important role for the proofs of both Theorems 1.1 and 1.2.

We start by discussing a few consequences of our setup from Section 1, see above (V_α). In the sequel, in order to simplify notations, we typically consider

$$0, \text{ an arbitrary point in } G \tag{2.1}$$

and abbreviate $B_r = B(0,r)$. All results hold uniformly in the choice of 0. Similarly as in [13, Lemma 6.1], we introduce under our assumptions on $\mathcal{G}$ (note in particular that controlled weights, see above (V_α), corresponds to the condition denoted by (p_0) in [13, Section 2]) the approximate renormalized lattice $\Lambda(L)$ with the properties that there is a constant $C_3 < \infty$ such that for all $x \in G$, $L \geqslant 1$ and $N \geqslant 1$:

$$\begin{gathered} 0 \in \Lambda(L), \bigcup_{y\in\Lambda(L)} B(y,L) = G, \text{ the balls } B(y,\tfrac{L}{2}),\ y \in \Lambda(L), \text{ are disjoint,} \\ \text{and } |\Lambda(L) \cap B(x,LN)| \leqslant C_3 N^\alpha. \end{gathered} \tag{2.2}$$

The asserted disjointness follow from an inspection of the proof of [13, Lemma 6.1]. We call $\pi = (x_i)_{1\leqslant i\leqslant M}$ a path in $\Lambda(L)$ from 0 to $A \subset G$ if $x_1 = 0$, $x_M \in A$, $x_i \in \Lambda(L)$ for all $1 \leqslant i \leqslant M$, and for each $1 \leqslant i \leqslant M-1$, there exist $x \in B(x_i,L)$ and $y \in B(x_{i+1},L)$ such that x and y are neighbors in $\mathcal{G}$.

We write $X = (X_t)_{t\geqslant 0}$ for the diffusion on the cable system (or metric graph) $\widetilde{\mathcal{G}}$ associated to $\mathcal{G}$ and $Z = (Z_n)_{n\geqslant 0}$ for the discrete skeleton of the trace of X on G, see [15, Section 2.1] for details. Furthermore, we denote by P_x, $x \in \widetilde{\mathcal{G}}$, the canonical law of X with $X_0 = x$. If $x \in G$, the law of Z under P_x is that of the discrete time simple random walk on the weighted graph $\mathcal{G} = (G,\lambda)$. For all $U \subset \widetilde{\mathcal{G}}$ open and $x,y \in \widetilde{\mathcal{G}}$, we denote by $g_U(x,y)$ the Green's function of X killed outside U between x and y, and abbreviate $g(x,y) = g_{\widetilde{\mathcal{G}}}(x,y)$. When $x,y \in G$, then $g(x,y)$ simply corresponds to the Green's function associated to the walk Z. For $U \subset G$ we write $H_U = H_U(Z) = \inf\{n \geqslant 0 : Z_n \in U\}$ for the entrance time in U and $T_U = H_{G\setminus U}$ for the exit time of Z from U. We also denote by e_U the equilibrium measure of a finite set $U \subset G$ and by $\mathrm{cap}(U)$ its capacity, which satisfies

$$P_x(H_U < \infty) = \sum_{y\in G} g(x,y)e_U(y) \text{ for all } x \in G; \tag{2.3}$$

this identity immediately follows from a last exit decomposition for the walk, cf. [47, (1.57)] for the finite graph setting. One further knows, see e.g. [16, (5.7)], that uniformly in $x \in G$ and $R \geqslant 1$,

$$cR^\nu \leqslant \mathrm{cap}(B(x,R)) \leqslant CR^\nu. \tag{2.4}$$

Under the standing assumptions on $\mathcal{G}$ (see Section 1), one also has that the following elliptic Harnack inequality holds. On account of [13, (3.3)] (cf. also references therein), such that for all $\zeta \geqslant C_4$, $x \in G$, $R \geqslant 1$, and $h : G \to [0,\infty)$ which are L-harmonic in $B(x,\zeta R)$,

$$\sup_{y\in B(x,R)} h(y) \leqslant c_3 \inf_{y\in B(x,R)} h(y). \tag{2.5}$$

We now introduce the obstacles that will play a role in the sequel. We call a set $\mathcal{O} \subset G$ a (L,R,n,κ)-good (or simply good when the parameters are clear from the context) obstacle if for each path π in $\Lambda(L)$ from 0 to B_R^{c}, there exists a set $A \subset \mathrm{range}(\pi \cap B_R)$ with $|A| \geqslant n$ such that $\mathrm{cap}(\mathcal{O} \cap B(y,L)) \geqslant \kappa$ for all $y \in A$. Recall our convention regarding constants c, C from the end of the previous section.

Lemma 2.1. *There exists a constant $C_5 \in (0,\infty)$ such that for all $\kappa, L \geqslant 1$, $R \geqslant L$, integer $n \geqslant 1$, $x \in B^{\mathsf{c}}_{R+C_5L}$, and for any (L,R,n,κ)-good obstacle set $\mathcal{O}$, one has*

$$P_0\left(H_x < H_{\mathcal{O}}\right) \leqslant Cd(x, B_{R+C_5L})^{-\nu} \exp\Big\{-\frac{c\kappa n}{L^\nu}\Big\}.$$

Proof. By a slight generalization of (2.3), see for instance [42, (2.17)], there exist constants $C_6 < \infty$ and $c_4 > 0$ such that for any $y \in G$ with $\mathrm{cap}(\mathcal{O} \cap B(y,L)) \geqslant \kappa$, we have

$$P_z\big(H_{\mathcal{O}} < T_{B(y,C_6L)}\big) \geqslant \frac{c_4\kappa}{L^\nu} \text{ for all } z \in B(y,L). \tag{2.6}$$

Let us introduce recursively a random sequence of vertices $y_1, \dots, y_M$ depending on Z under P_0 as follows: y_i is the first vertex $y \in \Lambda(L) \cap B_R$ visited by $B(Z_k, L)$, $k \in \mathbb{N}$, (with an arbitrary rule for splitting ties), and such that

- $\mathrm{cap}(B(y,L) \cap \mathcal{O}) \geqslant \kappa$, and
- $B(y,L) \cap B(y_j, C_6L) = \varnothing$ for all $j \leqslant i-1$.

We stop the recursion the first time $M = M(L,R,\mathcal{O},\kappa)$ after which such a vertex y as described before does not exist. Since $\mathcal{O}$ is a (L,R,n,κ)-good obstacle, there are at least $n-1$ different vertices $y \in \Lambda(L) \cap B_R$ such that Z hits $B(y,L)$ and $\mathrm{cap}(\mathcal{O} \cap B(y,L)) \geqslant \kappa$. One can easily deduce from (2.2) that there exists $c > 0$ such that $M \geqslant cn$ almost surely. Let us denote by H_i the first time Z hits $B(y_i, L)$, and by T_i the first time Z exits $B(y_i, C_6L)$ after H_i. Choosing C_5 large enough we then have

$$\begin{aligned} P_0(H_x < H_{\mathcal{O}}) &\leqslant P_0\Big(\mathcal{O} \cap \{Z_k, H_i \leqslant k < T_i\} = \varnothing\, \forall\, 1 \leqslant i \leqslant \lceil cn \rceil, T_{\lceil cn \rceil} \leqslant H_x < \infty\Big) \\ &\leqslant \Big(1 - \frac{c_4\kappa}{L^\nu}\Big)^{cn} \cdot Cd(x, B_{R+C_5L})^{-\nu}; \end{aligned} \tag{2.7}$$

here, the last inequality is obtained as follows: we first apply the strong Markov property of Z at time $T_{\lceil cn \rceil}$ combined with (G_ν), and noting that we have $d(x, Z_{T_{\lceil cn \rceil}}) \geqslant d(x, B_{R+CL})$, in order to obtain the second factor in the last line of the right-hand side of (2.7). We then apply the strong Markov property recursively at times $H_{\lceil cn \rceil - i}$, $0 \leqslant i \leqslant \lceil cn \rceil - 1$, together with the bound (2.6). Using the inequality $1 - t \leqslant e^{-t}$ for all $t \geqslant 0$ we obtain the first factor and can conclude. □

We now discuss a consequence of the above result for the Gaussian free field, which is tailored to our later purposes. Extending the definition from Section 1, we write φ from here on to denote the Gaussian free field on the metric graph $\widetilde{\mathcal{G}}$ and continue to write $\mathbb{P}$ for its canonical law. More generally, for $U \subset \widetilde{\mathcal{G}}$ open, we denote by $\mathbb{P}_U$ the law under which $(\varphi_x)_{x \in U}$ is a Gaussian free field on U, that is a centered Gaussian field with covariance $g_U(x,y)$, $x, y \in \widetilde{\mathcal{G}}$; we refer to [37] and [15] for further details regarding the cable system Gaussian free field. We further write '$x \leftrightarrow y$ in A' if there is a continuous path in $A \subset \widetilde{\mathcal{G}}$ from x to y, which is consistent with the notation introduced below (1.8) when identifying edges with their respective cables in $\widetilde{\mathcal{G}}$.

We now present a consequence of Lemma 2.1 and [37, Proposition 5.2], which is one of the key observations for the proofs of Theorems 1.1 and 1.2.

