Noisy bounded confidence models for opinion dynamics: the effect of boundary conditions on phase transitions

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September 7, 2020

Abstract

We study SDE and PDE models for opinion dynamics under bounded confidence, for a range of different boundary conditions, with and without the inclusion of a radical population. We perform exhaustive numerical studies with pseudospectral methods to determine the effects of the boundary conditions, suggesting that the no-flux case most faithfully reproduce the underlying mechanisms in the associated deterministic models of Hegselmann and Krause. We also compare the SDE and PDE models, and use tools from analysis to study phase transitions, including a systematic description of an order parameter.

1 Introduction and Previous Work

This work focuses on bounded confidence models for opinion dynamics under social influence \cite{10, 56, 57}, which is part of the larger field of mathematical modelling in the social sciences \cite{10, 64, 65}. In contrast to graph-based models in which communication occurs only between connected individuals, here communication is instead limited by the difference in their opinions, which is treated as a continuous variable. A typical example of a continuous opinion is an individual’s political orientation, which is not restricted to a few discrete choices, such as extreme left or right, but rather can vary across a spectrum. The motivation for bounded confidence models comes from ‘biased assimilation’ \cite{36} which postulates that individuals are more strongly influenced by others with similar opinions to their own. A related consideration is that genuine discussion generally only occurs between individuals who already share some common ground, i.e., they already have sufficiently close opinions. This can be modelled by the coupling between individuals increasing as their opinions become more similar.

Here we consider models with a hard cut-off in opinion space; individuals with opinions differing by more than the confidence bound $R$ do not influence each other. Such models were originally proposed as deterministic, discrete-time processes \cite{15, 68, 28}, which have also been extended to include noise or randomness \cite{52, 53, 54, 60}, modelling uncertainty in observations or external influences. Related models include continuous time ODEs \cite{4, 70}, SDEs, and (in the limit of many individuals) PDEs \cite{67, 34}. See \cite{10, 56, 57} for comprehensive reviews. In the below, due to the inclusion of noise in the SDE and (implicitly in) the PDE models, we refer to them as non-deterministic. More general formulations arise in mathematical biology as the Keller-Segel model \cite{33} for slime mold, and other similar models \cite{44, 43, 62}. Since the individuals which interact changes according to the dynamics, such models are often described through ‘co-evolutionary networks’ \cite{57}, for example the Vicsek model of phase transitions \cite{66}, the Cucker-Smale model for flocking \cite{12, 13, 46}, and robotics \cite{32, 6, 7}. We also highlight recent mathematical work concerning the well-posedness and long-time behaviour of related mean-field models \cite{11, 8, 23}.

A typical question one asks about such models is how a uniform, or disordered, initial condition evolves under the interactions of individuals; typically the system is driven to a more ordered state \cite{71, 10, 35}. This has clear analogues with order–disorder phase transitions \cite{71}. Two popular models are due to Hegselmann and Krause (HK) \cite{28} and Deffaunt and Weisbuch (DW) \cite{15, 68}. See \cite{57} for a recent review. Both are discrete in time and rely on the idea of repeated averaging under bounded confidence. In DW agents interact in randomly chosen pairs who then either do or do not compromise (depending on the separation of their opinions), in HK an individual moves to the average opinion of all agents within a distance $R$ of themselves in opinion space. In such models, an
initially homogeneous or uniform distribution of opinions is unstable. This is due to boundary effects, in which those with extreme opinions can only be influenced by those with less extreme opinions, and hence tend to move towards more moderate values, causing clusters. Three possible outcomes are: (i) consensus (a single cluster); (ii) polarization (two distinct clusters); (iii) fragmentation (more than two distinct clusters, or no distinct clusters). It has been noted that the number of clusters is typically $1/(2R)$, which is related to the $2R$-conjecture [3, 10].

This convergence to a steady state is not normally observed in real-world systems [1, 72, 9]. Two possible factors missing from these models are (i) influence of external agents or information, such as ‘radicals’, leaders, or advertising; (ii) uncertainty in the dynamics. For (i), it is usually assumed that such influences are constant in time [16, 72, 69, 27, 40, 34], or weakly susceptible to other opinions [73]. Numerical simulations have demonstrated counterintuitive effects concerning the introduction of radials, such as increasing radical numbers decreasing the number of individuals sharing their opinion after long times. We will demonstrate similar effects in our nondeterministic models; see Section 4. For (ii), it has been argued [1] that diffusion is an essential element of opinion dynamics, allowing the modelling of realistic political systems with disorder-order transitions and complex lifecycles. This has close links to both mathematical environmental noise in statistical physics models, which can lead to phase transitions [14, 1, 52, 61, 53, 25, 54, 67, 34]. Additionally, in many applications, it is not only the equilibrium that is interesting, but also the dynamical path to that equilibrium [10]. For example, if the opinion measures political persuasion, then one would typically be interested in the distribution at a particular time (e.g., on an election date) than in the long-term equilibrium. In Sections 3 and 4 we systematically investigate both the effects of the noise strength and the dynamical paths to equilibrium.

Due to the non-linear, non-local nature of many opinion dynamics models, and the resulting challenges of analytical investigation, the use of careful and systematic computer simulations has been important in social dynamics for decades, see e.g., [26, 30, 59, 73, 67]. For $N$ individuals, the agent-based, ODE, and SDE models typically have computational costs that scale as $N^2$, which is the cost of determining the pairwise distances. An interesting regime, which is amenable to both mathematical analysis and decreased computational cost, is the mean-field limit where the number of individuals becomes very large, $N \to \infty$. Then the high-dimensional descriptions of individual opinions reduces to a 1+1 dimensional non-local, non-linear PDE, for which there exist a range of accurate and efficient numerical approaches [67, 49, 24, 34]. Such PDE models have been referred to by a variety of names, including density-based [38], continuum [4, 29], Eulerian [7, 42], hydrodynamic [46, 47], kinetic [16, 5], or mean-field [21, 67, 50, 34].

