EE2 Mathematics: Fourier and Laplace Transforms

http://www2.imperial.ac.uk/~nsjones/teaching.htm

These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes — you are therefore advised to attend lectures and take your own.

1. The material in them is dependent upon the material on complex variables in the second part of this course.
2. Handouts are:
   (a) on Fourier Transforms and a list of functions;
   (b) on Laplace Transforms.

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Fourier Transforms

1.1 Introduction

There are three definitions of the Fourier Transform (FT) of a function $f(t)$ – see Appendix A. The one used here, which is consistent with that used in your own Department, is

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt.$$  \hspace{1cm} (1.1)

where the frequency $\omega$ is real. Another common notation is to write $F(\omega)$ or $\mathcal{F}(\omega)$ for $\mathcal{F}(\omega)$. Given the spectrum $\mathcal{F}(\omega)$ the function $f(t)$ can be recovered through the inverse transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} \, d\omega.$$ \hspace{1cm} (1.2)

Note the factor of $1/2\pi$ in the coefficient. The interplay between the function of time $f(t)$ (or a sampled time series) and the FT $\mathcal{F}(\omega)$ is subtle. Clearly, it is possible that functions $f(t)$ could be chosen for which the integral (1.1) is infinite – which means that this transform does not exist. There are two conditions that must be satisfied for the FT to exist:

(i) $f(t)$ must be absolutely integrable: that is

$$\int_{-\infty}^{\infty} |f(t)| \, dt < \infty.$$ \hspace{1cm} (1.3)

(ii) If $f(t)$ has discontinuities then it must be finite at these.

The following is a list of common functions:

1. The sign-function

$$\text{sgn}(t) = \begin{cases} -1 & t \leq 0 \\ +1 & t \geq 0 \end{cases}.$$ \hspace{1cm} (1.4)

2. The triangle or tent function:

$$\Lambda(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 1 + t & -1 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$ \hspace{1cm} (1.5)

3. The rectangle function:

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$ \hspace{1cm} (1.6)

4. The filtering function:

$$\text{sinc}(t) = \frac{\sin(t/2)}{t/2}.$$ \hspace{1cm} (1.7)

The $\frac{1}{2}$-factor is unusual but is the natural definition for this definition of the FT.

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1 The overbar notation $\overline{f}$ should not be confused with complex conjugate.
5. The Heaviside step function:

\[ H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t \leq 0 \end{cases} \]  

(1.8)

6. The error function:

\[ \text{erf}(t) = \frac{1}{\sqrt{\pi}} \int_{-t}^{t} e^{-x^2} \, dx = \frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-x^2} \, dx . \]  

(1.9)

7. The normalized autocorrelation function:

\[ \gamma(t) = \frac{\int_{-\infty}^{\infty} f^*(u) f(t-u) \, du}{\int_{-\infty}^{\infty} |f(u)|^2 \, du} . \]  

(1.10)

1.2 The Dirac δ-function

Observing the list of functions in the previous section, it is clear that there is one missing. How can a spike be represented? For instance, it is intuitive that the spectrum \( \tilde{f}(\omega) \) of a single sine-wave \( f(t) = \sin \omega_0 t \) should be a spike at \( \omega_0 \) but the condition of absolute integrability (1.3) is not satisfied because of the infinite range of the integral. How can such an improper function be represented? One way of formalizing a spiky function is to introduce the Dirac Delta function \( \delta(t - t_0) \) by considering the properties of a box of unit area under a limiting process, as in the figure below:

A box of unit area: width \( h \) & height \( h^{-1} \) at a point \( t_0 \) on the \( t \)-axis which limits to a spike as \( h \to 0 \) but retains unit area. The curve \( f(t) \) is some other function: The product of the two is non-zero only within the range of the box.

From the picture we represent \( \delta(t - t_0) \) as

\[ \delta(t - t_0) = \lim_{h \to 0} \begin{cases} h^{-1} & t_0 \leq t \leq t_0 + h \\ 0 & \text{otherwise} \end{cases} \]  

(1.11)
with the property of unit area

\[
\text{Area} = \int_{-\infty}^{\infty} \delta(t - t_0) \, dt = 1.
\]  

(1.12)

In the limit \( h \to 0 \) the \( \delta \)-function\(^2\) acts as a ‘spike’ at \( t_0 \) : of course it is not a proper function at all but it possesses the powerful property

\[
\int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = \sum_{i=1}^{N} f(t_i) \delta(t_i - t_0) \Delta t_i
\]

\[
= \lim_{h \to 0} \left[ f(t_0) h^{-1} \right]
\]

\[
= f(t_0).
\]  

(1.13)

To express this in words, when multiplied on a function \( f(t) \) and integrated, the \( \delta \)-function simply picks out the value of \( f(t) \) at the point of the spike \( t_0 \). This result can be expressed in a more general way:

\[
\int_{-\infty}^{\infty} f(t') \delta(t' - t) \, dt' = f(t).
\]  

(1.14)

This will be used many times in future sections.