Corollary 2.2. *For all $\kappa, L \geqslant 1$, $R \geqslant L$, $n \in \mathbb{N}$, $x \in G \backslash B_{R+C_5L}$, $U \subset \widetilde{\mathcal{G}}$ open such that $\mathcal{O} \stackrel{\text{def.}}{=} G \cap U^{\mathsf{c}}$ is a (L,R,n,κ)-good obstacle, we have*

$$\mathbb{P}_U\big(0 \leftrightarrow x \text{ in } \{y \in U : \varphi_y \geqslant 0\}\big) \leqslant Cd(x, B_{R+C_5L})^{-\nu} \exp\Big\{-\frac{c\kappa n}{L^\nu}\Big\}.$$

Proof. If $0 \in \mathcal{O}$ or $x \in \mathcal{O}$ the statement is trivial. Otherwise, by the symmetry of the Gaussian free field and [37, Proposition 5.2] for the graph $\mathcal{G}$ with infinite killing on $\mathcal{O}$ we have

$$\mathbb{P}_U\big(0 \leftrightarrow x \text{ in } \{y \in U : \varphi_y \geqslant 0\}\big) = \frac{1}{\pi} \arcsin\Big(\frac{g_U(0,x)}{\sqrt{g_U(0,0)g_U(x,x)}}\Big). \tag{2.8}$$

Using the inequality $\arcsin(t) \leqslant \pi t/2$ for all $t \in [0,1]$, the right-hand side is upper bounded by

$$\frac{g_U(0,x)}{2\sqrt{g_U(0,0)g_U(x,x)}}.$$

Now note that $g_U(0,x) = g_U(x,x)P_0(H_x < H_{U^c})$ and $g_U(0,x) = g_U(0,0)P_x(H_0 < H_{U^c})$. Combining this with the fact that $P_0(H_x < H_{U^c}) \leqslant P_0(H_x < H_{\mathcal{O}})$ and that

$$P_x(H_0 < H_{U^c}) \leqslant P_x(H_0 < H_{\mathcal{O}}) = \lambda_0 P_0(H_x < H_{\mathcal{O}})/\lambda_x \leqslant CP_0(H_x < H_{\mathcal{O}}),$$

which is due to [47, (1.23)] and [13, (2.10)], one deduces that the right-hand side of (2.8) is bounded by $CP_0(H_x < H_{\mathcal{O}})$. The conclusion now follows using Lemma 2.1. □

3 Markovian loops

We now collect some useful properties concerning Markovian loop soups, as introduced for instance in [35, 34, 36, 37]. These properties will then be used in the next section to prove Theorem 1.1. Indeed, the upper bound (1.5) relies on a profound link between the Gaussian free field and a Poisson cloud of Markovian loops, which we now recall. The (rooted) loop soup $\mathcal{L}$ of parameter $\frac{1}{2}$ is a Poisson point process of (bounded, continuous and rooted) loops on $\widetilde{\mathcal{G}}$ under $\mathbb{Q}$ having intensity measure $\alpha\mu$, with $\alpha = \frac{1}{2}$ and a measure μ, which we proceed to introduce. The measure μ acts on the space of rooted loops, i.e. of continuous trajectories $\gamma : [0,T] \to \widetilde{\mathcal{G}}$ satisfying $\gamma(0) = \gamma(T)$, for some $T = T(\gamma) \in (0,\infty)$ called the duration of the loop. More precisely, we define

$$\mu(\,\cdot\,) \stackrel{\text{def.}}{=} \int_{\widetilde{\mathcal{G}}} \mathrm{d}m(x) \int_0^\infty \frac{\mathrm{d}t}{t} q_t(x,x) P^t_{x,x}(\,\cdot\,), \tag{3.1}$$

where $P^t_{x,x}(\,\cdot\,)$ is the time t bridge probability measure for the diffusion X on $\widetilde{\mathcal{G}}$ (introduced above (2.3)), m is the natural Lebesgue measure on $\widetilde{\mathcal{G}}$, which assigns length $1/(2\lambda_{x,y})$ to the cable joining x and y, and q_t is the transition density of X relative to m; see [25, Section 2] and [37, Section 2], where it is denoted by $\widetilde{\mathcal{L}}_{1/2}$, for details.

If $U \subset \widetilde{\mathcal{G}}$ is an open subset, the loop soup $\mathcal{L}_U$ is obtained by retaining only the loops in $\mathcal{L}$ whose range is contained in U. The restriction property of $\mathcal{L}$ (see [25, Theorem 6.1], or also [47, Proposition 3.6] in the discrete setting) asserts that

$$\mathcal{L}_U \text{ has law } \mathbb{Q}_U, \tag{3.2}$$

where $\mathbb{Q}_U$ now denotes the law of the metric graph loop soup with underlying graph U, that is with infinite killing on U^c. The loop soup $\mathcal{L}$ induces clusters in $\widetilde{\mathcal{G}}$ as follows. Two continuous loops belong to the same cluster if there exists a finite sequence of loops starting and ending with the two loops of interest, and such that the ranges of any two consecutive loops in the sequence intersect. Clusters of loops are connected components obtained in this way using loops in the support of $\mathcal{L}$. We will take advantage of the following link relating the loop soup $\mathcal{L}$ to the free field φ under $\mathbb{P}$. Considering

$$\mathcal{C} = \mathcal{C}(\mathcal{L}), \text{ the trace on } \widetilde{\mathcal{G}} \text{ of the cluster of loops (in } \mathcal{L}) \text{ containing } 0, \tag{3.3}$$

one knows (see for instance [37, Proposition 2.1] for a proof, which relies on an isomorphism of Le Jan, see [36, Theorem 13]) that

$$\text{(3.4)} \qquad \text{the cluster of 0 in } \{x \in \widetilde{\mathcal{G}} : |\varphi_x| > 0\} \text{ has the same law under } \mathbb{P} \text{ as } \mathcal{C} \text{ under } \mathbb{Q}.$$

Further notice that by [15, Lemma 4.1], if $\varphi_0 > 0$ then the closure of the cluster of 0 in $\{x \in \widetilde{\mathcal{G}} : |\varphi_x| > 0\}$ is almost surely equal to the cluster of 0 in $\{x \in \widetilde{\mathcal{G}} : \varphi_x \geqslant 0\}$, whose intersection with G has the same law as the cluster of 0 in $E^{\geqslant 0}$, see below (1.1), as explained around [16, (1.6)]. Combined with (3.4), this explains the relevance of $\mathcal{L}$ in our context.

We proceed to gather specific features of the latter that will be beneficial for us. To this effect it is convenient to consider the discrete time loop soup $\widehat{\mathcal{L}}$ on G induced by $\mathcal{L}$, which is obtained by considering the trace on G of all continuous loops in the support of $\mathcal{L}$ intersecting G, and only retaining non-trivial loops, i.e. any loop visiting more than one vertex of $\mathcal{G}$. As explained in [37, Section 2], combined with the same proof as in the finite setting of [47, (3.17)], the intensity measure of $\widehat{\mathcal{L}}$ can be described as follows. To any rooted loop γ on $\widetilde{\mathcal{G}}$ visiting at least two vertices of G, one associates $N = N(\gamma)$ the number of times ($\geqslant 2$) γ jumps to another vertex in G, and the corresponding discrete skeleton $Z_0 = Z_0(\gamma),\dots,Z_n = Z_n(\gamma) = Z_0$. Then for all $n \geqslant 2$ and $x_0, \dots, x_{n-1} \in G$,

$$\text{(3.5)} \qquad \mu(N = n, Z_0 = x_0, \dots Z_{n-1} = x_{n-1}) = \frac{1}{n} \prod_{0 \leqslant i < n} \frac{\lambda_{x_i, x_{i+1}}}{\lambda_{x_i}}, \quad (\text{with } x_n = x_0).$$

We now collect several useful properties of the measure μ in (3.1). In the special case of $\mathbb{Z}^\alpha$, $\alpha \geqslant 3$ integer (with unit weights), results of this kind have appeared in [9, Section 2]. These relied in the case $\alpha = 4$ on intersection results of [32] for two random walks. With hopefully obvious notation, we write $K \overset{\gamma}{\longleftrightarrow} U$ to denote the property that the loop γ intersects both K and U, for $K, U \subset G$.

Lemma 3.1 ($0 < \nu \leqslant \frac{\alpha}{2}$). *For all $\zeta \geqslant C$, $R \geqslant 1$, $x \in G$, and $K \subset B(x,R)$,*

$$\text{(3.6)} \qquad \mu\big(K \overset{\gamma}{\longleftrightarrow} B(x,\zeta R)^{\mathsf{c}}\big) \leqslant C \cdot \mathrm{cap}(K)(\zeta R)^{-\nu},$$

$$\text{(3.7)} \qquad \mu\big(K \overset{\gamma}{\longleftrightarrow} B(x,\zeta R)^{\mathsf{c}},\ \mathrm{cap}(\gamma) \geqslant c \cdot R^{\nu} \log(R)^{-b_2}\big) \geqslant c \cdot \mathrm{cap}(K)(\zeta R)^{-\nu},$$

where, with a slight abuse of notation $\mathrm{cap}(\gamma) = \mathrm{cap}(\mathrm{range}(\gamma) \cap G)$.

Proof. We start by showing (3.6). Abbreviate $B = B(x,\zeta R)$ and write $Z = Z(\gamma)$ for the discrete skeleton of the loop γ. Let $R_0 = H_K = \inf\{n \geqslant 0 : Z_n \in K\}$, with the convention $\inf \varnothing = \infty$, and for $k \geqslant 0$, let $D_{k+1} = R_k + T_B \circ \theta_{R_k}$, where we recall the notation $T_B = H_{B^{\mathsf{c}}}$, when $R_k < \infty$ and $D_k = \infty$ otherwise, and $R_{k+1} = D_{k+1} + H_K \circ \theta_{D_{k+1}}$ (possibly infinite). The random variable $\kappa = \kappa(\gamma) = |\{k \geqslant 1 : R_k \leqslant N(\gamma)\}|$, which denotes the number of times γ returns to K, is always positive (and finite) on the event appearing on the left-hand side of (3.6). Thus, using (3.5), decomposing over the value of κ and re-rooting the (discrete, rooted) loop Z at Z_{R_i}, where $i \in \{1, \dots, k\}$ is chosen uniformly at random, which produces a factor $\frac{n}{k}$, one obtains that

$$\text{(3.8)} \qquad \mu\big(K \overset{\gamma}{\longleftrightarrow} B(x,\zeta R)^{\mathsf{c}}\big) = \sum_{k \geqslant 1} \frac{1}{k} \sum_{y \in \partial K} P_y(R_k < \infty, Z_{R_k} = y),$$

with ∂K denoting the interior boundary of the set K. For $k \geqslant 1$, one has for all $y \in B(x,R)$, applying the Markov property at time D_{k-1} and noting that for $z \in K$ the function $u \mapsto P_u(R_1 <$

$\infty, Z_{R_1} = z)$ is harmonic in B,

$$\begin{aligned} &P_y(R_k < \infty, Z_{R_k} = z) \\ &= \sum_{u \in \partial K} P_y(R_{k-1} < \infty, Z_{R_{k-1}} = u) P_u(R_1 < \infty, Z_{R_1} = z) \\ &\overset{(2.5)}{\leqslant} \sum_{u \in \partial K} P_y(R_{k-1} < \infty, Z_{R_{k-1}} = u) c_3 P_x(R_1 < \infty, Z_{R_1} = z) \\ &\leqslant c_3 P_y(R_{k-1} < \infty) P_x(R_1 < \infty, Z_{R_1} = z). \end{aligned} \tag{3.9}$$

By a straightforward induction argument, one finds that $P_y(R_k < \infty, Z_{R_k} = z)$ is bounded from above by $(c_3 P_x(R_1 < \infty))^{k-1} P_x(R_1 < \infty, Z_{R_k} = z)$. We now choose $z = y$, sum over $y \in \partial K$ and plug the resulting estimate into (3.8). Consequently, using that $P_x(R_1 < \infty) \leqslant \sup_{z \in B^{\mathsf{c}}} P_z(H_K < \infty)$, which is at most $c \cdot \mathrm{cap}(K)(\zeta R)^{-\nu}$ by a last-exit decomposition and (G_ν), (in particular this implies that $c_3 P_x(R_1 < \infty) \leqslant 1 - c$ for some $c > 0$ upon possibly taking $\zeta \geqslant C$ and hence the convergence of the resulting geometric series in (3.8)), the bound (3.6) readily follows.