There can be crucial differences between the SDE and PDE models: Firstly, order-disorder phase transitions are rigorously defined only in the thermodynamic limit as only then can there be non-uniqueness of invariant measures. However, such terms are still used for similar behaviour in the finite-$N$ models [63]. Secondly, there are questions about how large the number of agents needs to be in order for the SDE and PDE models to be in good agreement. Many techniques used to study PDE models arise from statistical mechanics, but care must be taken when directly transferring them to the social sciences. This is due to the different notions of a ‘large’ number of individuals/particles; in molecular systems this could be of the order $10^{23}$, whereas in social systems, it is more likely to be in the hundreds or thousands. In such cases, finite size effects are likely to be important [63]; we demonstrate this in Supplementary Material Section SM3.2. However, these differences must be balanced by the relative computational complexities: as above, the computational cost of the SDE, agent-based model scales as $N^2$, whereas that of the PDE is independent of $N$, depending instead on the discretization.

When considering non-deterministic models, one must specify boundary conditions. Previous work has been restricted to periodic domains, presumably for mathematical and computational ease, see e.g., [21, 67, 34]. However, care must be taken when comparing such implementations to the original deterministic models. For example, in the original models, if all initial opinions lie in an interval $[a, b]$, then this holds for all time [57]. To aid comparison, it would be natural to treat the non-deterministic models in the same way, requiring the individual opinions to lie in some prescribed interval, which, without loss of generality, may be chosen to be $[0, 1]$. One criticism of periodic boundary conditions is that they confine the two extreme opinions at 0 and 1. Whilst the even 2-periodic choice in [34] overcomes this conflation, it does so at the expense or introducing a mirror system, which can strongly influence the dynamics and has (potentially) undesirable effects on quantities such as the order parameter; see Section 2.4. Here we introduce an additional choice of boundary condition, namely no-flux, or zero Neumann.

As in the existing two cases, this preserves the mass conservation property of the model, but does not confine extreme opinions nor require the introduction of an auxiliary system. We propose that no-flux boundary conditions are much more similar to the original, discrete HK models in terms of the mechanism of cluster formation from a uniform initial distribution. For example, as we will see in Section 3, large $R$ and small $\sigma$ causes an initially uniform distribution to develop a cluster in the middle of the interval, rather than remaining in a uniform steady state, which is predicted by the periodic boundary condition cases. The challenge with no-flux boundary conditions
is principally numerical. For SDE models, reflecting boundary conditions are known to be challenging [58, 51], and PDE models can no longer use efficient Fourier methods. Here we implement efficient and robust Fourier and Chebyshev pseudospectral methods, based on [49, 22].

Our main contributions are:

- a unification of existing and novel SDE and PDE models for bounded confidence opinion dynamics;
- a systematic numerical study of these models, under three different boundary condition and a wide range of parameter regimes, with and without a ‘radical’ population;
- a careful discussion of an ‘order parameter’;
- the insight that the no-flux boundary conditions most faithfully reproduce the underlying mechanisms of the original deterministic models.

The remainder of the paper is organised as follows: In Section 2 we introduce the models, as well as the associated order parameter, and briefly discuss the numerical methods. In Sections 3 and 4 we present the numerical experiments for systems without and with radicals, respectively. Section 5 contains our conclusions and a description of some open problems. The Supplementary Material contains more discussion on the order parameter, a detailed description of the numerical methods used, validation against existing results from the literature, a thorough comparison of the SDE and PDE models, as well as some further examples.

2 Model

2.1 Dynamics

The original model of Hegselmann and Krause (HK) is discrete in time and space. It considers a set of $N$ agents, with agent $i$ having opinion $x_i$ with $x = (x_1, \ldots, x_N)$. Bounded confidence is introduced by defining a confidence level $R$ and a set for each agent $I(i, x) = \{1 \leq j \leq N : |x_i - x_j| \leq R\}$, i.e. the set of all individuals whose opinion is within $R$ of that of individual $i$. At each time the opinion of individual $i$ is updated through

$$x_i(t + 1) = |I(i, x(t))|^{-1} \sum_{j \in I(i, x(t))} x_j(t),$$

where here $|I|$ denotes the number of elements of $I$. In words, an individual’s opinion at the next time step is given by the mean of the opinions of individuals within their confidence interval. It is clear to see here that if two groups are separated by a distance of $R$ or more then they will form decoupled subsystems, which then never interact. Note that there exist alternative models in which the attraction increases with separation [47].

In the SDE models [67, 34], which originate in statistical physics, the mean is taken not over the set $I$, but over all individuals, with zero weight on those outside $I$, i.e. $|I|$ is replaced by $N$ in the normalisation. This leads to a simpler mean-field PDE but it is also possible to retain the original normalisation [20]. The original choice of normalisation is perhaps more physically relevant, especially in swarming/flocking models, in which the bounded confidence is based on physical, rather than opinion, distance [12, 13]. In such cases, it is plausible to assume that an individual is completely unaware of those outside its confidence bound. In opinion dynamics models, it is perhaps more reasonable to assume that an individual polls the opinion of all other individuals (as if in a completely connected network) and simply ignores the opinion of those individuals outside their confidence bound. Introducing the $1/N$ scaling results in dynamics which are slowed down by the presence of individuals who do not interact, and are, in fact, unaware of each other [46]; other works have claimed that the results are insensitive to this choice [7, 20].

