
APPLIED STOCHASTIC PROCESSES

LECTURES 3/4

INTRODUCTION TO THE THEORY OF MARKOV PROCESSES

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MARKOV STOCHASTIC PROCESSES

Definition of a Markov Process

- Let (Ω, \mathcal{F}) be a measurable space and T an ordered set. Let $X = X_t(\omega)$ be a stochastic process from the sample space (Ω, \mathcal{F}) to the state space (E, \mathcal{G}) . It is a function of two variables, $t \in T$ and $\omega \in \Omega$.
 - For a fixed $\omega \in \Omega$ the function $X_t(\omega)$, $t \in T$ is the **sample path** of the process X associated with ω .
 - Let \mathcal{K} be a collection of subsets of Ω . The smallest σ -algebra on Ω which contains \mathcal{K} is denoted by $\sigma(\mathcal{K})$ and is called the **σ -algebra generated** by \mathcal{K} .
 - Let $X_t : \Omega \mapsto E$, $t \in T$. The smallest σ -algebra $\sigma(X_t, t \in T)$, such that the family of mappings $\{X_t, t \in T\}$ is a stochastic process with sample space $(\Omega, \sigma(X_t, t \in T))$ and state space (E, \mathcal{G}) , is called the **σ -algebra generated by** $\{X_t, t \in T\}$.
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Definition of a Markov Process

- A **filtration** on (Ω, \mathcal{F}) is a nondecreasing family $\{\mathcal{F}_t, t \in T\}$ of sub- σ -algebras of \mathcal{F} : $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$.
- We set $\mathcal{F}_\infty = \sigma(\cup_{t \in T} \mathcal{F}_t)$. The **filtration generated by X_t** , where X_t is a stochastic process, is

$$\mathcal{F}_t^X := \sigma(X_s; s \leq t).$$

- We say that a stochastic process $\{X_t; t \in T\}$ is **adapted** to the filtration $\{\mathcal{F}_t\} := \{\mathcal{F}_t, t \in T\}$ if for all $t \in T$, X_t is an \mathcal{F}_t -measurable random variable.

Definition 1 *Let $\{X_t\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with values in E and let \mathcal{F}_t^X be the filtration generated by $\{X_t\}$. Then $\{X_t\}$ is a **Markov process** if*

$$\mathbb{P}(X_t \in \Gamma | \mathcal{F}_s^X) = \mathbb{P}(X_t \in \Gamma | X_s) \quad (1)$$

for all $t, s \in T$ with $t \geq s$, and $\Gamma \in \mathcal{B}(E)$.

Definition of a Markov Process

- The filtration \mathcal{F}_t^X is generated by events of the form $\{\omega | X_{s_1} \in B_1, X_{s_2} \in B_2, \dots, X_{s_n} \in B_n, \}$ with $0 \leq s_1 < s_2 < \dots < s_n \leq s$ and $B_i \in \mathcal{B}(E)$. The definition of a Markov process is thus equivalent to the hierarchy of equations

$$\mathbb{P}(X_t \in \Gamma | X_{t_1}, X_{t_2}, \dots, X_{t_n}) = \mathbb{P}(X_t \in \Gamma | X_{t_n}) \quad \text{a.s.}$$

for $n \geq 1$, $0 \leq t_1 < t_2 < \dots < t_n \leq t$ and $\Gamma \in \mathcal{B}(E)$.

Definition of a Markov Process

- Roughly speaking, the statistics of X_t for $t \geq s$ are completely determined once X_s is known; information about X_t for $t < s$ is superfluous. In other words: **a Markov process has no memory**. More precisely: when a Markov process is conditioned on the present state, then there is no memory of the past. *The past and future of a Markov process are statistically independent when the present is known.*
 - A typical example of a Markov process is the random walk: in order to find the position $x(t+1)$ of the random walker at time $t+1$ it is enough to know its position $x(t)$ at time t : how it got to $x(t)$ is irrelevant.
 - A non Markovian process X_t can be described through a Markovian one Y_t by enlarging the state space: the additional variables that we introduce account for the memory in the X_t . This "Markovianization" trick is very useful since there are many more tools for analyzing Markovian process.
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The Chapman-Kolmogorov Equation

- With a Markov process $\{X_t\}$ we can associate a function $P : T \times T \times E \times \mathcal{B}(E) \rightarrow \mathbb{R}^+$ defined through the relation

$$\mathbb{P} [X_t \in \Gamma | \mathcal{F}_s^X] = P(s, t, X_s, \Gamma),$$

for all $t, s \in T$ with $t \geq s$ and all $\Gamma \in \mathcal{B}(E)$.