**Example:** The Shannon sampling function (see the non-examinable extra material on the Shannon sampling Theorem in Appendix B) is a sum of \( \delta \)-functions whose spikes occur at fixed times \( t_n \):

\[
\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t - t_n).
\]  

(1.15)

Its product with a signal \( f(t) \) samples the signal only at discrete points \( t_n \) and so the area under the sampled signal is

\[
\int_{-\infty}^{\infty} f(t) \text{III}(t) \, dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - t_n) \, dt = \sum_{n=-\infty}^{\infty} f(t_n).
\]  

(1.16)

### 1.3 Integral representation of the \( \delta \)-function

The definitions of the the inverse FT \( f(t) \) in (1.2) and the FT \( \mathcal{F}(\omega) \) in (1.1) can be put together to give the Dirichlet integral

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t') e^{-i\omega t'} \, dt' \right) e^{i\omega t} \, d\omega
\]

\[
= \int_{-\infty}^{\infty} f(t') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t - t')} \, d\omega \right) \delta(t - t')
\]  

(1.17)

\(^2\)Another way of defining a \( \delta \)-function is to take a Gaussian curve of half-width \( h \) in the limit \( h \to 0 \).
where the order of integration has been exchanged. As the underbrace in (1.17) shows, comparison with (1.14) gives

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t - t')} d\omega.$$ (1.18)

The reverse process gives

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega') e^{i\omega' t} d\omega' \right) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(\omega') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega - \omega') t} dt \right) d\omega'$$ (1.19)

where the order of integration has been exchanged. As the underbrace in (1.19) shows, comparison with (1.14) gives

$$\delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega - \omega') t} dt.$$ (1.20)

Thus the ‘integral representation’ of the $\delta$-function is:

$$\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\Omega \tau} d\Omega, \quad \text{or} \quad \delta(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\Omega \tau} d\tau.$$ (1.21)

Every student’s first reaction is to evaluate one of the integrals in (1.21)

$$\delta(\tau) = \frac{1}{2\pi} \lim_{a \to \infty} \int_{-a}^{a} e^{\pm i\Omega \tau} d\Omega$$

$$= \lim_{a \to \infty} \left( \frac{\sin \alpha \tau}{\pi \tau} \right),$$ (1.22)

but then it is observed that as $a$ increases the oscillations become faster so the limit does not formally exist. As a function it has no meaning, but nevertheless, the two integral representations in (1.21) are extremely useful.

**Example**: Consider a (complex) function of time with one frequency in the form

$$f(t) = f_0 e^{i\omega_0 t}$$ (1.23)

and so

$$\bar{f}(\omega) = f_0 \int_{-\infty}^{\infty} e^{-i(\omega - \omega_0) t} dt = 2\pi f_0 \delta(\omega - \omega_0).$$ (1.24)

Thus the spectrum is just a single frequency – a spike at $\omega = \omega_0$. The inverse transform is

$$f(t) = \frac{f_0}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{i\omega t} d\omega = f_0 e^{i\omega_0 t}.$$ (1.25)

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3Either sign $\pm$ in the exponent can be chosen: the $\delta$-function is the same either way; $\delta(t - t_0)$ or $\delta(t_0 - t)$. 

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1.4 Parseval’s Theorem and its generalization

Theorem 1 Given two (complex) functions of time \( f(t) \) and \( g(t) \)

\[
\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) \mathcal{G}^*(\omega) \, d\omega. \tag{1.26}
\]

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F}(\omega)|^2 \, d\omega. \tag{1.27}
\]

The physical interpretation of (1.27) is that energy in time-space equals energy in spectral-space, as it must.

Proof: Firstly take the LHS of (1.26) and write \( f(t) \) and \( g^*(t) \) as inverse FTs:

\[
\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} \, d\omega \left( \int_{-\infty}^{\infty} \mathcal{G}^*(\omega') e^{-i\omega' t} \, d\omega' \right) \, dt
\]

\[
= \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \mathcal{F}(\omega) \left\{ \int_{-\infty}^{\infty} \mathcal{G}^*(\omega') \left( \int_{-\infty}^{\infty} e^{i(\omega-\omega') t} \, dt \right) \, d\omega' \right\} \, d\omega. \tag{1.28}
\]

Now use the integral representation

\[
\int_{-\infty}^{\infty} e^{i(\omega-\omega') t} \, dt = 2\pi \delta(\omega - \omega')
\]

to re-write (1.28) as

\[
\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \mathcal{F}(\omega) \left\{ \int_{-\infty}^{\infty} \mathcal{G}^*(\omega') (2\pi \delta(\omega - \omega')) \, d\omega' \right\} \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) \mathcal{G}^*(\omega) \, d\omega, \tag{1.30}
\]

which is the advertised result. (1.27) follows immediately by writing \( g = f \). \( \square \)

1.5 The Fourier Convolution Theorem

Every transform — Fourier, Laplace, Mellin, & Hankel — has a convolution theorem which involves a convolution product between two functions \( f(t) \) and \( g(t) \). The (Fourier) convolution is defined as\(^4\)

\[
f(t) \ast g(t) = \int_{-\infty}^{\infty} f(t')g(t-t') \, dt'. \tag{1.31}
\]

The delay \( t - t' \) may be put in either function, to show this, write \( \tau = t - t' \). Then

\[
f(t) \ast g(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau) \, d\tau. \tag{1.32}
\]

For convenience let us introduce the notation

\[
\mathcal{F}[f(t)] \equiv \mathcal{F}(\omega). \tag{1.33}
\]

\(^4\)The \( \ast \) convolution product should not be confused with complex conjugate.
**Theorem 2 (Fourier convolution theorem)** The transform of the convolution product in time is the product of the transforms in frequency:

\[
\mathcal{F}[f(t) \ast g(t)] = \mathcal{F}(\omega) \mathcal{G}(\omega) \quad \text{or} \quad f(t) \ast g(t) = \mathcal{F}^{-1}[\mathcal{F}(\omega) \mathcal{G}(\omega)].
\] (1.34)