For a system of $N$ individuals with opinions $x_i$, and a confidence bound $R$, the dynamics, given a suitable initial condition, are described by [67]

$$dx_i = -\frac{1}{N} \sum_{j|x_i-x_j|\leq R} (x_i - x_j)dt + \sigma dW^{(i)}_t,$$

where $W^{(i)}_t$ are independent Wiener processes. The motivation for including noise comes from agents’ ‘free will’, or uncertainty in measurement and communication. Taking the mean-field limit of $N \to \infty$ results in a Fokker-Planck PDE for the density of opinions $\rho(x,t)$:

$$\partial_t \rho(x,t) = \partial_x \left( \rho(x,t) \int (x-y)\rho(y,t)\mathbb{1}_{|x-y|\leq R}dy \right) + \frac{\sigma^2}{2} \partial_{xx} \rho(x,t),$$

where $\rho$ corresponds to the empirical measure $\rho^N(x,t) = N^{-1} \sum_j \delta_{x_j}(dx)$ as $N \to \infty$. 

3
2.2 Boundary Conditions

It remains to discuss the boundary conditions (BC) imposed on (1) and (2). As discussed above, we consider three separate cases: (i) Periodic; (ii) No-Flux; (iii) Even 2-Periodic, all on [0, 1]. For BC (i) [Periodic] the natural interpretation is that \( \rho(x,t) \) is extended periodically to the whole of \( \mathbb{R} \) with the periodicity condition \( \rho(x+1,t) = \rho(x,t), \forall x \in \mathbb{R} \). We note that \( \rho(\cdot,t) \) is not a probability density on \( \mathbb{R} \), but does serve as a (normalised) probability density on \([0,1]\). For BC (ii) [No-Flux], it is helpful to rewrite (2) in terms of the flux, \( j \):

\[
\partial_t \rho(x,t) = -\partial_x j(x,t), \quad j(x,t) = -\rho(x,t) \int (x-y)\rho(y,t)1_{|x-y| \leq R}dy - \frac{\sigma^2}{2} \partial_x \rho(x,t).
\]

We then impose no-flux boundary conditions, i.e., \( j(0,t) = j(1,t) = 0 \). For BC (iii) [Even 2-Periodic] we follow [34]. Rather than considering \( \rho(x,t) \) on \( x \in [0,1] \), we consider its unique even, 2-periodic extension which satisfies \( \rho(-x,t) = \rho(x,t) \) and \( \rho(x+2,t) = \rho(x,t), \forall x \in \mathbb{R} \). Again, \( \rho(\cdot,t) \) is a probability density on \([0,1]\). For clarity, in this case the sum over \( j \) in (1) is over the full, even 2-periodic system with \( 2N \) individuals. We will show that the no-flux boundary conditions most faithfully reproduce the behaviour of the original HK models, for example, the instability of a uniform initial distribution, and the physical interpretation of the domain.

2.3 Radicals

As described in [27, 34], a natural extension of bounded confidence models is to include \( N_r > 0 \) radicals, or extreme groups. These should be thought of as individuals with fixed opinions who, nevertheless, affect the opinions of those who interact with them. As discussed in [27], such a formalism can also be used to model advertising, charismatic leaders, and other external effects. In the language of statistical mechanics, radicals act as an external potential.

For the SDE, we retain the indexing of the ‘normal’ individuals as \( 1, \ldots, N \), and add in radicals indexed by \( N + 1, \ldots, N + N_r \). The dynamics are then governed by

\[
dx_i = -\frac{1}{N} \sum_{j:x_i-x_j \leq R} (x_i-x_j)dt + \sigma dW_i^{(i)}, \quad i = 1, \ldots, N \\
dx_i = 0, \quad i = N + 1, \ldots, N + N_r.
\]

Note that the sum in the first equation now runs over \( j = 1, \ldots, N + N_r \). The corresponding Fokker-Planck PDE is

\[
\partial_t \rho(x,t) = -\partial_x j(x,t), \quad j(x,t) = -\rho(x,t) \int (x-y)(\rho(y,t) + M \rho_r(y))1_{|x-y| \leq R}dy - \frac{\sigma^2}{2} \partial_x \rho(x,t),
\]

where \( \rho_r \) determines the (fixed) distribution of the radicals. We find it convenient to fix \( \rho_r \) as a probability distribution (in the senses described above) and scale the mass with a parameter \( M \). For physical reasons, \( \rho_r \) should be non-negative (although it is interesting to consider negative/repulsive opinions), and also \( M \ll 1 \), otherwise the interpretation as \( \rho_r \) as the density of ‘radicals’ is lost. However, these restrictions are not intrinsic to the model itself. Note that, in the case of even 2-periodic boundary conditions, \( x \in [-1,1] \), the full space, and the convolution is also taken over this whole domain.

2.4 Order Parameter

To enable quantification of the resulting opinion densities, Wang et al [67] introduced the order parameter, which has both a discrete and continuum definition:

\[
\hat{Q}_d(t) = \frac{1}{N^2} \sum_{i,j=1}^N 1_{|x_i(t)-x_j(t)| \leq R}, \quad \hat{Q}_c(t) = \int \int \rho(x,t)\rho(y,t)1_{|x-y| \leq R}dxdy.
\]

In the sequel, we denote both quantities by \( \hat{Q} \), as their use is unambiguously defined by the data to which they refer. We will shortly explain the seemingly extraneous tilde notation.

The order parameter measures the order (or disorder) of the opinions. One interpretation is that it measures the proportion of pairs of individuals within a radius \( R \) of each other in opinion space. For a uniform distribution \( x_i = i/N \), for each \( i \) we find \( \sum_j |x_i-x_j| = 2RN \), and so \( \hat{Q} = 2R \), which provides an \( R \)-dependent lower bound for \( \hat{Q} \); the same holds for \( \rho(x,t) = 1, \forall x \in [0,1] \) in the continuum case. For a single cluster of individuals of width less than \( R \), we find \( \hat{Q} = 1 \), which is the maximum value of \( \hat{Q} \) for both the periodic and no-flux boundary conditions.
For \( n \) equal clusters, of width less than \( R \), and separated by at least \( R \), we find that \( \hat{Q} = 1/n \), suggesting that the order parameter is essentially the inverse of the number of well-separated clusters. Note that there is not an injective mapping between densities and order parameters.