- Assume that $X_s = x$. Since $\mathbb{P} [X_t \in \Gamma | \mathcal{F}_s^X] = \mathbb{P} [X_t \in \Gamma | X_s]$ we can write

$$P(\Gamma, t|x, s) = \mathbb{P} [X_t \in \Gamma | X_s = x].$$

- The **transition function** $P(t, \Gamma|x, s)$ is (for fixed t, x, s) a probability measure on E with $P(t, E|x, s) = 1$; it is $\mathcal{B}(E)$ -measurable in x (for fixed t, s, Γ) and satisfies the **Chapman-Kolmogorov equation**

$$P(\Gamma, t|x, s) = \int_E P(\Gamma, t|y, u)P(dy, u|x, s). \quad (2)$$

- for all $x \in E, \Gamma \in \mathcal{B}(E)$ and $s, u, t \in T$ with $s \leq u \leq t$.
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The Chapman-Kolmogorov Equation

- The derivation of the Chapman-Kolmogorov equation is based on the assumption of Markovianity and on properties of the conditional probability:

i) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, X a random variable from $(\Omega, \mathcal{F}, \mu)$ to (E, \mathcal{G}) and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$. Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2) = \mathbb{E}(X|\mathcal{F}_1). \quad (3)$$

ii) Given $\mathcal{G} \subset \mathcal{F}$ we define the function

$P_X(B|\mathcal{G}) = P(X \in B|\mathcal{G})$ for $B \in \mathcal{F}$. Assume that f is such that $\mathbb{E}(f(X)) < \infty$. Then

$$\mathbb{E}(f(X)|\mathcal{G}) = \int_{\mathbb{R}} f(x)P_X(dx|\mathcal{G}). \quad (4)$$

The Chapman-Kolmogorov Equation

Now we use the Markov property, together with equations (3) and (4) and the fact that $s < u \Rightarrow \mathcal{F}_s^X \subset \mathcal{F}_u^X$ to calculate:

$$\begin{aligned} P(\Gamma, t|x, s) &:= \mathbb{P}(X_t \in \Gamma | X_s = x) = \mathbb{P}(X_t \in \Gamma | \mathcal{F}_s^X) \\ &= \mathbb{E}(I_\Gamma(X_t) | \mathcal{F}_s^X) = \mathbb{E}(\mathbb{E}(I_\Gamma(X_t) | \mathcal{F}_s^X) | \mathcal{F}_u^X) \\ &= \mathbb{E}(\mathbb{E}(I_\Gamma(X_t) | \mathcal{F}_u^X) | \mathcal{F}_s^X) = \mathbb{E}(\mathbb{P}(X_t \in \Gamma | X_u) | \mathcal{F}_s^X) \\ &= \mathbb{E}(\mathbb{P}(X_t \in \Gamma | X_u = y) | X_s = x) \\ &= \int_{\mathbb{R}} P(\Gamma, t | X_u = y) P(dy, u | X_s = x) \\ &=: \int_{\mathbb{R}} P(\Gamma, t | y, u) P(dy, u | x, s). \end{aligned}$$

$I_\Gamma(\cdot)$ denotes the indicator function of the set Γ . We have also set $E = \mathbb{R}$.

The Chapman-Kolmogorov Equation

- The CK equation is an integral equation and is the fundamental equation in the theory of Markov processes. Under additional assumptions we will derive from it the Fokker-Planck PDE, which is the fundamental equation in the theory of diffusion processes, and will be the main object of study in this course.
- A Markov process is **homogeneous** if

$$P(t, \Gamma | X_s = x) := P(s, t, x, \Gamma) = P(0, t - s, x, \Gamma).$$

- We set $P(0, t, \cdot, \cdot) = P(t, \cdot, \cdot)$. The Chapman–Kolmogorov (CK) equation becomes

$$P(t + s, x, \Gamma) = \int_E P(s, x, dz) P(t, z, \Gamma). \quad (5)$$

The Chapman-Kolmogorov Equation

- Let X_t be a homogeneous Markov process and assume that the **initial distribution** of X_t is given by the probability measure $\nu(\Gamma) = P(X_0 \in \Gamma)$ (for deterministic initial conditions— $X_0 = x$ —we have that $\nu(\Gamma) = I_\Gamma(x)$).
- The transition function $P(x, t, \Gamma)$ and the initial distribution ν determine the finite dimensional distributions of X by

