EE1 MATHEMATICS

FINITE-DIFFERENCE METHODS

Two-Point boundary value problem

- Find a function $y = y(x)$ which is the solution of

$$\frac{d^2y}{dx^2} = f(x), \quad 0 < x < 1, \tag{1a}$$

$$y(0) = \alpha, \quad y(1) = \beta, \tag{1b}$$

- Where $f(x)$ is a given function.
- This is an example of a **two–point boundary value problem**.
- We want to develop a numerical method for solving this problem.
- We will use the method of **finite differences**.

Finite Differences

- We place a uniform net on the interval $[0, 1]$:

$$x_j = jh, \quad j = 0, 1, \ldots N + 1, \quad h = \frac{1}{N+1}.$$

- Define $y_j = y(x_j)$. We want to write a system of equations for y_j whose solution approximates the solution of (1). We will approximate the second order derivative on the left hand side of (1) by a finite difference:

$$\frac{d^2y}{dx^2} \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}. \tag{2}$$

- Notice that

$$\lim_{h\to 0} \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = \frac{d^2y}{dx^2}.$$

- We use (2) in (1) to obtain the system of equations

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = f_j, \quad j = 1, \dots, N,$$

- with $f_j = f(x_j)$, $j = 1, \dots, N$.
- Notice that we have

$$y_0 = y(0) = \alpha, \quad \text{and} \quad y_{N+1} = y(1) = \beta. \tag{3}$$

- We want to write the equation

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = f_j, \quad j = 1, \dots, N, \tag{4}$$

- as a linear system of equations for the unknown vector

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_1 \\ \dots \\ y_N \end{pmatrix}.$$

- We multiply equation (4) by $\frac{h^2}{2}$ to obtain

$$\frac{1}{2}y_{j-1} - y_j + \frac{1}{2}y_{j+1} = \frac{h^2}{2}f_j, \quad j = 1, \dots N.$$

  Notice that when $j = 1$ we have (since, by (3), $y_0 = \alpha$)

$$\frac{1}{2}y_2 - y_1 + \frac{1}{2}\alpha = \frac{h^2}{2}f_1.$$

  Similarly, when $j = N$ we have

$$\frac{1}{2}y_{N-1} - y_N + \frac{1}{2}\beta = \frac{h^2}{2}f_N.$$

We combine the above three equations to write the system of equations (4) as a linear system

$$\mathbf{A}\mathbf{y} = \mathbf{f}, \tag{5}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \frac{h^2}{2} f_1 - \frac{\alpha}{2} \\ \frac{h^2}{2} f_2 \\ \dots \\ \frac{h^2}{2} f_{N-1} \\ \frac{h^2}{2} f_N - \frac{\beta}{2} \end{pmatrix},$$

and

$$\mathbf{A} = \begin{pmatrix} -1 & \frac{1}{2} & 0 & \dots & \dots & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \dots & \dots & \dots & \frac{1}{2} & -1 \end{pmatrix}.$$

- The linear system (5) can be solved using the LU factorization.