We now note that the order parameter for the even 2-periodic case is somewhat different. We now have (at least) two choices: compute the order parameter for \( N \) individuals on \([0, 1]\), or for \( 2N \) individuals on \([-1, 1]\). In the first case, it is unclear how one should define the distance between two individuals on \([0, 1]\), since the dynamics are defined on \([-1, 1]\) with even/periodic boundary conditions. It is also desirable to use the same definition of \( I \) in both the dynamics and the order parameter; this is only possible when defining \( \hat{Q} \) using \( \rho \) on the whole domain \([-1, 1]\).

Consider a uniform distribution of \( 2N \) individuals on \([-1, 1]\) with periodic boundary conditions, which results in \( \sum_{i,j=1}^{2N} 1_{|x_i-x_j| \leq R} = 2N^2 R \). Hence, to obtain the same value of \( \hat{Q} \) as in the periodic case, one requires a prefactor of \( 1/(2N^2) \). The same normalisation gives the corresponding results for single and multiple clusters which are well-separated from each other and the domain boundaries. Motivated by this, we redefine the order parameter, removing tildes:

\[
Q_d(t) = c_{BC} \frac{1}{N^2} \sum_{i,j=1}^{N_{BC}} 1_{|x_i(t) - x_j(t)| \leq R}, \quad Q_c(t) = c_{BC} \int \int \rho(x,t)\rho(y,t)1_{|x-y| \leq R}dxdy,
\]

where \( c_{BC} = 1 \), \( N_{BC} = N \) for periodic and no-flux boundary conditions and \( c_{BC} = 1/2 \), \( N_{BC} = 2N \) for even 2-periodic boundary conditions. Note that, as above, if \( R \) is small, or the opinion distribution is located away from the boundaries of the domain, then our definition is equivalent to that used previously [34]. However, this leads to some non-standard results for \( Q \) in the even 2-periodic case. Consider a single cluster of particles in \([1 - R/2, 1]\), and the corresponding mirror cluster in \([-1, -1 + R/2]\); this leads to a value of \( Q = 2 \). We use \( Q > 1 \) as a signature that the even 2-periodic boundary conditions have had a significant effect on the dynamics. We give some examples of the behaviour of the order parameter in Supplementary Material Section SM1.

### 2.4.1 Time To Equilibrium

It is useful to have a measure of how quickly the solution of (2) converges to equilibrium. We have found that a robust measure of being close to equilibrium is that the maximum value of \( \partial_t \rho \) is lower than a given tolerance \( \epsilon_{eq} \). When this first happens defines an equilibrium time \( t_{eq} \) for the system. We note that this time clearly depends on the tolerance chosen, and so should be regarded as a measure of relative time to equilibrium for different parameter regimes, rather than a firm statement that we have reached equilibrium. Unless otherwise stated, we choose \( \epsilon_{eq} = 10^{-4} \).

### 2.5 Numerical Methods

We give details and validation of the numerical methods in Supplementary Material Sections SM2 and SM3. For the SDEs we use an Euler-Maruyama scheme; employing either periodic boundary condition is straightforward. The no-flux boundary conditions are applied with a standard reflective method [51]. For the PDEs, we use pseudospectral methods, as originally described in [49], and implemented via [22].

### 3 Numerical Experiments: Opinion Dynamics in the Absence of Radicals

We begin our numerical experiments by studying noisy opinion dynamics models without radicals. For a given boundary condition, there are only three remaining choices: the values of \( R \) and \( \sigma \), and the initial condition.

#### 3.1 Uniform Initial Condition

Our first system has a uniform initial condition, which is standard in many opinion dynamics models, see, e.g., [28, 55, 47, 27, 31, 67, 34]. This initial condition is a steady state for both the periodic and even 2-periodic boundary conditions, which can be seen in Figure 1, and easily shown using the stationary PDE. Here the different order parameters are due only to the intrinsic dependence of \( Q \) on \( R \). However, this is the first demonstration of the significant effect of the boundary conditions on the dynamics; the uniform distribution is not a steady state for the no-flux case, and the density tends to cluster in the centre of the interval. This is a result of the density being
zero outside the domain $[0, 1]$, so there tends to be an inwards net force on the individuals. There is a clear trend as $R$ and $\sigma$ change: increasing $R$ tends to enhance the cluster formation, whilst the opposite is true for increasing $\sigma$. This has a simple explanation: for larger $R$ there is more interaction, and hence the individuals tend towards what is essentially the global mean, rather than the local mean with a small confidence interval; increasing $\sigma$ adds more diffusion to the system, which tends to disperse clusters.

There is another striking feature of the middle plot in Figure 1, namely that there seems to be a sudden switch in behaviour as $R$ and/or $\sigma$ are varied, with the long-time density being either strongly peaked or almost flat. This is particularly noticeable when $\sigma$ is small. As such, in the middle panel of Figure 1 we zoom in to a transition region (left panel) and also consider smaller values of $\sigma$ (right panel). The bottom panels of Figure 1 show the time to equilibrium, as defined in Section 2.4.1, on a log scale. The dynamics take appreciably longer to reach equilibrium in regions where the long-time dynamics is particularly sensitive to the choice of parameters. Note that in the small-$\sigma$ case it was necessary to increase the equilibration tolerance to $\epsilon_{eq} = 5 \times 10^{-4}$; there is overlap between the two plots at $\sigma = 0.05$ to aid comparison.