$$\begin{aligned} & \mathbb{P}(X_0 \in \Gamma_0, X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) \\ &= \int_{\Gamma_0} \int_{\Gamma_1} \dots \int_{\Gamma_{n-1}} P(t_n - t_{n-1}, y_{n-1}, \Gamma_n) P(t_{n-1} - t_{n-2}, y_{n-2}, dy_{n-1}) \\ & \dots \times P(t_1, y_0, dy_1) \nu(dy_0). \end{aligned} \tag{6}$$

Theorem 1 (*Ethier and Kurtz 1986, Sec. 4.1*) Let $P(t, x, \Gamma)$ satisfy (5) and assume that (E, ρ) is a complete separable metric space. Then there exists a Markov process X in E whose finite-dimensional distributions are uniquely determined by (6).

The Chapman-Kolmogorov Equation

- Let X_t be a homogeneous Markov process with initial distribution $\nu(\Gamma) = P(X_0 \in \Gamma)$ and transition function $P(x, t, \Gamma)$. We can calculate the probability of finding X_t in a set Γ at time t :

$$\mathbb{P}(X_t \in \Gamma) = \int_E P(x, t, \Gamma) \nu(dx).$$

- Thus, the initial distribution and the transition function are sufficient to characterize a homogeneous Markov process.
 - Notice that they do not provide us with any information about the actual paths of the Markov process.
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The Chapman-Kolmogorov Equation

- The transition probability $P(\Gamma, t|x, s)$ is a probability measure. Assume that it has a density for all $t > s$:

$$P(\Gamma, t|x, s) = \int_{\Gamma} p(y, t|x, s) dy.$$

- Clearly, for $t = s$ we have $P(\Gamma, s|x, s) = I_{\Gamma}(x)$.
- The Chapman-Kolmogorov equation becomes:

$$\int_{\Gamma} p(y, t|x, s) dy = \int_{\mathbb{R}} \int_{\Gamma} p(y, t|z, u)p(z, u|x, s) dz dy,$$

- and, since $\Gamma \in \mathcal{B}(\mathbb{R})$ is arbitrary, we obtain the equation

$$p(y, t|x, s) = \int_{\mathbb{R}} p(y, t|z, u)p(z, u|x, s) dz. \quad (7)$$

- The transition probability density is a function of 4 arguments: the initial position and time x, s and the final position and time y, t .
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The Chapman-Kolmogorov Equation

- In words, the CK equation tells us that, for a Markov process, the transition from x, s to y, t can be done in two steps: first the system moves from x to z at some intermediate time u . Then it moves from z to y at time t . In order to calculate the probability for the transition from (x, s) to (y, t) we need to sum (integrate) the transitions from all possible intermediary states z .
- The above description suggests that a Markov process can be described through a **semigroup** of operators, i.e. a one-parameter family of linear operators with the properties

$$P_0 = I, \quad P_{t+s} = P_t \circ P_s \quad \forall t, s \geq 0.$$

- A semigroup of operators is characterized through its **generator**.
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The Chapman-Kolmogorov Equation

- Indeed, let $P(t, x, dy)$ be the transition function of a homogeneous Markov process. It satisfies the CK equation (5):

$$P(t + s, x, \Gamma) = \int_E P(s, x, dz)P(t, z, \Gamma).$$

- Let $X := C_b(E)$ and define the operator

$$(P_t f)(x) := \mathbb{E}(f(X_t) | X_0 = x) = \int_E f(y)P(t, x, dy).$$

- This is a linear operator with

$$(P_0 f)(x) = \mathbb{E}(f(X_0) | X_0 = x) = f(x) \Rightarrow P_0 = I.$$

The Chapman-Kolmogorov Equation

- Furthermore:

$$\begin{aligned}(P_{t+s}f)(x) &= \int f(y)P(t+s, x, dy) \\ &= \int \int f(y)P(s, z, dy)P(t, x, dz) \\ &= \int \left(\int f(y)P(s, z, dy) \right) P(t, x, dz) \\ &= \int (P_s f)(z)P(t, x, dz) \\ &= (P_t \circ P_s f)(x).\end{aligned}$$

- Consequently:

$$P_{t+s} = P_t \circ P_s.$$

The Generator of a Markov Processes

- Let (E, ρ) be a metric space and let $\{X_t\}$ be an E -valued homogeneous Markov process. Define the one parameter family of operators P_t through

$$P_t f(x) = \int f(y) P(t, x, dy) = \mathbb{E}[f(X_t) | X_0 = x]$$

for all $f(x) \in C_b(E)$ (continuous bounded functions on E).