Conversely, \(2\pi\) times the transform of the product in time is the convolution product of the transforms in frequency:

\[
2\pi \mathcal{F}[f(t)g(t)] = \mathcal{F}(\omega) \ast \mathcal{G}(\omega) \quad \text{or} \quad 2\pi f(t)g(t) = \mathcal{F}^{-1}[\mathcal{F}(\omega) \ast \mathcal{G}(\omega)].
\] (1.35)

**Proof of (1.34):**

\[
\mathcal{F}[f(t) \ast g(t)] = \int_{-\infty}^{\infty} e^{-i\omega t} \left( \int_{-\infty}^{\infty} f(t')g(t-t') dt' \right) dt.
\] (1.36)

Writing \(\tau = t - t'\) and reversing the order of integration\(^5\), (1.36) becomes

\[
\mathcal{F}[f(t) \ast g(t)] = \left( \int_{-\infty}^{\infty} f(t')e^{-i\omega t'} dt' \right) \left( \int_{-\infty}^{\infty} g(\tau)e^{-i\omega \tau} d\tau \right) = \mathcal{F}(\omega) \mathcal{G}(\omega).
\] (1.37)

**Proof of (1.35):**

\[
\frac{1}{2\pi} \mathcal{F}^{-1}[\mathcal{F}(\omega) \ast \mathcal{G}(\omega)] = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} e^{i\omega t} \left( \int_{-\infty}^{\infty} \mathcal{F}(\omega') \mathcal{G}(\omega - \omega') d\omega' \right) d\omega.
\] (1.38)

Writing \(\Omega = \omega - \omega'\) and reversing the order of integration the RHS becomes \(f(t)g(t)\). \(\square\)

### 1.6 Examples of Fourier Transforms

1. As in §1.1, the rectangle function \(\Pi(t)\) is defined as

\[
\Pi(t) = \begin{cases} 
1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\] (1.39)

Therefore

\[
\Pi(\omega) = \int_{-\infty}^{\infty} \Pi(t)e^{-i\omega t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t} dt
\]

\[
= \left. \frac{e^{\frac{1}{2}i\omega} - e^{-\frac{1}{2}i\omega}}{i\omega} \right|_{\frac{1}{2}\omega} = \sin \frac{1}{2}\omega = \text{sinc } \omega.
\] (1.40)

\(^5\)The \(t-t'\)-plane is infinite in all four directions.
The inverse is a little trickier:

\[
\Pi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \text{sinc} \omega \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left( e^{\frac{i\omega t}{2}} - e^{-\frac{i\omega t}{2}} \right) \frac{1}{i\omega} \, d\omega
\]

\[
= \frac{1}{2i\pi} (I_1 - I_2) \quad (1.41)
\]

where

\[
I_1 = \int_{-\infty}^{\infty} \frac{e^{ip_1\omega}}{\omega} \, d\omega \quad p_1 = t + \frac{1}{2} \quad (1.42)
\]

and

\[
I_2 = \int_{-\infty}^{\infty} \frac{e^{ip_2\omega}}{\omega} \, d\omega \quad p_2 = t - \frac{1}{2} \quad (1.43)
\]

When \(p_1\) and \(p_2\) have the same sign then \(I_1 = I_2\); that is when \(t > \frac{1}{2}\) and \(t < -\frac{1}{2}\), in which case \(\Pi(t) = 0\) through cancellation in (1.41). In the range \(-\frac{1}{2} < t < \frac{1}{2}\) \(I_1\) and \(I_2\) have opposite signs where \(I_1 = i\pi\) (see Complex Variable notes on integration when a pole is on the real axis) but \(I_2 = -I_1\). Altogether we have the correct result

\[
\Pi(t) = \begin{cases} 
1 & -\frac{1}{2} < t < \frac{1}{2} \\
0 & \text{otherwise} 
\end{cases} \quad (1.44)
\]

2. As in §1.1, the tent function \(\Lambda(t)\) is defined as

\[
\Lambda(t) = \begin{cases} 
1 - t & 0 \leq t \leq 1 \\
1 + t & -1 \leq t \leq 0 \\
0 & \text{otherwise} 
\end{cases} \quad (1.45)
\]

Thus we have

\[
\mathcal{K}(\omega) = \int_{-1}^{0} (1 + t)e^{-i\omega t} \, dt + \int_{0}^{1} (1 - t)e^{-i\omega t} \, dt \quad (1.46)
\]

Now we know that

\[
\int_{a}^{b} te^{-i\omega t} \, dt = \frac{i}{\omega} \int_{a}^{b} t \, d[e^{-i\omega t}] = \frac{i}{\omega} \left( [t e^{-i\omega t}]_{a}^{b} - \int_{a}^{b} e^{-i\omega t} \, dt \right) = \frac{i}{\omega} \left[ t e^{-i\omega t} - \frac{i}{\omega} e^{-i\omega t} \right]_{a}^{b} \quad (1.47)
\]

and

\[
\int_{a}^{b} e^{-i\omega t} \, dt = \frac{i}{\omega} [e^{-i\omega t}]_{a}^{b} \quad (1.48)
\]
Using these in (1.46)
\[
\Lambda(\omega) = \frac{i}{\omega} \left[ (1 - e^{i \omega t}) + (e^{-i \omega t} - 1) \right] + \frac{i}{\omega} \left[ -i - \left( -e^{i \omega} - \frac{i}{\omega} e^{i \omega} \right) \right] - \frac{i}{\omega} \left[ \left( e^{-i \omega} - \frac{i}{\omega} e^{-i \omega} \right) + \frac{i}{\omega} \right] \\
= -\frac{i}{\omega} (e^{i \omega} - e^{-i \omega}) + \frac{2}{\omega^2} + \frac{i}{\omega} (e^{i \omega} - e^{-i \omega}) - \frac{1}{\omega^2} (e^{i \omega} + e^{-i \omega}) \\
= 2 \left( 1 - \cos \omega \right) \omega^2 + 2 \omega^2 + i \omega (e^{i \omega} - e^{-i \omega}) - \frac{1}{\omega^2} (e^{i \omega} + e^{-i \omega}) \\
= \frac{4 \sin^2 \frac{1}{2} \omega}{\omega^2} = \text{sinc}^2 \omega. \\
\tag{1.49}
\]