In Figures 2 and 3 we show the dynamics of parameter pairs labelled in the middle and right subplots of Figure 1, respectively. We show snapshots of the density at various times (top) and the order parameter as a function of time (bottom), with coloured dots corresponding to the time–$Q$ values for the snapshots in the upper panels. For the cases A–E in Figure 2, we see how the behaviour changes for fixed $\sigma = 0.05$ and increasing $R$. For small $R$, a shallow, almost uniform, cluster slowly develops. As $R$ increases the dynamics become richer. At first, the density develops two peaks, with the order parameter rising to a plateau. The two peaks then move together, before merging into a single cluster, indicated by a larger $Q$. In particular, for case E, where $R = 0.21$, the initial state has $Q \approx 0.4 \approx 2R$, as expected, it then rises to $Q \approx 0.5$, indicating the presence of two clusters, before ending at $Q \approx 1$, and a single cluster. Cases (B, F, G) and (H, I, J) demonstrate the effects of fixing $R$ and increasing $\sigma$. In (B, F, G), we begin in the small-$\sigma$ regime (B), where a single, steep cluster forms almost directly from the uniform state, as $\sigma$ increases (F, G), diffusion dominates resulting in an approximately uniform distribution. Similar behaviour is observed for larger $R$ (H, I, J).

Figure 3 shows the corresponding dynamics for the small-$\sigma$ case, fixing $R = 0.125$ (see the right hand plot of Figure 1). Here we observe a much richer collection of possible long-time states and dynamics; to aid visualisation we plot the density as a function of time and space in the bottom panels. For small $\sigma = 0.0225$ (A) we observe a final state with three clusters, and final order parameter approximately $1/3$. For slightly larger $\sigma = 0.025$ (B), the final state has two clusters, with order parameter around $1/2$, but the dynamics clearly pass through a transient state with three, non-equal clusters (see the snapshots in panel B at times 100 and 500). Increasing $\sigma$ further next results in a direct transition to a two-cluster state (C), followed by direct formation of a single cluster (D), and eventually an essentially disordered/uniform long-time state (E). We note that there are similar transitions when fixing $\sigma$ and varying $R$ (not shown).

Here we find it informative to compare the right panel of Figure 1 with Figure 3 of the original Hegselmann-Krause paper [28], which shows the long-time equilibria (for noiseless dynamics) as $R$ is increased. For small $R$, the state is homogeneous, whilst increasing $R$ results first in two clusters and then a single, central cluster, with rapid transitions as $R$ increases. As stated in [28], as $R$ increases ‘we step from fragmentation (plurality) over polarisation (polarity) to consensus (conformity)’. This is a direct analogue of our results just described. In contrast, for the
Figure 2: Results for a uniform initial condition and no-flux boundary conditions. Letters correspond to labelling in the middle panel of Figure 1. In each case, we show snapshots of the densities at the indicated times (top) and the time evolution of the order parameter (bottom); coloured dots show the order parameter at the times of the corresponding snapshots in the top panel.
two periodic boundary conditions, the uniform initial condition is an equilibrium, and we see no such $R$-induced transitions. This suggests that no-flux boundary conditions more faithfully reproduce the results, and underlying mechanisms, of the original models.

We also compare to Figure 1 of [57], which demonstrates the dependence on $R$ (their $d$) of the final number of clusters and equilibration time. They note that both dependencies are non-monotonic, whereas it may be intuitively expected that increasing $R$ causes a reduction in the number of clusters and a decrease in the equilibration time. Such effects are also visible when, e.g., fixing $\sigma = 0.2$ and varying $R$ in the right panel of Figure 1. This phenomenon of ‘abnormally’ slow convergence has also been demonstrated in [37], who described the resulting states as ‘metastable’. Similar sensitivities have also been observed in a noisy DW model [9], which also noted the importance of the initial condition on determining the long-time dynamics; we will now investigate the further choices of initial condition. In Supplementary Material SM4, we show a comparison with the SDE for short times; the agreement is very good.

### 3.2 Single Gaussian Initial Condition

In this section we investigate the effects of a non-uniform initial condition with a single, relatively broad consensus. Following [67], we choose

$$\rho_0(y) = Z \exp \left( - C [d(y, y_0)]^2 \right),$$

where $C = 20$ and $Z$ is the normalisation constant. Here $d(x, y)$ denotes the 1-periodic distance between two points. In [67], $y_0$ was chosen to be 0.5 but this is irrelevant beyond visualisation in the periodic case. However, for the no-flux and even 2-periodic boundary conditions, the choice of $y_0$ can result in qualitatively different dynamics. For example, if $y_0 = 0.5$ then the initial condition is symmetric and the periodic and even 2-periodic cases are identical; for other $y_0$ this is not the case.

In the top panels of Figure 4 we display the final order parameters, densities, and equilibration times for a range of values of $R$ and $\sigma$, for all three boundary conditions, and $y_0$ equal to 0.3 (left), and 0.2 (right). We also show a zoom in parameter space for $y_0 = 0.3$ (bottom left), and results for an initial condition which is a Gaussians mixture (bottom right); see Section 3.3. White regions in the equilibration times denote simulations which have not converged.

In the top panels of Figure 4 we display the final order parameters, densities, and equilibration times for a range of values of $R$ and $\sigma$, for all three boundary conditions, and $y_0$ equal to 0.3 (left), and 0.2 (right). We also show a zoom in parameter space for $y_0 = 0.3$ (bottom left), and results for an initial condition which is a Gaussians mixture (bottom right); see Section 3.3. White regions in the equilibration times denote simulations which have not converged.

We first discuss some general trends. With the exception of $y_0 = 0.2$ and even 2-periodic boundary conditions (top right plot), the final result is either a single cluster, or an (almost) uniform state. This is to be expected as the only force which can break up the initial cluster is diffusion, which favours the uniform state. The formation of one large cluster on the left and one small cluster on the right for $y_0 = 0.2$, even 2-periodic boundary conditions, small
Figure 4: As Figure 1 but for a Gaussian initial condition (4) with $y_0 = 0.3$ (top left, bottom left) and $y_0 = 0.2$ (top right), and an initial condition which is a linear combination of Gaussians (5) (bottom right) for different boundary conditions. Note the additional colour bar scale from 1 to 2 for the even 2-periodic case. White denotes simulations which have not converged by the final time ($10^4$).

