Simple orthogonal block structures, nesting and marginality

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Abstract I

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But there are still misunderstandings.

- If there are 5 blocks of 4 plots each,
  should the plot factor have 4 levels or 20?
- What is the difference between nesting and marginality?
- What is the difference between a factor,
  the effect of that factor
  (this effect may be called an interaction in some cases),
  and the smallest model which includes that factor whilst
  respecting marginality?

Abstract II

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My take on this is that there are three different partial orders involved: I will try to explain the difference.

Labelling plots in blocks

Suppose that there are five blocks of four plots each.
How should we label them?

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Advantage of using Q: if the data are analysed by someone who did not design the experiment,
they cannot make the mistake of thinking that all plots ω with $P(\omega) = 1$ have something in common.

Terminology

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P and Q are different types of thing, and play different roles, so I shall call P a pre-factor and Q a factor, but many people confuse them, or use different terminology.

Pre-factors and nesting

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Every partially ordered set (poset)
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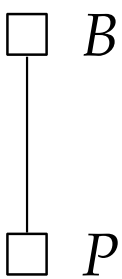

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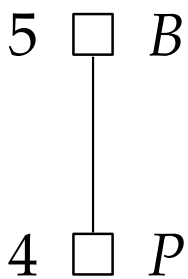

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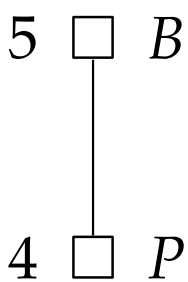

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If we have three rows (R) and eight columns (C) with no nesting then we get

3 □ R 8 □ C

Combining two factors or pre-factors

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Other notations: $A.B$ or $A : B$.

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Factors:	U with one level
	R with 3 levels (1st coordinate)
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From crossing and nesting to simple orthogonal block structures

The key ingredient of John Nelder's 1965 paper on 'Block structure and the null analysis of variance' was to realise that crossing and nesting could be iterated (maybe with some steps of each sort).

He developed an almost-complete theory, notation and algorithms based on this.

He called the resulting sets of experimental units with their factor lists simple orthogonal block structures.

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(For simplicity here, I am ignoring the possibility of aliasing.)

So we can show factors on a Hasse diagram too!

Crossing

Hasse diagram for pre-factors: $3 \square R$ $8 \square C$

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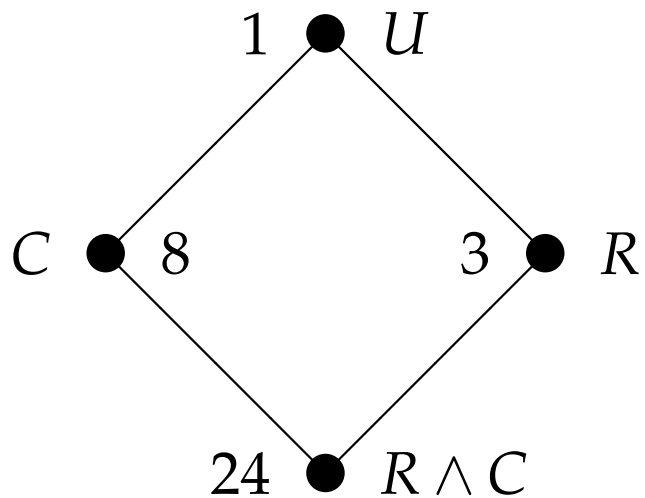

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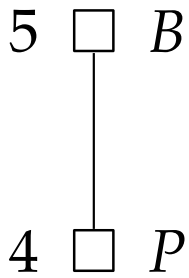

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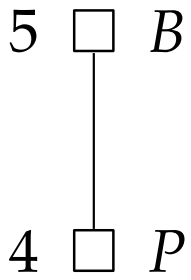