3. The auto-correlation function definition from (1.10) is
\[
\gamma(t) = \frac{\int_{\infty}^{\infty} f^*(u) f(t-u) du}{\int_{-\infty}^{\infty} |f(u)|^2 du} = \frac{f^*(t) \ast f(t)}{\int_{-\infty}^{\infty} |f(u)|^2 du}. \\
\tag{1.50}
\]

Using the Convolution Theorem, its FT is
\[
\tilde{\gamma}(\omega) = \frac{\mathcal{F} \left[ f^*(t) \ast f(t) \right]}{\int_{-\infty}^{\infty} |f(u)|^2 du} = \frac{|\mathcal{F}(\omega)|^2}{\int_{-\infty}^{\infty} |f(u)|^2 du}. \\
\tag{1.51}
\]

Notice that the denominator has been taken outside the integral because it is a number. Integrating the result w.r.t. \( \omega \) it is found that
\[
\int_{-\infty}^{\infty} \gamma(\omega) d\omega = \int_{-\infty}^{\infty} |\mathcal{F}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(u)|^2 du = 2 \pi , \\
\tag{1.52}
\]
where Parseval’s Theorem (1.27) has been used to obtain the last line.

4. For the Shannon sampling function in (1.15):
\[
\text{III}(t) = \sum_{n=\infty}^{\infty} \delta(t - t_n) \\
\tag{1.53}
\]

Then the FT of \( \text{III}(t) \) is
\[
\overline{\text{III}}(\omega) = \sum_{n=\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \omega t} \delta(t - t_n) dt = \sum_{n=\infty}^{\infty} e^{-i \omega t_n} \\
\tag{1.54}
\]

so the Convolution Theorem gives
\[
\mathcal{F} \left[ f(t) \ast \text{III}(t) \right] = \mathcal{F}(\omega) \overline{\text{III}}(\omega) = \sum_{n=\infty}^{\infty} \mathcal{F}(\omega) e^{-i \omega t_n}, \\
\tag{1.55}
\]
which is (correctly) the product of the transforms. Moreover, the FT of the ordinary product between \( f(t) \) and \( III(t) \) is

\[
2\pi \mathcal{F} [f(t)III(t)] = 2\pi \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\delta(t-t_n)\, dt
\]

\[
= 2\pi \sum_{n=-\infty}^{\infty} f(t_n)e^{-i\omega t_n}, \tag{1.56}
\]

which should be the convolution of \( \mathcal{F}(\omega) \) and \( III(\omega) \). To check this write

\[
 \mathcal{F}(\omega) \ast III(\omega) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(\omega')e^{-i(\omega-\omega')t_n}\, d\omega'
\]

\[
= 2\pi \sum_{n=-\infty}^{\infty} f(t_n)e^{-i\omega t_n} \tag{1.57}
\]

which agrees with (1.56). The convolution in time is

\[
f(t) \ast III(t) = \int_{-\infty}^{\infty} f(t-t')III(t')\, dt'
\]

\[
= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-t')\delta(t'-t_n)\, dt'
\]

\[
= \sum_{n=-\infty}^{\infty} f(t-t_n), \tag{1.58}
\]

and so

\[
\mathcal{F}[f(t) \ast III(t)] = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega t}f(t-t_n)\, dt
\]

\[
= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega(\tau_n+t_n)}f(\tau_n)\, d\tau_n
\]

\[
= \sum_{n=-\infty}^{\infty} e^{-i\omega t_n} \int_{-\infty}^{\infty} e^{-i\omega \tau_n}f(\tau_n)\, d\tau_n
\]

\[
= \sum_{n=-\infty}^{\infty} e^{-i\omega t_n} \mathcal{F}(\omega), \tag{1.59}
\]

which is (1.55), the ordinary product of the transforms.
2  Laplace Transforms

2.1  Introduction

For a function \( f(t) \) uniquely defined on \( 0 \leq t \leq \infty \), its Laplace transform (LT) is defined as

\[
\mathcal{L}[f(t)] = \overline{f}(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt ,
\]

where \( s \) may be complex. The LT may not exist if \( f(t) \) becomes singular in \([0, \infty)\). The LT is a one-sided transform in that it operates on \([0, \infty)\) and not, like the FT, on \((-\infty, \infty)\). For this reason, LTs are useful for initial value problems, such as circuit theory, where a function switches on at \( t = 0 \) and where \( f(0) \) has been specified.

Because \( s \) is a complex variable the inverse transform

\[
f(t) = \mathcal{L}^{-1} \overline{f}(s) = \oint_C e^{st} \overline{f}(s) \, ds
\]

is more difficult to handle because the contour \( C \) is a tricky infinite rectangle in the right-hand-half of the \( s \)-plane. Referred to as ‘Bromwich integrals’ the evaluation of these is beyond our present course. To circumvent this difficulty we resort firstly to a library of transforms (see Handout 7) for the standard functions and secondly to ways of piecing combinations of these together for those not in the list.