A and large $R$ is due to the periodicity of the initial condition. A small amount of mass is originally concentrated at the right end of the domain, near $y = 1$, and gets trapped due to the attraction to its periodic image near $y = -1$. Relatedly, we note the additional range of $Q$ for the even 2-periodic case, in particular for large $R$, where the density is significantly influenced by its periodic/even image. As expected, increasing $\sigma$ also causes a trend to disordered states, whereas increasing $R$ tends to increase the sharpness of the resulting cluster, with increased time-to-equilibrium in regions separating qualitatively different final densities.

There are also some unexpected observations. A feature of the no-flux boundary condition is the movement towards 0.5 of the final maximum opinion for fixed $R$, as $\sigma$ increases. This is likely a consequence of the competition between diffusion and attraction; for smaller values of $\sigma$, the noise is not sufficiently strong to disperse the original cluster, whereas larger noise can cause the cluster to move. Examples of this dynamics can be seen in Figures 5; case B ($R = 0.23, \sigma = 0.11$) shows a clear drift in the mean opinion over time for the no-flux boundary conditions. Parameter sets B ($R = 0.23, \sigma = 0.11$) and C ($R = 0.23, \sigma = 0.13$) for the even 2-periodic case demonstrate the significant effect that this choice of boundary condition has on the dynamics, with a strong cluster forming at the left hand edge. It is also interesting to note that the behaviour of the order parameter is not monotonic in a number of cases. We have found excellent agreement with the associated SDE agent-based model, at least for short times for which the computational cost of the SDEs is reasonable; see Supplementary Material Section SM4.

### 3.3 Two Gaussians Initial Condition

To demonstrate the effect of multi-modal initial conditions, we study a linear combination of two Gaussians, or a Gaussian mixture:

$$\rho_0(y) = Z \left[ \exp \left( -C[d(y,y_{0,1})]^2 \right) + \exp \left( -C[d(y,y_{0,2})]^2 \right) \right], \tag{5}$$

where $Z$ is the normalisation constant. We note that there are many parameters in this setup, and presumably also a correspondingly large number of interesting transitions between regimes as the parameters are varied, but for conciseness, we fix $C = 80$, $y_{0,1} = 0.2$, and $\sigma = 0.03$. In the bottom right plot of Figure 4 we show the final-time density, corresponding order parameter, and equilibration time as $R$ and $y_{0,2}$ are varied. We note a general trend,
Figure 5: As Figure 3 but for a Gaussian initial condition. Labels correspond to those in the bottom left panel of Figure 4, for A ($R = 0.23, \sigma = 0.07$), B ($R = 0.23, \sigma = 0.11$), C ($R = 0.23, \sigma = 0.13$), and D ($R = 0.23, \sigma = 0.15$).
as in previous cases, that (with a few notable exceptions) increasing $R$ for fixed $y_{0,2}$ increases the tendency for consensus to form, and also for consensus to be closer to the centre of the interval. For fixed $R$, increasing $y_{0,2}$ (i.e., separating the initial clusters) tends to favour the formation of two clusters, rather than a single consensus.

We focus on three particular pairs of parameters $(y_{0,2}, R)$, denoted A (0.65,0.3), B (0.7,0.3), and C (0.75,0.3) in Figures 4 and 6. In Figure 6 we plot the results up to $t_{2}$ separating the initial clusters) tends to favour the formation of two clusters, rather than a single consensus.

Increasing $y_{0,2}$ to 0.7 (C) demonstrates an additional effect of periodic boundary conditions. The long-term behaviour reverts to a single cluster, this time centred near 1, rather than towards the centre of this interval. The cause of this is the periodic nature of the domain – there are two ‘distances’ between the initial clusters, and the shorter of the two now crosses the 0–1 point. This causes the cluster to form to the right of the initial cluster at 0.7, rather than to the left. This is an issue for interpretation in terms of extreme opinions, and suggests that if this is an aim of the model then the no-flux boundary conditions are a more appropriate choice, both for stability and interpretability.

4 Numerical Experiments: The Effect of Radicals

In this section we will introduce a radical distribution, and investigate the associated sensitivity of the dynamics. Motivated by the observation that the most interesting dynamics occur for small $\sigma$ and a uniform initial distribution, we restrict to this regime here.

4.1 Uniform Initial Condition and Gaussian Radicals

We first consider a single Gaussian radical distribution of the form

$$M \rho_r(y) = MZ \exp\left(-C|d(y, y_0)|^2\right), \quad (6)$$

where $Z$ is a normalisation constant, and $M$ determines the mass of the radical population. We now have five parameters in the system: $R$ and $\sigma$, as before, and $y_0$, $C$, and $M$, which describe the mean, width and mass of the radicals. To reduce the number of parameters, and to set the radical distribution to be relatively narrow, we fix $C = 800$.

In Figure 7 we fix two of the remaining parameters, and vary the other two: $\sigma-R \ (y_0 = 0.7, M = 0.1)$, $M-R \ (\sigma = 0.02, y_0 = 0.7)$, $y_0-\sigma \ (R = 0.1, M = 0.1)$, and $y_0-R \ (\sigma = 0.02, M = 0.1)$. The corresponding snapshots are given in Figures 8–10. The radical populations are shown in red (Figure 7), or black (Figures 8–10) although due to their small size and strongly peaked normal distributions, they are sometimes hard to distinguish.

