Lectures on “Introduction to Geophysical Fluid Dynamics”

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• **Idea of the lectures** is to provide a relatively advanced-level course that builds up on the existing introductory-level fluid dynamics courses. The lectures target an audience of upper-level undergraduate students, graduate students, and postdocs.

• **Main topics:**

  1. Introduction
  2. Governing equations
  3. Geostrophic dynamics
  4. Quasigeostrophic theory
  5. Ekman layer
  6. Rossby waves
  7. Linear instabilities
  8. Ageostrophic motions
  9. Transport phenomena
  10. Nonlinear dynamics and wave-mean flow interactions

• **Suggested textbooks:**

  1. *Introduction to geophysical fluid dynamics* (Cushman-Roisin and Beckers);
  2. *Fundamentals of geophysical fluid dynamics* (McWilliams);
  3. *Geophysical fluid dynamics* (Pedlosky);
  4. *Atmospheric and oceanic fluid dynamics* (Vallis).
Motivations

- Main motivations for the recent rapid development of Geophysical Fluid Dynamics (GFD) include the following very important, challenging and multidisciplinary set of problems:
  - *Earth system modelling,*
  - *Predictive understanding of climate variability* (emerging new science!),
  - *Forecast of various natural phenomena* (e.g., weather),
  - *Natural hazards, environmental protection, natural resources,* etc.

What is GFD?

- Most of GFD is about dynamics of *stratified* and *turbulent* fluid on giant *rotating sphere.* On smaller scales GFD is just the classical fluid dynamics with geophysical applications.
  - Other planets and some astrophysical fluids (e.g., stars, galaxies) are also included in GFD.
- GFD combines applied math and theoretical physics.
  It is about *mathematical representation* and *physical interpretation* of geophysical fluid motions.
- Mathematics of GFD is *heavily computational,* even relative to other branches of fluid dynamics (e.g., modelling of the ocean circulation and atmospheric clouds are the largest computational problems in the history of science).
  - This is because lab experiments (i.e., analog simulations) can properly address only tiny fraction of interesting questions (e.g., small-scale waves, convection, microphysics).
- In geophysics theoretical advances are often GFD-based rather than experiment-based, because the real *field measurements* are extremely complex, difficult, expensive and often impossible.

*Let’s overview some geophysical phenomena of interest...*
An image of the Earth from space:

- Earth’s atmosphere and oceans are the main but not the only target of GFD
This is not an image of the Earth from space...

...but a visualized solution of the mathematical equations!
Atmospheric cyclones and anticyclones constitute most of the midlatitude weather.

This cyclone is naturally visualized by clouds:

- Modelling clouds is notoriously difficult problem in atmospheric science.
Tropical cyclones (hurricanes and typhoons) are a coupled ocean-atmosphere phenomenon. These are powerful storm systems characterized by low-pressure center, strong winds, heavy rain, and numerous thunderstorms.

Hurricane Katrina approaches New Orleans:
Ocean-atmosphere coupling: Ocean and atmosphere exchange momentum, heat, water, radiation, aerosols, and greenhouse gases.

*Ocean-atmosphere interface is a very complex two-sided boundary layer:*
Ocean currents are full of transient mesoscale eddies:

- *Mesoscale eddies* are dynamically similar to atmospheric cyclones and anticyclones, but much smaller and more abundant. They are the “oceanic weather”.

- Modelling the eddies and their effects is very important and challenging problem.
Submesoscale eddies around an island:

- Submesoscale motions are geostrophically and hydrostatically unbalanced, which means that they are less affected by the rotation and more 3D-like.

- Many GFD processes are influenced by coasts and topography (e.g., coastal currents, upwellings, tidal mixing, lee waves).

- Turbulence operates on all scales down to millimeters, but on smaller scales effects of planetary rotation and vertical stratification weaken and GFD turns into classical fluid dynamics.
• GFD deals with different waves operating on scales from meters to thousands of kilometers.
• Problem of internal gravity wave breaking is very important and challenging.

Breaking surface gravity waves:
• *Tsunami* is another example of the surface gravity wave.

*Evolution of a tsunami predicted by the high-accuracy shallow-water modelling:*
• GFD is involved in problems with formation and propagation of ice.
GFD is the basis for modelling material transport in the atmosphere:
...and in the ocean:

- Modelling biomass involves solving for concentrations of hundreds of mutually interacting species feeding on light, nutrients and each other.
• GFD applies beyond the Earth, to the atmospheres of other planets.

*Circulation of the Jupiter’s weather layer:*

This addon for the Celestia 3D Space Simulator can be found at www.celestiamotherlode.net
...and a closer look from the Cassini mission:
• Some theories argue that the alternating jets on giant gas planets are driven by deep convective plumes which feed upscale cascade of energy.
MagnetoHydroDynamics (MHD) of stars naturally extends the realm of GFD

Beautiful example of coronal rain on the Sun:
• **Representation of fluid flows**

Let’s consider a flow consisting of fluid particles. Each particle is characterized by its position \( r \) and velocity \( u \) vectors:

\[
\frac{dr(t)}{dt} = \frac{\partial r(a, t)}{\partial t} = u(r, t), \quad r(a, 0) = a
\]

**Trajectory** (pathline) of an individual fluid particle is “recording” of the path of this particle over some time interval. Instantaneous direction of the trajectory is determined by the corresponding instantaneous streamline.

**Streamlines** are a family of curves that are instantaneously tangent to the velocity vector of the flow \( u = (u, v, w) \). Streamline shows the direction a fluid element will travel in at any point in time.

A parametric representation of just one streamline (here \( s \) is coordinate along the streamline) at some moment in time is \( X_s(x_s, y_s, z_s) :\)

\[
\frac{dX_s}{ds} \times u(x_s, y_s, z_s) = 0 \quad \implies \quad i \left( w \frac{\partial y_s}{\partial s} - v \frac{\partial z_s}{\partial s} \right) - j \left( w \frac{\partial x_s}{\partial s} - u \frac{\partial z_s}{\partial s} \right) + k \left( v \frac{\partial x_s}{\partial s} - u \frac{\partial y_s}{\partial s} \right) = 0
\]

For 2D and non-divergent flows the velocity streamfunction can be used to plot streamlines:

\[
u = -\nabla \times \psi, \quad \psi = (0, 0, \psi), \quad u = (u, v, 0) \quad \implies \quad u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}
\]

Note, that \( u \cdot \nabla \psi = 0 \), hence, velocity vector \( u \) always points along the isolines of \( \psi(x, y) \), implying that these isolines are indeed the streamlines.

**Streakline** is the collection of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

**Timeline (material line)** is the line formed by a set of fluid particles that were marked at the same time, creating a line or a curve that is displaced in time as the particles move.

**Lagrangian framework**: Point of view such that fluid is described **by following fluid particles**. Interpolation problem, not optimal use of information.

**Eulerian framework**: Point of view such that fluid is described **at fixed positions in space**. Nonlinearity problem.
**GOVERNING EQUATIONS**

- **Complexity:** These equations are sufficient for finding a solution but are too complicated to solve; they are useful only as a starting point for GFD analysis.

- **Art of modelling:** Typically the governing equations are *approximated* analytically and, then, *solved approximately* (by analytical or numerical methods); one should always keep track of all main assumptions and approximations.

- **Continuity of mass:** Consider a fixed infinitesimal volume of fluid (Eulerian view) and flow of mass through its surfaces.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

\[
\frac{\mathrm{D} \rho}{\mathrm{D}t} = -\rho \nabla \cdot \mathbf{u} ;
\]

\[
\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\]

Note: if fluid is incompressible (i.e., \( \rho = \text{const} \)), then the continuity equation is reduced to \( \nabla \cdot \mathbf{u} = 0 \), which is its common form.

- **Material derivative** operating on \( X \) gives the rate of change of \( X \) with the time *following the fluid element* (i.e., subject to a space- and-time dependent velocity field). It is a link between the Eulerian \( (\partial / \partial t + \mathbf{u}) \cdot \nabla \) and Lagrangian \( (\mathrm{D} / \mathrm{D}t) \) descriptions of changes in the fluid.
The way to see that the material derivative describes the rate of change of any property \( F(t, x, y, z) \) following a fluid particle is by applying (i) the chain rule of differentiation and (ii) definition of velocity as the rate of change of particle position:

\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial t} = \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F
\]

- **Tendency term**, \( \partial X/\partial t \), represents the rate of change of \( X \) at a point which is fixed in space (and occupied by different fluid particles at different times). Changes of \( X \) are observed by a stand-still observer.

- **Advection term**, \( \mathbf{u} \nabla X \), represents changes of \( X \) due to movement with velocity \( \mathbf{u} \) (or, flow supply of \( X \) to a fixed point). Additional changes of \( X \) are experienced by an observer swimming with velocity \( \mathbf{u} \).

- **Material tracer equation** (evolution equation for composition): By similar argument, for any material tracer (e.g., chemicals, aerosols, gases) concentration \( \tau \) (amount per unit mass), the evolution equation is

\[
\frac{\partial (\rho \tau)}{\partial t} + \nabla \cdot (\rho \tau \mathbf{u}) = \rho S^{(\tau)},
\]

where \( S^{(\tau)} \) stands for all the non-conservative sources and sinks of \( \tau \) (e.g., boundary sources, molecular diffusion, reaction rate). The tracer diffusion is generally added and represented by \( \nabla \cdot (\kappa \nabla \tau) \), where \( \kappa \) is diffusivity (tensor) coefficient.

- **Momentum equation**: Let’s write the Newton’s Second Law in a fixed frame of reference, and for the infinitesimal volume of fluid \( \delta V \) and some force \( \mathbf{F} \) acting on a unit volume:

\[
\frac{D}{Dt}(\rho \mathbf{u} \delta V) = \mathbf{F} \delta V \quad \Rightarrow \quad \mathbf{u} \frac{D}{Dt}(\rho \delta V) + \rho \delta V \frac{D}{Dt} \mathbf{u} = \mathbf{F} \delta V \quad \Rightarrow \quad \frac{D\mathbf{u}}{Dt} = \frac{1}{\rho} \mathbf{F},
\]

where the first term on the llhs of the second equation is zero, because mass of the fluid element remains constant (i.e., no relativistic effects).

- **Pressure force** can be thought as the one arising thermodynamically (due to internal motion of molecules) from the pressure \( p(x, y, z) \) acting on 6 faces of the infinitesimal cubic volume \( \delta V \). Hence, the pressure force component in \( x \) is

\[
F_x \delta V = [p(x, y, z) - p(x + \delta x, y, z)] \delta y \delta z = -\frac{\partial p}{\partial x} \delta V \quad \Rightarrow \quad F_x = -\frac{\partial p}{\partial x} \quad \Rightarrow \quad \mathbf{F} = -\nabla p
\]

- **Frictional force** (due to internal motion of molecules) is typically approximated as \( \nu \nabla^2 \mathbf{u} \), where \( \nu \) is the kinematic viscosity.

- **Body force** \( \mathbf{F}_b \): most common examples are gravity and electromagnetic forces.
Coriolis force is a pseudo-force that only appears in a rotating frame of reference with the rotation rate $\Omega$: $F_c = -2\Omega \times u$.

(i) It acts to deflect each fluid particle at right angle to its motion.

(ii) It doesn’t do work on a particle, because it is perpendicular to the particle velocity.

(iii) Think about motion of a moving ball on a rotating table. Foucault pendulum.

(iv) Physics of the Coriolis force: particle on a sphere deflects because of the conservation of angular momentum (when moving to the smaller/larger latitudinal circle, it should be accelerated/decelerated to conserve its momentum).

(v) Watch some YouTube movies about the Coriolis force.

Let’s derive all the pseudo-forces in rotating coordinate systems. Rates of change of general vector $B$ in the inertial (fixed) and rotating (with $\Omega$) frames of reference (indicated by $i$ and $r$, respectively) are simply related:

$$\left[ \frac{dB}{dt} \right]_i = \left[ \frac{dB}{dt} \right]_r + \Omega \times B$$

Let’s apply this relationship to $r$ and $u_r$ and obtain

$$\left[ \frac{dr}{dt} \right]_i \equiv u_i = u_r + \Omega \times r$$, \hspace{1cm} (*)

$$\left[ \frac{d u_r}{dt} \right]_i = \left[ \frac{d u_r}{dt} \right]_r + \Omega \times u_r$$. \hspace{1cm} (**)

However, we need acceleration of $u_i$ in the inertial frame and expressed completely in terms of $u_r$ and in the rotating frame. Let’s (a) differentiate (*) with respect to time, and in the inertial frame of reference; and (b) substitute $[d u_r / dt]_i$ from (**):

$$\left[ \frac{d u_i}{dt} \right]_i = \left[ \frac{d u_r}{dt} \right]_r + \Omega \times u_r + \frac{d \Omega}{dt} \times r + \Omega \times \left[ \frac{dr}{dt} \right]_i$$

Now, we again substitute $[dr / dt]_i$ from (*) :

$$\frac{d \Omega}{dt} = 0 \quad \Rightarrow \quad \left[ \frac{d u_i}{dt} \right]_i = \left[ \frac{d u_r}{dt} \right]_r + 2\Omega \times u_r + \Omega \times (\Omega \times r)$$

The term disappearing due to the constant rate of rotation is the (minus) Euler force.

The last term is the (minus) centrifugal force. It acts a bit like gravity but in the opposite direction, hence, it can be incorporated in the gravity force field and be “forgotten”.


To summarize, the (vector) momentum equation is:

\[
\frac{Du}{Dt} + 2 \Omega \times u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + F_b
\]

Note, that in GFD the Coriolis force is traditionally kept on the lhs of the momentum equation.

**Equation of state** \( \rho = \rho(p, T, \tau_n) \) relates pressure \( p \) to the state variables — density \( \rho \), temperature \( T \), and chemical tracer concentrations \( \tau_n \), where \( n = 1, 2, \ldots \) is the tracer index. All the state variables are related to matter; therefore, the equation of state is a constitutive equation.

(a) Equations of state are often phenomenological and very different for different geophysical fluids, whereas so far other equations were universal.

(b) The most important \( \tau_n \) are humidity (i.e., water vapor concentration) in the atmosphere and salinity (i.e., concentration of diluted salt mix) in the ocean.

(c) Equation of state brings in temperature, which has to be determined thermodynamically [not part of these lectures!] from internal energy (i.e., energy needed to create the system), entropy (thermal energy not available for work), and chemical potentials corresponding to \( \tau_n \) (energy that can be available from changes of \( \tau_n \)).

(d) Example of equation of state (for sea water) involves empirically fitted coefficients of thermal expansion \( \alpha \), saline contraction \( \beta \), and compressibility \( \gamma \), which are all empirically determined functions of the state variables:

\[
\frac{d\rho}{\rho} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_{S,p} dT + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial S} \right)_{T,p} dS + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_{T,S} dp = -\alpha dT + \beta dS + \gamma dp
\]
• **Thermodynamic equation** is just one more way of writing the first law of thermodynamics, which is an expression of the conservation of total energy. (Recall that the second law is about “arrow of time”: direction of processes in isolated systems is such that the entropy only increases; in simple words, the heat doesn’t go from hot to cold objects.)

The thermodynamic equation can be written for \( T \) (i.e., \( DT/ Dt = \ldots \)), but in the GFD it is more convenient to write it for \( \rho \):

$$\frac{D\rho}{Dt} - \frac{1}{c_s^2} \frac{Dp}{Dt} = Q(\rho),$$

where \( c_s \) is *speed of sound*, and \( Q(\rho) \) is *source term* (both concepts have complicated expressions in terms of the state variables).

To summarize, our starting point (assuming one material tracer) is the following COMPLETE SET OF EQUATIONS:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \quad (1) \\
\frac{Du}{Dt} + 2 \Omega \times \mathbf{u} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_b \quad (2) \\
\rho &= \rho(p,T,\tau) \quad (3) \\
\frac{\partial (\rho \tau)}{\partial t} + \nabla \cdot (\rho \tau \mathbf{u}) &= \rho S^{(\tau)} \quad (4) \\
\frac{D\rho}{Dt} - \frac{1}{c_s^2} \frac{Dp}{Dt} &= Q(\rho) \quad (5)
\end{align*}
\]

(a) Momentum equation is for the flow velocity vector, hence, it can be written as 3 equations for the velocity components (scalars).

(b) We ended up with 7 equations and 7 unknowns (for only one tracer concentration): \( u, v, w, p, \rho, T, \tau \).

(c) **Boundary and initial conditions**: The governing equations (or their approximations) are to be solved subject to those.
• **Spherical coordinates** are natural for GFD: longitude $\lambda$, latitude $\theta$ and altitude $r$.

Material derivative for a scalar quantity $\phi$ in spherical coordinates is:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{u}{r \cos \theta} \frac{\partial \phi}{\partial \lambda} + \frac{v}{r} \frac{\partial \phi}{\partial \theta} + \frac{w}{r} \frac{\partial \phi}{\partial r},$$

where the flow velocity in terms of the corresponding unit vectors is:

$$u = iu + jv + kw, \quad (u, v, w) \equiv \left( r \cos \theta \frac{D\lambda}{Dt}, \; r \frac{D\theta}{Dt}, \; \frac{Dr}{Dt} \right).$$

Vector analysis provides differential operators in spherical coordinates acting on a field given by either scalar $\phi$ or vector $\mathbf{B} = iB^\lambda + jB^\theta + kB^r$:

$$\nabla \cdot \mathbf{B} = \frac{1}{\cos \theta} \left[ \frac{1}{r} \frac{\partial B^\lambda}{\partial \lambda} + \frac{1}{r} \frac{\partial (B^\theta \cos \theta)}{\partial \theta} + \cos \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial B^r}{\partial r} \right) \right],$$

$$\nabla \phi = i \frac{1}{r \cos \theta} \frac{\partial \phi}{\partial \lambda} + j \frac{1}{r} \frac{\partial \phi}{\partial \theta} + k \frac{\partial \phi}{\partial r},$$

$$\nabla^2 \phi \equiv \nabla \cdot \nabla \phi = \frac{1}{r^2 \cos \theta} \left[ \frac{1}{\cos \theta} \frac{\partial^2 \phi}{\partial \lambda^2} + \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial \phi}{\partial \theta} \right) + \cos \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) \right],$$

$$\nabla \times \mathbf{B} = \frac{1}{r^2 \cos \theta} \begin{vmatrix} i & j & k \\ r \cos \theta & \frac{j}{r} & \frac{k}{r} \\ \frac{\partial}{\partial \lambda} B^\lambda & \frac{\partial}{\partial \theta} B^\theta & \frac{\partial}{\partial r} B^r \end{vmatrix},$$

$$\nabla^2 \mathbf{B} = \nabla (\nabla \cdot \mathbf{B}) - \nabla \times (\nabla \times \mathbf{B}).$$

Writing down material derivative in spherical coordinates is a bit problematic, because the directions of the unit vectors $i$, $j$, $k$ change with changes in location of the fluid element; therefore, material derivatives of the unit vectors are not zeros. Note, that this doesn’t happen in Cartesian coordinates.
Material derivative in spherical coordinates:

\[
\frac{Du}{Dt} = \frac{Du}{Dt} i + \frac{Dv}{Dt} j + \frac{Dw}{Dt} k + \frac{Di}{Dt} i + \frac{Dj}{Dt} j + \frac{Dk}{Dt} k + \Omega_{flow} \times u,
\]

where \( \Omega_{flow} \) is rotation rate (relative to the centre of Earth) of the unit vector corresponding to the moving element of the fluid flow:

\[
\begin{align*}
\frac{Di}{Dt} &= \Omega_{flow} \times i, \\
\frac{Dj}{Dt} &= \Omega_{flow} \times j, \\
\frac{Dk}{Dt} &= \Omega_{flow} \times k.
\end{align*}
\]