- Assume for simplicity that $P_t : C_b(E) \rightarrow C_b(E)$. Then the one-parameter family of operators P_t forms a **semigroup** of operators on $C_b(E)$.
- We define by $\mathcal{D}(\mathcal{L})$ the set of all $f \in C_b(E)$ such that the strong limit

$$\mathcal{L}f = \lim_{t \rightarrow 0} \frac{P_t f - f}{t},$$

exists.

The Generator of a Markov Processes

Definition 2 *The operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \rightarrow C_b(E)$ is called the **infinitesimal generator** of the operator semigroup P_t .*

Definition 3 *The operator $\mathcal{L} : C_b(E) \rightarrow C_b(E)$ defined above is called the **generator of the Markov process** $\{X_t\}$.*

- The study of operator semigroups started in the late 40's independently by Hille and Yosida. Semigroup theory was developed in the 50's and 60's by Feller, Dynkin and others, mostly in connection to the theory of Markov processes.
 - Necessary and sufficient conditions for an operator \mathcal{L} to be the generator of a (contraction) semigroup are given by the Hille-Yosida theorem (e.g. Evans Partial Differential Equations, AMS 1998, Ch. 7).
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The Generator of a Markov Processes

- The semigroup property and the definition of the generator of a semigroup imply that, formally at least, we can write:

$$P_t = \exp(\mathcal{L}t).$$

- Consider the function $u(x, t) := (P_t f)(x)$. We calculate its time derivative:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{d}{dt}(P_t f) = \frac{d}{dt}(e^{\mathcal{L}t} f) \\ &= \mathcal{L}(e^{\mathcal{L}t} f) = \mathcal{L}P_t f = \mathcal{L}u. \end{aligned}$$

- Furthermore, $u(x, 0) = P_0 f(x) = f(x)$. Consequently, $u(x, t)$ satisfies the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \quad u(x, 0) = f(x). \quad (8)$$

The Generator of a Markov Processes

- When the semigroup P_t is the transition semigroup of a Markov process X_t , then equation (8) is called the **backward Kolmogorov equation**. It governs the evolution of an **observable**

$$u(x, t) = \mathbb{E}(f(X_t) | X_0 = x).$$

- Thus, given the generator of a Markov process \mathcal{L} , we can calculate all the statistics of our process by solving the backward Kolmogorov equation.
 - In the case where the Markov process is the solution of a stochastic differential equation, then the generator is a second order elliptic operator and the backward Kolmogorov equation becomes an initial value problem for a parabolic PDE.
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The Generator of a Markov Processes

- The space $C_b(E)$ is natural in a probabilistic context, but other Banach spaces often arise in applications; in particular when there is a measure μ on E , the spaces $L^p(E; \mu)$ sometimes arise. We will quite often use the space $L^2(E; \mu)$, where μ will be the **invariant measure** of our Markov process.
- The generator is frequently taken as the starting point for the definition of a homogeneous Markov process.
- Conversely, let P_t be a **contraction semigroup** (Let X be a Banach space and $T : X \rightarrow X$ a bounded operator. Then T is a contraction provided that $\|Tf\|_X \leq \|f\|_X \forall f \in X$), with $\mathcal{D}(P_t) \subset C_b(E)$, closed. Then, under mild technical hypotheses, there is an E -valued homogeneous Markov process $\{X_t\}$ associated with P_t defined through

$$\mathbb{E}[f(X(t)|\mathcal{F}_s^X)] = P_{t-s}f(X(s))$$

for all $t, s \in T$ with $t \geq s$ and $f \in \mathcal{D}(P_t)$.

