EE2 Mathematics

Example Sheet 2: Functions of a complex variable

Recall that for a complex function \( f(z) = u(x,y) + iv(x,y) \) the Cauchy-Riemann equations are \( u_x = v_y \) and \( u_y = -v_x \).

1. Do the following satisfy the Cauchy-Riemann equations:
   a) \( u = x; \quad v = y \),
   b) \( u = e^x \cos y; \quad v = -e^x \sin y \),
   c) \( u = x^3 - 3xy^2; \quad v = 3x^2y - y^3 \)

Pick your answers from i) Yes and ii) No.

2. Show that the following functions \( u(x,y) \) each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function \( v \). Find also \( f(z) = u + iv \).
   a) \( u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \), \( b) u = xy \).

Pick your answers from: i) \( 3x^2y - y^3 + 6xy + c \) ii) \( 3x^2y + y^3 + 6xy + c \) iii) \( 3x^2y + 3y^3 + 6xy + c \) iv) \( \frac{1}{2}(y^2 + x^2) + c \) v) \( \frac{1}{2}(y^2 - x^2) + c \) vi) \( f(z) = z^3 + 3z^2 + c \) vii) \( f(z) = z^3 + 2z^2 + c \)viii) \( f(z) = \frac{1}{2}z^2 + c \).

3. Find an analytic function of \( z = x + iy \) with an imaginary part that is \( (y \cos y + x \sin y) \exp x \)

Pick your answer from: i) \( f(z) = e^z + c \) ii) \( f(z) = z + c \) iii) \( f(z) = ze^{-z} + c \) iv) \( f(z) = ze^z + c \)

4. Consider the mapping \( w = \frac{1}{z-1} \) from the \( z \)-plane to the \( w \)-plane.
   a) Show that in the \( z \)-plane, the circle
      \( (x - 1)^2 + y^2 = 4 \)
      maps to a circle in the \( w \)-plane. What is the radius of this circle and where is its centre?
   b) To what curve does the line \( x = 0 \) in the \( z \)-plane map in the \( w \)-plane? Write out its equation.

Pick your answers from: i) \( \frac{1}{2} \) ii) \( \frac{1}{2} \) iii) \( \frac{1}{2} \) iv) \( (-\frac{1}{4}, -\frac{1}{2}) \) v) \( (-\frac{1}{2}, -\frac{1}{4}) \) vi) \( (-\frac{1}{2}, 0) \) vii) \( (0, 0) \) viii) Circle ix) Line ix) Archimedean Spiral x) Euler Spiral

5. a) Fixed points of a map \( w = f(z) \) occur when \( w = z \). Do the fixed points of \( w = \frac{4z^2}{z + 1} \) occur at \( z = 1 \) and \( z = 2 \)?
   b) For \( w = u + iv = \frac{4z^2}{z + 1} \) show that the image in the \( w \)-plane of the line \( x = 0 \) is the circle \( (u - 1)^2 + v^2 = 9 \). What is the image in the \( w \)-plane of the unit circle \( |z| = 1 \)? Write out its equation.

Pick your answers from: i) Yes ii) No iii) Circle iv) Line v) Archimedean Spiral vi) Euler Spiral vii) Fermat’s Spiral