Let’s find \( \Omega_{flow} \) by moving fluid particle in the direction of each unit vector and observing whether this motion generates any rotation. It is easy to see that motion in the direction of \( i \) makes \( \Omega_{||} \), motion in the direction of \( j \) makes \( \Omega_{\perp} \), and motion in the direction of \( k \) produces no rotation. Note (see left Figure), that \( \Omega_{||} \) is a rotation around the Earth’s rotation axis, and it can be written as:

\[
\Omega_{||} = \Omega_{||}(j \cos \theta + k \sin \theta).
\]

This rotation rate comes only from a zonally (i.e., along latitude) moving fluid element, and it can be estimated as the following:

\[
u \delta t = r \cos \theta \delta \lambda \quad \rightarrow \quad \Omega_{||} \equiv \frac{\delta \lambda}{\delta t} = \frac{u}{r \cos \theta} \quad \Rightarrow \quad \Omega_{||} = \frac{u}{r \cos \theta}(j \cos \theta + k \sin \theta) = j \frac{u}{r} + k \frac{u \tan \theta}{r}.
\]

Note: the rotation rate vector in the perpendicular to \( \Omega \) direction is aligned with \( i \) and given by

\[
\Omega_{\perp} = -i \frac{v}{r} \quad \Rightarrow \quad \Omega_{flow} = \Omega_{\perp} + \Omega_{||} = -i \frac{v}{r} + j \frac{u}{r} + k \frac{u \tan \theta}{r} \quad \Rightarrow
\]

\[
\begin{align*}
\frac{Di}{Dt} &= \Omega_{flow} \times i = -u \frac{v}{r \cos \theta}(j \sin \theta - k \cos \theta), \\
\frac{Dj}{Dt} &= -i \frac{u}{r \tan \theta} - k \frac{v}{r}, \\
\frac{Dk}{Dt} &= i \frac{u}{r} + j \frac{v}{r}
\end{align*}
\]

\[
(*) \quad \Rightarrow \quad \frac{Du}{Dt} = i \left( \frac{Du}{Dt} - \frac{uv \tan \theta}{r} + \frac{uw}{r} \right) + j \left( \frac{Dv}{Dt} - \frac{u^2 \tan \theta}{r} + \frac{vw}{r} \right) + k \left( \frac{Dw}{Dt} - \frac{u^2 + v^2}{r} \right)
\]

The additional quadratic terms are called metric terms.
**Coriolis force** also needs to be written in terms of the unit vectors of the spherical coordinates. The planetary rotation rate is always orthogonal to the unit vector \( \mathbf{i} \) (see Figure):

\[
\Omega = (0, \Omega_y, \Omega_z) = (0, \Omega \cos \theta, \Omega \sin \theta)
\]

However, the Coriolis force projects on all the unit vectors:

\[
2\Omega \times \mathbf{u} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 2\Omega \cos \theta & 2\Omega \sin \theta \\
u & v & w
\end{vmatrix} = \mathbf{i} (2\Omega w \cos \theta - 2\Omega v \sin \theta) + \mathbf{j} 2\Omega u \sin \theta - \mathbf{k} 2\Omega u \cos \theta.
\]

By combining the metric and Coriolis terms, we obtain the most complicated set of the governing equations (other equations are changed in the similar and trivial way) written in the spherical coordinates:

\[
\begin{align*}
\frac{Du}{Dt} - (2\Omega + \frac{u}{r \cos \theta})(v \sin \theta - w \cos \theta) &= -\frac{1}{\rho r \cos \theta} \frac{\partial p}{\partial \lambda}, \\
\frac{Dv}{Dt} + \frac{w v}{r} + \left(2\Omega + \frac{u}{r \cos \theta}\right) u \sin \theta &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta}, \\
\frac{ Dw }{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u \cos \theta &= -\frac{1}{\rho} \frac{\partial p}{\partial r} - g, \\
\frac{\partial \rho}{\partial t} + \frac{1}{r \cos \theta} \frac{\partial (u \rho)}{\partial \lambda} + \frac{1}{r \cos \theta} \frac{\partial (v \rho \cos \theta)}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 w \rho)}{\partial r} &= 0.
\end{align*}
\]

Metric terms are relatively small on the surface of a large planet \((r \to R_0)\) and, therefore, can be neglected for many process studies; Note, that the gravity acceleration \( g \) is included; viscous term and external forces can be also trivially added.

**Local Cartesian approximation.** Both for mathematical simplicity and for process studies, the governing equations can be written locally for a plane tangent to the planetary surface. Then, the momentum equations can be written as

\[
\begin{align*}
\frac{Du}{Dt} + 2 (\Omega \cos \theta w - \Omega \sin \theta v) &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\
\frac{Dv}{Dt} + 2 (\Omega \sin \theta u) &= -\frac{1}{\rho} \frac{\partial p}{\partial y}, \\
\frac{ Dw }{Dt} + 2 (-\Omega \cos \theta u) &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g.
\end{align*}
\]

(i) Neglect Coriolis force in the vertical momentum equation, because its effect (upward/downward deflection of fluid particles, also known as Eotvos effect), is small.

(ii) Neglect vertical velocity in the zonal momentum equation, because the corresponding component of the Coriolis force is small relative to the other one (vertical velocity components are often small relative to the horizontal ones).
Let’s introduce the Coriolis parameter $f \equiv 2\Omega^2 = 2\Omega \sin \theta$, which is a simple nonlinear function of latitude.

(a) Theoreticians often use $f$-plane approximation: $f = f_0$ (constant).

(b) Planetary sphericity is often accounted for by $\beta$-plane approximation: $f(y) = f_0 + \beta y$.

The resulting equations are:

$$
\begin{align*}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y}, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \\
\frac{D\rho}{Dt} + \rho \nabla u &= 0
\end{align*}
$$

These equations are to be combined with the other equations (thermodynamics, material tracer, etc.) written in the local Cartesian approximation, and even this system of equations is too difficult to solve. In order to simplify it further, we have to focus on specific classes of fluid motions. Our main focus will be on stratified incompressible flows.

• **Stratification.** Let’s think about density fields in terms of their dynamic anomalies due to fluid motion and pre-existing static fields:

$$
\rho(t, x, y, z) = \rho_0 + \bar{\rho}(z) + \rho'(t, x, y, z) = \rho_s(z) + \rho'(t, x, y, z)
$$

Later on, the static distribution of density will be represented in terms of stacked isopycnal (i.e., constant-density) and relatively thin fluid layers, and the dynamic density anomalies will be described by the deformations of these layers.

The pressure field can be also treated in terms of static and dynamic components:

$$
p(t, x, y, z) = p_s(z) + p'(t, x, y, z).
$$

We will use symbols $[\delta \rho']$ and $[\delta p']$ to describe the corresponding dynamic scales.

With this concept of fluid stratification, we are ready to make one more important approximation that will affect both thermodynamic and vertical momentum equations...

• **Boussinesq approximation.** It is used routinely for oceans and sometimes for atmospheres and invokes the following assumptions:

(1) Fluid incompressibility: $c_s = \infty$,

(2) Small variations of static density: $\bar{\rho}(z) \ll \rho_0 \implies$ only $\bar{\rho}(z)$ is neglected but not its vertical derivative.

(3) Anelastic approximation (used for atmospheres) is when $\bar{\rho}(z)$ is not neglected.

The thermodynamic equation in Boussinesq case ($D\rho/Dt = Q_\rho$) is traditionally written for buoyancy anomaly $b(\rho) \equiv -g\rho' / \rho_0$:

$$
(*) \quad \frac{D(\bar{b} + b)}{Dt} = Q_b,
$$

where $Q_b$ is source term proportional to $Q(\rho)$, and static buoyancy is $\bar{b}(z) \equiv -g\bar{\rho}/\rho_0$. 
Equation (*) is often written as
\[ \frac{D b}{D t} + N^2(z) w = Q_b, \]
\[ N^2(z) \equiv \frac{db}{dz} \]  

Buoyancy frequency \( N \) measures strength of the static (background) stratification in terms of its vertical derivative, in accord with (2).

NOTE: Primitive equations are often used in practice as approximation to (**)\, which in the realistic general circulation models is replaced by separate equations for thermodynamic variables, and, then, the buoyancy is found diagnostically from the equation of state:

\[ \frac{D T}{D t} = Q_T, \quad \frac{D S}{D t} = Q_S, \quad b = b(T, S, z) \]

Vertical momentum equation in the Boussinesq form is often written only for pressure anomaly (without the static part):

\[ p = p_s + p', \quad \rho = \rho_s + \rho', \quad -\frac{\partial p_s}{\partial z} = \rho_s g \quad \text{(static balance)}, \quad \frac{D w}{D t} = -\frac{1}{\rho} \frac{\partial \rho}{\partial z} - g \quad \text{(momentum)} \]

Let’s keep the static part for a while and rewrite the last equation in the Boussinesq approximation:

\[ \Rightarrow \quad (\rho_s + \rho') \frac{D w}{D t} = -\frac{\partial (p_s + p')}{\partial z} - (\rho_s + \rho') g \quad \Rightarrow \quad \rho_0 \frac{D w}{D t} = -\frac{\partial \rho'}{\partial z} - \rho' g \quad \Rightarrow \quad \frac{D w}{D t} = -\frac{1}{\rho_0} \frac{\partial \rho'}{\partial z} + b \]

Note, that in the vertical acceleration term \( \rho_s + \rho' \) is replaced by \( \rho_0 \), in accord with (2). Horizontal momentum equations are treated similarly.

To summarize the Boussinesq system of equations is (let’s drop primes, from now on, keeping in mind that \( p \) indicates dynamic pressure anomaly):

\[ \frac{D u}{D t} - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad \frac{D v}{D t} + f u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \quad \frac{D w}{D t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + b, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{D b}{D t} + N^2 w = Q_b, \]

- **Hydrostatic approximation.** For many fluid flows vertical acceleration is small relative to gravity, and gravity force is balanced by the vertical component of pressure gradient (we’ll come back to this approximation more formally):

\[ \frac{D w}{D t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \Rightarrow \quad \frac{\partial p}{\partial z} = -\rho g \]

Hydrostatic Boussinesq approximation is commonly used for many GFD phenomena.
• Buoyancy frequency \( N(z) \) appearing in the continuous stratification case has simple physical meaning. In a stratified fluid consider density difference \( \delta \rho \) between a fluid particle adiabatically lifted by \( \delta z \) and surrounding fluid \( \rho_s(z) \). Motion of the particle is determined by the buoyancy (Archimedes) force \( F \) and Newton’s second law:

\[
\delta \rho = \rho_{\text{particle}} - \rho_s(z + \delta z) = \rho_s(z) - \rho_s(z + \delta z) = -\frac{\partial \rho_s}{\partial z} \delta z \quad \rightarrow \quad F = -g \delta \rho = g \frac{\partial \rho_s}{\partial z} \delta z
\]

\[
\rightarrow \quad \rho_s \frac{\partial^2 \delta z}{\partial \eta^2} = g \frac{\partial \rho_s}{\partial z} \delta z \quad \rightarrow \quad \delta \ddot{z} + N^2 \delta z = 0
\]

(a) If \( N^2 > 0 \), then fluid is statically stable, and the particle will oscillate around its resting position with frequency \( N(z) \) (typical periods of oscillations are \( 10 - 100 \) minutes in the ocean, and about \( 10 \) times shorter in the atmosphere).

(b) In the atmosphere one should take into account how density of the lifted particle changes due to the local change of pressure. Then, \( N^2 \) is reformulated with potential density \( \rho_\theta \) rather than density itself.

• Rotation-dominated flows are in the focus of GFD. Such flows are slow, in the sense that they have advective time scales longer than the planetary rotation period: \( L/U \gg f^{-1} \).

Given typical observed flow speeds in the atmosphere \( (U_a \sim 1 - 10 \, \text{m/s}) \) and ocean \( (U_o \sim 0.1 \, U_a) \), the length scales of rotation-dominated flows are \( L_a \gg 10 - 100 \, \text{km} \) and \( L_o \gg 1 - 10 \, \text{km} \). Motions on these scales constitute most of the weather and strongly influence climate and climate variability.

Rotation-dominated flows tend to be hydrostatic (shown later).
Later on, we will use asymptotic analysis to focus on these scales and filter out less important faster and smaller-scale motions.

• Thin-layered framework describes fluid in terms of stacked, vertically thin but horizontally vast layers of fluid with slightly different densities (lighter towards the top, and heavier towards the bottom) — this is rather typical situation in GFD.

Let’s introduce the physical scales: \( L \) and \( H \) are the horizontal and vertical length scales such that \( L \gg H \); and \( U \) and \( W \) are the horizontal and vertical velocity scales, respectively, such that \( U \gg W \). From now on, we’ll focus mostly on motion with such scales.

Thin-layered flows tend to be hydrostatic (shown later).
Later on, we will formulate models that describe fluid in terms of properly scaled, vertically thin but horizontally vast fluid layers.
Summary. We considered the following sequence of simplified approximations:

\[ \text{Governing Equations (spherical coordinates)} \rightarrow \text{Local Cartesian} \rightarrow \text{Boussinesq} \rightarrow \text{Hydrostatic Boussinesq}. \]

Paid price for **local Cartesian**: simplified rotation and sphericity effects; neglected $\Omega y$.

Paid price for **Boussinesq**: incompressible, weakly stratified (i.e., static and dynamic densities); filtered motions include acoustics, shocks, bubbles, surface tension, inner Jupiter.

Paid price for **Hydrostatic**: small vertical accelerations; filtered motions include convection, breaking gravity waves, Kelvin-Helmholtz, density currents, double diffusion, tornadoes.

Let’s consider the simplest relevant thin-layered model, which is locally Cartesian, Boussinesq and hydrostatic, and try to focus on its rotation-dominated flow component...
BALANCED DYNAMICS

- **Shallow-water model** — our starting point — describes motion of a horizontal fluid layer with variable thickness, \( h(t, x, y) \). Density is a constant \( \rho_0 \) and vertical acceleration is neglected (hydrostatic approximation), hence:

\[
\frac{\partial p}{\partial z} = -\rho_0 g \quad \Rightarrow \quad p(t, x, y, z) = \rho_0 g [h(t, x, y) - z],
\]

where we took into account that \( p = 0 \) at \( z = h(t, x, y) \).

Note, that horizontal pressure gradient is independent of \( z \); hence, \( u \) and \( v \) are also independent of \( z \), and fluid moves in columns.

In local Cartesian coordinates:

\[
\frac{Du}{Dt} -fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} = -g \frac{\partial h}{\partial x}, \quad \frac{Dv}{Dt} +fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} = -g \frac{\partial h}{\partial y},
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}.
\]

Continuity equation is needed to close the system, but let’s note that vertical velocity component is related to the height of fluid column and derive the shallow-water continuity equation from the first principles. Recall that velocity does not depend on \( z \) and consider mass budget of a fluid column. The horizontal mass convergence (see earlier derivation of the continuity equation) into the column is (apply divergence theorem):

\[
M = -\int_S \rho_0 \mathbf{u} \cdot d\mathbf{S} = -\oint \rho_0 h \mathbf{u} \cdot d\mathbf{l} = -\int_A \nabla \cdot (\rho_0 h \mathbf{u}) \ dA,
\]

and this must be balanced by the local increase of the mass due to increase in height of fluid column:

\[
M = \frac{d}{dt} \int \rho_0 \ dV = \frac{d}{dt} \int_A \rho_0 h \ dA = \int_A \rho_0 \frac{\partial h}{\partial t} \ dA \quad \Rightarrow \quad \frac{\partial h}{\partial t} = -\nabla \cdot (h \mathbf{u}) \quad \Rightarrow \quad \frac{Dh}{Dt} + h \nabla \cdot \mathbf{u} = 0
\]

Note that the shallow-water continuity equation can be obtained by transformation \( \rho \to h \).
Relative vorticity is \[ \zeta = [\nabla \times \mathbf{u}]_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}; \quad \zeta > 0 \] is counterclockwise (cyclonic) motion, and \( \zeta < 0 \) is the opposite.

Vorticity equation is obtained from the momentum equations, by taking \( y \)-derivative of the first equation and subtracting it from the \( x \)-derivative of the second equation (remember to differentiate advection term of the material derivative; pressure terms cancel out):

\[
\frac{D\zeta}{Dt} + \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] (\zeta + f) + v \frac{df}{dy} = 0
\]

By using the shallow-water continuity equation we obtain:

\[
\frac{D\zeta}{Dt} - \frac{1}{h} (\zeta + f) \frac{Dh}{Dt} + v \frac{df}{dy} = 0 \quad \implies \quad \frac{1}{h} \frac{D(\zeta + f)}{Dt} - \frac{1}{h^2} (\zeta + f) \frac{Dh}{Dt} = 0 \quad \implies \quad \frac{D}{Dt} \left[ \frac{\zeta + f}{h} \right] = 0.
\]

- **Potential vorticity (PV) material conservation law:**

\[
\frac{Dq}{Dt} = 0, \quad q \equiv \frac{\zeta + f}{h}
\]

(a) This is a very powerful statement that reduces dynamical description of fluid motion to solving for evolution of materially conserved scalar quantity (analogy with electric charge).

(b) PV is controlled by changes in \( \zeta \), \( f(y) \), and \( h \) (stretching/squeezing of vortex tube).

(c) Under certain conditions (e.g., when the flow is rotation-dominated) the flow can be determined entirely from PV (e.g., when \( h = H = \text{const} \)).

(d) The above analyses can be extended to many layers and continuous stratification.

(e) More general formulation is referred to as Ertel PV: e.g., \( q = -g (\zeta + f) \frac{\partial \theta}{\partial p} \), where \( \theta \) is potential density.

- **Rossby number** is ratio of the scalings for material derivative (i.e., horizontal acceleration) and Coriolis forcing:

\[
\epsilon = \frac{U^2/L}{fU} = \frac{U}{fL}
\]

For rotation-dominated motions: \[ \epsilon \ll 1 \].

More conventional notation for Rossby number is \( Ro \), but we’ll use \( \epsilon \) to emphasize its smallness and apply the \( \epsilon \)-asymptotic expansion.

Using the smallness of \( \epsilon \), we can expand the governing equations in terms of the geostrophic (leading-order terms) and ageostrophic (first-order correction) motions:

\[
u = u_y + \epsilon u_a, \quad p' = p'_y + \epsilon p'_a, \quad \rho' = \rho'_y + \epsilon \rho'_a.
\]
Rossby number expansion: The goal is to be able to predict strong geostrophic motions, and this requires taking into account weak ageostrophic motions.

Let’s focus on the $\beta$-plane and mesoscales: $T = \frac{L}{U} = \frac{L}{\epsilon f_0 L} = \frac{1}{\epsilon f_0}$, $L/R_0 \sim \epsilon \Rightarrow [\beta y] \sim \frac{f_0}{R_0} L \sim \epsilon f_0$.

Let’s put the $\epsilon$-expansion in the horizontal momentum equations and see that only pressure gradient can balance Coriolis force:

$$
\frac{Du}{Dt} - f_0 (v_g + \epsilon v_a) - \beta y v_g + \epsilon^2 [...] = -\frac{1}{\rho_0} \frac{\partial p_g}{\partial x} - \frac{\epsilon}{\rho_0} \frac{\partial p_a}{\partial x}
$$

$$
\frac{Dv}{Dt} + f_0 (u_g + \epsilon u_a) + \beta y u_g + \epsilon^2 [...] = -\frac{1}{\rho_0} \frac{\partial p_g}{\partial y} - \frac{\epsilon}{\rho_0} \frac{\partial p_a}{\partial y}
$$

$\epsilon f_0 U \quad f_0 U \quad \epsilon f_0 U \quad \epsilon^2 f_0 U \quad [p']/\rho_0 L \quad \epsilon [p']/\rho_0 L$

- **Geostrophic balance** is obtained from the horizontal momentum equations at the leading order:

$$
f_0 v_g = \frac{1}{\rho_0} \frac{\partial p_g}{\partial x}, \quad f_0 u_g = -\frac{1}{\rho_0} \frac{\partial p_g}{\partial y}
$$

(a) Proper scaling for pressure must be $[p'] \sim \rho_0 f_0 U L$.