4.1.1 $R-\sigma$ [Figures 7 (top left) and 8]

For small, fixed $R$, in all three boundary conditions we see a similar behaviour as in the case of no radicals; small $\sigma$ results in a single cluster, which is now centred around the mean of the radical distribution, whilst increasing $\sigma$ results in a uniform, or almost-uniform distribution. This is perhaps what one would expect intuitively. For larger $R$, the long-time dynamics are less intuitive. For the periodic and no-flux boundary conditions, fixing $R$ and increasing $\sigma$ causes a transition from multiple clusters (two or three) to a single one; one cluster is always centred around the radical distribution. For the even 2-periodic boundary conditions, there are three qualitatively different long-time distributions. For small sigma, two clusters emerge, as for the other boundary conditions, and for larger $\sigma$, there is once again a single cluster. However, for intermediate $\sigma$ there is a different state with a cluster centred around the radical distribution, and another narrow cluster at zero; once again this additional state is due to the attractive nature of the mirror population. As before, we observe much longer equilibration times for regions of phase space on the border of different long-time distributions.

In Figure 8 we focus on two pairs of parameters with $(R, \sigma)$ equal to (0.1,0.02) [A] and (0.1,0.03) [B]. For A, the boundary condition has a strong effect on the dynamics. The periodic case rapidly forms three clusters, corresponding to an order parameter of around 1/3. As can be seen from the white region in in Figure 7 this
Figure 6: As Figure 3 but for an initial condition which is a linear combination of Gaussians. Labels correspond to those in the bottom right panel of Figure 4, for A ($R = 0.3, y_{0.2} = 0.65$), B ($R = 0.3, y_{0.2} = 0.7$), and C ($R = 0.3, y_{0.2} = 0.75$). Additionally, we show the solutions of the corresponding SDEs with $10^4$ particles. Densities shown by circles, order parameter by dashed lines and squares, and time-space plots in the bottom row of each panel.
Figure 7: As Figure 1 but for a uniform initial condition and a Gaussian radical distribution (6). Here we vary: (top left) the strength of the noise ($\sigma$) and the size of the confidence interval ($R$); (top right) the mass of the radicals ($M$) and the confidence bound ($R$); (bottom left) the mean position of the radical distribution ($y_0$) and the strength of the noise ($\sigma$); (bottom right) the mean position of the radical distribution ($y_0$) and the confidence bound ($R$). Radical distributions are shown in red.

Simulation has not reached our definition of equilibrium, so it is possible that this is not the final state. However, slightly increasing $R$ appears to make the three-cluster state stable. In the no-flux case, there are initially four clusters, two of which are relatively weak, which eventually merge into two strong clusters, once centred around the radicals, and one at the other side of the interval. As with the other boundary conditions, this suggests that, with these parameter values, the radicals cause an initial division in the population, but do not have enough influence to cause a single consensus to form. Finally, in the even 2-periodic case, the dynamics are even more complex: this is well demonstrated by the order parameter. The four clusters at times around time $10^2$, look similar to those in the no-flux case, and once again the final state has two clusters. However, here the additional clusters persist for much longer, and the second cluster is near to zero, rather than centred at approximately 0.2.

For B, the final state for all three boundary conditions is relatively similar, with a single, strong cluster centred around the radicals. However, once again, the dynamical path to this distribution depends on the boundary conditions. In all three cases, at times up to around 100, a secondary cluster on the left of the interval is visible. In the periodic case this then rapidly merges with the final cluster, whilst in the other two cases it is longer-lived, first moving away from the final cluster, before eventually merging into it. At time 500 the secondary cluster in the no-flux case is located near zero, as in case A; here we suggest that a larger value of $\sigma$ increases the diffusion to a point at which the weaker clusters are dispersed and can be captured by the strong cluster centred around the radicals. The clusters, especially the large one, are also slightly wider with the increased value of $\sigma$, which may aid this coalescence.

4.1.2 $M$–$R$ [Figures 7 (top right) and 9 (left)]

From Figure 7, it is clear that the dependence on $M$ is very weak in the two periodic cases, unless $R$ is very small. For larger $R$, and no-flux boundary conditions, varying $M$ can produce unintuitive results. Comparing A and B on the left of 9 we see that increasing $M$ causes the long-time distribution to switch from a single cluster around the radical distribution (which is something one would expect), to a bimodal distribution, in which the increase in the
number of radicals has caused polarisation. A possible explanation for this is that a larger radical cluster attracts
the nearby normal population more quickly, causing a split in the distribution after short times; this can be seen in
both cases A and B. In case A, the two clusters move towards each other (with the one near the radicals actually
moving away from the radical population), before forming a single cluster away from the radical distribution, which
then migrates towards the radicals. In case B, it appears that the radicals are now sufficiently strong to prevent
the initial cluster around them from moving towards the centre of the interval, leaving the two polarised clusters.

4.1.3 $y_0 - \sigma$ [Figures 7 (bottom left) and 9 (right)]

As is clear from Figure 7, the results of the periodic case are independent of $y_0$, up to a shift in the opinion axis.
In the other cases, the final order parameter and shape of the distribution are essentially independent of $y_0$ for all
but small $\sigma$, or for large (or, by symmetry, small) $y_0$ when the no-flux and even 2-periodic BCs become influential.
As $y_0$ increases, there is a clear, corresponding shift in position of final-time cluster, and, as before, increasing $\sigma$
leads to disorder. Case A corresponds to Case A in Figure 8. Case B (see right of Figure 9) corresponds to slightly
larger $\sigma$ and increased $y_0$, for which the dynamics are strongly-dependent on the choice of boundary conditions.
The periodic case has a single cluster, whilst no-flux has two, which are almost the same size, and even 2-periodic
has two, but with one near zero and significantly taller.

4.1.4 $y_0 - R$ [Figures 7 (bottom right) and 10]

In Figure 7 we see that the different regimes of the no-flux and even 2-periodic cases are much richer, with multiple
parameter regions with qualitatively different long-time dynamics. As could be expected, increasing $R$, and hence
the interaction range, leads to bigger differences between the results for different boundary conditions. In Figure 10,
we show some representative dynamics, demonstrating the richness and complexity as the parameters are varied.