2.2  Library of Laplace Transforms

1. **The constant function** \( f(t) = 1 \):

\[
f(t) = 1 ; \quad \overline{f}(s) = \frac{1}{s} \quad \text{Re} \, s > 0
\]

**Proof:**

\[
\overline{f}(s) = \int_{0}^{\infty} e^{-st} \, dt = \left[ \frac{e^{-st}}{s} \right]_{0}^{\infty} = \frac{1}{s} ,
\]

provided \( \text{Re} \, s > 0 \).

2. **The exponential-function** \( f(t) = e^{at} \):

\[
f(t) = \exp(at) ; \quad \overline{f}(s) = \frac{1}{s-a} ; \quad \text{Re} \, s > a
\]

**Proof:**

\[
\overline{f}(s) = \int_{0}^{\infty} e^{-(s-a)t} \, dt = \left[ \frac{e^{-(s-a)t}}{s-a} \right]_{0}^{\infty} = \frac{1}{s-a} ,
\]

provided \( \text{Re}(s-a) > 0 \).
3. The sine function:

\[ f(t) = \sin(at) ; \quad \mathcal{F}(s) = \frac{a}{s^2 + a^2} ; \quad \text{Re} \, s > 0 \]  

**Proof**: Take both the sine and cosine functions in combination: \( \cos at + i \sin at = e^{iat} \)

\[ \mathcal{L}(e^{iat}) = \int_0^\infty e^{-(s-ia)t} \, dt = \frac{1}{s-ia} = \frac{s + ia}{s^2 + a^2} \]

provided \( \text{Re}(s) > 0 \). Then the imaginary (real) part gives the result for sine (cosine).

4. The cosine function

\[ f(t) = \cos(at) ; \quad \mathcal{F}(s) = \frac{s}{s^2 + a^2} ; \quad \text{Re} \, s > 0 \]

5. The polynomial function \( f(t) = t^n \):

\[ f(t) = t^n ; \quad \mathcal{F}(s) = \frac{n!}{s^{n+1}} ; \quad (n \geq 0) ; \quad \text{Re} \, s > 0 \]

**Proof**: Define the LT as \( \mathcal{F}(s) = I_n \) as

\[ I_n = \int_0^\infty e^{-st} t^n \, dt = -\frac{1}{s} \int_0^\infty t^n \, d[e^{-st}] \]

\[ = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt = \frac{n}{s} I_{n-1} \]  

provided \( \text{Re}(s) > 0 \). With \( n = 0 \) and \( \mathcal{L}[1] = s^{-1} \) we obtain \( I_1 = s^{-2} \) and end up with

\[ \mathcal{F}(s) = I_n = \frac{n!}{s^{n+1}} . \]

6. The Heaviside function:

\[ f(t) = H(t - t_0) ; \quad \mathcal{F}(s) = \frac{\exp(-st_0)}{s} ; \quad \text{Re} \, s > 0 \]

**Proof**: For \( \text{Re} \, s > 0 \)

\[ \mathcal{L}[H(t - t_0)] = \int_0^\infty e^{-st} H(t - t_0) \, dt \]

\[ = \int_{t_0}^\infty e^{-st} \, dt = \frac{e^{-st_0}}{s} . \]

7. The Dirac \( \delta \)-function:

\[ f(t) = \delta(t - t_0) ; \quad \mathcal{F}(s) = \exp(-st_0) ; \quad t_0 \geq 0 \]

**Proof**: \( t_0 \) needs to reside within the positive range of \( t \)

\[ \int_0^\infty e^{-st} \delta(t - t_0) \, dt = \begin{cases} e^{-st_0} & t_0 \geq 0, \\ 0 & t_0 < 0. \end{cases} \]
8. **Shift theorem:**  
\[
\mathcal{L} [\exp(at)f(t)] = f(s-a) 
\]  
\[(2.17)\]

**Proof:** Provided \(\text{Re}(s - a) > 0\)

\[
\mathcal{L} [\exp(at)f(t)] = \int_0^\infty e^{-(s-a)t} f(t) \, dt = f(s-a).  
\]  
\[(2.18)\]

9. **Second shift theorem:**
\[
\mathcal{L} [H(t-a)f(t-a)] = \exp(-sa) f(s)  
\]  
\[(2.19)\]

**Proof:** let \(\tau = t - a\). Then

\[
\mathcal{L} [H(t-a)f(t-a)] = \int_0^\infty e^{-st} H(t-a) f(t-a) \, dt = e^{-sa} \int_{-a}^\infty e^{-s\tau} f(\tau) \, d\tau = e^{-sa} \int_0^\infty e^{-s\tau} f(\tau) \, d\tau = e^{-sa} f(s).  
\]  
\[(2.20)\]

10. **Convolution theorem:**
\[
\mathcal{L} \{f \ast g\} = f(s)g(s)  
\]  
\[(2.21)\]

where the convolution between two functions \(f(t)\) and \(g(t)\) is defined as

\[
f \ast g = \int_0^t f(t')g(t-t') \, dt'.  
\]  
\[(2.22)\]

Note that the convolution is over \([0, t]\) and not \([-\infty, \infty]\) as for the FT. The convolution integral on the RHS can also be written with \(f\) and \(g\) reversed: that is \(f \ast g = \int_0^t f(t-t')g(t') \, dt'\).