(b) Counterintuitive dynamics: Induced local pressure anomaly results in a circular flow around it, rather than in a classical fluid flow response along the pressure gradient.

(c) It follows from the geostrophic balance, that $u_g$ is nondivergent: $\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$ (below it is shown that $w_g = 0$).

(d) Geostrophic flow is 2D and nondivergent, hence, it can be described by a velocity streamfunction; note, that pressure in the geostrophic balance acts as the streamfunction in disguise!

(e) Geostrophic balance is not a prognostic equation; the next order of the $\epsilon$-expansion is needed to determine the flow evolution.

Now, let’s prove that geostrophically balanced flows are also hydrostatically balanced...
• **Hydrostatic balance.** Vertical acceleration is typically small for large-scale geophysical motions, because they are thin-layered and rotation-dominated. Let’s prove this formally:

\[
\frac{Dw}{Dt} = -\frac{1}{\rho_s + \rho_g} \frac{\partial (p_s + p_g)}{\partial z} - g, \quad \frac{Dw}{Dt} \sim 0, \quad \frac{\partial p_s}{\partial z} = -\rho_s g \implies \frac{\partial p_g}{\partial z} = -\rho_g g \quad (\star)
\]

Use the corresponding scalings \( W = U H / L, \quad T = L / U, \quad [p'] = \rho_0 f_0 U L, \quad U = \epsilon f_0 L \) to identify the validity bound for the leading-order hydrostatic balance:

\[
\frac{Dw}{Dt} \ll \frac{1}{\rho_0} \frac{\partial p_g}{\partial z} \implies \frac{H U^2}{L^2} \ll \frac{\rho_0 f_0 U L}{\rho_0 H} \implies \left( \frac{H}{L} \right)^2 \ll 1
\]

If the last inequality is true, then vertical acceleration can be neglected — this situation routinely happens for large-scale geophysical flows.

• **Scaling for geostrophic-flow density anomaly.** From the hydrostatic balance \((\star)\) and geostrophic scaling for pressure \([p']\), we find scaling for geostrophic dynamic density anomaly \(\rho_g\):

\[
[p_g] \equiv [p'] \sim \frac{[p']}{g H} = \frac{\rho_0 f_0 L}{g H} = \rho_0 \epsilon \frac{f_0^2 L^2}{g H} = \rho_0 \epsilon F, \quad F \equiv \frac{f_0^2 L^2}{g H} = \left( \frac{L}{L_d} \right)^2, \quad L_d \equiv \sqrt{g H} f_0 \sim O(10^4 \text{ km}),
\]

where \(L_d\) is the external deformation scale, and \(F\) is Froude number (ratio of characteristic flow velocity to the fastest wave speed). For many geophysical scales of interest: \(F \ll 1\), therefore, it is safe to assume that

\[
F \sim \epsilon \implies [\rho_g] = \rho_0 \epsilon^2
\]

Thus, ubiquitous and powerful, double-balanced (geostrophic and hydrostatic) motions correspond to nearly flat isopycnals.

• **Continuity for ageostrophic flow.** Let’s now turn attention to the continuity equation and also \(\epsilon\)-expand it:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0, \quad \rho = \rho_s + \rho_g, \quad u = u_g + \epsilon u_a, \quad v = v_g + \epsilon v_a, \quad w = w_g + \epsilon w_a \rightarrow \frac{\partial \rho}{\partial t} + (\rho_s + \rho_g) \left( \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) + u_g \frac{\partial \rho_g}{\partial x} + v_g \frac{\partial \rho_g}{\partial y} + \epsilon \rho_s \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) + \epsilon^2 \left[ \ldots \right] + \frac{\partial}{\partial z} \left( w_g \rho_s + \epsilon \rho_a \rho_s + w_g \rho_g + \epsilon \rho_a \rho_g \right) = 0
\]

Use \(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0\) and \(\rho_g \sim \epsilon^2\) to obtain at the leading order: \(\frac{\partial (w_g \rho_s)}{\partial z} = 0 \implies w_g \rho_s = \text{const}\)

Because of the BCs, somewhere in the water column \(w_g(z)\) has to be zero \(\implies w_g = 0\), \(w = \epsilon w_a\), \([w] = W = \epsilon U \frac{H}{L}\).
At the next order of the $\epsilon$-expansion we recover the continuity equation for ageostrophic flow component:

$$\frac{\partial (u_a \rho_s)}{\partial z} + \rho_s \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) = 0.$$ 

Let’s keep this in mind and use it in the derivation of geostrophic vorticity equation.

- **Geostrophic (absolute) vorticity equation** is obtained by going to the next order of $\epsilon$ in the shallow-water momentum equations:

\[
\begin{align*}
\frac{D_g u_g}{Dt} - (\epsilon f_0 v_a + v_g \beta y) &= -\epsilon \frac{1}{\rho_s} \frac{\partial p_a}{\partial x}, \\
\frac{D_g v_g}{Dt} + (\epsilon f_0 u_a + u_g \beta y) &= -\epsilon \frac{1}{\rho_s} \frac{\partial p_a}{\partial y}, \\
\frac{D_g}{Dt} &= \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}.
\end{align*}
\]

(i) Take curl of the above equations (i.e., subtract $y$-derivative of the first equation from $x$-derivative of the second equation) and mind complexity of the material derivative;

(ii) Use nondivergence of the geostrophic velocity;

(iii) Use continuity equation for ageostrophic flow to replace horizontal ageostrophic velocity divergence.

Thus, we obtain the geostrophic vorticity equation:

\[
\begin{align*}
\frac{D_g}{Dt} \zeta_g + \beta v_g &= \frac{D_g}{Dt} [\zeta_g + \beta y] = \epsilon f_0 \frac{\partial (\rho_s u_a)}{\partial z}, \\
\zeta_g &= \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}.
\end{align*}
\]

(a) This looks almost as PV material conservation law, but unfortunately it is not one, because of the rhs term. Can it be reformulated in the missing PV component?

(b) Evolution of the absolute vorticity is determined by divergence of the vertical mass flux, due to tiny vertical velocity. This is the process of squeezing or stretching the isopycnals, which is related to the form drag mechanism discussed below. Can this term be determined from the leading-order fields?

(c) Quasigeostrophic theory expresses the rhs in terms of vertical movement of isopycnals, then, it relates this movement to pressure, which is the streamfunction in disguise.

(d) On the other hand, evolution of the horizontal absolute vorticity produces the squeezing or stretching deformations, which induce motions in the neighbouring layers.

(e) If $\rho_s$ is constant within the layer (i.e., thin-layered framework), then, it cancels out from the rhs, and we are left with the vertical component of velocity divergence.
Form drag is a horizontal pressure-gradient force associated with variable isopycnal layer thickness, which in turn can arise due to squeezing or stretching of isopycnal-layer thicknesses.

Geostrophic motions are very efficient in terms of redistributing horizontal momentum vertically, through the form drag mechanism.

Let’s consider a constant-density fluid layer confined by two interfaces, \( h_1(x, y) \) and \( h_2(x, y) \). The zonal pressure-gradient force acting on a volume of fluid is

\[
F_x = -\frac{1}{L} \int_0^L \left( \frac{\partial p}{\partial x} \right) h_1 \, dx = -\frac{1}{L} \int_0^L \left[ \frac{\partial p}{\partial x} \right]_{h_2} ^{h_1} \, dx = -h_1 \frac{\partial p}{\partial x} + h_2 \frac{\partial p}{\partial x} = p \frac{\overline{\partial h_1}}{\partial x} - p \frac{\overline{\partial h_2}}{\partial x},
\]

where \( p_1 \) and \( p_2 \) are pressures on the interfaces; \( \partial p/\partial x \) does not depend on vertical position within a layer; \( L \) is taken to be a circle of latitude, and overline denotes zonal averaging. The force acting on fluid within the layer is zero, if its boundaries \( \eta_1 \) and \( \eta_2 \) are flat. The above statement can be reversed: if the isopycnal boundaries of a fluid layer are deformed (e.g., by squeezing or stretching), the layer can be accelerated or decelerated by the corresponding form drag force.

Thus, if a geostrophic motion in some isopycnal layer squeezes or stretches it, the underlying layer is also deformed, and the resulting pressure-gradient force accelerates fluid in the underlying layer.
QUASIGEOSTROPHIC THEORY

- **Two-layer shallow-water model** is a natural extension of the single-layer shallow-water model. It illuminates effects of isopycnal deformations on the geostrophic vorticity. This model can be straightforwardly extended to many isopycnal (i.e., constant-density) layers, thus, producing the family of *isopycnal models*.

The model assumes geostrophic and hydrostatic balances, and usual Boussinesq treatment of density:

\[ \Delta \rho \equiv \rho_2 - \rho_1 \ll \rho_1, \rho_2, \quad \rho_1 \approx \rho_2 \approx \rho. \]

All notations are introduced on the sketch.

The layer thicknesses and pressures consist of the static and dynamic components:

\[ \begin{align*}
    h_1(t, x, y) &= H_1 + H_2 + \eta_1(t, x, y), \\
    h_2(t, x, y) &= H_2 + \eta_2(t, x, y), \\
    p_1 &= \rho_1 g (H_1 + H_2 - z) + p_1'(t, x, y), \\
    p_2 &= \rho_1 g H_1 + \rho_2 g (H_2 - z) + p_2'(t, x, y),
\end{align*} \]

Here, the shallow-water dynamic pressure anomalies are independent of \( z \), as we have seen, and the static pressures are obtained by integrating the hydrostatic balance \( \partial p / \partial z = -\rho g \), and taking the integration constants such that the static pressure is zero at the upper surface and continuous on the internal interface between the layers.

- **Continuity boundary condition for pressure** is just a component of the continuity boundary condition for stress tensor (sometimes, this involves surface tension); here, it allows to relate dynamic pressure anomalies and isopycnal deformations.

In the two-layer model this boundary condition is equivalent to saying that:

(a) pressure at the upper surface must be zero (more generally, it must be equal to the atmospheric pressure),

(b) pressure on the internal interface must be continuous, i.e., \( p_1 = p_2 = P \).

Note, that in the absence of motion \( (p'_1 = p'_2 = 0) \) both of these conditions are automatically satisfied for the static pressure component:

\[ p_1|_{z=H_1+H_2} = 0, \quad p_1|_{z=H_2} = p_2|_{z=H_2} = \rho_1 g H_1. \]

In the presence of motion, the upper-surface pressure continuity statement \( p_1|_{z=\eta_1+H_1+H_2} = 0 \) translates into

\[ p_1'(t, x, y) = \rho_1 g \eta_1(t, x, y). \]
On the internal interface, the pressure continuity statement is:

\[ P = p_1|_{z=\eta_2 + H_2} = \rho_1 g (H_1 - \eta_2) + p_1', \quad P = p_2|_{z=\eta_2 + H_2} = \rho_1 g H_1 - \rho_2 g \eta_2 + p_2' \implies p_2'(t, x, y) = p_1'(t, x, y) + g \Delta \rho \eta_2(t, x, y) \]

Thus, by using expression for the upper-layer pressure, we obtain:

\[ p_2'(t, x, y) = \rho_1 g \eta_1(t, x, y) + g \Delta \rho \eta_2(t, x, y) \]

- **Geostrophy** at the leading order links horizontal velocities and slopes of the isopycnals (interfaces) in the upper and deep layers:

\[-f_0 v_1 = -g \frac{\partial \eta_1}{\partial x}, \quad f_0 u_1 = -g \frac{\partial \eta_1}{\partial y}; \quad -f_0 v_2 = -g \frac{\rho_1}{\rho_2} \frac{\partial \eta_1}{\partial x} - g \frac{\Delta \rho \partial \eta_2}{\rho_2} \frac{\partial \eta_2}{\partial x}, \quad f_0 u_2 = -g \frac{\rho_1}{\rho_2} \frac{\partial \eta_1}{\partial y} - g \frac{\Delta \rho \partial \eta_2}{\rho_2} \frac{\partial \eta_2}{\partial y} \]

Next, we recall that \( \rho_1 \approx \rho_2 \approx \rho \) (Boussinesq argument), introduce the **reduced gravity** \( g' \equiv g \Delta \rho / \rho \), and, thus, simplify the second-layer equations:

\[-f_0 v_2 = -g \frac{\partial \eta_1}{\partial x} - g' \frac{\partial \eta_2}{\partial x}, \quad f_0 u_2 = -g \frac{\partial \eta_1}{\partial y} - g' \frac{\partial \eta_2}{\partial y} \]

- **Geostrophic vorticity equations.**

Now, let’s take a look at the full system of the two-layer shallow-water equations:

\[ \frac{D u_1}{Dt} - f v_1 = -g \frac{\partial \eta_1}{\partial x}, \quad \frac{D v_1}{Dt} + f u_1 = -g \frac{\partial \eta_1}{\partial y}, \quad \frac{\partial (h_1 - h_2)}{\partial t} + \nabla \cdot ((h_1 - h_2) u_1) = 0, \]

\[ \frac{D u_2}{Dt} - f v_2 = -g \frac{\partial \eta_1}{\partial x} - g' \frac{\partial \eta_2}{\partial x}, \quad \frac{D v_2}{Dt} + f u_2 = -g \frac{\partial \eta_1}{\partial y} - g' \frac{\partial \eta_2}{\partial y}, \quad \frac{\partial h_2}{\partial t} + \nabla \cdot (h_2 u_2) = 0. \]

As we have argued, at the leading order the momentum equations are geostrophic; at the \( \epsilon \)-order, we can formulate the layer-wise vorticity equations with the additional rhs terms responsible for vertical deformations. For this purpose:

(a) Expand the momentum equations in terms of \( \epsilon \),

(b) take curl of the momentum equations \( \frac{1}{\partial x} \cdot (\partial (2)/\partial x - \partial (1)/\partial y) \),

(c) replace divergence of the horizontal ageostrophic velocity \( (u_a, v_a) \) with the vertical divergence of \( w_a \).

The resulting equations are:

\[ \frac{D_n \zeta_n}{Dt} + \beta v_n = f_0 \frac{\partial w_n}{\partial z}, \quad \frac{D_n}{Dt} = \frac{\partial}{\partial t} + u_n \frac{\partial}{\partial x} + v_n \frac{\partial}{\partial y}, \quad \zeta_n \equiv \frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y}, \quad n = 1, 2. \]
Within each layer horizontal velocity does not depend on $z$, therefore, vertical integrations of the vorticity equations across each layer yield (here, we assume nearly flat isopycnals everywhere by replacing $h_1 - h_2 \approx H_1$ and $h_2 \approx H_2$ on the lhs):

$$H_1 \left( \frac{D_1 \zeta_1}{D t} + \beta v_1 \right) = f_0 \left( w_1(h_1) - w_1(h_2) \right), \quad H_2 \left( \frac{D_2 \zeta_2}{D t} + \beta v_2 \right) = f_0 w_2(h_2),$$

(*)

Thus, we extended the assumption of nearly flat isopycnals to everywhere, beyond the scale of motions. Note, that in (*) we took $w_2(bottom) = 0$, but this is true only for the flat bottom (along topographic slopes vertical velocity can be non-zero, as only normal-to-boundary velocity component vanishes).

- **Vertical movement of isopycnals in terms of pressure** can be obtained, and this step closes the equations. For that, we use kinematic boundary condition, which comes from considering fluid elements on the fluid interface or surface, such that the vertical coordinates of these elements are given by $z = h(t, x, y)$. Next, let’s consider functional $F(t, x, y, z) = h(t, x, y) - z$, and acknowledge, that it is always zero for a fluid elements sitting on the interface or surface; hence, its material derivative is zero:

$$\frac{DF}{Dt} = 0 = \frac{Dh}{Dt} - w \frac{\partial z}{\partial z} \rightarrow w = \frac{Dh}{Dt}$$

By combining the kinematic boundary condition with the Boussinesq argument ($\rho_1 \approx \rho_2 \approx \rho$), we obtain:

$$w_n(h_n) = \frac{D_n h_n}{D t} = \frac{D_n \eta_n}{D t} \implies w_1(h_1) = \frac{1}{\rho g} \frac{D_1 p'_1}{D t}, \quad w_{1,2}(h_2) = \frac{1}{\Delta \rho g} \frac{D_1,2(p'_2 - p'_1)}{D t}$$

(**)

- **Pressure is streamfunction in disguise.**

In each layer geostrophic velocity streamfunction is linearly related to dynamic pressure anomaly, as follows from the geostrophic momentum balance:

$$f_0 v_n = \frac{1}{\rho} \frac{\partial p'_n}{\partial x}, \quad f_0 u_n = -\frac{1}{\rho} \frac{\partial p'_n}{\partial y} \implies \psi_n = \frac{1}{f_0 \rho} p'_n, \quad u_n = -\frac{\partial \psi_n}{\partial y}, \quad v_n = \frac{\partial \psi_n}{\partial x}$$

(***)

Relative vorticity $\zeta$ is always conveniently expressed in terms of $\psi$:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$$

Let’s now combine (*), (**) and (***) to obtain the fully closed equations predicting evolution of the leading-order streamfunction.
• Two-layer quasigeostrophic (QG) model.

\[
\frac{D_1 \zeta_1}{Dt} + \beta v_1 - \frac{f_0^2}{gH_1} \left( \frac{\rho}{\Delta \rho} \frac{D_1}{Dt} (\psi_1 - \psi_2) + \frac{D_1 \psi_1}{Dt} \right) = 0,
\]

\[
\frac{D_2 \zeta_2}{Dt} + \beta v_2 - \frac{f_0^2}{gH_2} \frac{\rho}{\Delta \rho} \frac{D_2}{Dt} (\psi_2 - \psi_1) = 0
\]

(a) Note that \( \Delta \rho \ll \rho \), therefore the last term of the first equation is neglected (i.e., the rigid-lid approximation is taken; it states that the surface elevation is much smaller than the internal interface displacement).

(b) Familiar reduced gravity is \( g' \equiv g \Delta \rho / \rho \), and stratification parameters are defined as

\[
S_1 = \frac{f_0^2}{g' H_1}, \quad S_2 = \frac{f_0^2}{g' H_2}.
\]

(c) Dimensionally, \([S_1] \sim [S_2] \sim L^{-2} \rightarrow\) QG (i.e., a double-balanced) motion of a stratified fluid operates on the internal deformation scales: \( R_1 = 1/\sqrt{S_1} \), and \( R_2 = 1/\sqrt{S_2} \), which are \( O(100km) \) in the ocean and about 10 times larger in the atmosphere (fundamentally, \( R_n \ll L_d = f_0^2 / gH \), because \( g' \ll g \)).

With the above information taken into account, we obtain the final set of the QG equations:

\[
\frac{D_1}{Dt} \left[ \nabla^2 \psi_1 - S_1 (\psi_1 - \psi_2) \right] + \beta v_1 = 0, \quad \frac{D_2}{Dt} \left[ \nabla^2 \psi_2 - S_2 (\psi_2 - \psi_1) \right] + \beta v_2 = 0
\]

Potential vorticity anomalies are defined as

\[
q_1 = \nabla^2 \psi_1 - S_1 (\psi_1 - \psi_2), \quad q_2 = \nabla^2 \psi_2 - S_2 (\psi_2 - \psi_1)
\]

Note: These expressions for the PV anomalies can be obtained by linearization of the full shallow-water PV (given without proof).
- **Potential vorticity (PV) material conservation law.**

  *(Absolute) PV is defined as* \( \Pi_1 = q_1 + f = q_1 + f_0 + \beta y, \quad \Pi_2 = q_2 + f = q_2 + f_0 + \beta y \).