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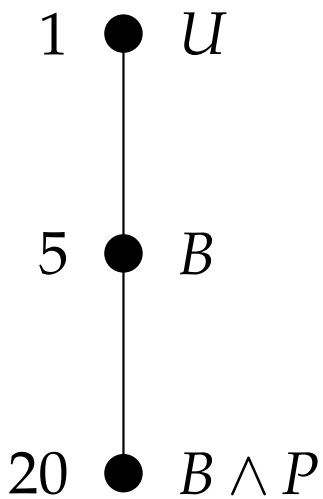
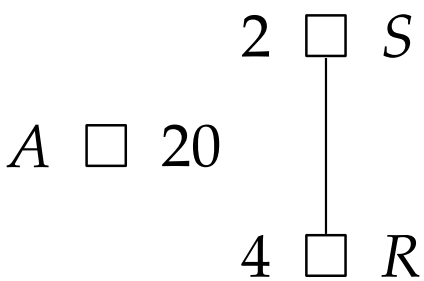


Iteration: $(20\ \text{Athletes} * ((2\ \text{Sessions}) / (4\ \text{Runs}))$



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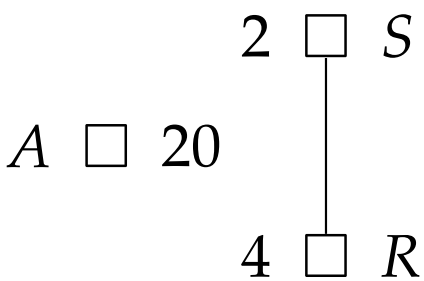

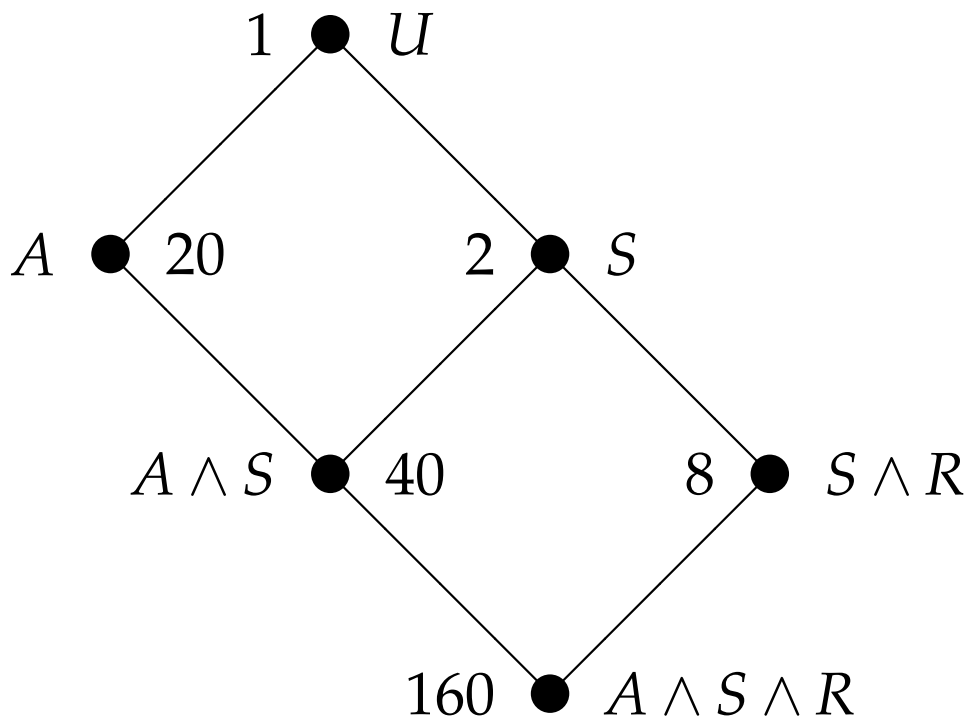

Start with the first poset

Terry Speed and I found that
you can start with the nesting poset and use it to directly construct the set Ω of experimental units and its factors.