To deduce (3.7) one first observes, recalling the discussion that led to (3.8), that $\kappa(\gamma) \geqslant 1$ on the event appearing on the left-hand side of (3.7) if γ is rooted in K. To obtain a lower bound, one simply retains loops γ with $\kappa(\gamma) = 1$, re-roots such loops similarly as above (3.8), and requires the required capacity to be generated 'on the way back', i.e. after time D_1 and before exiting the ball of radius R around Z_{D_1} (which occurs before hitting K if $\lambda_0 > 2$), to find that for all $t \geqslant 1$,

$$\begin{aligned} &\mu\Big(K \overset{\gamma}{\longleftrightarrow} B(x, \zeta R)^{\mathsf{c}}, \, \mathrm{cap}(\gamma) \geqslant \frac{R^\nu}{t \log(R)^{b_2}}\Big) \\ &\qquad \geqslant \inf_{z \in \partial_{\mathrm{out}} B} P_z\Big(H_K < \infty, \mathrm{cap}\big(Z_{[0, T_{B(z,R)}]}\big) \geqslant \frac{R^\nu}{t \log(R)^{b_2}}\Big), \end{aligned} \tag{3.10}$$

where $Z_{[0,s]} = \{x \in G : Z_t = x \text{ for some } t \in [0, s]\}$ and $\partial_{\mathrm{out}} B = \{x \in B^{\mathsf{c}} : \exists\, y \in B \text{ with } y \sim x\}$. Applying the Markov property at time $T_{B(z,R)}$ and noting that $X_{T_{B(z,R)}} \in B(x, (\zeta + C)R)$ for all $z \in \partial_{\mathrm{out}} B$ by [13, (2.8)], the probability in the second line of (3.10) is bounded from below by

$$\inf_{z \in G} P_z\Big(\mathrm{cap}\big(Z_{[0, T_{B(z,R)}]}\big) \geqslant \frac{R^\nu}{t \log(R)^{b_2}}\Big) \cdot \inf_{z' \in B(x, (\zeta + C)R)} P_{z'}(H_K < \infty).$$

Regarding the second factor, by a similar computation as above involving a last-exit decomposition and the lower bounds in (G_ν), it is bounded from below by $c \cdot \mathrm{cap}(K)(\zeta R)^{-\nu}$. With respect to the first factor, observe that applying [16, Lemma 5.3] with K as appearing therein chosen as $K = \varnothing$ and $t = C$ sufficiently large, one infers that the infimum over z is bounded away from zero by $c > 0$, whence (3.7) follows. □

4 Critical one-arm probability

In this section we prove Theorem 1.1. We start with some preparation and consider the scales

$$R = C_7(\ell + 1)L, \qquad \text{for } \ell, L \geqslant 2, \tag{4.1}$$

with C_7 to be chosen momentarily (see (4.4) below). Recall the approximate lattice $\Lambda(L)$ from the beginning of Section 2. We define $\mathcal{A}_k \subset \Lambda(L)$ for $1 \leqslant k \leqslant \ell$ as

$$\mathcal{A}_k = \big\{x \in \Lambda(L) : B(x, L) \cap \partial B_{C_7 k L} \neq \varnothing\big\}, \tag{4.2}$$

and the associated 'annulus'

$$\mathbb{A}_k = \bigcup_{x \in \mathcal{A}_k} B(x, L) \ (\subset G). \tag{4.3}$$

Henceforth, the constant C_7 in (4.1) is fixed so that for all $\ell, L \geqslant 2$ and $k, k' \in \{1, \dots, \ell\}$,

$$\mathbb{A}_k \subset B_R \text{ and } d(\mathbb{A}_k, \mathbb{A}_{k'}) \geqslant |k - k'|L, \tag{4.4}$$

which is always possible by (4.2) and [13, (2.8)]. We now consider the continuous loop soup $\mathcal{L}$ of intensity $\alpha = \frac{1}{2}$ on $\widetilde{\mathcal{G}}$ with canonical law $\mathbb{Q}$, see Section 3 for details, and in the sequel we refer to as 'loop' any element in the support of its intensity measure. For given $\delta > 0$ and $L \geqslant 1$, we will call a loop *small* if

$$\mathrm{cap}(\gamma) < \delta L^{\nu} \log(L)^{-b_2}, \tag{4.5}$$

and *big* otherwise. With this terminology, the point measures $\mathcal{L}_k^b$ for $1 \leqslant k \leqslant \ell$ are defined as follows. If $\mathcal{L} = \sum_i \delta_{\gamma_i}$ refers to a generic realization, then

$$\mathcal{L}_k^b = \sum_i \delta_{\gamma_i} 1\{\gamma_i \text{ is big and range}(\gamma_i) \subset \widetilde{B}(x, L) \text{ for some } x \in \mathcal{A}_k\}, \tag{4.6}$$

where with hopefully obvious notation, $\widetilde{B}(x, L) \subset \widetilde{\mathcal{G}}$ is obtained by adding all the cables between neighboring vertices in $B(x, L)$. Recalling from (3.3) the cluster $\mathcal{C}$ of the origin in the loop soup $\mathcal{L}$, we consider the events

$$\mathbf{B}_k = \left\{ \begin{array}{c} \mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing \text{ and } \mathcal{C} \text{ does not contain the} \\ \text{trace of any loop in the support of } \mathcal{L}_k^b \end{array} \right\}. \tag{4.7}$$

Note that (4.7) depends implicitly on the choice of parameters δ, ℓ and L (furthermore, $\mathbf{B}_k$ is measurable since the restrictions in (4.5) and (4.6) induce measurable constraints). Intuitively, the event $\mathbf{B}_k$ refers to an annulus in which certain big loops are avoided by $\mathcal{C}$ (which we think of as being bad). Our interest will be in the quantity

$$N = N_{\delta, \ell, L} \stackrel{\text{def.}}{=} \sum_{1 \leqslant k \leqslant \ell} 1_{\mathbf{B}_k}. \tag{4.8}$$

The following result is key to the proof of Theorem 1.1. It asserts that large families of bad annulis (i.e., k's such that $\mathbf{B}_k$ occurs) are typically rare. Importantly, the quantitative error bound is sufficiently sharp.

Proposition 4.1. *For all integers $\ell, L \geqslant 2$, $\delta, \rho \in (0, \frac{1}{2})$ such that $L^{-c} \leqslant \delta \leqslant c_5 \log(\ell)^{-\frac{\nu}{\alpha}}$ and $\rho \log(\rho^{-1}) \leqslant c_6 \delta \log(L)^{-b_2}$, one has*

$$\mathbb{Q}\big(N \geqslant (1 - \rho)\ell\big) \leqslant C(\ell L)^{\alpha} \exp\left\{ - \frac{c_7 \delta \ell}{\log(L)^{b_2}} \right\}. \tag{4.9}$$

We will prove Proposition 4.1 further below and first show how to deduce Theorem 1.1 from it. This involves an estimate on the capacity of the union of many big loops, such as the cluster $\mathcal{C}$ on the complement of the event in (4.9). The following lemma gives a lower bound which is not far from additive.

Lemma 4.2. *For all integers $\ell, L \geqslant 2$, the following holds. Let $I \subset \{1, \dots, \ell\}$ and assume that $S_k \subset \mathbb{A}_k$ with cap$(S_k) \geqslant \eta$ for each $k \in I$. Then, recalling b_1 from (1.17), one has*

$$\mathrm{cap}\Big(\bigcup_{k \in I} S_k\Big) \geqslant c_8 |I| \Big(L^{\nu} \log(1 + |I|)^{-b_1} \wedge \inf_{k \in I} \mathrm{cap}(S_k)\Big). \tag{4.10}$$

Proof. By a classical variational characterization of capacity, see for instance [47, Prop. 1.9] for a proof on finite graphs, for any $S \subset G$ one has that

$$\operatorname{cap}(S) = \Big(\inf_{\mu} \sum_{x,y \in G} \mu(x) g(x,y) \mu(y)\Big)^{-1}, \tag{4.11}$$

where the infimum ranges over all probability measures μ supported on S. Consider the measure

$$\mu(x) \stackrel{\text{def.}}{=} \frac{1}{|I|} \sum_{k \in I} \bar{e}_{S_k}(x), \quad x \in G, \tag{4.12}$$

where $\bar{e}_{S_k} = e_{S_k}/\operatorname{cap}(S_k)$ denotes the normalized equilibrium measure, so μ is indeed a probability measure supported on $\bigcup_{k \in I} S_k$. Evaluating the sum on the right-hand side of (4.11) at μ from (4.12), and noting that for all $x \in S_k$, $y \in S_j$, $1 \leqslant k \neq j \leqslant \ell$, the bound $g(x,y) \leqslant C d(\mathbb{A}_k, \mathbb{A}_j)^{-\nu} \leqslant C L^{-\nu} |j-k|^{-\nu}$ holds owing to (G_ν) and (4.4), it follows that

$$\begin{aligned}
\operatorname{cap}\Big(\bigcup_{k \in I} S_k\Big)^{-1} &\leqslant \frac{1}{|I|^2} \sum_{k,j \in I} \sum_{x \in S_k} \sum_{y \in C_j} \bar{e}_{S_k}(x) g(x,y) \bar{e}_{C_j}(y) \\
&\stackrel{(2.3)}{\leqslant} \frac{1}{|I|^2} \sum_{k \in I} \frac{1}{\operatorname{cap}(S_k)} + \frac{2}{|I|} \sum_{n=1}^{|I|} C(nL)^{-\nu},
\end{aligned}$$

from which (4.10) readily follows. □

Using Proposition 4.1 and Lemma 4.2, we now give the proof of our first main result.

Proof of Theorem 1.1. On account of (3.4) and below, and since φ and $-\varphi$ have the same law under $\mathbb{P}$, one has that

$$\mathbb{P}(0 \leftrightarrow B_R^{\mathsf{c}}) = \frac{1}{2} \mathbb{Q}(\mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing). \tag{4.13}$$

As a consequence, it is sufficient to upper bound the quantity on the right-hand side of (4.13). It is further sufficient to assume that $R \geqslant C$, which will be tacitly supposed from here on. In particular, no loss of generality is incurred by assuming that R is of the form (4.1) and by proving the statement for a suitable choice of $\ell > 1$ (see (4.18) below) and all $L \geqslant C$.