  (a) PV is materially conserved quantity:

  \[
  \frac{D_n}{Dt} \Pi_n = \frac{\partial \Pi_n}{\partial t} + \frac{\partial \psi_n}{\partial x} \frac{\partial \Pi_n}{\partial y} - \frac{\partial \psi_n}{\partial y} \frac{\partial \Pi_n}{\partial x} = 0, \quad n = 1, 2
  \]

  (b) PV can be considered as a “charge” advected by the flow; but this is *active* charge, as it defines the flow itself.

  (c) **PV inversion** brings in intrinsic and important spatial nonlocality of the velocity field around “elementary charge” of PV:

  \[
  \Pi_1 = \nabla^2 \psi_1 - S_1 (\psi_1 - \psi_2) + \beta y + f_0, \quad \Pi_2 = \nabla^2 \psi_2 - S_2 (\psi_2 - \psi_1) + \beta y + f_0
  \]

  (d) PV consists of of relative vorticity, density anomaly (resulting from isopycnal displacement), and planetary vorticity.

- **Continuous stratification** yields (without derivation) similar PV conservation law and PV inversion formula for the geostrophic fields:

  \[
  \psi = \frac{1}{f_0 \rho} \rho', \quad u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad \rho = -\rho_0 f_0 \frac{\partial \psi}{g \partial z}, \quad N^2(z) = -\frac{g}{\rho_s} \frac{d \rho_s}{dz}
  \]

  \[
  \frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Pi}{\partial x} = 0, \quad \Pi = \nabla^2 \psi + f_0 \frac{\partial}{\partial z} \left( \frac{1}{N^2(z)} \frac{\partial \psi}{\partial z} \right) + f_0 + \beta y
  \]

  Note, that density anomalies are now described by vertical derivative of velocity streamfunction, rather than by deformation of interface \( \eta \) that is related to (vertical) difference between the streamfunction values above and below it.

- **Boundary conditions** for QG equations.

  (a) On the lateral solid boundaries there is always no-normal-flow condition: \( \psi = C(t) \).

  (b) The other lateral boundary conditions are no-slip: \( \frac{\partial \psi}{\partial n} = 0 \), free-slip: \( \frac{\partial^2 \psi}{\partial n^2} = 0 \), partial-slip: \( \frac{\partial^2 \psi}{\partial n^2} + \frac{1}{\alpha} \frac{\partial \psi}{\partial n} \), periodic, double-periodic, etc.

  (c) There are also integral constraints on mass and momentum. For example, we can require that basin-averaged density anomaly integrates to zero in each layer:

  \[
  \int \int \rho \, dx \, dy = 0 \quad \rightarrow \quad \int \int \frac{\partial \psi}{\partial z} \, dx \, dy = 0.
  \]

  (d) Vertical velocities on the open surface and rigid bottom are determined from the *Ekman boundary layers* (discussed later!).
• **Ageostrophic circulation** (of the $\epsilon$-order) can be obtained with further efforts, and even *diagnostically*.

For example, vertical ageostrophic velocity is equal to material derivative of pressure, which is known from the QG solution:

\[
\begin{align*}
\left. w_1 \right|_{h_1} &= \frac{1}{\rho g} \frac{D_1 p'_1}{Dt}, \\
\left. w_1 \right|_{h_2} &= \frac{1}{\Delta \rho g} \frac{D_1 (p'_2 - p'_1)}{Dt}
\end{align*}
\]

**OTHER COMMENTS** on this section:

(a) *Midlatitude theory*: QG framework does not work at the equator, where $f = 0$.

(b) *Vertical control*: Nearly horizontal geostrophic motions are determined by vertical stratification, vertical component of $\zeta$, and vertical isopycnal stretching.

(c) *Four main assumptions* that have been made:

(i) Rossby number $\epsilon$ is small (hence, the expansion focuses on mesoscales);
(ii) $\beta$-plane approximation and small meridional variations of Coriolis parameter;
(iii) isopycnals are nearly flat ($[\delta \rho'] \sim \epsilon F \rho_0 \sim \epsilon^2 \rho_0$) everywhere;
(iv) hydrostatic Boussinesq balance.

• **Planetary-geostrophic equations** can be similarly derived for small-Rossby-number motions on scales that are much larger than internal deformation scale $R$ and for large meridional variations of Coriolis parameter.

Let’s start from the full shallow-water equations,

\[
\begin{align*}
\frac{Du}{Dt} - fu &= -g \frac{\partial h}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -g \frac{\partial h}{\partial y}, \\
\frac{Dh}{Dt} + h \nabla \cdot u &= 0,
\end{align*}
\]

and consider $F = L^2 / R^2 \sim \epsilon^{-1} \gg 1$.

Then, let’s reasonably assume that, for large scales of motion, fluid height variations are as large as the mean height of fluid:

\[
h = H (1 + \epsilon F \eta) = H (1 + \eta).
\]

Asymptotic expansions $u = u_0 + \epsilon u_1 + \ldots$ and $\eta = \eta_0 + \epsilon \eta_1 + \ldots$ yield:

\[
\begin{align*}
\epsilon \left[ \frac{\partial u_0}{\partial t} + u_0 \nabla u_0 - f v_1 \right] - f v_0 &= -g H \frac{\partial \eta_0}{\partial x} - \epsilon g H \frac{\partial \eta_1}{\partial x} + O(\epsilon^2), \\
\ldots, \\
\epsilon F \left[ \frac{\partial \eta_0}{\partial t} + u_0 \cdot \nabla \eta_0 \right] + (1 + \epsilon F \eta_0) \nabla \cdot u_0 &= 0.
\end{align*}
\]

Thus, only geostrophic balance is retained in the momentum equation, and all terms are retained in the continuity equation, and the resulting set of equations is:

\[
\begin{align*}
-f v &= -g \frac{\partial h}{\partial x}, \\
f u &= -g \frac{\partial h}{\partial y}, \\
\frac{Dh}{Dt} + h \nabla \cdot u &= 0
\end{align*}
\]
Vortex street behind obstacle

Meandering oceanic current
Observed atmospheric PV

Atmospheric PV from a model
Solutions of geostrophic turbulence
(PV snapshots)
EKMAN LAYERS

- Ekman surface boundary layer.

Boundary layers are governed by physical processes very different from those in the interior. Non-geostrophic effects at the free-surface and rigid-bottom boundary layers are responsible for transferring momentum from the wind and bottom stresses to the interior (large-scale) geostrophic currents. Let’s consider the corresponding Ekman layer at the ocean surface:

(a) Horizontal momentum is transferred down by vertical turbulent flux (its exact form is unknown due to complexity of many physical processes involved), which is commonly modelled by vertical friction (i.e., by diffusion of momentum) with constant turbulent viscosity coefficient:

\[
\overline{w'} \frac{\partial \overline{u'}}{\partial z} = A_v \frac{\partial^2 \overline{u}}{\partial z^2},
\]

where overbar and prime indicate the time mean and fluctuating flow components, respectively. Note, that the vertical friction must be balanced by some other term containing velocity, because the momentum diffusion creates flow velocity, and at the leading order only Coriolis force contains the velocity.

(b) Consider boundary layer correction, so that \( u = u_E + u_g \) in the thin layer with depth \( h_E \):

\[
-f_0(v_g + v_E) = -\frac{1}{\rho_0} \frac{\partial p_g}{\partial x} + A_v \frac{\partial^2 u_E}{\partial z^2}, \quad f_0(u_g + u_E) = -\frac{1}{\rho_0} \frac{\partial p_g}{\partial y} + A_v \frac{\partial^2 v_E}{\partial z^2}.
\]

The Ekman balance is

\[
-f_0 v_E = A_v \frac{\partial^2 u_E}{\partial z^2}, \quad f_0 u_E = A_v \frac{\partial^2 v_E}{\partial z^2} \quad (*)
\]

To make the friction term important in the balance, the Ekman layer thickness must be \( h_E \sim [A_v/f_0]^{1/2} \), therefore, let’s define:

\[ h_E = [2A_v/f_0]^{1/2}. \]

Typical values of \( h_E \) are \( \sim 1 \) km in the atmosphere and \( \sim 50 \) m in the ocean.

(c) If the Ekman number \( \text{Ek} \equiv \left( \frac{h_E}{H} \right)^2 = \frac{2A_v}{f_0 H^2} \) is small, i.e., \( \text{Ek} \ll 1 \), then, the boundary layer correction can be matched to the interior geostrophic solution.
(d) The boundary conditions for the Ekman flow are zero at the bottom of the boundary layer and the stress condition at the upper surface:

\[ A_v \frac{\partial u_E}{\partial z} = \frac{1}{\rho_0} \tau^x, \quad A_v \frac{\partial v_E}{\partial z} = \frac{1}{\rho_0} \tau^y \quad (**) \]

Let’s look for solution of (*) and (**) in the form:

\[ u_E = e^{z/h_E} \left[ C_1 \cos \left( \frac{z}{h_E} \right) + C_2 \sin \left( \frac{z}{h_E} \right) \right], \]
\[ v_E = e^{z/h_E} \left[ C_3 \cos \left( \frac{z}{h_E} \right) + C_4 \sin \left( \frac{z}{h_E} \right) \right], \]

and obtain the Ekman spiral solution:

\[ u_E = \sqrt{2} \frac{\rho_0}{f_0 h_E} e^{z/h_E} \left[ \tau^x \cos \left( \frac{z}{h_E} - \frac{\pi}{4} \right) - \tau^y \sin \left( \frac{z}{h_E} - \frac{\pi}{4} \right) \right], \]
\[ v_E = \sqrt{2} \frac{\rho_0}{f_0 h_E} e^{z/h_E} \left[ \tau^y \cos \left( \frac{z}{h_E} - \frac{\pi}{4} \right) + \tau^x \sin \left( \frac{z}{h_E} - \frac{\pi}{4} \right) \right]. \]

- **Ekman pumping.** Vertically integrated, horizontal Ekman transport \( U_E = \int u_E \, dz \) can be divergent, and it satisfies:

\[-f_0 V_E = A_v \left( \frac{\partial u_E}{\partial z} \bigg|_{\text{top}} - \frac{\partial u_E}{\partial z} \bigg|_{\text{bottom}} \right) = \frac{1}{\rho_0} \tau^x, \]
\[ f_0 U_E = A_v \left( \frac{\partial v_E}{\partial z} \bigg|_{\text{top}} - \frac{\partial v_E}{\partial z} \bigg|_{\text{bottom}} \right) = \frac{1}{\rho_0} \tau^y. \]

The bottom stress terms vanish due to the exponential decay of the boundary layer solution. In order to obtain vertical Ekman velocity at the bottom of the Ekman layer, let’s integrate the continuity equation

\[-(w_E \bigg|_{\text{top}} - w_E \bigg|_{\text{bottom}}) = w \bigg|_{\text{bottom}} \equiv w_E = \frac{\partial U_E}{\partial x} + \frac{\partial V_E}{\partial y} + \frac{\partial}{\partial x} \int u_g \, dz + \frac{\partial}{\partial y} \int v_g \, dz. \]

Recall the non-divergence of the geostrophic velocity and use the above-derived integrated Ekman transport components to obtain

\[ w_E = \frac{\partial U_E}{\partial x} + \frac{\partial V_E}{\partial y} + \int \left( \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) \, dz = \frac{\partial U_E}{\partial x} + \frac{\partial V_E}{\partial y} = \frac{1}{f_0 \rho_0} \nabla \times \tau \]
Thus, the Ekman pumping can be found from the wind curl:

\[ w_E = \frac{1}{f_0 \rho_0} \nabla \times \tau \]

**Conclusion:** Ekman pumping \( w_E \) provides external forcing for the interior geostrophic motions by vertically squeezing or stretching isopycnal layers; it can be viewed as transmission of an external stress into the geostrophic forcing.

- **Ekman bottom boundary layer** can be solved for in a similar way (see Practical Problems).
ROSSBY WAVES

- In the broad sense, Rossby wave is inertial wave propagating on the background PV gradient. First discovered in the Earth’s atmosphere.
- Oceanic Rossby waves are more difficult to observe (e.g., altimetry, in situ measurements)

- Sea surface height anomalies propagating to the west are signatures of baroclinic Rossby waves.

- To what extent transient flow anomalies can be characterized as waves rather than isolated coherent vortices remains unclear.
Flow speed from the high resolution computation shows many eddies/waves.

- Many properties of the flow fluctuations can be interpreted in terms of the linear (Rossby) waves.

Visualization of oceanic eddies/waves by virtual tracer.
• General properties of waves.

(a) Waves provide interaction mechanism which is both long-range and fast relative to flow advection.

(b) Waves are observed as periodic propagating (and standing) patterns, e.g.,
\[ \psi = \text{Re}\{A \exp[i(kx + ly + mz - \omega t + \phi)]\} \]
characterized by amplitude, wavenumbers, frequency, and phase.

Wavevector is defined as the ordered set of wavenumbers: \( \mathbf{K} = (k, l, m) \).

(c) Dispersion relation comes from the dynamics and connects frequency and wavenumbers, and, thus, yields phase speeds and group velocity \( C_g \).

(d) Phase speeds along the axes of coordinates are rates at which intersections of the phase lines with each axis propagate along this axis:
\[
C_p^{(x)} = \frac{\omega}{k}, \quad C_p^{(y)} = \frac{\omega}{l}, \quad C_p^{(z)} = \frac{\omega}{m};
\]
these speeds do not form a vector (note that phase speed along an axis increases with decreasing projection of \( \mathbf{K} \) on this axis).

(e) The fundamental phase speed \( C_p = \omega/K \), where \( K = |\mathbf{K}| \), is defined along the wavevector. This is natural, because waves described by complex exponential functions have instantaneous phase lines perpendicular to \( \mathbf{K} \).

The fundamental phase velocity (vector) is defined as
\[
C_p = \frac{\omega}{|\mathbf{K}|} = \frac{\omega}{K^2} \mathbf{K}.
\]

(f) The group velocity (vector) is defined as
\[
C_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right).
\]

(g) Propagation directions: phase propagates in the direction of \( \mathbf{K} \); energy (hence, information!) propagates at some angle to \( \mathbf{K} \).

(h) If frequency \( \omega = \omega(x, y, z) \) is spatially inhomogeneous, then trajectory traced by the group velocity is called ray, and the path of waves is found by ray tracing methods.
Mechanism of Rossby waves. Consider the simplest 1.5-layer (a.k.a. the equivalent barotropic) QG PV model, which is obtained by considering $H_2 \to \infty$ in the two-layer QG PV model:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Pi}{\partial x} = 0, \quad \Pi = \nabla^2 \psi - \frac{1}{R^2} \psi + \beta y,$$

where $R^{-2} = S_1$ is the stratification parameter written in terms of the inverse length scale parameter $R$. By introducing the Jacobian operator $J(A, B) = A_x B_y - A_y B_x$, the corresponding equivalent-barotropic equation can be written as

$$\frac{\partial}{\partial t} \left( \nabla^2 \psi - \frac{1}{R^2} \psi \right) + J\left( \psi, \nabla^2 \psi - \frac{1}{R^2} \psi \right) + \beta \frac{\partial \psi}{\partial x} = 0.$$  (*)

Note, that in the limit $R \to \infty$ the dynamics becomes purely 2D and deformations of the layer thickness become infinitesimal; this is equivalent to $g' \to g$.

We are interested in small-amplitude flow disturbances around the state of rest; the corresponding linearized equation (*) is

$$\frac{\partial}{\partial t} \left( \nabla^2 \psi - \frac{1}{R^2} \psi \right) + \beta \frac{\partial \psi}{\partial x} = 0$$

$$\rightarrow \quad \psi \sim e^{i(kx + ly - \omega t)} \quad \rightarrow$$

$$-i\omega \left( -k^2 - l^2 - \frac{1}{R^2} \right) + i\beta k = 0$$

Thus, the resulting Rossby waves dispersion relation is

$$\omega = \frac{-\beta k}{k^2 + l^2 + R^{-2}}.$$  

Plot dispersion relation, discuss zonal, phase and group speeds...

Consider a timeline in the fluid at rest, then, perturb it (see Figure): the resulting westward propagation of Rossby waves is due to the $\beta$-effect and material PV conservation.
**Energy equation.** Multiply the equivalent-barotropic equation by \(-\psi\) and use the identity 
\[-\psi \nabla^2 \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} \left( \nabla \psi \right)^2 - \nabla \cdot \psi \nabla \frac{\partial \psi}{\partial t}\]
to obtain the (mechanical) flow energy equation:

\[
\frac{\partial E}{\partial t} + \nabla \cdot S = 0, \quad E = \frac{1}{2} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right] + \frac{1}{2} R^2 \psi^2, \quad S = -\left( \psi \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\beta}{2} \psi^2, \psi \frac{\partial^2 \psi}{\partial y \partial t} \right)
\]

where \( E \) is energy (density), consisting of the kinetic (first term) and potential (second term) components; and \( S \) is energy flux (vector).

(a) It can be shown (see Practical Problems), that the mean energy \( \langle E \rangle \) of a wave packet propagates according to:

\[
\frac{\partial \langle E \rangle}{\partial t} + C_g \cdot \nabla \langle E \rangle = 0.
\]

(b) The energy equation for the corresponding nonlinear equivalent-barotropic equation is derived similarly; its energy flux vector is

\[
S = -\left( \psi \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\beta}{2} \psi^2, \psi \frac{\partial^2 \psi}{\partial y \partial t} \right).
\]

**Mean-flow effects.** Consider small-amplitude flow disturbances around some background flow given by its streamfunction \( \Psi(x, y, z) \).

What happens with the dispersion relation and, hence, with the waves?

To simplify the problem, let’s stay with the 1.5-layer QG PV model, consider uniform, zonal background flow \( \Psi = -Uy \), and substitute:

\[
\psi \to -Uy + \psi, \quad \Pi \to \left( \beta + \frac{U}{R^2} \right) y + \nabla^2 \psi - \frac{1}{R^2} \psi.
\]

The linearized dynamics is the following:

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left( \nabla^2 \psi - \frac{1}{R^2} \psi \right) + \frac{\partial \psi}{\partial x} \left( \beta + \frac{U}{R^2} \right) = 0 \quad \to \quad \psi \sim e^{i(kx + ly - \omega t)} \quad \to \quad \omega = kU - \frac{k \left( \beta + UR^{-2} \right)}{k^2 + l^2 + R^{-2}}
\]

(a) In the dispersion relation, the first term \( kU \) is Doppler shift, which is due to advection of the wave by the background flow;

(b) The second term contains effect of the altered background PV;

(c) There are also corresponding changes in the group velocity;

(d) Complicated 2D and 3D background flows profoundly influence Rossby waves properties, but the corresponding dispersion relations are difficult to obtain.
Two-layer Rossby waves. Consider now the two-layer QG PV equations linearized around the state of rest:

$$\frac{\partial}{\partial t} \left[ \nabla^2 \psi_1 - \frac{1}{R_1^2} (\psi_1 - \psi_2) \right] + \beta \frac{\partial \psi_1}{\partial x} = 0, \quad \frac{\partial}{\partial t} \left[ \nabla^2 \psi_2 - \frac{1}{R_2^2} (\psi_2 - \psi_1) \right] + \beta \frac{\partial \psi_2}{\partial x} = 0, \quad R_1^2 = \frac{g' H_1}{f_0^2}, \quad R_2^2 = \frac{g' H_2}{f_0^2}$$

Diagonalization of the dynamics: the governing equations can be decoupled from each other by the linear transformation of variables from the layer-wise streamfunctions to the streamfunctions of the vertical modes.