**Proof:** The LT of the convolution product in (2.22) is written down and then the order of the integrals is exchanged, as in the figure, using \(\tau = t - t'\).
The region of integration \( \mathbb{R} \) can be read from the figure: the \( t' \)-integration is taken in the vertical direction to cover \( \mathbb{R} \) but to cover this in the reverse order, the \( t \)-integration is taken in the horizontal direction.

\[ \mathcal{L}(f * g) = \int_{0}^{\infty} e^{-st} \left( \int_{0}^{t} f(t')g(t - t') \, dt' \right) \, dt \]
\[ = \int_{0}^{\infty} \left( \int_{t=t'}^{\infty} e^{-st}g(t - t') \, dt \right) f(t') \, dt' \]
\[ = \int_{0}^{\infty} e^{-st} \left( \int_{\tau=0}^{\infty} e^{-s\tau}g(\tau) \, d\tau \right) f(t') \, dt' \]
\[ = \overline{f}(s) \overline{g}(s). \]  

(2.23)

11. **Integral:**

\[ \mathcal{L} \left( \int_{0}^{t} f(t') \, dt' \right) = \frac{\overline{f}(s)}{s} \]  

(2.24)

**Proof:** The integral in (2.24) is a convolution product between \( f(t) \) and \( g(t) = 1 \). Thus \( \overline{g}(s) = 1/s \), giving the result from (2.23).

12. **Derivative:**

\[ \mathcal{L} \left[ \dot{f}(t) \right] = s\overline{f}(s) - f(0) \]  

(2.25)

**Proof:** Noting that \( f(0) \) means \( f(t=0) \)

\[ \mathcal{L} \left[ \dot{f} \right] = \int_{0}^{\infty} e^{-st} \dot{f} \, dt \]
\[ = \int_{0}^{\infty} e^{-st} \, d\dot{f} = [e^{-st}f(t)]_{0}^{\infty} + s \int_{0}^{\infty} e^{-st} f \, dt \]
\[ = s\overline{f}(s) - f(0) \]  

(2.26)

provided Re \( s > 0 \).

13. **Second derivative:** Noting that \( \dot{f}(0) \) means \( \dot{f}(t=0) \)

\[ \mathcal{L} \left[ \ddot{f}(t) \right] = s^{2}\overline{f}(s) - sf(0) - \dot{f}(0) \]  

(2.27)

**Proof:**

\[ \mathcal{L} \left[ \ddot{f} \right] = \int_{0}^{\infty} e^{-st} \ddot{f} \, dt \]
\[ = \int_{0}^{\infty} e^{-st} \, d\ddot{f} = [e^{-st}\ddot{f}(t)]_{0}^{\infty} + s \int_{0}^{\infty} e^{-st} \ddot{f} \, dt \]
\[ = s\mathcal{L} \left[ \dot{f} \right] - \dot{f}(0) \]
\[ = s^{2}\overline{f}(s) - sf(0) - \dot{f}(0), \]  

(2.28)

provided Re \( s > 0 \).
2.3 Using the Convolution Theorem to find inverses

If we are given an inverse LT as a function \( F(s) \) which is too complicated to appear in the Library above but can be split into composite functions \( F(s) = \overline{f}(s) \overline{g}(s) \) where \( \overline{f}(s) \) and \( \overline{g}(s) \) do belong to the Library, then the Convolution Theorem allows us to write

\[
F(t) = \mathcal{L}^{-1}\left( \overline{f}(s) \overline{g}(s) \right) = f(t) * g(t).
\]

**Example 1:** Find \( \mathcal{L}^{-1}\left[ \frac{1}{s(s^2+1)} \right] \). We identify \( f(s) = s^{-1} \) and \( g(s) = (s^2+1)^{-1} \). The Library tell us that \( f(t) = 1 \) and \( g(t) = \sin t \). Thus

\[
F(t) = 1 * \sin t = \int_0^t \sin t' \, dt' = 1 - \cos t.
\]

**Example 2:** Find \( \mathcal{L}^{-1}\left[ \frac{s}{(s^2+a^2)^2} \right] \). Identify

\[
\overline{f}(s) = \frac{s}{s^2 + a^2} \quad \overline{g}(s) = \frac{1}{s^2 + a^2}.
\]

The Library tell us that \( f(t) = \cos at \) and \( g(t) = a^{-1} \sin at \), and so

\[
F(t) = a^{-1} \sin at * \cos at = a^{-1} \int_0^t \sin(at') \cos a(t-t') \, dt'.
\]

Using \( \sin(A + B) + \sin(A - B) = 2 \sin A \cos B \) we find

\[
\sin(at') \cos a(t-t') = \frac{1}{2} [\sin at + \sin a(2t' - t)]
\]

and so from (2.32)

\[
F(t) = \frac{1}{2a} \int_0^t [\sin(at) + \sin a(2t' - t)] \, dt'
= \frac{1}{2a} \left[ t \sin at - \frac{1}{2a} \{ \cos at - \cos at \} \right]
= \frac{t}{2a} \sin at.
\]

**Example 3:** Find \( \mathcal{L}^{-1}\left[ \frac{a^2}{(s^2+a^2)^2} \right] \). Identify \( F(s) = |\overline{f}(s)|^2 \) where

\[
\overline{f}(s) = \frac{a}{s^2 + a^2} \quad \overline{g}(s) = \overline{f}(s).
\]