As we now explain, for all $\ell > 1$, $L \geqslant C$, $\delta \in (0,c)$ and $\rho \in (0,1)$, with

$$\eta = c_8 \rho \ell L^{\nu} \Big(\frac{\delta}{\log(L)^{b_2}} \wedge \frac{1}{\log(1+\rho\ell)^{b_1}}\Big) \tag{4.14}$$

(see Lemma 4.2 regarding c_8), one has the inclusion

$$\big\{\mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing, \operatorname{cap}(\mathcal{C}) < \eta\big\} \subset \{N \geqslant (1-\rho)\ell\} \tag{4.15}$$

with the random variable N as in (4.8). To see this, suppose that $\{N < (1-\rho)\ell\}$. We now argue that the intersection of this event with the one on the left-hand side of (4.15) is empty, from which the desired inclusion in (4.15) follows. For this purpose, consider the (random) set $I = \{k : \mathbf{B}_k \text{ does not occur}\} \subset \{1, \dots, \ell\}$, so that

$$|I| = \ell - N > \rho \ell \tag{4.16}$$

on the event $\{N < (1-\rho)\ell\}$. By definition of $\mathbf{B}_k$ in (4.7) and the defining property of big loops, see (4.5), one has on the event $\{\mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing\}$ that $\mathcal{C} \supset \bigcup_{k \in I} S_k$, where $S_k \subset \mathbb{A}_k$

comprises at least the range of one loop of capacity at least $\delta L^{\nu}(\log L)^{-b_2}$ for each $k \in I$, whence $\mathrm{cap}(S_k) \geqslant \delta L^{\nu}(\log L)^{-b_2}$ by monotonicity of the capacity. Applying Lemma 4.2, this yields that

$$\mathrm{cap}(\mathcal{C}) \geqslant c_8(\ell - N)L^{\nu}\big((\delta \log(L)^{-b_2}) \wedge \log(1+\ell-N)^{-b_1}\big) \overset{(4.16),(4.14)}{\geqslant} \eta,$$

and (4.15) follows.

Using (4.13), (4.15), the tail bound $\mathbb{Q}(\mathrm{cap}(\mathcal{C}) > t) \leqslant Ct^{-1/2}$ (which holds by (3.4)) and [16, Corollary 1.3], it follows from (4.9) that under the assumptions of Proposition 4.1,

$$\begin{aligned} \mathbb{P}(0 \leftrightarrow B_R^{\mathsf{c}}) &\leqslant \mathbb{Q}(\mathrm{cap}(\mathcal{C}) \geqslant \eta) + \mathbb{Q}(N \geqslant (1-\rho)\ell) \\ &\leqslant C\big(\eta^{-1/2} + (\ell L)^{\alpha} \exp\{-c_7 \log(L)^{-b_2}\delta\ell\}\big). \end{aligned} \tag{4.17}$$

We now choose the parameters δ, ρ and ℓ so that they satisfy the assumptions of Proposition 4.1, and that the second line of (4.17) is small. For $\gamma \in (0,1)$ to be specified in a moment, let

$$\begin{aligned} \delta &= c_5(\log \ell)^{-\nu/\alpha}, \\ \rho &= \gamma(\log L)^{-b_2}(\log\log L)^{-(b_2+\frac{\nu}{\alpha})}(\log\log\log L)^{-1+b_2}, \\ \ell &= \gamma^{-1}(\log L)^{1+b_2}(\log\log L)^{\nu/\alpha}. \end{aligned} \tag{4.18}$$

Observe that $\gamma\log(1/\gamma) \to 0$ as $\gamma \to 0$. Choosing γ small enough one can thus ensure that $\rho\log(1/\rho) \leqslant c_6\delta\log(L)^{-b_2}$ for all $L \geqslant C$, and hence by our choice of δ and ℓ, the assumptions of Proposition 4.1 are satisfied upon possibly further increasing L. In particular, (4.17) is in force and, possibly reducing the value of γ even further so that $\frac{c_7c_5}{\gamma} > \alpha + \frac{\nu}{2}$, the second term in brackets in the second line of (4.17) is $O(L^{-\nu/2-\varepsilon})$ for some $\varepsilon > 0$ as $L \to \infty$. In view of (4.14), (4.17) and (4.18), this means that the term $\eta^{-1/2}$ dominates, and one obtains from (4.14), noting further that $\delta\log(L)^{-b_2} \geqslant \log(1+\rho\ell)^{-b_1}$ if and only if $\nu = 1$, that for all $L \geqslant C$,

$$\begin{aligned} &\mathbb{P}(0 \leftrightarrow B_R^{\mathsf{c}}) \\ &\quad\leqslant C(\log\log\log L)^{\frac{1-b_2}{2}}(\log\log L)^{\frac{b_1+b_2}{2}+\frac{(1+\nu-b_1)\nu}{2\alpha}}(\log L)^{\frac{\nu-1+b_2(\nu+1)}{2}} \cdot (\ell L)^{-\frac{\nu}{2}}. \end{aligned} \tag{4.19}$$

The claim now follows since one can safely replace L by ℓL inside the log and $\log\log$ factors by definition of ℓ in (4.18) and the exponents of $\log\log R$ in (1.6) arise using that $\nu < \alpha/2$ in the second case and that $\alpha \geqslant \nu + 2 = 3$ when $\nu = 1$. □

We now prove Proposition 4.1.

Proof of Proposition 4.1. Recall the event $\mathbf{B}_k$ from (4.8), and for arbitrary $D \subset \{1,\dots,\ell\}$ consider the event $\mathbf{B}_D = \bigcap_{k\in D}\mathbf{B}_k$. For all $D \subset \{1,\dots,\ell\}$ with $|D| \geqslant (1-\rho)\ell$, and δ, ℓ, L, ρ as appearing in the statement, we will argue that if $L^{-c} \leqslant \delta \leqslant (c_5\log(\ell)^{-\frac{\nu}{\alpha}})$,

$$\mathbb{Q}\big(\mathbf{B}_D\big) \leqslant C(\ell L)^{\alpha}\exp\Big\{-\frac{c(1-\rho)\delta\ell}{\log(L)^{b_2}}\Big\}. \tag{4.20}$$

Once this is shown, applying a union bound over subsets of $\{1,\dots,\ell\}$ of cardinality $\lceil(1-\rho)\ell\rceil$ and using the bound $\binom{\ell}{\lceil(1-\rho)\ell\rceil} \leqslant e^{C\rho\log(1/\rho)\ell}$ valid for all $\rho \in (0,\frac{1}{2})$, which is a consequence of Stirling's formula, the bound (4.9) directly follows when $\rho\log(\rho^{-1}) \leqslant c_6\delta\log(L)^{-b_2}$ for c_6 small enough.

We now prove (4.20). With $\mathcal{L}_k^b$ as in (4.6) let

$$\mathcal{L}_D^b = \sum_{k\in D}\mathcal{L}_k^b. \tag{4.21}$$

In words, $\mathcal{L}_D^b$ collects all the big loops contained in a box $\widetilde{B}(x,L)$ for some $x \in \bigcup_{k\in D} \mathcal{A}_k$. Upon intersection with G, their union forms the set $\mathcal{O}$, i.e., writing $\mathcal{L}_D^b = \sum_i \delta_{\gamma_i}$ we define

$$\mathcal{O} \stackrel{\text{def.}}{=} G \cap \widetilde{\mathcal{I}}_D^b, \quad \text{where } \widetilde{\mathcal{I}}_D^b = \bigcup_i \text{range}(\gamma_i). \tag{4.22}$$

Recall the notion of a good obstacle set from above Lemma 2.1. We first isolate the following result.

Lemma 4.3. *For $\ell, L \geqslant 2$, $L^{-c} \leqslant \delta \leqslant c_5 \log(\ell)^{-\frac{\nu}{\alpha}}$, $D \subset \{1,\dots,\ell\}$ with $|D| \geqslant \frac{\ell}{2}$, R as in (4.1) and $\kappa = \delta L^{\nu}(\log L)^{-b_2}$, letting $\mathbf{G} = \{\mathcal{O} \text{ is a } \big(L, R, \frac{|D|}{2}, \kappa\big)\text{-good obstacle set}\}$, one has*

$$\mathbb{Q}(\mathbf{G}) \geqslant 1 - \exp\big\{ - c\delta^{-\frac{\alpha}{\nu}} |D|\big\}. \tag{4.23}$$

We delay the proof of Lemma 4.3 for a few lines. Since the point measures $\mathcal{L}_k^b$ have disjoint support as k varies, owing to (4.6), (4.3) and (4.4) the event $\mathbf{B}_D$ can be recast in view of (4.7) and (4.21) as

$$\mathbf{B}_D = \big\{\mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing, \mathcal{C} \cap \widetilde{\mathcal{I}}_D^b = \varnothing\big\} \subset \big\{0 \leftrightarrow B_R^{\mathsf{c}} \text{ in } \mathcal{L}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}\big\} \tag{4.24}$$

(recall from above (3.2) that $\mathcal{L}_U$ denotes the restriction of $\mathcal{L}$ to loops contained in U). One issue with the event $\mathbf{B}_D$ rendered visible by (4.24) is that $\mathcal{L}$ is not independent from $\widetilde{\mathcal{I}}_D^b$—in particular, (3.2) is not directly applicable in this context with $U = \widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b$. To deal with this issue, we proceed as follows. Let $\check{\mathcal{L}}_D^b \stackrel{\text{law}}{=} \mathcal{L}_D^b$ denote a copy independent of $\mathcal{L}$ defined under $\mathbb{Q}$ by suitably enlarging the underlying probability space. Let $\mathcal{L}' = \mathcal{L} - \mathcal{L}_D^b$ and define

$$\check{\mathcal{L}} = \check{\mathcal{L}}_D^b + \mathcal{L}'.$$

Since $\mathcal{L}'$ is independent from $\mathcal{L}_D^b$ and using the definition of $\check{\mathcal{L}}_D^b$, one infers that $\check{\mathcal{L}} \stackrel{\text{law}}{=} \mathcal{L}$ and that $\check{\mathcal{L}}$ and $\mathcal{L}_D^b$ are independent under $\mathbb{Q}$. In particular, since $\widetilde{\mathcal{I}}_D^b$ is plainly $\mathcal{L}_D^b$-measurable in view of (4.22), one obtains that

$$\check{\mathcal{L}} \text{ and } \widetilde{\mathcal{I}}_D^b \text{ are independent under } \mathbb{Q}. \tag{4.25}$$