The barotropic mode $\phi_1$ and the first baroclinic mode $\phi_2$ are defined as

$$\phi_1 \equiv \psi_1 \frac{H_1}{H_1 + H_2} + \psi_2 \frac{H_2}{H_1 + H_2}, \quad \phi_2 \equiv \psi_1 - \psi_2.$$

These modes represent separate (i.e., governed by different dispersion relations) families of Rossby waves:

$$\frac{\partial}{\partial t} \nabla^2 \phi_1 + \beta \frac{\partial \phi_1}{\partial x} = 0 \quad \Rightarrow \quad \omega_1 = -\frac{\beta k}{k^2 + l^2}$$

$$\frac{\partial}{\partial t} \left[ \nabla^2 \phi_2 - \frac{1}{R_D^2} \phi_2 \right] + \beta \frac{\partial \phi_2}{\partial x} = 0, \quad R_D \equiv \left[ \frac{1}{R_1^2} + \frac{1}{R_2^2} \right]^{-1/2} \quad \Rightarrow \quad \omega_2 = -\frac{\beta k}{k^2 + l^2 + R_D^{-2}}$$

where $R_D$ is referred to as the first baroclinic Rossby radius.

(a) The diagonalizing layers-to-modes transformation and its inverse (modes-to-layers) transformation are linear operations. The (pure) barotropic mode can be written in terms of layers as

$$\psi_1 = \psi_2 = \phi_1,$$

therefore, it is vertically uniform and actually describes the vertically averaged flow. Barotropic waves are fast (typical periods are several days in the ocean and 10 times faster in the atmosphere); their dispersion relation does not depend on the stratification.

(b) The (pure) baroclinic mode can be written in terms of layers as

$$\psi_1 = \phi_2 \frac{H_2}{H_1 + H_2}, \quad \psi_2 = -\phi_2 \frac{H_1}{H_1 + H_2} \quad \Rightarrow \quad \psi_2 = -\frac{H_1}{H_2} \psi_1.$$

therefore, it changes sign vertically, and its vertical integral is zero. Baroclinic waves are slow (typical periods are several months in the ocean and 10 times faster in the atmosphere); they can be viewed as propagating anomalies of the pycnocline (thermocline), because the streamfunction has vertical derivative.
• Continuously stratified Rossby waves.

Continuously stratified model is a natural extension of the isopycnal model with a large number of layers. The corresponding linearized QG PV dynamics is given by

\[
\frac{\partial}{\partial t} \left[ \nabla^2 \psi + \frac{f_0^2}{\rho_s} \frac{\partial}{\partial z} \left( \frac{\rho_s}{N^2(z)} \frac{\partial \psi}{\partial z} \right) \right] + \beta \frac{\partial \psi}{\partial x} = 0
\]

\[
\rightarrow \quad \psi \sim \Phi(z) e^{i(kx+ly-\omega t)} \rightarrow
\]

\[
\frac{f_0^2}{\rho_s} \frac{d}{dz} \left( \frac{\rho_s}{N^2(z)} \frac{d\Phi(z)}{dz} \right) = \left( k^2 + l^2 + \frac{k\beta}{\omega} \right) \Phi(z) \equiv \lambda \Phi(z) \quad (\ast)
\]

Boundary conditions at the top and bottom are to be specified, e.g., by imposing zero density anomalies:

\[
\rho \sim \frac{d\Phi(z)}{dz} \bigg|_{z=0,-H} = 0. \quad (**)
\]

Combination of (\ast) and (**) is an eigenvalue problem that can be solved for a discrete spectrum of eigenvalues and eigenmodes.

(a) Eigenvalues \( \lambda_n \) yield dispersion relations \( \omega_n = \omega_n(k,l) \), and the corresponding eigenmodes \( \phi_n(z) \) are the vertical normal modes, like the familiar barotropic and first baroclinic modes in the two-layer case.

(b) The Figure illustrates the first, second and third baroclinic modes for the oceanic stratification.

(c) The corresponding baroclinic Rossby deformation radius \( R_D^{(n)} \equiv \lambda_n^{-1/2} \) characterizes horizontal length scale of the \( n \)th vertical mode. The higher is the mode, the more oscillatory it is in vertical, and the slower it propagates.

(d) The (zeroth) barotropic mode has \( R_D^{(0)} = \infty \) and \( \lambda_0 = 0 \).

(e) The first Rossby deformation radius \( R_D^{(1)} \) is the most important fundamental length scale of the geostrophic turbulence.
LINEAR INSTABILITIES

- *Linear stability analysis* is the first step toward understanding turbulent flows. Sometimes it can predict some patterns and properties of flow fluctuations.

These Figures illustrate different regimes of thermal convection.
Linear stability analysis is very useful for simple flows (convective rolls), somewhat useful for intermediate-complexity flows (convective plumes), and completely useless in highly developed turbulence.

- *Small-amplitude behaviours* can be predicted by linear stability analysis very well, and some of the linear predictions carry on to turbulent flows.
- *Nonlinear effects* become increasingly more important in more complex turbulent flows.
Shear instability occurs on flows with sheared velocity...

Eventually, there is substantial stirring and mixing of material and vorticity \( \Rightarrow \)
Instabilities of jet streams

Developed instabilities of idealized jet

Tropical instability waves
• **Barotropic instability** is *horizontal-shear instability* of geophysical flows.

What is the **necessary condition** for this instability?

Let’s consider 1.5-layer QG PV model configured in a zonal channel \((-L < y < +L)\) and linearized around some zonally uniform and meridionally sheared background flow \(U(y)\):

\[
\left( \frac{\partial}{\partial t} + U(y) \frac{\partial}{\partial x} \right) \left[ \nabla^2 \psi - \frac{1}{R^2} \psi \right] + \frac{\partial \psi}{\partial y} \frac{d\Pi}{dy} = 0 , \quad \frac{d\Pi}{dy} = \beta - \frac{d^2 U}{dy^2} + \frac{U}{R^2} ,
\]

where \(\Pi\) is the background potential vorticity. Let’s look for the usual wave solutions:

\[
\psi \sim \phi(y) e^{ik(x-ct)} , \quad c = c_r + i \frac{\omega_i}{k} \rightarrow (U - c) \left( -k^2 \phi + \phi_{yy} - \frac{1}{R^2} \phi \right) + \phi \left( \beta - U_{yy} + \frac{U}{R^2} \right) = 0
\]

\[
\rightarrow \phi_{yy} - \phi \left( k^2 + \frac{1}{R^2} \right) + \phi \frac{d\Pi/dy}{U - c} = 0 ,
\]

Multiply the governing equation by (complex conjugated) \(\phi^*\), integrate it in \(y\) using the simple identity

\[
\phi^* \phi_{yy} = \frac{\partial}{\partial y} (\phi^* \phi_y) - \phi^*_y \phi_y ,
\]

and take into account that the integral of the \(y\)-derivative is zero, because of the boundary conditions on the channel walls:

\[
\phi(-L) = \phi(L) = 0.
\]

The resulting integrated equation,

\[
\int_{-L}^{L} \left( \left| \frac{d\phi}{dy} \right|^2 + |\phi|^2 \left( k^2 + \frac{1}{R^2} \right) \right) dy - \int_{-L}^{L} |\phi|^2 \frac{d\Pi/dy}{U - c} dy = 0 ,
\]

can be written so, that its first integral \([\ldots]\) is real, and the second integral is complex:

\[
\rightarrow \ldots + \frac{i \omega_i}{k} \int_{-L}^{L} |\phi|^2 \frac{d\Pi/dy}{|\phi|^2} dy = 0 .
\]

If the last integral is non-zero, then, **necessarily**: \(\omega_i = 0\), and the normal mode \(\phi(y)\) is **neutral**; this results in the following theorem...

**Necessary condition for barotropic instability** states that \(\omega_i\) can be nonzero (hence, instability has to occur for \(\omega_i > 0\)), only if the integral is zero, hence, **ONLY IF** the background PV gradient \(d\Pi/dy\) changes sign somewhere in the domain (note: this is equivalent to the existence of inflection point in the velocity profile in the case of \(\beta = 0\) and pure 2D dynamics). This is also true for non-zonal parallel flows.
Baroclinic instability is vertical-shear instability of geophysical flows. What is the necessary condition for this instability? Consider a channel with vertically and meridionally sheared but zonally uniform background flow $U(y, z)$; and apply the continuously stratified QG PV model:

$$\Pi = \beta y - \frac{\partial U}{\partial y} - \frac{\partial}{\partial z} \left[ \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \int U(y, z) \, dy \right], \quad \frac{\partial \Pi}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial z} \left[ \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \right],$$

where $\Pi$ is the background potential vorticity. The linearized PV equation is:

$$\left( \frac{\partial}{\partial t} + U(y, z) \frac{\partial}{\partial x} \right) \left[ \nabla^2 \psi + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] + \frac{\partial \psi}{\partial x} \frac{\partial \Pi}{\partial y} = 0 \quad (\ast)$$

Conservation of density (sum of dynamic density anomaly and background density) on material particles can be written as (first, in the full form; then, in the linearized form):

$$\frac{D_g \rho}{Dt} = \frac{D_g (\rho_g + \rho_b)}{Dt} = 0 \rightarrow \frac{\partial \rho_g}{\partial t} + \frac{\partial \rho_b}{\partial t} + (U + u) \frac{\partial \rho_g}{\partial x} + (U + u) \frac{\partial \rho_b}{\partial y} + v \frac{\partial \rho_b}{\partial y} + w \frac{\partial \rho_b}{\partial z} = 0.$$

By linearizing out the quadratic terms and taking into account that the background density is stationary and $x$-independent, we obtain:

$$\rightarrow \frac{\partial \rho_g}{\partial t} + U \frac{\partial \rho_g}{\partial x} + v \frac{\partial \rho_b}{\partial y} + w \frac{\partial \rho_b}{\partial z} = 0.$$

Let’s consider the bottom and top rigid boundaries, hence $w = 0$:

$$\frac{\partial \rho_g}{\partial t} + U \frac{\partial \rho_g}{\partial x} + v \frac{\partial \rho_b}{\partial y} = 0 \quad \text{at} \quad z = 0, H.$$

Then, in the continuously stratified fluid this statement translates into

$$\rho_g = \frac{\rho_0 f_0}{g} \frac{\partial \psi}{\partial z}, \quad \rho_b = \frac{\rho_0 f_0}{g} \frac{\partial}{\partial z} \int (-U) \, dy \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial t \partial z} + U \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial \psi}{\partial x} \frac{\partial U}{\partial z} = 0 \quad (**).$$

With the usual wave solution $\psi \sim \phi(y, z) e^{ik(x-ct)}$, the linearized PV equation $(\ast)$ and boundary conditions $(**)$ become:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \phi}{\partial z} \right) - k^2 \phi + \frac{1}{U - c} \frac{\partial \Pi}{\partial z} \phi = 0; \quad (U - c) \frac{\partial \phi}{\partial z} - \frac{\partial U}{\partial z} \phi = 0, \quad \text{at} \quad z = 0, H.$$
Let's multiply the above equation by $\phi^*$ and integrate over $z$ and $y$. Vertical integration of the second term involves the boundary conditions:

$$
\int_0^H \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \phi}{\partial z} \right) \phi^* \, dz = - \int_0^H \frac{f_0^2}{N^2} 1 \frac{\partial |\phi|^2}{\partial z} \, dz + \left[ \frac{f_0^2}{N^2} \frac{\partial \phi}{\partial z} \phi^* \right]^H_0 = \ldots + \left[ \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \frac{|\phi|^2}{U - c} \right]^H_0
$$

Taking the above into account, full integration of the $\phi^*$-multiplied equation yields the following imaginary part equal to zero:

$$
\frac{\omega_i}{k} \int_{-L}^{L} \left( \int_0^H \frac{\partial \Pi}{\partial y} \frac{|\phi|^2}{|U - c|^2} \, dz + \left[ \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \frac{|\phi|^2}{|U - c|^2} \right]^H_0 \right) \, dy = 0
$$

In the common situation: $\frac{\partial U}{\partial z} = 0$ at $z = 0, H \implies$ the necessary condition for baroclinic instability is that $\frac{\partial \Pi(y, z)}{\partial y}$ changes sign at some depth. In practice, vertical change of the PV gradient sign always indicates baroclinic instability.
• Eady model (Eric Eady was PhD graduate from ICL) is a classical, continuously stratified model of baroclinically unstable atmosphere.

Let’s assume:
(i) $f$-plane ($\beta = 0$),
(ii) linear stratification ($N(z) = \text{const}$),
(iii) constant vertical shear $U(z) = U_0 z / H$,
(iv) rigid boundaries at $z = 0, H$.

NOTE: Background PV is zero, hence, the necessary condition for instability is satisfied.

The linearized QG PV equation and boundary conditions are:

\[
\frac{\partial}{\partial t} + z U_0 \frac{\partial}{\partial x} \left[ \nabla^2 \psi + f_0^2 \frac{\partial^2 \psi}{N^2 \partial z^2} \right] = 0; \quad \frac{\partial^2 \psi}{\partial t \partial z} + \frac{z U_0}{H} \frac{\partial^2 \psi}{\partial x \partial z} - \frac{U_0}{H} \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad z = 0, H.
\]

Look for the wave-like solution in horizontal plane to obtain the vertical-structure equation and the corresponding boundary conditions:

\[
\psi \sim \phi(z) e^{i(k(x-ct)+lt)} \rightarrow \left( \frac{z U_0}{H} - c \right) \left[ \frac{f_0^2}{N^2} \frac{d^2 \phi}{dz^2} - (k^2 + l^2) \phi \right] = 0; \quad \left( \frac{z U_0}{H} - c \right) \frac{d \phi}{dz} - \frac{U_0}{H} \phi = 0 \quad \text{at} \quad z = 0, H \quad (*)
\]

For $c \neq U_0 \frac{z}{H}$, we obtain linear ODE with characteristic vertical scale $H/\mu$:

\[
H^2 \frac{d^2 \phi}{dz^2} - \mu^2 \phi = 0, \quad \mu \equiv \frac{N H}{f_0} \sqrt{k^2 + l^2} = R_D^{(1)} \sqrt{k^2 + l^2}
\]

Look for solution of the above ODE in the form $\phi(z) = A \cosh(\mu z / H) + B \sinh(\mu z / H)$, substitute it in the top and bottom boundary conditions $(*)$ and obtain 2 linear equations for $A$ and $B$ that yield:

\[
B = -A \frac{U_0}{\mu c}, \quad c^2 - U_0 c + U_0^2 \left( \frac{1}{\mu} \coth \mu - \frac{1}{\mu^2} \right) = 0 \quad \Rightarrow \quad c = \frac{U_0}{2} \pm \frac{U_0}{\mu} \left[ \frac{1}{2} \coth \frac{\mu}{2} \left( \frac{\mu}{2} - \tanh \frac{\mu}{2} \right) \right]^{1/2}
\]

The second bracket under the square root is always positive, hence, the normal modes grow ($\omega_i > 0$) if $\mu$ satisfies:

\[
\frac{\mu}{2} < \coth \frac{\mu}{2}
\]

which is the region to the left of the dashed curve (see Figure below).

(a) The maximum growth rate occurs at $\mu = 1.61$, and it is estimated to be $0.31 \frac{U_0}{R_D^{(1)}}$.

(b) For any $k$ the most unstable wave has $l = 0$; and this wave is characterized by $k_{\text{crit}} = 1.6 / R_D^{(1)}$ ($L_{\text{crit}} \approx 4 R_D^{(1)}$).

NOTE: Both of the corresponding time and length scales are consistent with the observed synoptic scale variability.
(c) Eady solution can be interpreted as a pair of phase-locked edge waves (upper panel: $\phi$, middle panel: $\rho = \partial \phi / \partial z$, and bottom panel: $v = \partial \phi / \partial x$).

(d) Assumptions of the Eady model are quite unrealistic, as well as the absence of PV gradients, but nevertheless it is a good starting point for analyses and one of the classical models illustrating the baroclinic instability mechanism.

Figure illustrating Eady’s solution in terms of the phase-locked edge waves:

- **Phillips model** is another simple model of the baroclinic instability mechanism. It describes two-layer fluid with the uniform background zonal velocities $U_1$ and $U_2$, and with $\beta$-effect (see Problem Sheet). In this situation background PV gradient is nonzero, thus, making the problem more relevant.

(a) *Stabilizing effect of $\beta$*: Phillips model has critical shear $U_1 - U_2 \sim \beta R_D^2$.

(b) If the upper layer is thinner than the deep layer (ocean-like situation), then the eastward critical shear is larger than the westward one.
**Mechanism of baroclinic instability.**

Illustrated by the *Eady* and *Phillips models*, it feeds geostrophic turbulence (i.e., synoptic scale variability in the atmosphere and dynamically similar mesoscale eddies in the ocean), and, therefore, it is fundamentally important.

(a) *Available potential energy* (APE) is part of potential energy released as a result of isopycnal flattening due to the baroclinic instability. In this process APE of the large-scale background flow is converted into the *eddy kinetic energy* (EKE).

*Figure to the right:* Consider a fluid particle, initially positioned at *A*, that migrates to either *B* or *C*. If it moves along levels of constant pressure (in QG: streamfunction), then no work is done on the particle $\Rightarrow$ full mechanical energy of the particle remains unchanged. However, its APE can be converted in the EKE, and the other way around.

(b) Consider the following exchanges of fluid particles:

- $A \leftrightarrow B$ leads to *accumulation* of APE (the heavier particle goes “up”, and the lighter particle goes “down”),
- $A \leftrightarrow C$ leads, on the opposite, to *release* of APE.

That is, if $\alpha > \gamma$ (steep tilt of isopycnals, relative to tilt of pressure isolines), then APE is released into EKE. This is a situation of the positive *baroclinicity*:

$$\nabla p \times \nabla \rho > 0,$$

which routinely happens in geophysical fluids because of the prevailing *thermal winds*.

*Thermal wind* situation is a consequence of double, geostrophic and hydrostatic balance:

$$-f_0v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad f_0u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{g}{\rho_0 f_0} \frac{\partial \rho}{\partial y}, \quad \frac{\partial v}{\partial z} = \frac{g}{\rho_0 f_0} \frac{\partial \rho}{\partial x}.$$

Consider a thermal wind situation with $\frac{\partial p}{\partial z} < 0$ and $u > 0$; prove that it is baroclinically unstable (i.e., has positive baroclinicity):

$$\frac{\partial \rho}{\partial y} > 0 \quad \Rightarrow \quad \frac{\partial u}{\partial z} > 0 \quad \text{and} \quad u > 0 \quad \Rightarrow \quad \frac{\partial p}{\partial y} < 0 \quad \Rightarrow \quad \nabla p \times \nabla \rho = \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial \rho}{\partial y} > 0.$$
Energetics of barotropically and baroclinically unstable flows.

Can we quantify amounts of potential energy transferred from a baroclinically unstable flow to the growing perturbations? Can we quantify amounts of kinetic energy transferred from a barotropically unstable flow to the growing perturbations?