Given pre-factors $P_1, \ldots, P_m$ with $n_1, \ldots, n_m$ levels,
and a nesting relation $\sqsubset$:

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m \quad \text{where } \Omega_i = \{1, 2, \ldots, n_i\}.$$

If $\mathcal{A}$ is any subset of $\{1, 2, \ldots, m\}$ satisfying

$$\text{if } i \in \mathcal{A} \text{ and } P_i \sqsubset P_j \text{ then } j \in \mathcal{A}$$

then include the factor $\bigwedge_{i \in \mathcal{A}} P_i$.

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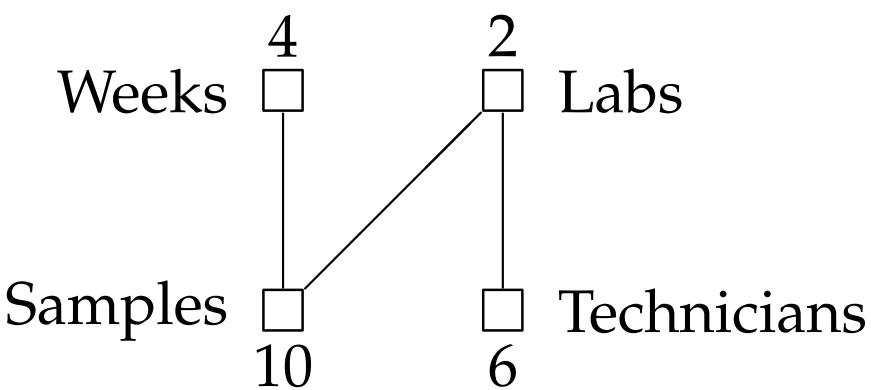

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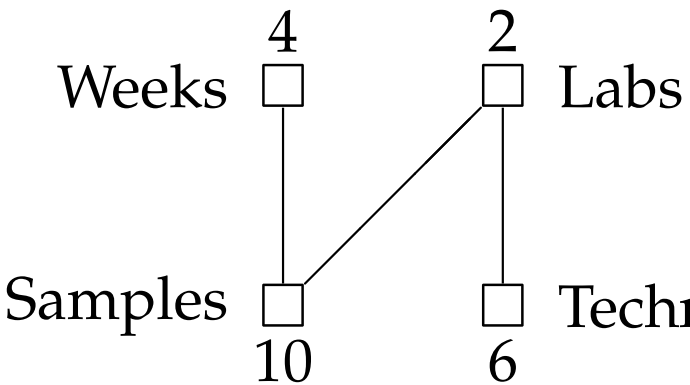