Moreover, $\mathcal{L}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b} \leqslant \mathcal{L}'$ by definition of $\widetilde{\mathcal{I}}_D^b$ and therefore $\mathcal{L}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b} \leqslant \check{\mathcal{L}}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}$ by monotonicity. Thus, returning to (4.24), observing that the event $\mathbf{G}$ defined above (4.23) is $\widetilde{\mathcal{I}}_D^b$-measurable and that, conditionally on $\widetilde{\mathcal{I}}_D^b$, the loop soup $\check{\mathcal{L}}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}$ has law $\mathbb{Q}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}$ owing to (4.25) and the restriction property (3.2), it follows that

$$\mathbb{Q}(\mathbf{B}_D, \mathbf{G}) \leqslant \mathbb{Q}\big(0 \leftrightarrow B_R^{\mathsf{c}} \text{ in } \check{\mathcal{L}}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}, \mathbf{G}\big) = 2\mathbb{E}^{\mathbb{Q}}\big[\mathbb{P}_{\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b}(0 \leftrightarrow B_R^{\mathsf{c}}) \cdot 1_{\mathbf{G}}\big], \tag{4.26}$$

where the last step also uses the formula (4.13) applied to $\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{I}}_D^b$ instead of $\widetilde{\mathcal{G}}$. Applying a union bound over points on the boundary of $G \setminus B_R$ connected to 0 on the right-hand side of (4.26) and subsequently applying Corollary 2.2, which is in force on the event $\mathbf{G}$, with the choices $n = (1-\rho)\frac{\ell}{2}$ and κ as in Lemma 4.3, one deduces that

$$\mathbb{Q}(\mathbf{B}_D, \mathbf{G}) \leqslant C(\ell L)^{\alpha} \exp\bigg\{ - \frac{c(1-\rho)\delta\ell}{\log(L)^{b_2}}\bigg\}.$$

Together with the bound on $\mathbb{Q}(\mathbf{G}^{\mathsf{c}})$ provided by Lemma 4.3, the upper bound (4.20) immediately follows, which completes the proof. □

It remains to provide the proof of Lemma 4.3.

Proof of Lemma 4.3. The event $\mathbf{G}$ in question refers to the existence of a set $A \subset \mathrm{range}(\pi \cap B_R)$ with $|A| \geqslant \frac{|D|}{2}$ for every path π in $\Lambda(L)$ from 0 to B_R^{c} with the property that

$$\mathrm{cap}\big(\widetilde{\mathcal{I}}_D^b \cap B(y,L)\big) \geqslant \delta L^\nu \log(L)^{-b_2}, \quad y \in A. \tag{4.27}$$

Let π be any such path. By definition of $\mathcal{A}_k$ in (4.2), the range of π intersects $\mathcal{A}_k$ for any $k \in D$. We pick one such point of intersection for every $k \in D$, which defines the set A'. In view of (2.2) and (4.1), applying a union bound over the possible choices for A' yields a factor bounded by

$$|B_R \cap \Lambda(L)|^{|D|} \leqslant e^{C\log(\ell)|D|}. \tag{4.28}$$

As we now explain, it is enough to consider a fixed set $A' = \{x_k : k \in D\}$ with $x_k \in \mathcal{A}_k$ for all $k \in D$ and to show for

$$A \overset{\text{def.}}{=} \{x_k : \widetilde{B}(x_k, L) \text{ contains a big loop in } \mathcal{L}\} \tag{4.29}$$

that for $\delta \geqslant L^{-c}$,

$$\mathbb{Q}\big(|A| < \tfrac{|D|}{2}\big) \leqslant e^{-c\delta^{-\frac{\alpha}{\nu}}|D|}. \tag{4.30}$$

Indeed the set A defined by (4.29) satisfies (4.27) by definition of $\widetilde{\mathcal{I}}_D^b$ and of big loops, see (4.5), and A has the required cardinality on the complement of the event in (4.30). Furthermore, for $\delta \leqslant c_5 \log(\ell)^{-\frac{\nu}{\alpha}}$, the entropy term (4.28) gets absorbed in the exponential from (4.30).

To show (4.30) one observes that the events $E_k = \{\widetilde{B}(x_k, L) \text{ contains a big loop in } \mathcal{L}\}$ are independent as k varies since the sets $\widetilde{B}(x_k, L)$ are disjoint by construction; see (4.4). Hence $|A|$ dominates a binomial random variable with parameters $|D|$ and $p = \inf_k \mathbb{Q}(E_k)$. We will now argue that for all $\delta \in (0,1)$ and $\delta \geqslant L^{-c}$,

$$p \geqslant 1 - e^{-c\delta^{-\frac{\alpha}{\nu}}}. \tag{4.31}$$

Once this is shown, a union bound gives that $\mathbb{Q}(|A| < n) \leqslant 2^n(1-p)^n$ for $n = |D|/2$ and (4.30) follows for $\delta \in (0,c)$ for some small enough constant $c > 0$.

To argue that (4.31) holds, we use Lemma 3.1. Let $x \in G$. By (3.7) applied with $\zeta = C$ to $K = B(x, C_8\delta^{1/\nu}L)$ for suitable choice of C_8, we can ensure that

$$\mathbb{Q}\left(\begin{array}{c} \exists\gamma \in \mathrm{supp}(\mathcal{L}) \text{ s.t. } \mathrm{range}(\gamma) \cap B(x, C_8\delta^{1/\nu}L) \neq \varnothing \\ \text{and } \mathrm{cap}(\gamma) \geqslant \delta L^\nu \log(\delta^{1/\nu}L)^{-b_2} \end{array} \right) \geqslant c_9 \tag{4.32}$$

whenever $\delta^{1/\nu}L \geqslant 1$. Requiring that $\delta \geqslant L^{-c}$, one further ensures that $\log(\delta^{1/\nu}L) \geqslant c\log L$ (along with $\delta^{1/\nu}L \geqslant 1$), which effectively allows to remove the factor $\delta^{1/\nu}$ from the logarithm in (4.32). Now, applying (3.6) with ζ a large enough constant, we can further ensure that with probability at most $\frac{c_9}{2}$, a loop in $\mathrm{supp}(\mathcal{L})$ will intersect both K and the complement of $B(x, \zeta C_8\delta^{1/\nu}L)$. Combining this with (4.32) yields that

$$\mathbb{Q}\left(\begin{array}{c} \exists\gamma \in \mathrm{supp}(\mathcal{L}) \text{ s.t. } \mathrm{range}(\gamma) \cap B(x, C_8\delta^{1/\nu}L) \neq \varnothing, \\ \mathrm{range}(\gamma) \subset B(x, \zeta C_8\delta^{1/\nu}L) \text{ and } \mathrm{cap}(\gamma) \geqslant \delta L^\nu \log(L)^{-b_2} \end{array} \right) \geqslant \frac{c_9}{2}. \tag{4.33}$$

One now considers for a given x_k as above, the set $\widetilde{\Lambda}_k = B(x_k, L) \cap \Lambda(2\zeta C_8\delta^{1/\nu}L)$, so that the boxes $B(x, \zeta C_8\delta^{1/\nu}L)$ are disjoint as x varies in $\widetilde{\Lambda}_k$ by (2.2), and forms the subset $\Lambda_k \subset \widetilde{\Lambda}_k$ by keeping only those points x such that $B(x, \zeta C_8\delta^{1/\nu}L) \subset B(x_k, L)$. By (2.2) and ($V_\alpha$), $|\Lambda_k| \geqslant c\delta^{-\frac{\alpha}{\nu}}$ and the events in (4.33) are independent as x varies in Λ_k by construction. The claim (4.31) now follows, since the occurrence of at least one of these events already implies E_k. $\square$

Remark 4.4. If one is only interested in proving (1.7), that is obtaining (1.5) for some subpolynomial function q, the above proof of Theorem 1.1 can be simplified as follows. One replaces the events $\mathbf{B}_k$ from (4.7) by a single event

$$\mathbf{B} = \{\mathcal{C} \cap B_R^{\mathsf{c}} \neq \varnothing, \mathcal{C} \cap B_R \text{ does not contain any big loop}\},$$

where we recall the definition of big loops from below (4.5), and notice that by a similar reasoning as in (4.17)

$$\mathbb{P}(0 \leftrightarrow B_R^{\mathsf{c}}) \leqslant \mathbb{P}(\mathbf{B}) + \mathbb{P}(\mathcal{C} \geqslant \delta L^{\nu} \log(L)^{-b_2}) \leqslant \mathbb{P}(\mathbf{B}) + C\delta^{-1/2} L^{-\nu/2} \log(L)^{b_2/2}.$$

One can then define $\mathcal{O}$ as the set of vertices in B_R hit by a big loop, show similarly as in Lemma 4.3 that $\mathcal{O}$ is a $(L, R, \ell/2, \kappa)$-good obstacle with high probability, and hence proceeding similarly as in (4.26) and below one deduces that $\mathbb{P}(\mathbf{B}) \leqslant (L\ell)^{\alpha} \exp\{-c\delta \log(L)^{-b_2}\ell\}$. Taking $\delta = c$ and $\ell = \log(L)^{1+b_2}$, it follows that

$$\mathbb{P}(0 \leftrightarrow B_R^{\mathsf{c}}) \leqslant CL^{-\nu/2} \log(L)^{b_2/2} \leqslant CR^{-\nu/2} \log(R)^{(\nu+b_2(\nu+1))/2}$$

by (4.1). When $\nu > 1$ this simply corresponds to a higher log power in (1.6), but when $\nu = 1$, for instance on $\mathbb{Z}^3$, one obtains $q(R) = \log(R)^{1/2}$ instead of $\log\log R$ in (1.6). Incidentally, this factor $\log(R)^{1/2}$ on $\mathbb{Z}^3$ already appeared multiple times before, see [11, 16], and it can also be deduced from [15, (1.9)], hence the interest of improving it to $\log\log R$. Intuitively, this improvement comes from asking to meet many big loops in Proposition 4.1, whose union have a much bigger capacity than a single loop by Lemma 4.2.

5 Bounds on the two-point function

In this section, we prove Theorem 1.2 and Corollary 1.3, and at the very end provide a short derivation of (1.9). One important tool will be the random interlacements model, originally introduced on $\mathbb{Z}^d$ in [45] (cf. also [17] for an introduction to the model), extended to general transient graphs in [50], and to the cable system in [37]. We refer for instance to [15, Section 2.5] for a brief introduction of the model at the level of generality needed in the present context.