In the continuously stratified QG PV model, the kinetic and available potential energy densities of flow perturbations are:

\[ K(t, x, y, z) = \frac{1}{2} |\nabla \psi|^2, \quad P(t, x, y, z) = \frac{1}{2} f_0^2 \left( \frac{N^2}{2} \frac{\partial \psi}{\partial z} \right)^2 \]

Let’s consider the continuously stratified QG PV equation linearized around some background zonal flow \( U(y, z) \):

\[
\left( \frac{\partial}{\partial t} + U(y, z) \frac{\partial}{\partial x} \right) \left[ \nabla^2 \psi + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] + \frac{\partial \psi}{\partial x} \frac{\partial \Pi}{\partial y} = 0 \quad (*)
\]

Energy equation is obtained by multiplying (*) with \(-\psi\) and, then, by mathematical manipulation (like we have done for equivalent-barotropic QG dynamics):

\[
\frac{\partial}{\partial t} (K + P) + \nabla \cdot S - \frac{\partial}{\partial z} \left[ \psi \left( \frac{f_0^2}{N^2} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \frac{\partial \psi}{\partial z} \right) \right] = \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \frac{\partial U}{\partial y} + \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} f_0^2 \frac{\partial U}{\partial z} \quad (**) \]

**Vertical energy flux** is in square brackets on the rhs, and it is due to the form drag arising from isopycnal deformations.

**Horizontal energy flux**:

\[ S = -\psi \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla \psi + \left[ -f_0^2 \psi \frac{\partial \Pi}{\partial y} \frac{\psi^2}{2} + U (K + P) + \psi \frac{\partial \psi}{\partial y} \frac{\partial U}{\partial y} + f_0^2 \psi \frac{\partial \psi}{\partial z} \frac{\partial U}{\partial z} \right] \]

Integration of (**) over the domain removes horizontal and vertical flux divergences, and the total energy equation is obtained:

\[
\frac{\partial}{\partial t} \int \int \int (K + P) dV = \int \int \int \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \frac{\partial U}{\partial y} dV + \int \int \int \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} f_0^2 \frac{\partial U}{\partial z} dV \quad (***)
\]

**Energy conversion terms** on the rhs of (*** ) have clear physical interpretations:

(a) *Reynolds-stress energy conversion* term can be written as integral of \(-u'v' \frac{\partial U}{\partial y}\), where primes remind that we deal with the flow fluctuations around \( U(y, z) \).

This conversion is positive (and associated with barotropic instability of horizontally sheared flow), if the Reynolds stress \( u'v' \) acts against the velocity shear (see left panel of Figure below), that is, \( u'v' < 0 \). In this case the background flow feeds growing instabilities at the rate given by the energy conversion.
(b) Form-stress energy conversion term involves the form stress \( v' \rho' \). The integrand can be rewritten using thermal wind relations and

\[
\frac{\partial \psi}{\partial z} = -\frac{\rho' g}{\rho_0 f_0}, \quad N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}, \quad \frac{d\bar{\rho}}{dz} < 0 :
\]

\[
v' \left( -\frac{\rho' g}{\rho_0 f_0} \right) \frac{f_0^2}{N^2} \left( \frac{g}{\rho_0 f_0} \frac{\partial \bar{\rho}}{\partial y} \right) = v' \rho' \frac{g}{\rho_0} \left[ \frac{\partial \bar{\rho}}{\partial y} / \frac{d\bar{\rho}}{dz} \right] = \frac{g}{\rho_0} v' \rho' \left[ -\frac{dz}{dy} \right] = \frac{g}{\rho_0} v' \rho' \left[ -\tan \alpha \right] \approx \frac{g}{\rho_0} v' \rho' \left[ -\alpha \right] \sim -v' \rho'
\]

This conversion term is positive (and associated with baroclinic instability), if the form stress is negative: \( v' \rho' \). This implies flattening of tilted isopycnals (right panel of Figure below shows \( -v' \rho' \) and isopycnals; the situation has negative density anomalies moving northward).
AGEOSTROPHIC MOTIONS

(a) Geostrophy filters out all types of (very fast) gravity waves, which are important for many geophysical processes.

(b) Geostrophy doesn’t work near the equator (where: \( f = 0 \)), because the Coriolis force becomes too small.

Let’s consider, first, gravity waves and, then, equatorial waves, that are both important ageostrophic fluid motions.

• **Linearized shallow-water model.** Let’s consider a layer of fluid with constant density, \( f \)-plane approximation, and deviations of the free surface \( \eta \):

\[
\frac{\partial u}{\partial t} - f_0 v = -g \frac{\partial \eta}{\partial x}, \quad \frac{\partial v}{\partial t} + f_0 u = -g \frac{\partial \eta}{\partial y}, \quad p = -\rho_0 g (z - \eta), \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\]

The last equation can be vertically integrated, using the linearized kinematic boundary condition on the free surface:

\[
w(z = h) = \frac{\partial \eta}{\partial t} \rightarrow \frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0; \quad (*)
\]

alternatively this equation can be obtained by linearization of the shallow-water continuity equation.

Take **curl of the momentum equations**, substitute the velocity divergence taken from \((*)\) into the Coriolis term and obtain:

\[
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{f_0}{H} \frac{\partial \eta}{\partial t} = 0 \quad (**)
\]

Take **divergence of the momentum equations**, substitute the velocity divergence taken from \((*)\) in the tendency term and obtain:

\[
\frac{1}{H} \frac{\partial^2 \eta}{\partial t^2} + f_0 \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - g \nabla^2 \eta = 0 \quad (***)
\]

From \((**)\) and \((***)\), by the time differentiation we obtain

\[
\frac{\partial}{\partial t} \left[ \nabla^2 \eta - \frac{1}{c_0^2} \frac{\partial^2 \eta}{\partial t^2} - \frac{f_0^2}{c_0^2} \eta \right] = 0, \quad c_0^2 = gH
\]

Let’s integrate it in time and choose the integration constant so, that \( \eta = 0 \) is a solution; the resulting **free-surface evolution equation** is also known as the **Klein-Gordon equation**:  

\[
\nabla^2 \eta - \frac{1}{c_0^2} \frac{\partial^2 \eta}{\partial t^2} - \frac{f_0^2}{c_0^2} \eta = 0 \quad (***)
\]
This equation needs lateral boundary conditions, which are to be obtained from the velocity boundary conditions.

**Velocity-component equations.** Let’s take the $u$-momentum equation, differentiate it with respect to time, and add it to the $v$-momentum equation multiplied by $f_0$; similarly, let’s take time derivative of the $v$-momentum equation and subtract from it the $u$-momentum equation multiplied by $f_0$:

$$
\frac{\partial^2 u}{\partial t^2} + f_0^2 u = -g \left( \frac{\partial^2 \eta}{\partial x \partial t} + f_0 \frac{\partial \eta}{\partial y} \right), \quad \frac{\partial^2 v}{\partial t^2} + f_0^2 v = -g \left( \frac{\partial^2 \eta}{\partial y \partial t} - f_0 \frac{\partial \eta}{\partial x} \right).
$$

Let’s consider solid boundary at $x=0$ (ocean coast). On the boundary: $u = 0$, therefore:

$$
\frac{\partial^2 \eta}{\partial x \partial t} + f_0 \frac{\partial \eta}{\partial y} = 0 \quad \text{at} \quad x = 0.
$$

Let’s now look for the wave solution $\eta = \tilde{\eta}(x) e^{i (ly - \omega t)}$ of both $\ast \ast \ast \ast$ and the above boundary condition:

$$
\frac{d^2 \tilde{\eta}}{dx^2} + \left[ \frac{\omega^2}{c_0^2} - \frac{f_0^2}{c_0^2} - l^2 \right] \tilde{\eta} = 0, \quad - \frac{\omega}{f_0} \frac{d\tilde{\eta}}{dx}(0) + l \tilde{\eta}(0) = 0.
$$

The main equation can be written as:

$$
\frac{d^2 \tilde{\eta}}{dx^2} = \lambda^2 \tilde{\eta}, \quad \text{where (dispersion relation):} \quad \lambda^2 = -\frac{\omega^2}{c_0^2} + \frac{f_0^2}{c_0^2} + l^2 \quad \rightarrow \quad \tilde{\eta} = e^{-\lambda x}
$$

It supports solutions that are either oscillatory (imaginary $\lambda$) or decaying (real $\lambda$) in $x$. Let’s consider them separately.

- **Poincare (gravity-inertial) waves** are the oscillatory solutions in $x$:

  $$
  \lambda = ik, \quad \tilde{\eta} = A \cos kx + B \sin kx, \quad x = 0: \quad A = B \frac{\kappa \omega}{lf_0}, \quad \omega^2 = f_0^2 + c_0^2 (k^2 + l^2)
  $$

  (a) These are very fast waves: For wavelength $\sim 1000$ km and $H \sim 5$ km, the phase speed is $c_0 = \sqrt{gH} \sim 300$ m s$^{-1}$ (compare this tsunami-like speed to the slow speed $\sim 0.2$ m s$^{-1}$ for the oceanic baroclinic Rossby wave).

  (b) In the long-wave limit: $\omega = f_0$. These waves are called the inertial oscillations; they are characterized by circular motions (see Problem Sheet).

  (c) In the short-wave limit, the effects of rotation vanish, and these are the common (nondispersive) non-rotating shallow-water surface gravity waves (note their difference from the deep-ocean waves considered in the Problem Sheet!).

  (d) Poincare waves are isotropic: their propagation properties are the same in any direction (in the flat-bottom $f$-plane case that we considered).
● Kelvin waves are the decaying solutions (edge waves!); on the western (eastern) walls they correspond to different signs of $k$ (let’s take $k > 0$):

$$\lambda = k \ (= -k), \quad \tilde{\eta} = Ae^{-kx} \ (= Ae^{kx}), \quad x = 0: \ k = -\frac{f_0l}{\omega} \ (= \frac{f_0l}{\omega}) \quad (*)$$

In the northern hemisphere, positive $k$ at the western wall implies $C^y_\rho = \omega/l < 0$, hence the Kelvin wave will propagate to the south. Thus, the meridional phase speed is northward at the eastern wall and southward at the western wall, that is, the coast is always to the right of the Kelvin wave propagation direction. Note, that $f_0$ changes sign in the southern hemisphere, and this modifies the Kelvin wave so, that it has the coast always to the left (see Figure).

With (*) the Kelvin wave dispersion relation becomes:

$$\left(\omega^2 - f_0^2\right) \left(1 - \frac{c_0^2}{\omega^2} l^2\right) = 0$$

Its first root, $\omega = \pm f_0$, is just another class of inertial oscillations.

Its second root corresponds to the (nondispersive) Kelvin wave exponentially decaying away from the boundary:

$$\omega = \mp c_0 l, \quad k = \pm \frac{f_0}{c_0} \quad \Rightarrow \quad \eta = Ae^{\pm x f_0/c_0} e^{i(y + c_0 t)}$$

Substitute this into the rhs of the normal to the wall velocity equation, and find that this velocity component is zero everywhere:

$$\frac{\partial^2 u}{\partial t^2} + f_0^2 u = -g \left(\frac{\partial^2 \eta}{\partial x \partial t} + f_0 \frac{\partial \eta}{\partial y}\right) = 0 \quad \rightarrow \quad u = Ae^{f_0 t} \quad \rightarrow \quad A = 0 \quad \Rightarrow \quad u = 0. \quad (*)$$

because at the boundary it is always true that $u(t, 0, y) = 0$. Note, that this equation has oscillatory solutions, but they are not allowed by the boundary condition.

Because of (*), the along-wall velocity component of the Kelvin wave is in the geostrophic balance:

$$\frac{\partial u}{\partial t} - f_0 v = -g \frac{\partial \eta}{\partial x} \quad \Rightarrow \quad -f_0 v = -g \frac{\partial \eta}{\partial x},$$

hence, Kelvin wave is a boundary-trapped “centaur” that is simultaneously ageostrophic (gravity) and geostrophic wave.

(a) There are Kelvin waves running around islands (in the proper direction); they are often phase-locked to tides.
(b) Kelvin waves can be further subdivided into the barotropic and baroclinic modes.
• **Geostrophic adjustment** is a powerful and ubiquitous process, in which a fluid in an initially unbalanced state naturally evolves toward a state of geostrophic balance. Let's focus on the *linearized shallow-water* dynamics, which contains both geostrophically balanced and unbalanced motions:

\[
\frac{\partial u}{\partial t} - f_0 v = -g \frac{\partial \eta}{\partial x}, \quad \frac{\partial v}{\partial t} + f_0 u = -g \frac{\partial \eta}{\partial y}, \quad \frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0,
\]

and consider a manifestly unbalanced initial state: discontinuity in fluid height. In non-rotating flow any initial disturbance will be radiated away by the gravity waves, characterized by phase speed \( c_0 = \sqrt{gH} \), and the final state will be the *state of rest*. In rotating fluid there is geostrophic balance that can trap the fluid in it, because it has absolutely no time dependence!

*Effect of rotation* is crucial for geostrophic adjustment, because:

(a) PV conservation provides a powerful constraint on the fluid evolution;
(b) There is fully adjusted steady state which is not the state of rest.

Let's start with the corresponding PV description of the dynamics:

\[
\frac{\partial \Pi}{\partial t} + \mathbf{u} \cdot \nabla \Pi = 0, \quad \Pi = \zeta + f_0 h = \frac{(\zeta + f_0)H}{1 + \eta/H},
\]

and linearize both PV and its conservation law:

\[
\Pi_{LIN} \approx \frac{1}{H} \left( \zeta + f_0 \right) \left( 1 - \frac{\eta}{H} \right) \approx \frac{1}{H} \left( \zeta + f_0 - \frac{f_0 \eta}{H} \right) \quad \Rightarrow \quad q = \zeta - f_0 \frac{\eta}{H}, \quad \frac{\partial q}{\partial t} = 0
\]

where \( q \) is the linearized PV anomaly. Let's consider a discontinuity in fluid height: \( \eta(x, 0) = \eta_0 \), \( x < 0 \); \( \eta(x, 0) = -\eta_0 \), \( x > 0 \).

The initial distribution of the linearized PV anomaly is:

\[
q(x, y, 0) = -f_0 \frac{\eta_0}{H}, \quad x < 0; \quad q(x, y, 0) = +f_0 \frac{\eta_0}{H}, \quad x > 0.
\]

During the *geostrophic adjustment* process, the height discontinuity will become smeared out into a slope by radiating gravity waves; through the geostrophic balance this slope must maintain a geostrophic flow current that will necessarily emerge during the adjustment process.

First, let's introduce the (final) geostrophic flow streamfunction:

\[
f_0 u = -g \frac{\partial \eta}{\partial y}, \quad f_0 v = g \frac{\partial \eta}{\partial x} \quad \Rightarrow \quad \Psi = \frac{g \eta}{f_0}
\]
Since PV is conserved on the fluid particles, the particles are only redistributed along the $y$-axis (this is based on physical reasoning; alternative argument comes from the symmetry of the problem). The final steady state is the solution of the equation described by monotonically changing $\Psi \sim \eta$ and sharp jet concentrated along this slope:

\[ \zeta - f_0 \frac{\eta}{H} = q(x,y) \quad \implies \quad \left( \nabla^2 - \frac{1}{R_D^2} \right) \Psi = q(x,y), \quad R_D = \frac{\sqrt{gH}}{f_0} \quad \implies \quad \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{R_D^2} \Psi = \frac{f_0 \eta_0}{H} \text{sign}(x) \]

\[ \implies \Psi = -\frac{g\eta_0}{f_0} (1 - e^{-x/R_D}), \quad x > 0 ; \quad \Psi = +\frac{g\eta_0}{f_0} (1 - e^{+x/R_D}), \quad x < 0 \]

\[ \implies \quad u = 0 , \quad v = -\frac{g\eta_0}{f_0 R_D} e^{-|x|/R_D} \]

(a) PV constrains the adjustment within the deformation radius from the initial disturbance.

(b) Excessive initial energy (which can be estimated; see Problem Sheet) is radiated away by gravity waves. The underlying processes which transfer energy from (initially) unbalanced flows to gravity waves remain poorly understood.
• Equatorial waves are the special class of linear waves populating the equatorial zone.

Let’s assume the equatorial β-plane and write the momentum, continuity, and PV equations (and recall that $c_0 = gH$):

\[
\frac{\partial u}{\partial t} - \beta y v = -g \frac{\partial \eta}{\partial x} \quad \Rightarrow \quad -\beta y \frac{\partial^2 u}{\partial t^2} - \beta y^2 \frac{\partial^2 v}{\partial t} = -\frac{g \beta y}{c_0^2} \frac{\partial^2 \eta}{\partial x \partial t} - \frac{\beta y}{H} \frac{\partial^2 \eta}{\partial x \partial t} \quad (*)
\]

\[
\frac{\partial v}{\partial t} + \beta u = -g \frac{\partial \eta}{\partial y} \quad \Rightarrow \quad \frac{\partial^3 v}{\partial t^3} + \frac{\beta y}{c_0^2} \frac{\partial^2 u}{\partial t^2} = -\frac{g \beta y}{c_0^2} \frac{\partial^3 \eta}{\partial y \partial t^2} = \frac{1}{H} \frac{\partial^3 \eta}{\partial y \partial t^2} \quad (**)\]

\[
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad \Rightarrow \quad \frac{\partial^2 \eta}{\partial x \partial t} - \frac{\partial^2}{\partial y \partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (***)
\]

\[
\frac{\partial}{\partial t} \left( \zeta - \frac{\beta y}{H} \eta \right) + \beta v = 0 \quad \Rightarrow \quad \frac{\partial^2}{\partial x \partial t} \left( \zeta - \frac{\beta y}{H} \eta \right) - \beta \frac{\partial v}{\partial x} = 0 \quad (****)
\]

Add up (*) and (**), and use (***) and (****) to get rid of η:

\[
-\frac{1}{c_0^2} \frac{\partial^3 v}{\partial t^3} - \frac{\beta y^2}{c_0^2} \frac{\partial v}{\partial t} = \frac{\partial^2 \zeta}{\partial x \partial t} + \beta \frac{\partial v}{\partial x} + \frac{\partial^2}{\partial y \partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

Substitute $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ to obtain the meridional-velocity equation:

\[
\frac{\partial}{\partial t} \left[ \frac{1}{c_0^2} \left( \frac{\partial^2 v}{\partial t^2} + (\beta y)^2 v \right) - \nabla^2 v \right] - \beta \frac{\partial v}{\partial x} = 0
\]

Let’s look for the wave solution:

\[
v = \tilde{v}(y) e^{ikx - \omega t} \quad \Rightarrow \quad \frac{\partial^2 \tilde{v}}{\partial y^2} + \tilde{v} \left[ \frac{\omega^2}{c_0^2} - k^2 - \frac{(\beta y)^2}{c_0^2} - \frac{\beta k}{\omega} \right] = 0
\]

Solutions of this inhomogeneous ODE are symmetric around the equator and are given by the set of Hermite polynomials $H_n$, which multiply the exponential:

\[
\tilde{v}_n(y) = A_n H_n \left( \frac{y}{L_{eq}} \right) \exp \left[ -\frac{1}{2} \left( \frac{y}{L_{eq}} \right)^2 \right],
\]

where $L_{eq} = \sqrt{c_0^2/\beta}$ is called the equatorial barotropic radius of deformation ($\sim 3000$ km; the equatorial baroclinic deformation radii are much shorter and can be obtained by considering a multi-layer problem and projecting it on the vertical modes).
Let’s obtain the dispersion relation by recalling the following recurrence relations for the Hermite polynomials:

\[ H'_n = 2nH_{n-1}, \quad H'_{n-1} = 2yH_{n-1} - H_n, \]

and by considering \( v_n = H_n \exp[-y^2/2] \):

\[
v'_n = (H'_n - yH_n) e^{-y^2/2} = (2nH_{n-1} - yH_n) e^{-y^2/2}, \quad v''_n = \left(2nH'_{n-1} - H_n - yH'_n - y(2nH_{n-1} - yH_n)\right) e^{-y^2/2} = -(2n+1-y^2) e^{-y^2/2}
\]

\[ \rightarrow v''_n + (2n+1-y^2)v_n = 0 \]

Therefore, by comparing the above equation with the governing ODE, and by considering \( y \to y/L \), we obtain the resulting dispersion relation for equatorial waves:

\[
\omega_n^2 = c_0^2 \left( k^2 + \frac{(2n+1)}{L_{eq}^2} \right) + \frac{\beta k}{\omega_n} c_0^2
\]

Let’s now analyze this dispersion relation by considering its frequency limits and effects of lateral boundaries:

(a) If \( \omega_n \) is large, then: \( \omega_n^2 = c_0^2 \left( k^2 + \frac{(2n+1)}{L_{eq}^2} \right) \).