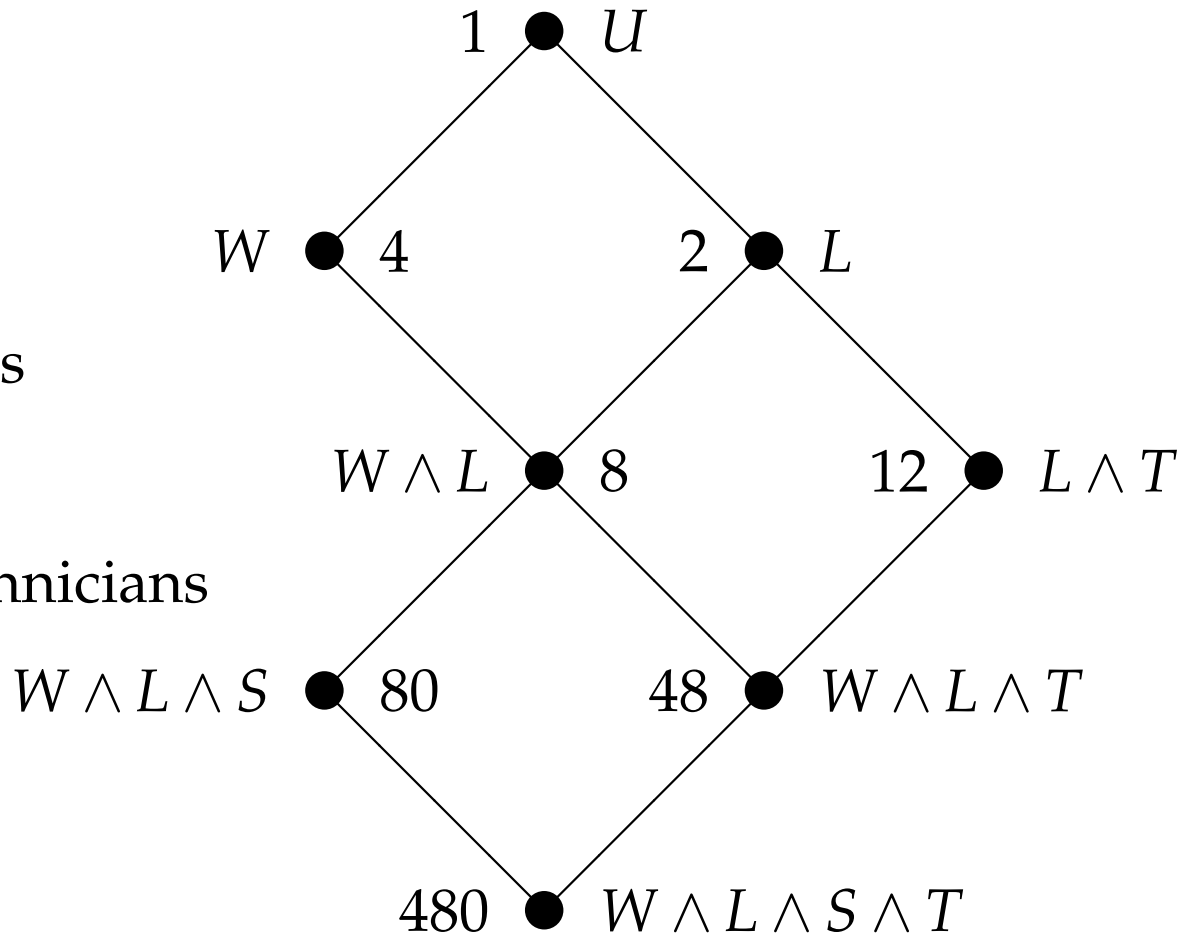

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Some software thinks that $B \wedge Q$ has 100 levels,
and tries to make 100×100 matrices to deal with this.

Other orthogonal block structures

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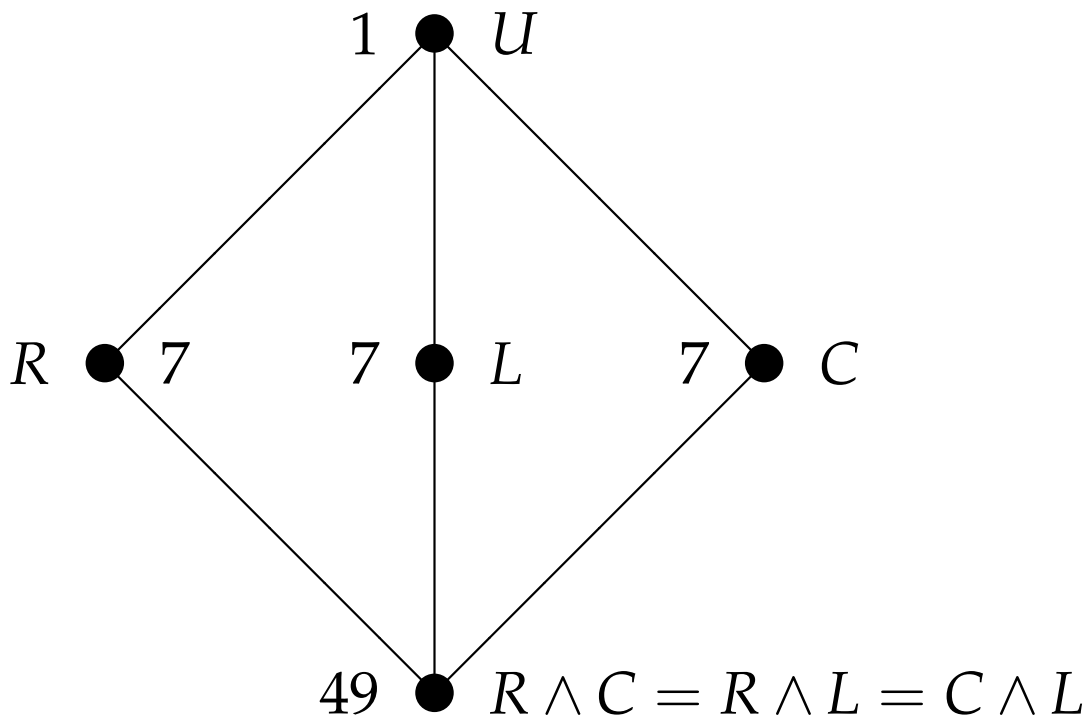
There are still other collections of mutually orthogonal factors which obey most of the theory but do not come from pre-factors.
For example, the Rows (R), Columns (C) and Letters (L) of a 7×7 Latin square give the following.