Under the probability $\overline{\mathbb{P}}$, we denote by $\widetilde{\mathcal{I}}^u$ the closure of the interlacement set at level $u > 0$ on $\widetilde{\mathcal{G}}$, and by $\widetilde{\mathcal{V}}^u \overset{\text{def.}}{=} \widetilde{\mathcal{G}} \backslash \widetilde{\mathcal{I}}^u$ the corresponding vacant set on $\widetilde{\mathcal{G}}$. We further abbreviate by $\mathcal{I}^u \overset{\text{def.}}{=} \widetilde{\mathcal{I}}^u \cap G$ the (usual) interlacement set on G. With the notation introduced above Corollary 2.2, conditionally on $\widetilde{\mathcal{I}}^u$, the measure $\mathbb{P}_{\widetilde{\mathcal{V}}^u}$ is the law of a Gaussian free field killed on $\widetilde{\mathcal{I}}^u$. The interest of random interlacement in the study of the Gaussian free field is due to an isomorphism theorem between the two objects, first observed in [46], and then improved in [37, 49, 15]. Our first result is a useful consequence of this isomorphism combined with the restriction property (3.2) and the loop soup isomorphism, in the (weak) form (3.4). For $a \in \mathbb{R}$, let $\widetilde{\mathcal{K}}^a = \widetilde{\mathcal{K}}^a(\varphi) \subset \widetilde{\mathcal{G}}$ denote the connected component of 0 (see (2.1)) in $\{x \in \widetilde{\mathcal{G}} : \varphi_x \geqslant a\}$.

Lemma 5.1. *For each $u > 0$, there exists a coupling of $(\varphi, \widetilde{\mathcal{V}}^u, \gamma)$, with φ having law $\mathbb{P}$, $\widetilde{\mathcal{V}}^u$ a vacant set with the same law as under $\overline{\mathbb{P}}$, and γ with law $\mathbb{P}_{\widetilde{\mathcal{V}}^u}$ conditionally on $\widetilde{\mathcal{V}}^u$, such that*

$$\widetilde{\mathcal{K}}^{\sqrt{2u}} \subset \{x \in \widetilde{\mathcal{V}}^u : \gamma_x \geqslant 0\}.$$

Proof. By [15, Theorem 1.1 and Lemma 3.4,(2)], the isomorphism (Isom) on p.259 therein is satisfied for any graph satisfying (G_ν), such as $\mathcal{G}$. Using the symmetry of the Gaussian free field, one readily deduces from this isomorphism that

$$\widetilde{\mathcal{K}}^{\sqrt{2u}} \text{ has the same law under } \mathbb{P} \text{ as } \widetilde{\mathcal{K}}^0 1\{\widetilde{\mathcal{K}}^0 \subset \widetilde{\mathcal{V}}^u\} \text{ under } \mathbb{P} \otimes \overline{\mathbb{P}}, \tag{5.1}$$

where $A1\{E\}$ is the set which is equal to A if the event E occurs, and is equal to $\varnothing$ otherwise. By (3.4) and the symmetry of $\mathbb{P}$ under $\varphi \mapsto -\varphi$, the set $\widetilde{\mathcal{K}}^0$ is either empty with probability $\frac{1}{2}$, or has the same law as the cluster $\mathcal{C}$ of 0 in $\mathcal{L}$ under $\mathbb{Q}$ otherwise. Moreover, since $\widetilde{\mathcal{V}}^u$ and $\mathcal{L}$ are independent, it follows from (3.2) that, conditionally on $\widetilde{\mathcal{V}}^u$, the loops $\mathcal{L}_{\widetilde{\mathcal{V}}^u}$ which are entirely contained in $\widetilde{\mathcal{V}}^u$ have the same law as a loop soup under $\mathbb{Q}_{\widetilde{\mathcal{V}}^u}$. Therefore, conditionally on $\widetilde{\mathcal{V}}^u$ the set $\widetilde{\mathcal{K}}^{\sqrt{2u}}$ is stochastically dominated by a set which is either empty with probability $\frac{1}{2}$, or is the cluster of 0 in a loop soup under $\mathbb{Q}_{\widetilde{\mathcal{V}}^u}$ otherwise (to see this, notice that whenever the latter set is non-empty, the inclusion from (5.1) translates into the requirement that the cluster of 0 in $\mathcal{L}_{\widetilde{\mathcal{V}}^u}$ is equal to the cluster of 0 in $\mathcal{L}$). Using (3.4) but for the graph $\widetilde{\mathcal{G}}$ with infinite killing on $\widetilde{\mathcal{I}}^u$ and the symmetry of the Gaussian free field again, the claim follows. □

As we detail below, one obtains from the first passage percolation upper bound in [42, Theorem 5.4] that the set $\mathcal{I}^u$ is a (L, R, n, κ)-good obstacle set (see above Lemma 2.1) with high probability for an appropriate choice of L, n, κ, and combining this with Corollary 2.2 and Lemma 5.1 we now deduce Theorem 1.2.

Proof of Theorem 1.2. The statement is trivial for $a = 0$ and by [15, Lemma 4.3], we may assume without loss of generality that $a > 0$. Abbreviate $u = a^2/2$, let $\eta \in (0,1)$, and recall τ_a^{tr} from (1.8). For $a > 0$ the truncation $\{x \nleftrightarrow \infty \text{ in } E^{\geqslant a}\}$ has probability one, hence applying Lemma 5.1 we have that

$$\tau_a^{\mathrm{tr}}(0,x) = \tau_{\sqrt{2u}}^{\mathrm{tr}}(0,x) \leqslant \overline{\mathbb{E}}\Big[\mathbb{P}_{\widetilde{\mathcal{V}}^u}\big(0 \leftrightarrow x \text{ in } \{y \in \widetilde{\mathcal{V}}^u : \varphi_y \geqslant 0\}\big)\Big]. \tag{5.2}$$

To bound the right-hand side of (5.2), we use Corollary 2.2. Recall the notion of a good obstacle set introduced above Lemma 2.1. By [42, (6.4)] (see also below [42, (2.22)] regarding the function G_ν appearing therein), which holds without the condition [42, (2.5)] for similar reasons as explained in [42, Remark 6.6], there exist $c, c_{10} > 0$ and $C < \infty$, depending on η, such that letting $L = Cu^{-\frac{1}{\nu}}$, $n = cRu^{\frac{1}{\nu}}$ and $\kappa = u^{-1}(\log(\frac{1}{u}) \vee 1)^{-b_2}$, if $u^{\frac{1}{\nu}}R \geqslant C$, then

$$\overline{\mathbb{P}}\big(\mathcal{I}^u \text{ is a } (L, R(1-\eta), n, \kappa)\text{-good obstacle}\big) \geqslant 1 - \exp\Big\{-\frac{c_{10}u^{\frac{1}{\nu}}R(1-\eta)}{\log(u^{\frac{1}{\nu}}R(1-\eta))^{b_1}}\Big\}. \tag{5.3}$$

We now set $R = d(0,x)$ in what follows and first assume that $u^{\frac{1}{\nu}}d(0,x) \geqslant C$. Using that $d(x, B_{R(1-\eta)+CL}) \geqslant (\eta/2)d(x,0)$ if $Ra^{\frac{2}{\nu}}$ is large enough, that $\mathbb{P}_{\widetilde{\mathcal{V}}^u}\big(0 \leftrightarrow x \text{ in } \{y \in \widetilde{\mathcal{V}}^u : \varphi_y \geqslant 0\}\big) \leqslant Cd(0,x)^{-\nu}$ $\overline{\mathbb{P}}$-a.s., and that $\tau_0^{\mathrm{tr}}(0,x) \geqslant cd(0,x)^{-\nu}$ by (G_ν) and [37, Proposition 5.2], combining Corollary 2.2 (with $R(1-\eta)$ in place of R therein), (5.2) and (5.3) for $\eta = 1/2$, one obtains (1.11) when $a^{\frac{2}{\nu}}R \geqslant C$ by our choices of u, L, n and κ. Finally when $a^{\frac{2}{\nu}}R \leqslant C$, one can easily check that (1.11) remains valid by monotonicity of $a \mapsto \tau_a^{\mathrm{tr}}(0,x)$ modulo adapting the constant C_2. □

Proof of Corollary 1.3. Throughout this proof we assume that $d(x,y) = |x-y|_2$ is the Euclidian distance on $\mathbb{Z}^3$, and similarly as before we can assume without loss of generality that $a > 0$. Furthermore, while $\nu = 1$ in the current setting, we still write ν for enhanced comparison with the general setting of [16], to which we frequently refer in the sequel. For the upper bound, one notices that the constant c_{10} from (5.3) can be taken equal to $(1-\eta)\pi/3$ when $Ru^{\frac{1}{\nu}}$ is large enough since by definition it is equal to $(1-\eta)$ times the constant $c_{38} = c_4$ from [42, Theorem 5.4], and that $c_4 = \pi/3$ on $\mathbb{Z}^3$ by [42, (2.6)]. Noting that when λ is large enough, then (5.3) with the choices $R = \lambda\xi = \lambda a^{-\frac{2}{\nu}}$ and $u = a^2/2$ constitutes the main contribution to the bound (1.11) with $R = d(x,y)$, one readily concludes the upper bound after a change of variable for η.

