EE2 Mathematics

Example Sheet 5: LAPLACE TRANSFORMS

1. a) For the coupled ODEs

\[ 2\dot{x} + \dot{y} + x + 6 = 0 \quad \dot{x} + 2\dot{y} + y = 0 \]

where \( x(0) = \dot{x}(0) = 1 \), show that

\[ y(s) = \frac{3(s + 3)}{(s + 1)(3s + 1)} \quad \text{and} \quad x(s) = \frac{3(s^2 - 3s - 2)}{s(s + 1)(3s + 1)} . \]

Split these expressions into partial fractions & invert to find \( x(t) \) and \( y(t) \).

Pick your answers from i) \(-6 + 3e^{-t} + 4e^{-\frac{1}{2}t}\) ii) \(-6 + 3e^{-2t} + 4e^{-\frac{1}{2}t}\) iii) \(3e^{-t} - 4e^{-\frac{1}{2}t}\) iv) \(-3e^{-t} + 4e^{-\frac{1}{2}t}\)

(b) In the same manner as part a), use Laplace transforms to solve

\[ \dot{x} + 5x + 2y = e^{-t}, \quad \dot{y} + 2x + 2y = 0, \quad x(0) = 1, \quad y(0) = 0. \]

Pick your answers from i) \(6e^{-5t} + \frac{2}{3}(t + \frac{2}{3})e^{-t}\) ii) \(\frac{16}{25}e^{-6t} + \frac{1}{5}(t + \frac{2}{5})e^{-t}\); iii) \(\frac{8}{25}e^{-6t} - \frac{1}{5}(\frac{8}{5} + 2t)e^{-t}\); iv) \(\frac{8}{25}e^{-6t} + \frac{1}{5}(\frac{8}{5} + 2t)e^{-t}\);

2. A function \( f(t) \) has a Laplace transform \( \mathcal{L}\{f(t)\} \equiv \mathcal{F}(s) \). Use the ‘shift property’ \( \mathcal{L}\{e^{at}f(t)\} = \mathcal{F}(s-a) \), where \( a \) is a constant, and the ‘second shift property’; \( \mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\mathcal{F}(s) \)

to show that the solution of the SHM equation with discontinuous driving terms

\[ \ddot{x} + x = H(t - \pi) - H(t - 2\pi) \]

and with initial conditions \( x(0) = \dot{x}(0) = 0 \), is

\[ x = 0 \quad 0 \leq t \leq \pi \]
\[ x = 1 + \cos t \quad \pi \leq t \leq 2\pi \]
\[ x = 2 \cos t \quad 2\pi \leq t \]

where \( H(t) \) is the Heaviside step function.

3. If a function \( f(t) \) is periodic in time \( t \) with fixed period \( T \) such that \( f(t) = f(t - T) \) with \( T > 0 \) show that for \( s > 0 \)

\[ \mathcal{F}(s) = \frac{1}{1 - e^{-st}} \int_0^T f(t)e^{-st} \, dt . \]

Note that this enables a Laplace Transform to be found by performing the integral only over the period \((0, T)\) for which \( f(t) \) is defined.

4. Use the result of Q3 to find the Laplace transform of the ‘saw-tooth’ function

\[ f(t) = t \quad 0 \leq t \leq 1 \]
\[ f(t) = f(t - 1) \quad 1 \leq t \]

Pick your answers from i) \( \mathcal{F}(s) = s^{-2} + s^{-1} (e^{-s} + e^{-2s} + e^{-3s} \ldots ) \).
ii) \( \mathcal{F}(s) = s^{-2} - s^{-1} (e^{-s} + e^{-2s} + e^{-3s} \ldots ) \).
iii) \( \mathcal{F}(s) = s^{-2} - 2s^{-1} (e^{-s} + e^{-2s} + e^{-3s} \ldots ) \).

The function \( f(t) \) is often used in electronics for representing discontinuous voltages.