This is identical to the dispersion relation for midlatitude Poincare waves, if we take \( f_0 = 0 \) and \( l = \sqrt{2n+1}/L_{eq} \).

(b) If \( \omega_n \) is small, then: \( \omega_n = -\frac{\beta k}{k^2 + (2n+1)/L_{eq}^2} \).

This is identical to the dispersion relation for midlatitude Rossby waves, if we take \( l = \sqrt{2n+1}/L_{eq} \).

(c) Mixed Rossby-gravity (Yanai) wave corresponds to \( n = 0 \). It behaves like Rossby/gravity wave for low/high frequencies.

(d) Equatorial Kelvin wave is the edge wave for which the equator plays the role of solid boundary. Let’s take \( v = 0 \), and use \((*)\), \((***)\), and \((****)\):

\[
\frac{\partial u}{\partial t} = -y \frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0, \quad (*)
\]

\[-\frac{\partial^2 u}{\partial t \partial y} + \beta y \frac{\partial u}{\partial x} = 0 \quad (***)
\]
From (⋆) we obtain the zonal-velocity equation and its D’Alembert solution:
\[
\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad u = A \, G_-(x - c_0 t, y) + B \, G_+(x + c_0 t, y),
\]
and notice, that this solution has to satisfy the PV constraint (⋆⋆). Substitute the D’Alembert solution in (⋆⋆), introduce pair of propagating-wave variables \( \xi = x \pm c_0 t \), and recall that \( L_{eq} = \sqrt{c_0/\beta} \):
\[
\frac{\partial}{\partial \xi} \left( -c_0 \frac{\partial G_-}{\partial y} - \beta y G_- \right) = 0, \quad \frac{\partial}{\partial \xi} \left( c_0 \frac{\partial G_+}{\partial y} - \beta y G_+ \right) = 0 \quad \rightarrow \quad -c_0 \frac{\partial G_-}{\partial y} - \beta y G_- = 0, \quad c_0 \frac{\partial G_+}{\partial y} - \beta y G_+ = 0
\]
\[
G_- = A_- e^{-(y/L_{eq})^2} F_-(\xi), \quad G_+ = A_+ e^{(y/L_{eq})^2} F_+(\xi) \quad \rightarrow \quad G_- = A_- e^{-(y/L_{eq})^2} F_-(x-c_0 t), \quad G_+ = A_+ e^{(y/L_{eq})^2} F_+(x+c_0 t)
\]
Only \( G_- \) remains finite away from the equator, hence, \( A_+ = 0 \), and Kelvin wave propagates only to the east.

**Vertical modes**: In continuously stratified case, the flow solution can be split in the set of vertical baroclinic modes. Each baroclinic mode has its own Poincare, Rossby, Yanai and Kelvin waves and dispersion relations.

**Note**: Equatorial waves play the key role in the global, coupled ocean-atmosphere phenomenon called ENSO (see Figure).
Schematic of El Nino Southern Oscillation (ENSO) “delayed oscillator” mechanism

El Nino and La Nina and occur interannually and cause extreme floods and droughts in many regions of the world.

- Normal state is perturbed; weakening of trade winds
- “Warm” Kelvin wave radiates to the east and “cold” Rossby wave radiates to the west (their basin-crossing times are about 70 and 220 days).
- When Kelvin wave reaches the boundary, it warms the upper ocean and “El Nino” phenomenon occurs.
- “Cold” Rossby wave reflects from the western boundary as “cold” Kelvin wave; then, it propagates to the east, terminates El Nino, and initiates “La Nina” event.
Stokes drift is a nonlinear phenomenon that illustrates the difference between the average Lagrangian velocity (i.e., velocity estimated following fluid particles) and the average Eulerian velocity (i.e., velocity estimated at fixed spatial positions).

Essential physics: Stokes drift may occur only when the flow is both time-dependent and spatially inhomogeneous.

Let’s consider the textbook example of deep-water linear gravity waves (see Figure and Problem Sheet) and derive the Stokes drift velocity.

Lagrangian motion of a fluid particle is described as:

\[ \mathbf{x} = \xi(a, t), \quad \frac{\partial \xi}{\partial t} = \mathbf{u}(\xi, t), \quad \xi(a, 0) = a, \]

where \( \mathbf{u} \) is the Eulerian velocity (at a fixed position), and \( \frac{\partial \xi}{\partial t} \) is the Lagrangian velocity (found along the particle trajectory).

Let’s compare time averages of these velocities (denoted by overlines) and assume they are not the same:

\[ \overline{\mathbf{u}}_E = \mathbf{u}(\mathbf{x}, t), \quad \overline{\mathbf{u}}_L = \frac{\partial \xi(a, t)}{\partial t} = \mathbf{u}(\xi(a, t), t) \rightarrow \mathbf{u}_S = \overline{\mathbf{u}}_L - \overline{\mathbf{u}}_E. \]

Let’s now consider a sinusoidal plane wave on the free surface of fluid: \( \eta = A \cos(kx - \omega t) \). The corresponding interior flow solution (see Problem Sheet) is given in terms of the velocity potential \( \phi \), which is harmonic (i.e., \( \nabla^2 \phi = 0 \)); and the corresponding (nonlinear) dispersion relation of the deep-water waves:

\[ \phi = A \frac{\omega}{k} e^{kz} \sin(kx - \omega t) \quad \omega^2 = gk \]

Let’s focus on the horizontal \( \xi_x \) and vertical \( \xi_z \) components of the Lagrangian position vector \( \xi \):

\[ \frac{\partial \xi_x}{\partial t} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \xi_z}{\partial t} = \frac{\partial \phi}{\partial z}, \]

and integrate it near some point \( \mathbf{x} = (x, z) \). Within the linear theory this yields:

\[ \xi_x = x + \int \frac{\partial \phi}{\partial x} \, dt = x - A e^{kz} \sin(kx - \omega t), \quad \xi_z = z + \int \frac{\partial \phi}{\partial z} \, dt = z + A e^{kz} \cos(kx - \omega t). \]
The **central idea** is to calculate velocity component in the direction of wave propagation following $\xi$ by Taylor-expanding the Eulerian velocity field around $x$:

$$\eta = u(\xi, t) - u(x, t) = \left[ u(x, t) + (\xi - x) \frac{\partial u(x, t)}{\partial x} + (\xi - z) \frac{\partial u(x, t)}{\partial z} + \ldots \right] - u(x, t)$$

$$\approx (\xi - x) \frac{\partial^2 \phi(x, t)}{\partial x^2} + (\xi - z) \frac{\partial^2 \phi(x, t)}{\partial x \partial z} = \ldots$$

$$= [-A e^{kz} \sin(kx - \omega t)] [-\omega k A e^{kz} \sin(kx - \omega t)] + [A e^{kz} \cos(kx - \omega t)] [-\omega k A e^{kz} \cos(kx - \omega t)]$$

$$= \omega k A^2 e^{2kz} \left[ \sin^2(kx - \omega t) + \cos^2(kx - \omega t) \right] = \omega k A^2 e^{2kz} \Rightarrow \eta = 4\pi^2 A^2 e^{4\pi z/\lambda}$$

(a) Stokes drift speed $\eta$ is a nonlinear (quadratic) quantity in terms of the wave amplitude $A$.

(b) Stokes drift decays exponentially with depth and depends on frequency and wavenumber of the flow fluctuations.

(c) **Darwin drift** (permanent displacement of mass after the passage of a body through a fluid) is a related phenomenon.
Homogeneous turbulent diffusion is a theory for describing dispersion of passive tracer (or Lagrangian particles) by spatially homogeneous and stationary turbulence; let’s also for simplicity assume that the turbulence is isotropic.

Take $C$ as the passive tracer concentration, and $u$ as the turbulent velocity field. Let’s consider large-scale (coarse-grained) quantities: passive tracer concentration $\overline{C}$ and velocity field $\overline{u}$; so that the corresponding small-scale (turbulent) fluctuations are $C'$ and $u'$. Let’s assume the complete scale separation between the large and small scales (i.e., $\overline{C'} = 0$ and $\overline{u'} = 0$) and coarse-grain the advection-diffusion tracer equation by taking its time average:

$$\partial C / \partial t + u \cdot \nabla C = \text{molecular diffusion} + \text{sources/sinks} \quad \Rightarrow \quad \partial(\overline{C} + C') / \partial t + (\overline{u} + u') \cdot \nabla (\overline{C} + C') = ...$$

Can we find a simple mathematical model (parameterization, closure) for the turbulent stress term on the rhs?

Lagrangian point of view on turbulent diffusion. For this purpose let’s consider dispersion (i.e., spreading) of an ensemble of Lagrangian particles. Concentration of the particles is equivalent to $C$, and displacement of each particle from its initial position is given by the integral of its Lagrangian velocity:

$$x(t) - x(0) = \int_0^t u_L(t') \, dt'$$

Standard functions characterizing evolution of the Lagrangian particles ensemble are single-particle dispersion $D(t)$ and Lagrangian velocity autocorrelation function $R(\tau)$. These functions are obtained by ensemble averaging (i.e., over many flow realizations):

$$D(t) \equiv \langle (x(t) - x(0))^2 \rangle, \quad R(t - t') \equiv \frac{\langle u_L(t) \cdot u_L(t') \rangle}{\langle u^2 \rangle}$$

and they are mathematically connected with each other. Notice, that

$$\int_0^t R(t' - t) \, dt' = \langle [x(t') - x(0)]_0^t \frac{u_L(t)}{u^2} \rangle,$$

therefore:

$$\frac{d}{dt} D(t) = 2 \langle [x(t) - x(0)] u_L(t) \rangle = 2 \langle u^2 \rangle \int_0^t R(t' - t) \, dt' = 2 \langle u^2 \rangle \int_0^t R(\tau) \, d\tau = 2 \langle u^2 \rangle \int_0^t R(\tau) \, d\tau.$$
Next, recall the formula for differentiation under integral sign,
\[
F(x) = \int_{a(x)}^{b(x)} f(x, t) \, dt \quad \implies \quad \frac{d}{dx} F(x) = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) \, dt,
\]
and find:
\[
D(t) = 2 \langle u^2 \rangle \int_0^t (t - \tau) R(\tau) \, d\tau \quad (**)
\]
Prove the above formula by differentiating it and, eventually, by obtaining (⋆):
\[
\frac{dD}{dt} = 2 \langle u^2 \rangle \left( (t - t) R(t) - 0 + \int_0^t R(\tau) \, d\tau \right)
\]

Asymptotic limits: Let’s consider the short- and long-time asymptotic limits of \( D(t) \) by focusing on (⋆):

(a) **Ballistic limit**: \( t \to 0 \).
Then, \( \tau \approx 0, \quad R(\tau) \approx 1 \quad \implies \quad D \sim t^2 \)

(b) **Diffusive limit**: \( t \to \infty \).
Introduce Lagrangian decorrelation time \( T_L = \int_0^\infty R(\tau) \, d\tau \).

\[
\implies \quad \left. \frac{dD}{dt} \right|_\infty = 2 T_L \langle u^2 \rangle \quad \implies \quad D \sim t
\]

In the diffusive limit, the area occupied by particles (or passive tracer) grows linearly in time, as in the molecular diffusion process with the eddy diffusivity equal to:
\[
\kappa = \langle u^2 \rangle T_L
\]

Let’s prove the diffusion equation analogy by considering the one-dimensional diffusion equation and by focusing on the mean-square displacement of the tracer concentration (it is equivalent to the single-particle dispersion!):
\[
\frac{\partial C}{\partial t} = \kappa \frac{\partial^2 C}{\partial x^2}, \quad D(t) = \left[ \int_{-\infty}^{\infty} x^2 C(x) \, dx \right] \left[ \int_{-\infty}^{\infty} C(x) \, dx \right]^{-1}
\]
Let’s differentiate $D(t)$ and replace the tendency term by the rhs of the diffusion equation:

$$\frac{\partial D}{\partial t} \sim \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x^2 C \, dx = \kappa \int_{-\infty}^{\infty} x^2 \frac{\partial^2 C}{\partial x^2} \, dx = (\text{by parts}) = 2\kappa \int_{-\infty}^{\infty} C \, dx = 2\kappa$$

Thus, in the diffusion process analogy, the tracer-containing area grows linearly in time.

NOTE: the same diffusion process analogy in 2D and 3D cases yields $4\kappa$ and $6\kappa$ on the rhs, respectively.
Nonlinear flow interactions become fundamentally important when growing flow instabilities reach significant amplitude and become finite-amplitude nonlinear eddies and currents.

- Weakly nonlinear analysis can predict slowly evolving amplitude of nearly monochromatic nonlinear waves through derivation of an amplitude equation.
- Dynamical systems framework (bifurcations, attractors, etc.) can be useful for describing transition to turbulence.
- Exact analytic solutions of nonlinear flows are known (e.g., solitary waves), but remain simple and exceptional.
- Statistical wave turbulence framework (resonant triads, kinetic equations, etc.) can be useful, when the underlying linear dynamics is relatively simple and wave coherency is weak.
- Stochastic modelling of turbulence is an emerging field, but it is poorly constrained by physics.
- Numerical modelling is presently the most useful (in terms of the new knowledge!) approach for theoretical analysis of nonlinear flows, but under the relaxed scientific standards it can be intoxicating and detrimental.

**Illustration: Stages of nonlinear evolution of the growing instabilities in the Phillips model**
• **Turbulence modelling** is the process of construction and use of a model aiming to predict effects of broadly defined spatio-temporally complex nonlinear flow dynamics, which is referred to as fluid “turbulence”.

• **Closure problem** is a dream (or a modern alchemy?) to predict coarse-grained flow evolution by expressing important dynamical effects of unresolved flow features in terms of the coarse-grained flow fields.

Let’s consider some velocity field consisting of the coarse-grained (i.e., obtained by some spatio-temporal filtering) and fluctuating (i.e., small-scale) components:

\[ u = \overline{u} + u', \quad \overline{u'} = 0. \]

Let’s assume the following dynamics:

\[ \frac{du}{dt} + uu + Au = 0 \quad \rightarrow \quad \frac{d\overline{u}}{dt} + \overline{uu} + A\overline{u} = 0 \quad \rightarrow \quad \frac{d\overline{u}}{dt} + \overline{uu} + \overline{u'\overline{u'}} + A\overline{u} = 0 \]

To close the equation for \( \overline{u} \), let’s obtain the equation for \( \overline{uu} = \overline{u}\overline{u} + \overline{u'\overline{u'}} \) by multiplying \((*)\) with \( u \) and by coarse-graining:

\[ \frac{1}{2} \frac{d\overline{u^2}}{dt} + \overline{uuu} + A\overline{u^2} = 0 \]

What are we going to do with the cubic term? An equation determining it will contain a quartic term \( \overline{uuuu} \), and so on...

Let’s imagine a magic “philosopher’s stone” relationship that makes the closure:

\[ \overline{uuuu} = \alpha \overline{uu} \overline{uu} + \beta \overline{uuu} \]

Many theoreticians are looking for various “philosopher’s stone” relationships that will be laughed at a century from now, but by doing this a great deal of physical knowledge is obtained and many mathematical instruments are developed.

• **Reynolds Decomposition.** Common example of coarse-graining is decomposition of a turbulent flow into the time-mean and fluctuation (i.e., “eddy”) components:

\[ u(t, x) = \overline{u}(x) + u'(t, x), \quad p(t, x) = \overline{p}(x) + p'(t, x), \quad \rho(t, x) = \overline{\rho}(x) + \rho'(t, x). \]

For example, let’s apply the Reynolds decomposition to the \( x \)-momentum equation and, then, average this equation over time (as denoted by overline):

\[ \frac{\partial \overline{u}}{\partial t} + \overline{u} \cdot \nabla \overline{u} = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x} - \nabla \cdot \overline{u'\overline{u'}} = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x} - \frac{\partial}{\partial x} \overline{u' u'} - \frac{\partial}{\partial y} \overline{u' v'} - \frac{\partial}{\partial z} \overline{u' w'}. \]
The last group of terms is the first component of divergence of the *Reynolds stress tensor* $\overline{u'_i u'_j}$.

(a) Components of nonlinear stress $\overline{u'\phi'}$ are usually called *eddy flux* components of $\phi$. Divergence of an eddy flux can be interpreted as internally and nonlinearly generated *eddy forcing* exerted on the coarse-grained flow.

(b) It is very tempting to assume that nonlinear stress can be related to the corresponding time-mean gradient, for example:

$$\overline{u'\phi'} = -\nu \frac{\partial \phi}{\partial x}$$

This *flux-gradient assumption* is often called the *eddy diffusion* or *eddy viscosity* (closure). Note, that this flux-gradient relation is exactly true for the real viscous stress (only in Newtonian fluids!) arising due to the molecular dynamics.

(c) The flux-gradient assumption is common in models and theories, but it is often either inaccurate or fundamentally wrong, because fluid dynamics is different from molecular dynamics.

(d) Turbulent QG PV dynamics can be also coarse-grained to yield diverging eddy fluxes, because $\phi$ can stand for PV. Since PV anomalies consist of the relative-vorticity and buoyancy parts, the PV eddy flux $\overline{u'q'}$ can be straightforwardly split into the Reynolds stress (i.e., eddy vorticity flux) and form stress (i.e., eddy buoyancy flux) components.

**Parameterization of unresolved eddies**: The above coarse-graining approach can be extended beyond the Reynolds decomposition into the time mean and fluctuations by decomposing flow into a large-scale and slowly evolving component and the small-scale residual eddies. For example, consider the equivalent-barotropic model with eddy viscosity:

$$\Pi = \nabla^2 \psi - \frac{1}{R^2} \psi + \beta y, \quad \frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Pi}{\partial x} = \nu \nabla^2 \zeta = \nu \nabla^4 \psi,$$

here it is assumed that the model solves for the large-scale flow, and the viscous term represents effects of unresolved eddies.

(a) Molecular viscosity of water is $\sim 10^{-6}$ m$^2$ s$^{-1}$, but typical values of $\nu$ used in geophysical models are 100–1000 m$^2$ s$^{-1}$. What do these numbers imply?

Typical viscosities (in m$^2$ s$^{-1}$): honey $\sim 0.005$, peanut butter $\sim 0.25$, basaltic lava $\sim 1000$.

In simple words, oceans in modern theories and models are made of basaltic lava rather than water...

(Similar analogy holds for the atmosphere; although kinematic viscosity is about 20 times larger in the air.)