Combining two factors: II

If A and B are factors then their infimum $A \wedge B$ satisfies:

- $A \wedge B$ is finer than A, and $A \wedge B$ is finer than B;
- if any other factor is finer than A and finer than B then it is finer than $A \wedge B$.

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Each level of factor $A \vee B$ combines levels of A
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Each level of factor $A \vee B$ combines levels of A and also combines levels of B, and has replication as small as possible subject to this.

I claim that the supremum is even more important than the infimum in designed experiments and data analysis.

Factorial treatments plus control

		Chemical				
		Z	N	S	K	M
Dose	0	✓				
	1		✓	✓	✓	✓
	2		✓	✓	✓	✓

$$\text{Dose} \vee \text{Chemical} = \text{Fumigant},$$

which is the two-level factor distinguishing zero treatment from the rest.

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If you do not fit Fumigant, its effect will be included in whichever of Dose and Chemical you fit first.

Hasse diagram including supremum

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		Z	N	S	K	M
Dose	0	✓				
	1		✓	✓	✓	✓
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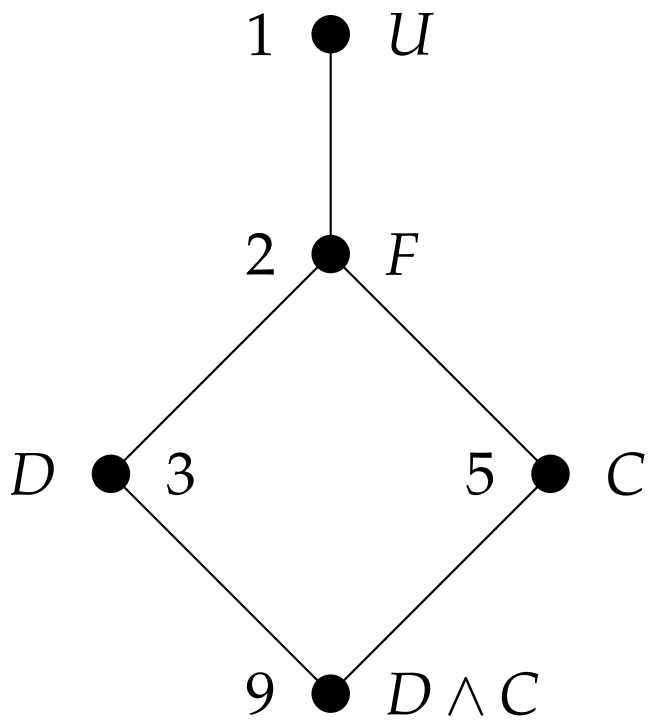