For the lower bound, we follow a strategy similar to the proof of [16, (8.3)], which however requires an adaptation to obtain the exact constant $\pi/6$ in (1.13), that we are now going to detail. Let $a>0$, $R=\lambda\xi$, $h_{K\cup K'}(x)=P_x(H_{K\cup K'}<\infty)$ for $K,K'\subset\widetilde{\mathcal{G}}$ and

$$A(K,K',a)=\Big\{K\leftrightarrow K' \text{ in } \{x\in\widetilde{\mathcal{G}}\backslash(K\cup K'):\,\varphi_x\geqslant a(1-\eta-h_{K\cup K'}(x))\}\Big\},$$

where $K\leftrightarrow K'$ in A means here that $K\cup K'\cup A$ is a connected set in $\widetilde{\mathcal{G}}$. Then similarly as in [16, (6.17)], it follows from the Markov property of the Gaussian free field that for all $s,t>0$

$$\tau^{\mathrm{tr}}_{(1-\eta)a}(0,[Re])\geqslant\mathbb{E}\big[1\{\mathrm{cap}(\widetilde{\mathcal{K}}^a_{s\xi})\geqslant t\xi^\nu,\mathrm{cap}(\widetilde{\mathcal{K}}^a_{s\xi}(R))\geqslant t\xi^\nu\}\mathbb{P}_{\mathcal{U}}\big(A(\widetilde{\mathcal{K}}^a_{s\xi},\widetilde{\mathcal{K}}^a_{s\xi}(R),a)\big)\big],\tag{5.4}$$

where $\widetilde{\mathcal{K}}^a_{s\xi}$, respectively $\widetilde{\mathcal{K}}^a_{s\xi}(R)$, denotes the cluster of 0, respectively $[Re]$, in $\{x\in B(0,s\xi):\varphi_x\geqslant a\}$, respectively $\{x\in B([Re],s\xi):\varphi_x\geqslant a\}$, and $\mathcal{U}=\widetilde{\mathcal{G}}\backslash(\widetilde{\mathcal{K}}^a_{s\xi}\cup\widetilde{\mathcal{K}}^a_{s\xi}(R))$. One can control the first term in (5.4) by the FKG inequality and an adaptation of the proof of [16, Lemma 6.2], which implies that there exists $c_{11}>0$ such that for all $0\leqslant a\leqslant c$,

$$\mathbb{P}\big(\mathrm{cap}(\widetilde{\mathcal{K}}^a_{s\xi})\geqslant c_{11}s^\nu\xi^\nu,\mathrm{cap}(\widetilde{\mathcal{K}}^a_{s\xi}(R))\geqslant c_{11}s^\nu\xi^\nu\big)\geqslant c\xi^{-\nu}q(\xi)^{-2}\exp\{-Cq(\xi)\},\tag{5.5}$$

for some constants $c=c(s)>0$ and $C=C(s)<\infty$. Let us now bound the probability on the right-hand side of (5.4) when $t=c_{11}s^\nu$. For $M\geqslant1$ to be chosen later, we abbreviate $\ell=M\xi\log(R/\xi)^{(2\nu+1)/\nu}$ as in [16, (6.19)], write c_{12} for the constant which is equal to the constant c_{13} from [16, Theorem 5.1], recall the definition of the balls $\widetilde{B}(x,L)\subset\widetilde{\mathcal{G}}$ from below (4.6), and denote by

$$\mathcal{L}_\ell\stackrel{\mathrm{def.}}{=}\bigcup_{i=0}^{\lceil 2R/\ell\rceil}\widetilde{B}([(i\ell/2)e],c_{12}\ell),\tag{5.6}$$

the set which corresponds to the one introduced in [16, (6.13)] (the set in (5.6) should not be confused with the loop soup which plays no role here). Similarly as above [16, (8.2)], let

$$\begin{aligned}\mathcal{L}'_\ell&\stackrel{\mathrm{def.}}{=}\mathcal{L}_\ell\cup\widetilde{B}(0,\sigma'\ell)\cup\widetilde{B}([Re],\sigma'\ell),\\ \mathcal{L}''_\ell&\stackrel{\mathrm{def.}}{=}\mathcal{L}'_\ell\backslash(\widetilde{B}(0,\sigma\xi)\cup\widetilde{B}([Re],\sigma\xi)),\end{aligned}$$

where $\sigma'\geqslant c_{12}$ and $\sigma\geqslant s$ are constants we will fix later. For $K\subset\widetilde{B}(0,s\xi)$ and $K'\subset\widetilde{B}([Re],s\xi)$, abbreviating $U=\widetilde{\mathcal{G}}\backslash(K\cup K')$, let us also denote by $\mathbb{P}^{a,\ell}_U$ the law of $(\varphi_x+a\overline{h}_\ell(x))_{x\in U}$ under $\mathbb{P}_U$, see above Corollary 2.2, where $\overline{h}_\ell(x)=P_x(H_{\mathcal{L}''_\ell}<H_{K\cup K'})$. Then, similarly as in [16, (6.17)], by a change of measure one has that

$$\mathbb{P}_U\big(A(K,K',a)\big)\geqslant\mathbb{P}^{a,\ell}_U\big(A(K,K',a)\big)\exp\Big\{-\frac{a^2\mathrm{cap}_U(\mathcal{L}''_\ell)+1/e}{2\mathbb{P}^{a,\ell}_U(A(K,K',a))}\Big\},\tag{5.7}$$

where $\mathrm{cap}_U(\mathcal{L}''_\ell)$ denotes the capacity of $\mathcal{L}''_\ell$ associated to the diffusion X on $\widetilde{\mathcal{G}}$ killed on hitting U^{c} (see below[16, Corollary 5.2] for a rigorous definition). As we will soon explain, similarly as in [16, (8.2)], uniformly in $\sigma\geqslant s$, one can fix $s=s(\eta)\geqslant1$, $\sigma'\geqslant c_{12}$ and $M\geqslant\sigma'$ such that for all $K\subset\widetilde{B}(0,s\xi)$, $K'\subset\widetilde{B}([Re],s\xi)$ with $\mathrm{cap}(K)\geqslant c_{11}s^\nu\xi^\nu$ and $\mathrm{cap}(K')\geqslant c_{11}s^\nu\xi^\nu$, still abbreviating $U=\widetilde{\mathcal{G}}\backslash(K\cup K')$, if $R/\xi\geqslant C=C(\sigma)$ then

$$\mathbb{P}^{a,\ell}_U\big(A(K,K',a)\big)\geqslant1-\eta.\tag{5.8}$$

Moreover, proceeding similarly as in [16, (7.7) and (7.8)] one can fix $\sigma=\sigma(s)$ large enough so that

$$\mathrm{cap}_{K\cup K'}(\mathcal{L}''_\ell)\leqslant\mathrm{cap}(\mathcal{L}_\ell\cap\mathbb{Z}^3)+C(s^\nu\xi^\nu+(\sigma')^\nu\ell^\nu)\leqslant\frac{\pi}{3}(1+\eta)^2\frac{R}{\log(\frac{2R}{\ell})},\tag{5.9}$$

where the last equality follows from [42, (2.24)] with $n = P = [2R/\ell] + 1$ and $N = R$ therein, and upon taking $R \geqslant C\xi$ for some constant $C = C(\eta, \sigma', s, M)$. Note that the constant $C_{13} = C_4$ appearing therein is equal to $\pi/3$ on $\mathbb{Z}^3$ by [42, (2.6)], and we refer to [26, Lemma 2.2] and [10, Lemma 2.1] for similar statements proved directly on $\mathbb{Z}^3$. Combining (1.6), (5.4), (5.5), (5.7), (5.8) and (5.9) and noting that $\log(\frac{2R}{\ell}) \geqslant (1-\eta)\log(\lambda)$ if λ is large enough by our choice of R and ℓ, one easily deduces the lower bound in (1.13) after a change of variable in η and a.

It remains to prove (5.8). By definition of $\mathbb{P}_U^{a,\ell}$ and $A(K, K', a)$ we have

$$\begin{aligned} \mathbb{P}_U^{a,\ell}\big(A(K,K',a)\big) &\geqslant \mathbb{P}_U\left(K \leftrightarrow K' \text{ in } \{x \in U : \varphi_x \geqslant a(1-\eta-h_{K\cup K'}(x)) - \overline{h}_\ell(x))\}\right) \\ &\geqslant \mathbb{P}_U\left(K \leftrightarrow K' \text{ in } \{x \in \mathcal{L}'_\ell \cap U : \varphi_x \geqslant -\eta a\}\right) \end{aligned} \tag{5.10}$$

since $h_{K\cup K'}(x) + \overline{h}_\ell(x) = P_x(H_{K\cup K'} < \infty) + P_x(H_{\mathcal{L}''_\ell} < H_{K\cup K'}) \geqslant 1$ for all $x \in \mathcal{L}'_\ell$ by definition as long as $\sigma'\ell > \sigma\xi$, which holds as long as R/ξ is large enough, depending on σ, σ'. For σ' and $M \geqslant \sigma'$ large enough, depending only on s and η, one can bound from below the last probability in (5.10) by

$$(1-\eta/2)\big(1-\exp\{-Ca^2\mathrm{cap}(K)\}\big)\big(1-\exp\{-Ca^2\mathrm{cap}(K')\}\big)$$

for some constants $c = c(\eta) > 0$ and $C = C(\eta) < \infty$ using a reasoning similar to the proof of [16, (8.2)]. Indeed, the proof essentially consists of defining three independent random interlacements, each at level $(\eta a)^2/6$, whose union is included in $\{\varphi \geqslant -\eta a\}$ via the isomorphism theorem, such that the following events occur: the first interlacement has a trajectory hitting K and going to ∞, the second interlacement has a trajectory hitting K' and going to ∞, and the last interlacement has a connected component in $\mathcal{L}'_\ell \cap U$ which intersects both the first and second interlacement. The probability of the intersection of the two first events can be bounded by $(1-\exp\{-Ca^2\mathrm{cap}(K)\})(1-\exp\{-Ca^2\mathrm{cap}(K')\})$ owing to [16, Lemma 7.2], and the last one can be lower bounded by $1-\eta/2$ upon taking σ' and M large enough; we refer to [16, (7.17),(7.20)] for as to why and leave the details to the reader. The inequality (5.8) then follows readily when $\mathrm{cap}(K) \geqslant c_{11}s^\nu\xi^\nu$, $\mathrm{cap}(K') \geqslant c_{11}s^\nu\xi^\nu$ and $s = s(\eta)$ is a sufficiently large constant. □

Remark 5.2. As should be clear from (5.7), (5.9) and (5.10), see also [42, Proposition 3.2] for the upper bound, the intuitive reason one is able to obtain the exact constant $\pi/6$ in Corollary 1.3 is that

$$\log\left(\frac{\tau_a^{\mathrm{tr}}(0,[Re])}{\tau_0^{\mathrm{tr}}(0,[Re])}\right) \sim -\frac{a^2\mathrm{cap}(\mathcal{L}_\ell)}{2} \tag{5.11}$$

as $R \to \infty$, where $\mathcal{L}_\ell$ is a 'tube' (see (5.6)) from 0 to Re of length R and width ℓ; and that by [42, Lemma 2.1], see also [10, Lemma 2.1] for a similar statement directly on $\mathbb{Z}^3$,

$$\mathrm{cap}(\mathcal{L}_\ell) \sim \frac{\pi R}{3\log(R/\ell)} \sim \frac{\pi R}{3\log(R/\xi)}$$

as long as ℓ is roughly of order ξ. One could similarly obtain an explicit constant for graphs satisfying (V_α) and (G_ν) with $\nu \leqslant 1$ as long as the function $g(x,y)d(x,y)^{-\nu}$ converges as $d(x,y) \to \infty$, see [42, (2.2) and Lemma 2.1]. However when $\nu > 1$, for instance on $\mathbb{Z}^\alpha$, $\alpha > 3$, even if (5.11) was true with ℓ of order ξ, one would not obtain the correct constant anymore as $\mathrm{cap}(\mathcal{L}_\ell) \asymp R\ell^{\nu-1}$, see [42, Lemma 2.1], and thus depends on the exact choice of the constant ℓ/ξ, which is a priori not clear. Note that ℓ is actually of order $\xi\log(r/\xi)^C$ in the proof of (1.9), hence the additional logarithmic factor therein.

