Proof of the Jacobi Identity

First, we establish a relationship for later use: Let \( f, g \) be functions \( f, g \in \{ u, v, w \} \) with \( f \neq g \) and \( a \in \{ p_1, \ldots, p_N, q_1, \ldots, q_N \} \) such that \( f \) and \( g \) depend partially on \( a \).

\[
\frac{\partial}{\partial a} \{ f, g \} = \frac{\partial}{\partial a} \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)
\]

\[
= \sum_{i=1}^{N} \left( \frac{\partial^2 f}{\partial a \partial q_i} \frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial q_i} \frac{\partial^2 g}{\partial a \partial p_i} - \frac{\partial^2 f}{\partial a \partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial a \partial q_i} \right)
\]

\[
= \left\{ \frac{\partial f}{\partial a}, g \right\} + \left\{ f, \frac{\partial g}{\partial a} \right\}
\]

\((\dagger)\)

Now we can prove the Jacobi identity:

\[ J = \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \{ w, \{ u, v \} \} \]

expand the last term...

\[
= \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \left\{ w, \sum_{i=1}^{N} \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right\}
\]

by linearity...

\[
= \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \sum_{i=1}^{N} \left( \left\{ w, \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} \right\} - \left\{ w, \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right\} \right)
\]

by the product rule...

\[
= \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \sum_{i=1}^{N} \left( \frac{\partial u}{\partial q_i} \{ w, \frac{\partial v}{\partial p_i} \} + \frac{\partial v}{\partial p_i} \{ w, \frac{\partial u}{\partial q_i} \} - \frac{\partial u}{\partial p_i} \{ w, \frac{\partial v}{\partial q_i} \} - \frac{\partial v}{\partial q_i} \{ w, \frac{\partial u}{\partial p_i} \} \right)
\]

by \((\dagger)\)

\[
= \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \sum_{i=1}^{N} \left( \frac{\partial u}{\partial q_i} \left( \frac{\partial}{\partial p_i} \{ w, v \} - \left\{ \frac{\partial w}{\partial p_i}, v \right\} \right) + \frac{\partial v}{\partial p_i} \left( \frac{\partial}{\partial q_i} \{ w, u \} - \left\{ \frac{\partial w}{\partial q_i}, u \right\} \right) \right)
\]

grouping terms \( A \) and terms \( B \)...

\[
= \{ u, \{ v, w \} \} + \{ v, \{ w, u \} \} + \sum_{i=1}^{N} \left( \frac{\partial u}{\partial q_i} \left( \frac{\partial}{\partial p_i} \{ w, v \} - \left\{ \frac{\partial w}{\partial p_i}, v \right\} \right) - \frac{\partial v}{\partial q_i} \left( \frac{\partial}{\partial p_i} \{ w, u \} - \left\{ \frac{\partial w}{\partial p_i}, u \right\} \right) \right)
\]

expanding these remaining terms...

\[
= - \sum_{i,j=1}^{N} \frac{\partial u}{\partial q_i} \frac{\partial^2 w}{\partial q_j \partial p_j} \frac{\partial v}{\partial p_j} + \sum_{i,j=1}^{N} \frac{\partial u}{\partial q_i} \frac{\partial^2 w}{\partial p_i \partial q_j} \frac{\partial v}{\partial q_j} - \sum_{i,j=1}^{N} \frac{\partial v}{\partial q_i} \frac{\partial^2 w}{\partial q_j \partial p_j} \frac{\partial u}{\partial p_j} + \sum_{i,j=1}^{N} \frac{\partial v}{\partial q_i} \frac{\partial^2 w}{\partial p_i \partial q_j} \frac{\partial u}{\partial q_j}
\]

\[
+ \sum_{i,j=1}^{N} \frac{\partial u}{\partial p_i} \frac{\partial^2 w}{\partial q_j \partial p_j} \frac{\partial v}{\partial p_j} - \sum_{i,j=1}^{N} \frac{\partial u}{\partial p_i} \frac{\partial^2 w}{\partial p_i \partial q_j} \frac{\partial v}{\partial q_j} + \sum_{i,j=1}^{N} \frac{\partial v}{\partial p_i} \frac{\partial^2 w}{\partial q_j \partial p_j} \frac{\partial u}{\partial p_j} - \sum_{i,j=1}^{N} \frac{\partial v}{\partial p_i} \frac{\partial^2 w}{\partial p_i \partial q_j} \frac{\partial u}{\partial q_j}
\]

We note that since each term is summed over all \( i, j \), then each term is symmetric in \( i, j \). Hence each pair of terms \( C, D, E, F \) cancels and we get

\[ J = 0 \]