The Library tell us that \( f(t) = g(t) = \sin at \) so \( \sin at \) is convolved with itself

\[
F(t) = \sin at * \sin at
= \int_0^t \sin at' \sin a(t-t') \, dt'
= \frac{1}{2a} [\sin at - at \cos at].
\]

having used the trig-identity \( \cos(A - B) - \cos(A + B) = 2 \sin A \sin B \).
2.4 Examples involving partial fractions and the Shift theorem

Example 1: Find \( f(t) \) when

\[
\mathcal{F}(s) = \frac{6s^2 + 10s + 2}{s(s^2 + 3s + 2)}.
\]  

(2.37)

Noting that \( s^2 + 3s + 2 = (s + 1)(s + 2) \) (2.37) can be split by Partial Fractions (PFs) into

\[
\mathcal{F}(s) = \frac{6s^2 + 10s + 2}{s(s + 1)(s + 2)} = \frac{1}{s} + \frac{2}{s + 1} + \frac{3}{s + 2}.
\]  

(2.38)

Thus, using the Library

\[
f(t) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{s + 1} + \frac{3}{s + 2}\right)
= 1 + 2e^{-t} + 3e^{-2t}.
\]  

(2.39)

Example 2: Find \( f(t) \) when

\[
\mathcal{F}(s) = \frac{2}{s(s - 2)}.
\]  

(2.40)

in which case

\[
\mathcal{F}(s) = -\frac{1}{s} + \frac{1}{s - 2},
\]  

(2.41)

and so

\[
f(t) = -1 + e^{2t}.
\]  

(2.42)

Example 3: Find \( f(t) \) when \( \mathcal{F}(s) = (s - 1)^{-4} \). From the Library,

\[
\mathcal{L}[t^3] = \frac{3!}{s^4}
\]  

(2.43)

therefore \( \mathcal{L}^{-1}[s^{-4}] = t^3/6 \). With the application of the Shift Theorem with \( a = 1 \) we have

\[
\mathcal{L}^{-1}[(s - 1)^{-4}] = \frac{1}{6} t^3 e^t.
\]  

(2.44)

2.5 Solving ODEs using Laplace Transforms

Many textbook methods are given to solve 2nd order ODEs of the type

\[
\ddot{x} + \alpha \dot{x} + \omega_0^2 x = f(t),
\]  

(2.45)

but only the LT-method can handle those cases when the forcing function is not smooth. Examples might be voltage inputs of the square wave or saw-tooth type. To approach this using LTs, the transform is taken of (2.45)

\[
(s^2 \mathcal{F}(s) - sx_0 - \dot{x}_0) + \alpha (s \mathcal{F}(s) - x_0) + \omega_0^2 \mathcal{F}(s) = \mathcal{F}(s).
\]  

(2.46)
where \( x_0 = x(0) \) and \( \dot{x}_0 = \dot{x}(0) \). This re-organizes into

\[
(s^2 + \alpha s + \omega_0^2) \bar{\pi}(s) = \bar{f}(s) + (s + \alpha)x_0 + \dot{x}_0.
\]  (2.47)

Note that the final expression for \( \bar{\pi}(s) \) divides conveniently into two parts corresponding to the Complementary Function and the Particular Integral

\[
\bar{\pi}(s) = \frac{\bar{f}(s)}{s^2 + \alpha s + \omega_0^2} + \frac{(s + \alpha)x_0 + \dot{x}_0}{s^2 + \alpha s + \omega_0^2}
\]  (2.48)

The initial conditions appear in \( x_0 \) and \( \dot{x}_0 \) as part of the Complementary Function. How to take the inverse depends on whether the denominator has real or complex roots. These we consider by example.

**Example 1:** Solve \( \ddot{x} + \dot{x} - 2x = e^t \) with \( x_0 = 3 \) and \( \dot{x}_0 = 0 \).

(2.47) becomes

\[
(s^2 + s - 2) \bar{\pi}(s) = \frac{1}{s - 1} + 3(s + 1).
\]  (2.49)

Noting that \( s^2 + s - 2 = (s - 1)(s + 2) \) we have

\[
\bar{\pi}(s) = \frac{1}{(s - 1)^2(s + 2)} + \frac{3(s + 1)}{(s - 1)(s + 2)}
\]  (2.50)

Using PFs

\[
\bar{\pi}(s) = \frac{1}{3(s - 1)^2} + \frac{17}{9(s - 1)} + \frac{10}{9(s + 2)}
\]  (2.51)

and so the Library gives us

\[ x(t) = \frac{1}{3} te^t + \frac{17}{9} e^t + \frac{10}{9} e^{-2t}. \]  (2.52)

**Example 2:** Solve \( \ddot{x} + 16x = \sin 2t \) with \( x_0 = 0 \) and \( \dot{x}_0 = 1 \).

(2.47) becomes

\[
(s^2 + 16) \bar{\pi}(s) = 1 + \frac{2}{s^2 + 4},
\]  (2.53)

and so

\[
\bar{\pi}(s) = \frac{1}{s^2 + 16} + \frac{2}{(s^2 + 4)(s^2 + 16)}
\]  
\[
= \frac{5}{6(s^2 + 16)} + \frac{1}{6(s^2 + 4)}
\]  
\[
= \frac{5}{24} \left( \frac{4}{s^2 + 4^2} \right) + \frac{1}{12} \left( \frac{2}{s^2 + 2^2} \right). \]  (2.54)
Therefore, from the Library

\[ x(t) = \frac{5}{24} \sin 4t + \frac{1}{12} \sin 2t. \quad (2.55) \]

**Example 3 (real roots):** Solve \( \ddot{x} + 3 \dot{x} + 2x = f(t) \) with \( x_0 = 1 \) and \( \dot{x}_0 = -2 \). In this example \( f(t) \) has not been specified although it must be assumed that its LT exists.