We conclude with the brief:

Proof of (1.9). We start with observing that [16, (8.3)] in combination with [42, Remark 6.5,4)] entails that for our choices of parameters we have

$$\tau^{\mathrm{tr}}_{a/C}(x,y) \geqslant \xi^{-\nu} q(\xi)^{-2} \exp\Big\{ -Cq(\xi) - \frac{C(d(x,y)/\xi)}{\log(d(x,y)/\xi)^b} \Big\}, \tag{5.12}$$

where q is as in Theorem 1.1, ξ as in (1.12), $b = 1$ if $\nu = 1$, and $b = 1-\nu$ for $\nu \in (1, \alpha/2)$. Now due to (G_ν), we have that [37, Proposition 5.2] entails $\tau^{\mathrm{tr}}_0(x,y) \leqslant Cd(x,y)^{-\nu}$, and we furthermore note that the term $q(\xi)^{-2}\exp\{-Cq(\xi)\}$ appearing in (5.12) is now negligible by Theorem 1.1 and our condition on $d(x,y)$; inequality (1.9) follows. □

Acknowledgments: This research was supported through the programs 'Oberwolfach Research Fellows' and 'Oberwolfach Leibniz Fellows' by the Mathematisches Forschungsinstitut Oberwolfach in 2023. The research of AD has been supported by the Deutsche Forschungsgemeinschaft (DFG) grant DR 1096/2-1. AP has been supported by the Engineering and Physical Sciences Research Council (EPSRC) grant EP/R022615/1, Isaac Newton Trust (INT) grant G101121, European Research Council (ERC) starting grant 804166 (SPRS), and the Swiss NSF. PFR thanks the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center located in Kyoto University, for its hospitality.

References

[1] S. Andres and A. Prévost. First passage percolation with long-range correlations and applications to random Schrödinger operators. *Preprint*, arXiv:2112.12096, 2021.

[2] J. Aru, T. Lupu, and A. Sepúlveda. The first passage sets of the 2D Gaussian free field: convergence and isomorphisms. *Comm. Math. Phys.*, 375(3):1885–1929, 2020.

[3] M. T. Barlow. Which values of the volume growth and escape time exponent are possible for a graph? *Rev. Mat. Iberoamericana*, 20(1):1–31, 2004.

[4] M. T. Barlow. *Random walks and heat kernels on graphs*, volume 438 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2017.

[5] M. T. Barlow and R. F. Bass. Random walks on graphical Sierpinski carpets. In *Random walks and discrete potential theory (Cortona, 1997)*, Sympos. Math., XXXIX, pages 26–55. Cambridge Univ. Press, Cambridge, 1999.

[6] D. J. Barsky and M. Aizenman. Percolation critical exponents under the triangle condition. *Ann. Probab.*, 19(4):1520–1536, 1991.

[7] J. Bricmont, J. L. Lebowitz, and C. Maes. Percolation in strongly correlated systems: the massless Gaussian field. *J. Statist. Phys.*, 48(5-6):1249–1268, 1987.

[8] Z. Cai and J. Ding. One-arm exponent of critical level-set for metric graph Gaussian free field in high dimensions. *Preprint*, arXiv:2307.04434, 2023.

[9] Y. Chang and A. Sapozhnikov. Phase transition in loop percolation. *Probab. Theory Relat. Fields*, 164(3-4):979–1025, 2016.

[10] A. Dembo and I. Okada. Capacity of the range of random walk: The law of the iterated logarithm. *Preprint*, arXiv:2208.02184, 2022.

[11] J. Ding and M. Wirth. Percolation for level-sets of Gaussian free fields on metric graphs. *Ann. Probab.*, 48(3):1411–1435, 05 2020.

[12] A. Drewitz, O. Elias, A. Prévost, J. Tykesson, and F. Viklund. Percolation for two-dimensional excursion clouds and the discrete Gaussian free field. *Preprint*, arXiv:2205.15289, 2022.

[13] A. Drewitz, A. Prévost, and P.-F. Rodriguez. Geometry of Gaussian free field sign clusters and random interlacements. *Preprint*, arXiv:1811.05970, 2018.

[14] A. Drewitz, A. Prévost, and P.-F. Rodriguez. The sign clusters of the massless Gaussian free field percolate on $\mathbb{Z}^d, d \geqslant 3$ (and more). *Comm. Math. Phys.*, 362(2):513–546, 2018.

[15] A. Drewitz, A. Prévost, and P.-F. Rodriguez. Cluster capacity functionals and isomorphism theorems for Gaussian free fields. *Probab. Theory Related Fields*, 183(1-2):255–313, 2022.

[16] A. Drewitz, A. Prévost, and P.-F. Rodriguez. Critical exponents for a percolation model on transient graphs. *Invent. Math.*, 232(1):229–299, 2023.

[17] A. Drewitz, B. Ráth, and A. Sapozhnikov. *An introduction to random interlacements.* SpringerBriefs in Mathematics. Springer, Cham, 2014.

[18] A. Drewitz and P.-F. Rodriguez. High-dimensional asymptotics for percolation of Gaussian free field level sets. *Electron. J. Probab.*, 20:no. 47, 39, 2015.

[19] H. Duminil-Copin, S. Goswami, P.-F. Rodriguez, and F. Severo. Equality of critical parameters for percolation of Gaussian free field level sets. *Duke Math. J.*, 172(5):839–913, 2023.

[20] H. Duminil-Copin, S. Goswami, P.-F. Rodriguez, F. Severo, and A. Teixeira. A characterization of strong percolation via disconnection. *Preprint*, arXiv:2308.07920, 2023.

[21] H. Duminil-Copin, S. Goswami, P.-F. Rodriguez, F. Severo, and A. Teixeira. Finite range interlacements and couplings. *Preprint*, arXiv:2308.07303, 2023.

[22] H. Duminil-Copin, S. Goswami, P.-F. Rodriguez, F. Severo, and A. Teixeira. Phase transition for the vacant set of random walk and random interlacements. *Preprint*, arXiv:2308.07919, 2023.

[23] H. Duminil-Copin, K. K. Kozlowski, D. Krachun, I. Manolescu, and M. Oulamara. Rotational invariance in critical planar lattice models. *Preprint*, arXiv:2012.11672, 2020.

[24] R. Fitzner and R. van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electronic Journal of Probability*, 22(none):1 – 65, 2017.

[25] P. J. Fitzsimmons and J. S. Rosen. Markovian loop soups: permanental processes and isomorphism theorems. *Electron. J. Probab.*, 19:no. 60, 30, 2014.

[26] S. Goswami, P.-F. Rodriguez, and F. Severo. On the radius of Gaussian free field excursion clusters. *Ann. Probab.*, 50(5):1675–1724, 2022.

[27] A. Grigor'yan and A. Telcs. Sub-Gaussian estimates of heat kernels on infinite graphs. *Duke Math. J.*, 109(3):451–510, 2001.

[28] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, 128(2):333–391, 1990.

[29] T. Hara, R. van der Hofstad, and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.*, 31(1):349–408, 2003.

[30] A. Jego, T. Lupu, and W. Qian. Crossing exponent in the Brownian loop soup. *Preprint*, arXiv:2303.03782, 2023.

[31] G. Kozma and A. Nachmias. Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.*, 24(2):375–409, 2011.

[32] G. F. Lawler. The probability of intersection of independent random walks in four dimensions. *Comm. Math. Phys.*, 86(4):539–554, 1982.

[33] G. F. Lawler, O. Schramm, and W. Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.*, 7:no. 2, 13, 2002.

[34] G. F. Lawler and J. A. Trujillo Ferreras. Random walk loop soup. *Trans. Am. Math. Soc.*, 359(2):767–787, 2007.

[35] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.

[36] Y. Le Jan. Markov loops and renormalization. *Ann. Probab.*, 38(3):1280–1319, 2010.

[37] T. Lupu. From loop clusters and random interlacements to the free field. *Ann. Probab.*, 44(3):2117–2146, 2016.

[38] T. Lupu. Convergence of the two-dimensional random walk loop-soup clusters to CLE. *J. Eur. Math. Soc. (JEMS)*, 21(4):1201–1227, 2019.

[39] T. Lupu. An equivalence between gauge-twisted and topologically conditioned scalar Gaussian free fields. *Preprint*, arXiv:2209.07901, 2022.

[40] S. Muirhead. Percolation of strongly correlated Gaussian fields II. sharpness of the phase transition. *Preprint*, arXiv:2206.10724, 2022.

[41] S. Muirhead and F. Severo. Percolation of strongly correlated Gaussian fields I. decay of subcritical connection probabilities. *Preprint*, arXiv:2206.10723, 2022.

[42] A. Prévost. First passage percolation, local uniqueness for interlacements and capacity of random walk. *Preprint*, arXiv:2309.03880, 2023.

[43] P.-F. Rodriguez and A.-S. Sznitman. Phase transition and level-set percolation for the Gaussian free field. *Comm. Math. Phys.*, 320(2):571–601, 2013.

[44] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

[45] A.-S. Sznitman. Vacant set of random interlacements and percolation. *Ann. Math. (2)*, 171(3):2039–2087, 2010.

[46] A.-S. Sznitman. An isomorphism theorem for random interlacements. *Electron. Commun. Probab.*, 17:no. 9, 9, 2012.

[47] A.-S. Sznitman. *Topics in Occupation Times and Gaussian free fields.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[48] A.-S. Sznitman. Disconnection and level-set percolation for the Gaussian free field. *J. Math. Soc. Japan*, 67(4):1801–1843, 2015.

[49] A.-S. Sznitman. Coupling and an application to level-set percolation of the Gaussian free field. *Electron. J. Probab.*, 21:Paper No. 35, 26, 2016.

[50] A. Teixeira. Interlacement percolation on transient weighted graphs. *Electron. J. Probab.*, 14:no. 54, 1604–1628, 2009.

[51] W. Werner. On clusters of Brownian loops in d dimensions. In *In and out of equilibrium 3. Celebrating Vladas Sidoravicius*, volume 77 of *Progr. Probab.*, pages 797–817. Birkhäuser/Springer, Cham, 2021.