The Generator of a Markov Processes

Example 4 *The Poisson process is a homogeneous Markov process.*

Example 5 *The one dimensional Brownian motion is a homogeneous Markov process. The transition function is the Gaussian defined in the example in Lecture 2:*

$$P(t, x, dy) = \gamma_{t,x}(y)dy, \quad \gamma_{t,x}(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

The semigroup associated to the standard Brownian motion is the heat semigroup $P_t = e^{\frac{t}{2} \frac{d^2}{dx^2}}$. The generator of this Markov process is $\frac{1}{2} \frac{d^2}{dx^2}$.

- Notice that the **transition probability density** $\gamma_{t,x}$ of the one dimensional Brownian motion is the fundamental solution (Green's function) of the heat (diffusion) PDE

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}.$$

The Adjoint Semigroup

- The semigroup P_t acts on bounded measurable functions.
- We can also define the adjoint semigroup P_t^* which acts on probability measures:

$$P_t^* \mu(\Gamma) = \int_{\mathbb{R}} \mathbb{P}(X_t \in \Gamma | X_0 = x) d\mu(x) = \int_{\mathbb{R}} p(t, x, \Gamma) d\mu(x).$$

- The image of a probability measure μ under P_t^* is again a probability measure. The operators P_t and P_t^* are adjoint in the L^2 -sense:

$$\int_{\mathbb{R}} P_t f(x) d\mu(x) = \int_{\mathbb{R}} f(x) d(P_t^* \mu)(x). \quad (9)$$

The Adjoint Semigroup

- We can, formally at least, write

$$P_t^* = \exp(\mathcal{L}^* t),$$

- where \mathcal{L}^* is the L^2 -adjoint of the generator of the process:

$$\int \mathcal{L} f h \, dx = \int f \mathcal{L}^* h \, dx.$$

- Let $\mu_t := P_t^* \mu$. This is the **law** of the Markov process and μ is the initial distribution. An argument similar to the one used in the derivation of the backward Kolmogorov equation (8) enables us to obtain an equation for the evolution of μ_t :

$$\frac{\partial \mu_t}{\partial t} = \mathcal{L}^* \mu_t, \quad \mu_0 = \mu.$$

The Adjoint Semigroup

- Assuming that $\mu_t = \rho(y, t) dy$, $\mu = \rho_0(y) dy$ this equation becomes:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^* \rho, \quad \rho(y, 0) = \rho_0(y). \quad (10)$$

- This is the **forward Kolmogorov** or **Fokker-Planck** equation. When the initial conditions are deterministic, $X_0 = x$, the initial condition becomes $\rho_0 = \delta(y - x)$.
- Given the initial distribution and the generator of the Markov process X_t , we can calculate the transition probability density by solving the Forward Kolmogorov equation. We can then calculate all statistical quantities of this process through the formula

$$\mathbb{E}(f(X_t) | X_0 = x) = \int f(y) \rho(t, y; x) dy.$$

- We will derive rigorously the backward and forward Kolmogorov equations for Markov processes that are defined as solutions of stochastic differential equations later on.
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The Adjoint Semigroup

- We can study the evolution of a Markov process in two different ways:
- Either through the evolution of **observables** (Heisenberg/Koopman)

$$\frac{\partial(P_t f)}{\partial t} = \mathcal{L}(P_t f),$$

- or through the evolution of **states** (Schrödinger/Frobenius-Perron)

$$\frac{\partial(P_t^* \mu)}{\partial t} = \mathcal{L}^*(P_t^* \mu).$$

- We can also study Markov processes at the level of trajectories. We will do this after we define the concept of a stochastic differential equation.
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Ergodic Markov processes

- A very important concept in the study of limit theorems for stochastic processes is that of **ergodicity**.
- This concept, in the context of Markov processes, provides us with information on the long-time behavior of a Markov semigroup.

Definition 6 *A Markov process is called **ergodic** if the equation*

$$P_t g = g, \quad g \in C_b(E) \quad \forall t \geq 0$$

has only constant solutions.