We obtain

\[ x(s) = \frac{\tilde{f}(s)}{s^2 + 3s + 2} + \frac{x_0(s + 3) + \dot{x}_0}{s^2 + 3s + 2} \quad (2.56) \]

so from (2.48) with the specified initial conditions

\[ x(s) = \frac{\tilde{f}(s)}{(s + 1)(s + 2)} + \frac{1}{s + 2}. \quad (2.57) \]

Using PFs we find

\[ \tilde{x}(s) = \frac{\tilde{f}(s)}{s+1} - \frac{\tilde{f}(s)}{s+2} + \frac{1}{s+2} \]

\[ \equiv \tilde{f}(s)\tilde{g}_1(s) - \tilde{f}(s)\tilde{g}_2(s) + \tilde{g}_2(s) \quad (2.58) \]

where \( \tilde{g}_1(s) = (s + 1)^{-1} \) and \( \tilde{g}_2(s) = (s + 2)^{-1} \). From these definitions it is clear that \( g_1(t) = e^{-t} \) and \( g_2(t) = e^{-2t} \). From the Convolution Theorem we have

\[ x(t) = \int_0^t \left[ e^{-t-t'} - e^{-2(t-t')} \right] f(t') \, dt' + e^{-2t} \quad (2.59) \]

The power of the LT-method can be seen here in that it solves, in principle, an ODE with any forcing, provided \( \tilde{f}(s) \) exists.

**Example 4 (complex roots):** Solve \( \ddot{x} + 2 \dot{x} + 2x = f(t) \) with \( x_0 = 1 \) and \( \dot{x}_0 = 0 \). In this example \( f(t) \) has not been specified although it must be assumed that its LT exists.

From (2.48) the next step comes out to be

\[ \tilde{x}(s) = \frac{\tilde{f}(s)}{(s+1)^2 + 1} + \frac{s + 2}{(s+1)^2 + 1} \quad (2.60) \]

where it has been noted that \( s^2 + 2s + 2 \) does not have real roots. Now define

\[ \tilde{g}_1(s) = \frac{1}{(s+1)^2 + 1} \quad \tilde{g}_2(s) = \frac{s + 1}{(s+1)^2 + 1}. \quad (2.61) \]

Therefore \( \tilde{x}(s) \) can be re-expressed as

\[ \tilde{x}(s) = \tilde{f}(s)\tilde{g}_1(s) + \tilde{g}_2(s) + \tilde{g}_1(s). \quad (2.62) \]

Inverse transforms can be found from the Shift Theorem and the Library

\[ g_1(t) = e^{-t} \sin t \quad g_2(t) = e^{-t} \cos t. \quad (2.63) \]

The Convolution Theorem gives the final result

\[ x(t) = \int_0^t f(t-t')e^{-t'} \sin t' \, dt' + e^{-t} [\cos t + \sin t]. \quad (2.64) \]
A Appendix: Three definitions of the Fourier Transform

1. **Definition 1:** (used in these notes)

\[ \mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt, \quad (A.1) \]

with the inverse Fourier transform written as

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega t} \, d\omega. \quad (A.2) \]

2. **Definition 2:** This definition is sometimes used in signal processing:

\[ \mathcal{F}(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist} \, dt, \quad (A.3) \]

with the inverse Fourier transform written as

\[ f(t) = \int_{-\infty}^{\infty} \mathcal{F}(s)e^{2\pi ist} \, ds. \quad (A.4) \]

Hence \( s \) acts like the frequency with \( \omega = 2\pi s \).

3. **Definition 3:** This next definition is used more in mathematical physics because of the symmetry in the coefficients of both the transform and its inverse:

\[ \mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt, \quad (A.5) \]

with the inverse Fourier transform written as

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega t} \, d\omega. \quad (A.6) \]

B Appendix: The Nyquist-Shannon Sampling Theorem

Not examinable: this result belongs to your own Signals Processing course. Let \( \omega_s = 2\pi / T \) be the sampling rate of a band-width-limited signal which is centred around zero with bandwidth \([\omega_{max}, \omega_{max}]\).

**Theorem 3** When sampling a signal, the sampling frequency must be greater than twice the bandwidth in order to reconstruct the signal perfectly from the sampled version.

**Proof:** Let \( f(t) \) be a continuous signal and let \( III(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \). Then we consider

\[ f^{(s)}(t) = f(t)III(t) \quad (B.1) \]
with $\omega_s = 2\pi/T$. Then
\[
\overline{f(s)}(\omega) = \mathcal{F}\left[ f(t) \sum_{n=-\infty}^{n=\infty} \delta(t - nT) \right] \\
= \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \overline{f}(\omega) \ast e^{-i\omega nT} \\
= \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} \overline{f}(\omega')e^{-i(\omega'-\omega)nT} \, d\omega' \\
= \sum_{n=-\infty}^{n=\infty} f(nT) e^{i\omega nT} \\
= \sum_{n=-\infty}^{n=\infty} \overline{f}(\omega - n\omega_s)
\]
where the Poisson summation formula has been used in the last step. The signal bandwidth is $2\omega_{max}$ so in order for a replicated $\overline{f}(\omega)$, shifted by $\omega_s$, not to overlap then the condition $\omega_s > 2\omega_{max}$ must hold. If $\omega_s$ is not large enough then overlap occurs with aliasing. \hfill \Box