(b) *Reynolds number* measures relative importance of nonlinear and viscous terms (*Peclet number* is similar but for diffusion term):

$$Re = \frac{U^2/L^2}{\nu U/L^3} = \frac{UL}{\nu}, \quad Pe = \frac{UL}{\kappa}$$

Modern general circulation models strive to achieve larger and larger $Re$ (and $Pe$) by employing better numerical algorithms and faster supercomputers.
Triad interactions in turbulence are the elementary nonlinear interactions that transfer energy between scales. Let’s consider a double-periodic domain with the following forced and dissipative 2D dynamics:

\[ \frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi. \quad (\ast) \]

All flow fields can be expanded in Fourier series (summation is over all negative and positive wavenumbers):

\[ \psi(x,y,t) = \sum_k \tilde{\psi}(k,t) e^{ikx}, \quad \zeta(x,y,t) = \sum_k \tilde{\zeta}(k,t) e^{ikx}, \quad k = ik_1 + jk_2, \quad \tilde{\zeta} = -K^2 \tilde{\psi}, \quad K^2 = k_1^2 + k_2^2. \]

Substituting the Fourier expansions in \( \ast \) yields

\[ -\frac{\partial}{\partial t} \sum_k K^2 \tilde{\psi}(k,t) e^{ikx} = \left[ \sum_p p_1 \tilde{\psi}(p,t) e^{ipx} \right] \left[ \sum_q q_2 \tilde{\zeta}(q,t) e^{iqx} \right] - \left[ \sum_p p_2 \tilde{\psi}(p,t) e^{ipx} \right] \left[ \sum_q q_1 \tilde{\zeta}(q,t) e^{iqx} \right] + \sum_k \tilde{F}(k,t) e^{ikx} + \nu \sum_k K^4 \tilde{\psi}(k,t) e^{ikx}, \]

where \( k, p \) and \( q \) are 2D wavevectors.

Wavevector evolution equation is obtained for each spectral coefficient \( \tilde{\psi}(k,t) \) by multiplying the above equation with \( \exp(-ikx) \), by integrating over the domain, using \( Q^2 = q_1^2 + q_2^2 \), and by noting that the Fourier modes are orthogonal:

\[ \int e^{ipx} e^{iqx} \, dA = L^2 \delta(p + q) \quad \rightarrow \]

\[ \frac{\partial}{\partial t} \tilde{\psi}(k,t) = \sum_{p,q} \frac{-Q^2}{-K^2} \left( p_1 q_2 - p_2 q_1 \right) \delta(p + q - k) \tilde{\psi}(p,t) \tilde{\psi}(q,t) + \frac{1}{-K^2} \tilde{F}(k,t) - \nu K^2 \tilde{\psi}(k,t) \quad (***) \]

This can be reformulated for evolution of the complex amplitude \( |\tilde{\psi}(k,t)| \) by multiplying the equation with the complex conjugate spectral coefficient \( \tilde{\psi}^*(k,t) \). Note, that there are as many equations \( *** \) involved, as wavevectors \( k \) considered.

Interaction coefficient weighs the nonlinear term according to the dynamics, and it is nonzero only for the interacting wavevector triads that must satisfy: \( p + q = k \), because of the \( \delta \)-function involved.

Hermitian (conjugate) symmetry property (i.e., \( \tilde{\psi} \) is Hermitian function) states that

\[ \tilde{\psi}(k_1,k_2,t) = \tilde{\psi}^*(-k_1,-k_2,t), \]

because \( \psi \) is real function.
Note the following about the triad interactions:

(a) *Redistribution of spectral energy density.*
Suppose, there are initially only two Fourier modes, with wavevectors \( p \) and \( q \), and with the Fourier coefficients \( \tilde{\psi}(p, t) \) and \( \tilde{\psi}(q, t) \). Due to the conjugate symmetry, these modes must have their conjugate-symmetric partners at \( -p \) and \( -q \), which are described by the Fourier coefficients \( \tilde{\psi}^*(-p, t) \) and \( \tilde{\psi}^*(-q, t) \); thus, the initial combination of the “two modes” are actually the “four modes” organized in 2 conjugate-symmetric pairs. Nonlinear interactions involving the initial 2 pairs will generate 2 more pairs,

\[
\begin{align*}
k &= p + q, \\
l &= -p - q, \\
m &= p - q, \\
n &= -p + q,
\end{align*}
\]

and the subsequent nonlinear generation of the new wavevectors will continue to infinity.

(b) *Locality.* Nonlinear triad interactions are labeled as local \( (k \sim p \sim q) \) and non-local \( (k \sim p \ll q) \), depending on the difference between involved scales (see Figure).

(c) *Cascades* in turbulence are energy transfers between scales based on local interactions.

(d) *Fourier spectral descriptions* are popular, because the modes are simple and orthogonal, and in spatially homogeneous situations they even satisfy the linearized dynamics. Other spectral descriptions are not only possible but can be even more useful.

(e) *Fourier expansion in time* allows to talk about nonlinear interactions of individual waves rather than wavevectors. If phases of these waves are approximately random, then the problem can be approached by wave turbulence theory; if the phases are coherent, then people talk about coherent structures.
• Homogeneous and stationary, non-rotating 3D turbulence is characterized by energy transfers from the larger to smaller scales. These transfers can involve both local and nonlocal interactions; however, the “forward energy cascade” is a popular concept (conjecture) stating that the energy is transferred only between similar scales.

Forward energy cascade assumes the following:

(a) At large length scales there is some energy input (e.g., due to instabilities of the large-scale flow), all dissipation happens on short length scales, and on the intermediate length scales the turbulence is controlled by conservation of energy.

(b) Dissipation acts on very short length scales, such that the fluid motion is characterized by $Re \leq 1$. These are the scales on which the cascading energy is drained out.

(c) The turbulence within the cascade is characterized by self-similarity.

Let’s consider:

isotropic wavenumber, $k$,
energy spectral density, $E(k)$, and
energy input rate, $\epsilon$.

Energy within a spectral interval is $E(k)\delta k$.

The physical dimensions are:

$[k] = \frac{1}{L}$, $[E] = L U^2 = \frac{L^3}{T^2}$, $[\epsilon] = \frac{U^2}{T} = \frac{L^2}{T^3}$

Velocity scale and advective time scale are:

$v_k = [kE(k)]^{1/2}$,
$\tau_k = (kv_k)^{-1} = [k^3E(k)]^{-1/2}$.

In the assumed inertial spectral range the kinetic energy is conserved; it is neither produced nor dissipated. Energy input in and output from each spectral interval, on the one hand, is $\epsilon$, and, on the other hand, should scale with $v_k$ and $\tau_k$ only:

$$\epsilon \sim \frac{v_k^2}{\tau_k} = \frac{kE(k)}{\tau_k} = k^{5/2}E(k)^{3/2} \implies E(k) \sim \epsilon^{2/3}k^{-5/3}$$

Kolmogorov “minus-five-thirds” spectrum
Dissipative length scale $L_{\text{visc}}$ can be obtained by equating the advective time scale $\tau_k$ and the viscous time scale $\tau_{\text{visc}} = [k^2 \nu]^{-1}$ for the corresponding wavenumber $k_{\text{visc}}$:

$$\tau_k = k^{-3/2} E^{-1/2} \sim \epsilon^{-1/3} k^{-2/3} \quad \rightarrow \quad \tau_k = \tau_{\text{visc}} \quad \Rightarrow \quad k_{\text{visc}} \sim \epsilon^{1/4} \nu^{-3/4} \quad \Rightarrow \quad \frac{1}{k_{\text{visc}}} \equiv L_{\text{visc}} \sim \epsilon^{-1/4} \nu^{3/4}
$$

Alternatively, we can find the power law scalings like this:

$$k_{\text{visc}} \sim L_{\text{visc}}^{-1} \sim \epsilon^\alpha \nu^\beta \sim \frac{L_{\text{visc}}^{2\alpha}}{T^{3\alpha}} \frac{L_{\text{visc}}^{2\beta}}{T^{\beta}} \quad \Rightarrow \quad 2\alpha + 2\beta = -1, \quad 3\alpha + \beta = 0 \quad \rightarrow \quad \alpha = \frac{1}{4}, \quad \beta = -\frac{3}{4}
$$

**2D homogeneous turbulence** is controlled by conservation of not only energy but also enstrophy $Z = \zeta^2$:

$$\frac{\partial}{\partial t} \zeta^2 = 2\zeta \frac{\partial \zeta}{\partial t} = -2\zeta \cdot \nabla \zeta = -\nabla \cdot (\mathbf{u} \zeta^2) + \zeta^2 \nabla \cdot \mathbf{u}, \quad (*)$$

where the rhs in $(*)$ vanishes, because we assume nondivergent flow and no-flow-through boundaries, i.e., $\mathbf{u} \cdot d\mathbf{S} = 0$, therefore:

$$\frac{\partial}{\partial t} \int_A \zeta^2 \, dA = \int_A \frac{\partial}{\partial t} \zeta^2 \, dA = - \int_A \nabla \cdot (\mathbf{u} \zeta^2) \, dA = - \int_S \mathbf{u} \zeta^2 \, dS = 0 \quad \Rightarrow \quad \text{Conservation of Enstrophy}
$$

The 2D turbulence is characterized by the following:

(a) Energy is transferred to larger scales (hence, “inverse energy cascade” concept is valid) and ultimately removed by some other physical processes; the Kolmogorov spectrum $E(k) \sim k^{-5/3}$ is preserved.

(b) Enstrophy is transferred to smaller scales (i.e., there is “forward enstrophy cascade”) and ultimately removed by viscous dissipation.

(c) Upscale energy transfer occurs often through 2D vortex mergers.

(d) Downscale enstrophy transfer (i.e., enstrophy cascade) occurs often through irreversible stretching, filamentation and stirring of relative vorticity.

To obtain its spectral law, the enstrophy cascade can be treated similarly to the energy cascade. Let’s assume that enstrophy input rate $\eta$ produces enstrophy that cascades through the inertial spectral range to the dissipation-dominated scales:
Now, let’s recall that the advective time scale is  
\[ \tau_k = k^{-3/2} E(k)^{-1/2}, \]
\[ \Rightarrow \eta \sim \frac{\zeta^2}{\tau_k} = \frac{(k v_k)^2}{\tau_k} = \frac{k^3 E(k)}{\tau_k} = \frac{k^{9/2} E(k)^{3/2}}{\eta} \Rightarrow E(k) \sim \eta^{2/3} k^{-3} \]

Let’s now use (**) to get rid of \( E(k) = \eta \)

Equate this to the viscous time scale to obtain the dissipative length scale for enstrophy:
\[ \tau_{visc} \sim [k^2 \nu]^{-1} = \eta^{-1/3} \Rightarrow k_{visc} \sim \eta^{1/6} \nu^{-1/2} \Rightarrow \frac{1}{k_{visc}} = L_{visc} \sim \eta^{-1/6} \nu^{1/2} \]

Instead of engaging into detailed analysis of the 2D vortex mergers, let’s consider an alternative explanation of the energy transfer to the larger scales... Vorticity is conserved, but it is also being stretched and filamented (e.g., consider a circular patch of vorticity that evolves and becomes elongated as a spaghetti). The corresponding streamfunction is obtained by the vorticity inversion \( \nabla^2 \psi = \zeta \), therefore, its length scale will be controlled by the elongated vorticity scale, hence, the streamfunction scale will keep increasing. Therefore, the total kinetic energy will become dominated by larger scales.

- **Effects of rotation and stratification on the 3D turbulence** are such, that they suppress vertical motions, and, therefore, create and maintain quasi-2D turbulence.

The \( \beta \)-effect or other horizontal inhomogeneities of the background PV make quasi-2D turbulences anisotropic. E.g., one of such phenomena is the emergence of multiple alternating jets (such as zonal bands in the atmosphere of Jupiter). Length scales controlling widths of the multiple jets are the Rhines scale \( L_R = (U/\beta)^{1/2} \) (here, \( U \) is characteristic eddy velocity scale) and baroclinic Rossby radius \( R_D \).
When people talk about *homogeneous 3D turbulence*, they usually discuss this kind of solutions... (shown are vertical vorticity isolines)

There are many types of *inhomogeneous 3D turbulence*, characterized by some broken spatial symmetries

Turbulent convection (heavy fluid on the top)
2D turbulence is characterized by interacting and long-living coherent vortices.

These vortices are materially conserved vorticity extrema.
Merger of two same-sign vortices (*snapshots from early to late time*)

**In 2D turbulence:**

- Upscale energy transfer occurs through *vortex mergers*
- Downscale enstrophy transfer occurs through irreversible *filamentation and stirring of vorticity anomalies*
- **Transformed Eulerian Mean** (TEM) is a useful transformation of the equations of motion (for predominantly zonal eddying flows, like atmospheric storm track or oceanic Circumpolar Current). TEM framework:

(a) eliminates eddy fluxes in the thermodynamic equation,
(b) in a simple form collects all eddy fluxes in the zonal momentum equation,
(c) highlights the role of eddy PV flux.

Let’s start with the Boussinesq system of equations,

\[
\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + F, \quad \frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \quad \frac{Dw}{Dt} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - b, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{Db}{Dt} + N^2 w = Q_b,
\]

assume geostrophic and ageostrophic velocities and focus on the \(\epsilon\)-order terms in the zonal momentum and thermodynamic equations:

\[
\frac{\partial u_a}{\partial t} + u_a \frac{\partial u_a}{\partial x} + v_g \frac{\partial u_a}{\partial y} - f_0 v_a = F, \quad \frac{\partial b}{\partial t} + u_g \frac{\partial b}{\partial x} + v_g \frac{\partial b}{\partial y} + N^2 w_a = Q_b.
\]

These equations can be rewritten in the flux divergence form:

\[
\frac{\partial u_a}{\partial t} + \frac{\partial u_a u_a}{\partial x} + \frac{\partial v_a u_a}{\partial y} - f_0 v_a = F, \quad \frac{\partial b}{\partial t} + \frac{\partial u_a b}{\partial x} + \frac{\partial v_a b}{\partial y} + N^2 w_a = Q_b.
\]

Next, assume conceptual model of eddies evolving on zonally symmetric mean flow and feeding back on this flow. Separate eddies from the mean flow by applying zonal \(x\)-averaging (denoted by overline; \(\overline{f} = 0\) ):

\[
u_g = \overline{u_g(t,y,z)} + u'_g(t,x,y,z), \quad v_g = \overline{v'_g(t,x,y,z)} \quad \rightarrow \quad \frac{\partial \overline{u_g}}{\partial t} = f_0 \overline{v_a} - \frac{\partial}{\partial y} \left( u'_g v'_g \right) + F \quad (*)
\]

Note, that zonal integration of any \(\partial (\text{flux})/\partial x\) term yields zero, because of the zonal symmetry.

Similar decomposition of the buoyancy yields:

\[
\frac{\partial B}{\partial t} = -N^2 \overline{w_a} - \frac{\partial}{\partial y} \left( \overline{v'_g v'_g} \right) + Q_b \quad (**)\n\]

Equations (*) and (**) are coupled by the thermal wind relations, and because of this coupling, effects of the momentum and heat fluxes cannot be clearly separated from each other — this is a fundamental nature of the geostrophic turbulence.

Progress can be made by recognizing that \(\overline{v_a}\) and \(\overline{w_a}\) are related by mass conservation (i.e., non-divergent 2D field). Hence, we can define ageostrophic meridional streamfunction, \(\psi_a\), such that

\[
\overline{v_a} = -\frac{\partial \psi_a}{\partial z}, \quad \overline{w_a} = \frac{\partial \psi_a}{\partial y}.
\]
Meridional eddy buoyancy flux can be easily incorporated in $\psi_a$, and we can define the residual mean meridional streamfunction,

$$\psi^* \equiv \psi_a + \frac{1}{N^2} v_g b$$

$$\implies \quad \overline{\psi^*} = -\frac{\partial \psi^*}{\partial z} = \overline{\psi_a} - \frac{\partial}{\partial z} \left( \frac{1}{N^2} v_g b \right), \quad \overline{w^*} = \frac{\partial \psi^*}{\partial y} = \overline{\psi_a} + \frac{\partial}{\partial y} \left( \frac{1}{N^2} v_g b \right),$$

that by construction describes non-divergent 2D flow $(\overline{\psi^*}, \overline{w^*})$.

(a) Thus, $\psi^*$ combines the (ageostrophic) Eulerian mean circulation with the eddy-induced (Lagrangian) circulation.

The eddy-induced circulation can be understood as a Stokes drift phenomenon.

(b) These circulations tend to compensate each other, hence, mean zonal flow feels their residual effect.

With the definition of $\psi^*$, the momentum equation (*) can be written as

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v^*} - \frac{\partial}{\partial y} \overline{u_g v_g} + \frac{\partial}{\partial z} f_0 \frac{v_g b}{N^2} + \overline{F} = f_0 \overline{v^*} + \nabla yz \cdot \mathbf{E} + \overline{F}, \quad \mathbf{E} \equiv (0, -\overline{u_g v_g}, \frac{f_0}{N^2} \frac{v_g b}{v_g b}),$$

where we introduced the Eliassen-Palm flux $\mathbf{E}$.

Next, let’s take into account that $\nabla yz \cdot \mathbf{E} = \overline{v_g q_g}$ (see Problem Sheet), and obtain the Transformed Eulerian Mean (TEM) equations:

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v^*} + \overline{v_g q_g} + \overline{F}, \quad \frac{\partial \overline{b}}{\partial t} = -\overline{N^2} \overline{w^*} + \overline{Q_b}, \quad \frac{\partial \overline{v^*}}{\partial y} + \frac{\partial \overline{w^*}}{\partial z} = 0, \quad f_0 \frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y} \quad (***)$$

where the last equation is just the thermal wind balance.

Let’s eliminate the left-hand sides from the first two equations by differentiating them with respect to $z$ and $y$, respectively. The outcome is equal by the last equation from (**), and the resulting diagnostic equation is

$$-f_0^2 \frac{\partial \overline{v^*}}{\partial z} + N^2 \frac{\partial \overline{w^*}}{\partial y} = f_0 \frac{\partial}{\partial z} \overline{v_g q_g} + f_0 \frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{Q_b}}{\partial y}.$$

Now we can take into account definition of $\psi^*$ and obtain the final diagnostic equation:

$$f_0^2 \frac{\partial^2 \psi^*}{\partial z^2} + N^2 \frac{\partial^2 \psi^*}{\partial y^2} = f_0 \frac{\partial}{\partial z} \overline{v_g q_g} + f_0 \frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{Q_b}}{\partial y} \quad (***)$$
(a) If we know the eddy PV flux, the TEM equations allow us to solve for the complete circulation pattern. This can be done by solving the elliptic problem (****) for $\psi^*$, at every time (step).

(b) Eddy PV flux still has to be found dynamically, but the theory allows for many dynamical insights.

(c) The TEM framework can be extended to non-QG flows.

(d) Non-Acceleration Theorem states that under certain conditions eddies (or waves) have no net effect on the zonally averaged flow. Let's prove it by considering zonally averaged QG PV equation (with a non-conservative rhs $\overline{D}$):

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial \overline{v'q'}}{\partial y} = \overline{D}, \quad \overline{q} = \frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \overline{\psi}}{\partial z} \right) + \beta y.$$ 

Let's differentiate $\left( \partial / \partial y \right)$ the QG PV equation:

$$\frac{\partial^2}{\partial t \partial y} \left[ \frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \overline{\psi}}{\partial z} \right) \right] = - \frac{\partial^2}{\partial y^2} \overline{v'q'} + \frac{\partial \overline{D}}{\partial y},$$ 

and recall that

$$\overline{v'q'} = \overline{v' \overline{q'}} = \nabla_{yz} \cdot \mathbf{E} \rightarrow \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \right) \right] \frac{\partial \overline{q}}{\partial t} = \frac{\partial^2 (\nabla_{yz} \cdot \mathbf{E})}{\partial y^2} - \frac{\partial \overline{D}}{\partial y}.$$

*Theorem:* If there is no eddy PV flux (i.e., Eliassen-Palm flux is non-divergent) in stationary and conservative situation, then the flow cannot get accelerated ($\partial \overline{q} / \partial t = 0$), because the "Eulerian mean" and "eddy-induced" circulations completely cancel each other.