Proof of the Jacobi Identity

First, we establish a relationship for later use: Let f, g be functions $f, g \in \{u, v, w\}$ with $f \not\equiv g$ and $a \in \{p_1, ..., p_N, q_1, ..., q_N\}$ such that f and g depend partially on a.

$$\begin{aligned}\frac{\partial}{\partial a}\{f, g\} &= \frac{\partial}{\partial a}\sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)\\ &= \sum_{i=1}^{N}\left(\frac{\partial^2 f}{\partial a \partial q_i}\frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial a \partial p_i} - \frac{\partial^2 f}{\partial a \partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial a \partial q_i}\right)\\ &= \left\{\frac{\partial f}{\partial a}, g\right\} + \left\{f, \frac{\partial g}{\partial a}\right\} \end{aligned} \tag{†}$$

Now we can prove the Jacobi identity:

$J = \{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\}$

expand the last term...

$$= \{u, \{v, w\}\} + \{v, \{w, u\}\} + \left\{w, \sum_{i=1}^{N}\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right\}$$

by linearity...

$$= \{u, \{v, w\}\} + \{v, \{w, u\}\} + \sum_{i=1}^{N}\left(\left\{w, \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right\} - \left\{w, \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right\}\right)$$

by the product rule...

$$= \{u, \{v, w\}\} + \{v, \{w, u\}\} + \sum_{i=1}^{N}\left(\frac{\partial u}{\partial q_i}\left\{w, \frac{\partial v}{\partial p_i}\right\} + \frac{\partial v}{\partial p_i}\left\{w, \frac{\partial u}{\partial q_i}\right\} - \frac{\partial u}{\partial p_i}\left\{w, \frac{\partial v}{\partial q_i}\right\} - \frac{\partial v}{\partial q_i}\left\{w, \frac{\partial u}{\partial p_i}\right\}\right)$$

by †

$$\begin{aligned} = \{u, \{v, w\}\} + \{v, \{w, u\}\} + \sum_{i=1}^{N}\Bigg(&\frac{\partial u}{\partial q_i}\left(\underbrace{\frac{\partial}{\partial p_i}\{w, v\}}_{A} - \left\{\frac{\partial w}{\partial p_i}, v\right\}\right) + \frac{\partial v}{\partial p_i}\left(\underbrace{\frac{\partial}{\partial q_i}\{w, u\}}_{B} - \left\{\frac{\partial w}{\partial q_i}, u\right\}\right)\\ &- \frac{\partial u}{\partial p_i}\left(\underbrace{\frac{\partial}{\partial q_i}\{w, v\}}_{A} - \left\{\frac{\partial w}{\partial q_i}, v\right\}\right) - \frac{\partial v}{\partial q_i}\left(\underbrace{\frac{\partial}{\partial p_i}\{w, u\}}_{B} - \left\{\frac{\partial w}{\partial p_i}, u\right\}\right)\Bigg) \end{aligned}$$

grouping terms A and terms B...

$$= \cancel{\{u, \{v, w\}\}} + \cancel{\{v, \{w, u\}\}} - \overbrace{\cancel{\{u, \{v, w\}\}}}^{A} - \overbrace{\cancel{\{v, \{w, u\}\}}}^{B} + \sum_{i=1}^{N}\left(-\frac{\partial u}{\partial q_i}\left\{\frac{\partial w}{\partial p_i}, v\right\} - \frac{\partial v}{\partial p_i}\left\{\frac{\partial w}{\partial q_i}, u\right\} + \frac{\partial u}{\partial p_i}\left\{\frac{\partial w}{\partial q_i}, v\right\} + \frac{\partial v}{\partial q_i}\left\{\frac{\partial w}{\partial p_i}, u\right\}\right)$$

expanding these remaining terms...

$$\begin{aligned} = &-\underbrace{\sum_{i,j=1}^{N}\frac{\partial u}{\partial q_i}\frac{\partial^2 w}{\partial p_i \partial q_j}\frac{\partial v}{\partial p_j}}_{C} + \underbrace{\sum_{i,j=1}^{N}\frac{\partial u}{\partial q_i}\frac{\partial^2 w}{\partial p_i \partial p_j}\frac{\partial v}{\partial q_j}}_{D} - \underbrace{\sum_{i,j=1}^{N}\frac{\partial v}{\partial p_i}\frac{\partial^2 w}{\partial q_i \partial q_j}\frac{\partial u}{\partial p_j}}_{E} + \underbrace{\sum_{i,j=1}^{N}\frac{\partial v}{\partial p_i}\frac{\partial^2 w}{\partial q_i \partial p_j}\frac{\partial u}{\partial q_j}}_{C}\\ &+ \underbrace{\sum_{i,j=1}^{N}\frac{\partial u}{\partial p_i}\frac{\partial^2 w}{\partial q_i \partial q_j}\frac{\partial v}{\partial p_j}}_{E} - \underbrace{\sum_{i,j=1}^{N}\frac{\partial u}{\partial p_i}\frac{\partial^2 w}{\partial q_i \partial p_j}\frac{\partial v}{\partial q_j}}_{F} + \underbrace{\sum_{i,j=1}^{N}\frac{\partial v}{\partial q_i}\frac{\partial^2 w}{\partial p_i \partial q_j}\frac{\partial u}{\partial p_j}}_{F} - \underbrace{\sum_{i,j=1}^{N}\frac{\partial v}{\partial q_i}\frac{\partial^2 w}{\partial p_i \partial p_j}\frac{\partial u}{\partial q_j}}_{D} \end{aligned}$$

We note that since each term is summed over all i, j, then each term is symmetric in i, j. Hence each pair of terms C, D, E, F cancels and we get

$$J \equiv 0$$

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