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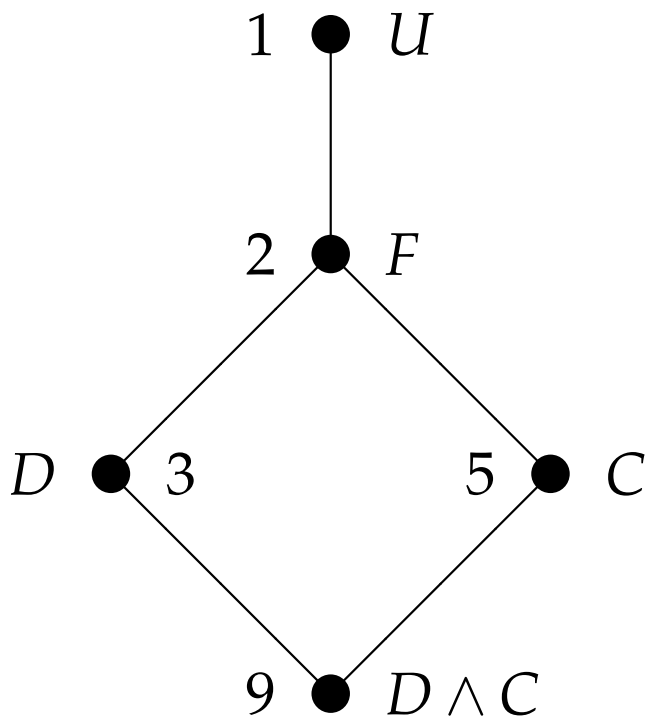

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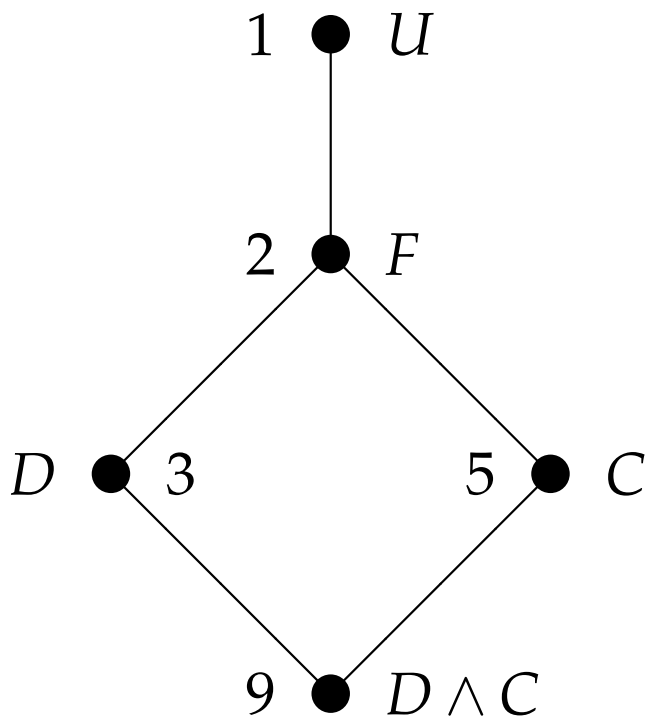

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Heiko Großmann's software includes suprema
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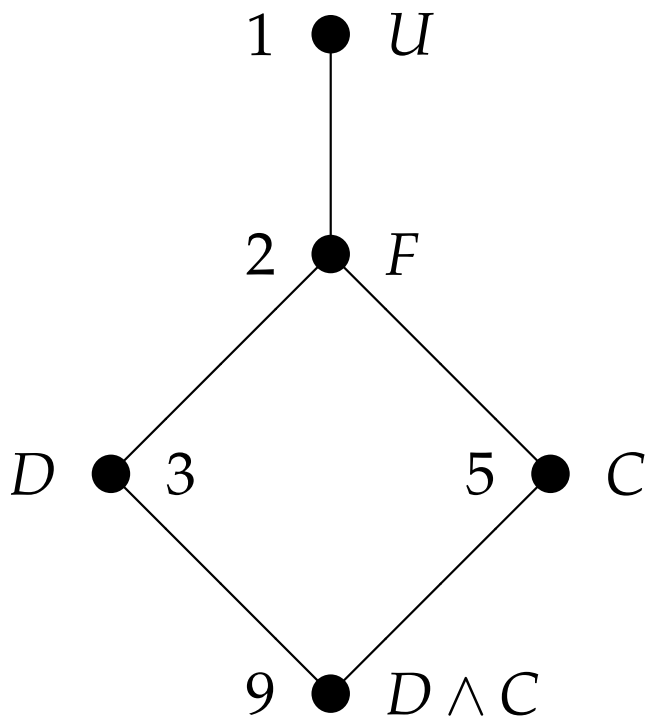