- Roughly speaking, ergodicity corresponds to the case where the semigroup P_t is such that $P_t - I$ has only constants in its null space, or, equivalently, to the case where the generator \mathcal{L} has only constants in its null space. This follows from the definition of the generator of a Markov process.
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Ergodic Markov processes

- Under some additional compactness assumptions, an ergodic Markov process has an *invariant measure* μ with the property that, in the case $T = \mathbb{R}^+$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t g(X_s) ds = \mathbb{E}g(x),$$

- where \mathbb{E} denotes the expectation with respect to μ .
- This is a physicist's definition of an ergodic process: **time averages equal phase space averages.**
- Using the adjoint semigroup we can define an invariant measure as the solution of the equation

$$P_t^* \mu = \mu.$$

- If this measure is unique, then the Markov process is ergodic.
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Ergodic Markov processes

- Using this, we can obtain an equation for the invariant measure in terms of the adjoint of the generator L^* , which is the generator of the semigroup P_t^* . Indeed, from the definition of the generator of a semigroup and the definition of an invariant measure, we conclude that a measure μ is invariant if and only if

$$\mathcal{L}^* \mu = 0$$

- in some appropriate generalized sense ($(\mathcal{L}^* \mu, f) = 0$ for every bounded measurable function).
- Assume that $\mu(dx) = \rho(x) dx$. Then the **invariant density** satisfies the **stationary Fokker-Planck equation**

$$\mathcal{L}^* \rho = 0.$$

- The invariant measure (distribution) governs the long-time dynamics of the Markov process.
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Stationary Markov Processes

- If X_0 is distributed according to μ , then so is X_t for all $t > 0$. The resulting stochastic process, with X_0 distributed in this way, is **stationary** .
 - In this case the transition probability density (the solution of the Fokker-Planck equation) is independent of time:
$$\rho(x, t) = \rho(x).$$
 - Consequently, the statistics of the Markov process is independent of time.
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Stationary Markov Processes

Example 7 *The one dimensional Brownian motion is not an ergodic process: The null space of the generator $\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2}$ on \mathbb{R} is not one dimensional!*

Example 8 *Consider a one-dimensional Brownian motion on $[0, 1]$, with periodic boundary conditions. The generator of this Markov process \mathcal{L} is the differential operator $\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2}$, equipped with periodic boundary conditions on $[0, 1]$. This operator is self-adjoint. The null space of both \mathcal{L} and \mathcal{L}^* comprises constant functions on $[0, 1]$. Both the backward Kolmogorov and the Fokker-Planck equation reduce to the heat equation*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2}$$

with periodic boundary conditions in $[0, 1]$. Fourier analysis shows that the solution converges to a constant at an exponential rate.

Stationary Markov Processes

Example 9 • *The one dimensional **Ornstein-Uhlenbeck (OU) process** is a Markov process with generator*

$$\mathcal{L} = -\alpha x \frac{d}{dx} + D \frac{d^2}{dx^2}.$$

- *The null space of \mathcal{L} comprises constants in x . Hence, it is an ergodic Markov process. In order to calculate the invariant measure we need to solve the stationary Fokker–Planck equation:*

$$\mathcal{L}^* \rho = 0, \quad \rho \geq 0, \quad \|\rho\|_{L_1(\mathbb{R})} = 1. \quad (11)$$

- *Let us calculate the L^2 -adjoint of \mathcal{L} . Assuming that f, h decay sufficiently fast at infinity, we have:*

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{L} f h \, dx &= \int_{\mathbb{R}} [(-\alpha x \partial_x f) h + (D \partial_x^2 f) h] \, dx \\ &= \int_{\mathbb{R}} [f \partial_x (\alpha x h) + f (D \partial_x^2 h)] \, dx =: \int_{\mathbb{R}} f \mathcal{L}^* h \, dx, \end{aligned}$$

Stationary Markov Processes

- *where*

$$\mathcal{L}^* h := \frac{d}{dx} (axh) + D \frac{d^2 h}{dx^2}.$$

- *We can calculate the invariant distribution by solving equation (11).*
- *The invariant measure of this process is the Gaussian measure*

$$\mu(dx) = \sqrt{\frac{\alpha}{2\pi D}} \exp\left(-\frac{\alpha}{2D} x^2\right) dx.$$

- *If the initial condition of the OU process is distributed according to the invariant measure, then the OU process is a stationary Gaussian process.*
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Stationary Markov Processes

- Let X_t be the 1d OU process and let $X_0 \sim \mathcal{N}(0, D/\alpha)$. Then X_t is a mean zero, Gaussian second order stationary process on $[0, \infty)$ with correlation function

$$R(t) = \frac{D}{\alpha} e^{-\alpha|t|}$$

and spectral density

$$f(x) = \frac{D}{\pi} \frac{1}{x^2 + \alpha^2}.$$

Furthermore, the OU process is **the only** real-valued mean zero Gaussian second-order stationary Markov process defined on \mathbb{R} .