4.2 Uniform Initial Condition and Double-Gaussian Radicals

As a second example we consider a double-Gaussian distribution of radicals

$$M \rho_r(y) = M Z \left[ \lambda \exp \left( - C [d(y, y_{0,1})]^2 \right) + (1 - \lambda) \exp \left( - C [d(y, y_{0,2})]^2 \right) \right],$$

where, once again, $Z$ is the normalisation constant for the term in square brackets, and $M$ is the mass of the radical
distribution. Here, $\lambda \in [0,1]$ is a parameter describing the relative masses of the two Gaussians. The physical
Figure 9: As Figure 8 but for: (left) the labels in the top right panel of Figure 7 for A ($M = 0.0625$ and $R = 0.275$) and B ($M = 0.075$ and $R = 0.275$); (right) the labels in the bottom left panel of Figure 7 with B ($y_0 = 0.85$ and $\sigma = 0.025$).

Figure 10: As Figure 8 but for the labels in the bottom right panel of Figure 7, with A ($y_0 = 0.7$ and $R = 0.1$), and B ($y_0 = 0.7$ and $R = 0.3$).
interpretation here is that we have two competing groups of radicals. To simplify exposition, whilst demonstrating interesting effects, we first fix $C = 800$ (to ensure a concentrated radical distribution) and $M = 0.1$. We consider two choices of $\lambda$, namely $\lambda = 0.5$, which corresponds to a completely even split of radicals, and $\lambda = 0.499$, which is a slight bias towards one of the radical opinions.

In Figure 11, we choose $y_{0.1} = 1 - y_{0.2} = 0.2$, and vary $R$ and $\sigma$ with $\lambda = 0.5$ (left) and $\lambda = 0.499$ (right). By symmetry, the periodic and even 2-periodic cases are identical, for $\lambda = 0.5$. We note similar trends to before: increasing $R$ favours the formation of a single cluster, whilst increasing $\sigma$ tends to produce an almost uniform distribution. For the periodic and even 2-periodic cases, for moderate $R$, increasing $\sigma$ causes a transition from two clusters to a single one; we interpret this as the noise becoming large enough to overcome the attraction of the radical distribution. This does not occur in the no-flux case, at least for the range of parameters studied here. When a single cluster is formed, we see significant effects of the boundary conditions. For no-flux, the cluster is either centred in the middle of the interval ($\lambda = 0.5$), or on top of the dominant radical cluster ($\lambda = 0.499$). In contrast, the periodic and even 2-periodic cases result in clusters centred around 0/1, caused by the shortest distance between the radical clusters being across 0/1, rather than through 0.5. Physically, this means that two competing populations of radicals can result in either a moderate cluster (for no-flux), or two extreme clusters (for periodic and even 2-periodic). This is a significant effect of the boundary conditions, and suggests issues when interpreting such results without giving careful consideration to the modelling choices. Similar sensitivities with respect to the choices of $y_{0.1}$ and $y_{0.2}$ are demonstrated in Supplementary Material Section SM5.

In cases A ($\sigma = 0.035$ and $R = 0.25$) and B ($\sigma = 0.035$ and $R = 0.3$), we see that strong dependence on the choice of $\lambda$. In the periodic case with parameters A, the short-time dynamics are similar for both values of $\lambda$, but the symmetry breaking with $\lambda = 0.499$ is very clear in the long-time dynamics. For the no-flux case with parameters B, two initial clusters merge into a single cluster at the centre of the interval, with the asymmetry becoming visible only at longer times when the cluster migrates to the right with $\lambda = 0.499$. We highlight that these are far from the only interesting and non-intuitive results from the model. For parameters C ($\sigma = 0.025$ and $R = 0.1$) we show only the $\lambda = 0.5$ periodic case, as the other cases are very similar. Here we interpret the two radical clusters as competing political viewpoints. Whilst the long-time behaviour in all cases is a pair of equal clusters centred around the radicals, our interest here lies in the intermediate dynamics, where there is a clear third peak centred around zero. As discussed in [5] in the context of the Scottish independence referendum, this is a common feature of such situations in which the population mostly polarises, but leaves a number of ‘undecided’ individuals, who only move to one of the popular opinions after long times. It would be interesting to study such situations further.

5 Conclusions and Outlook

We have demonstrated the significant effect of the choice of boundary condition in SDE and PDE bounded confidence models, as well as the sensitivity of such models to small changes in various parameters, with and without the inclusion of radical distributions. In particular, the no-flux choice most faithfully recreates the underlying dynamics of the original deterministic models [28, 27]. There are many possible extensions, some of which have been studied in the literature for other models in opinion dynamics: asymmetric [28], heterogeneous [29, 41, 73], or time-dependent [45] confidence intervals; influence that is negative [57], or which increases with separation [47]: stub-
Figure 12: As Figure 3, but with a uniform initial condition and double Gaussian radical distribution. Labels correspond to parameter values in Figure 11, with A ($\sigma = 0.035$ and $R = 0.25$). B ($\sigma = 0.035$ and $R = 0.3$). C ($\sigma = 0.025$ and $R = 0.1$).

6 Data Availability

The supplementary material is available at https://www.maths.ed.ac.uk/~bgoddard/files/OD-SupplementaryMaterial.pdf

The full code used to produce the figures in the manuscript and Supplementary Material is available from https://bitbucket.org/bdgoddard/2dchebclassod_public/src/master/

7 Acknowledgments

The work of GAP was partially funded by the EPSRC, grant number EP/P031587/1, and by JP- Morgan Chase & Co. Any views or opinions expressed herein are solely those of the authors listed, and may differ from the views and opinions expressed by JPMorgan Chase & Co. or its affiliates. This material is not a product of the Research Department of J.P. Morgan Securities LLC. This material does not constitute a solicitation or offer in any jurisdiction.

BDG is grateful to Andrew Archer, Valerio Restocchi, and David Sibley for helpful discussions.

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