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Does yours?

Linear model for two factors

Given two treatment factors A and B, the linear model for response Y_ω on unit ω is often written as follows.
If $A(\omega) = i$ and $B(\omega) = j$ then

$$Y_\omega = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_\omega,$$

where the ε_ω are random variables with zero means and a covariance matrix whose eigenspaces we know.

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Some authors: "Too many parameters! Let's impose constraints."

(a) $\sum_i \alpha_i = 0$, and so on

(b) $\sum_i r_i \alpha_i = 0$, where $r_i = |\{\omega : A(\omega) = i\}|$, and so on.

Linear model with constraints: bad consequences

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Popular software allows both of these.

Say goodbye to linear models with constraints

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(I read this in one of his papers, but could not find it again when preparing these slides.)

RAB's approach to such linear models

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Let us parametrize these subspaces separately,
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This is the approach which I always use in teaching and in consulting, and in my 2008 book.

Expectation subspaces

$$\mathbb{E}(\mathbf{Y}) \in V_A \quad \Longleftrightarrow \quad \text{there are constants } \alpha_i \text{ such that } \mathbb{E}(Y_\omega) = \alpha_i \text{ whenever } A(\omega) = i.$$

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$$\mathbb{E}(\mathbf{Y}) \in V_{A\wedge B} \iff \text{there are constants } \gamma_{ij} \text{ such that } \mathbb{E}(Y_\omega) = \gamma_{ij} \text{ if } A(\omega) = i \text{ and } B(\omega) = j.$$

Dimensions

For general factors A and B:

$$\dim(V_A + V_B) = \dim(V_A) + \dim(V_B) - \dim(V_A \cap V_B).$$

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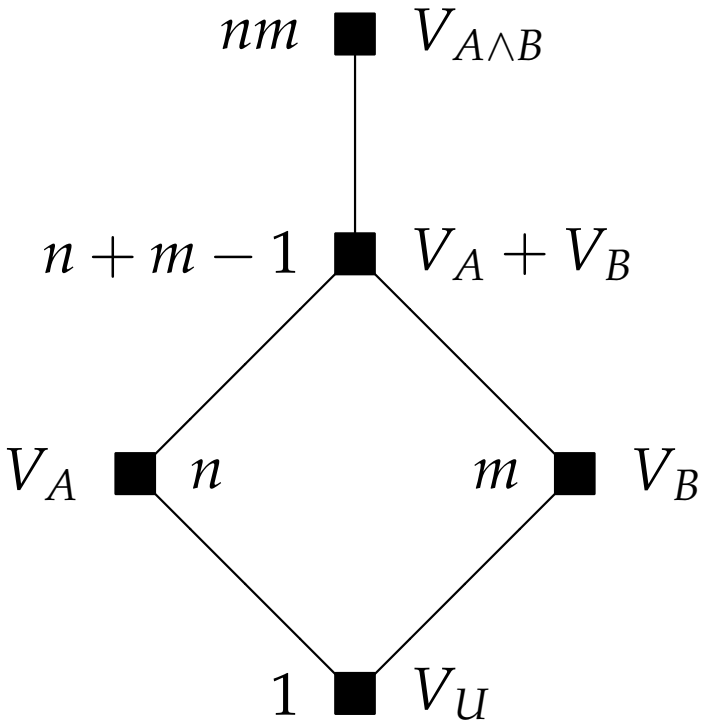
which has dimension 1, so

$$\dim(V_A + V_B) = \dim(V_A) + \dim(V_B) - 1.$$

Another partial order; another Hasse diagram

The relation "is contained in" gives a partial order on subspaces of a vector space.
So we can use a Hasse diagram to show the subspaces being considered to model the expectation of $\mathbf{Y}$.
Now it is helpful to show the dimension of each subspace on the diagram.

Hasse diagram for model subspaces



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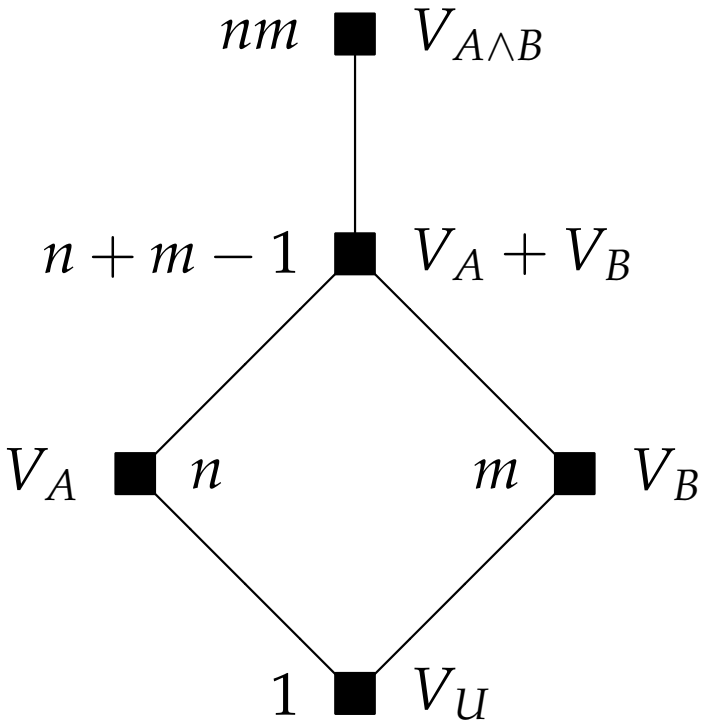
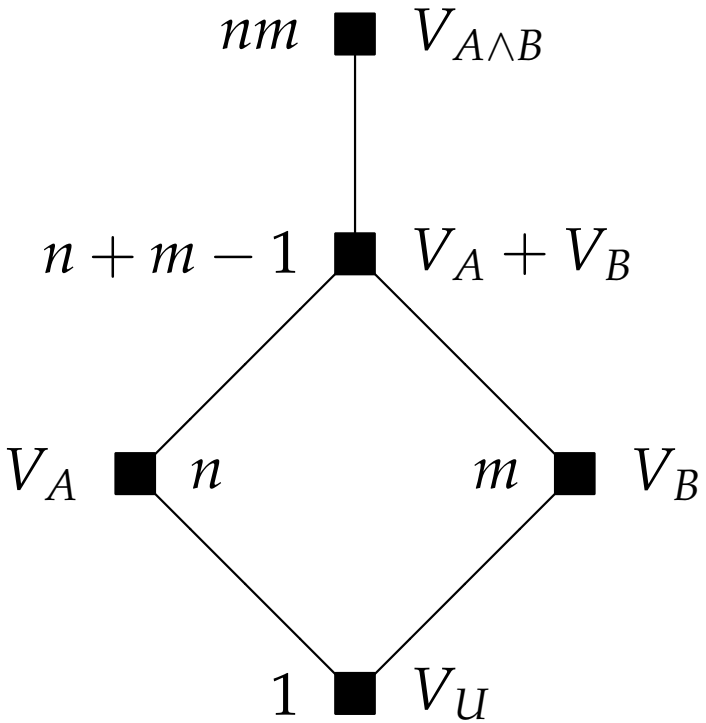


null model

Hasse diagram for model subspaces



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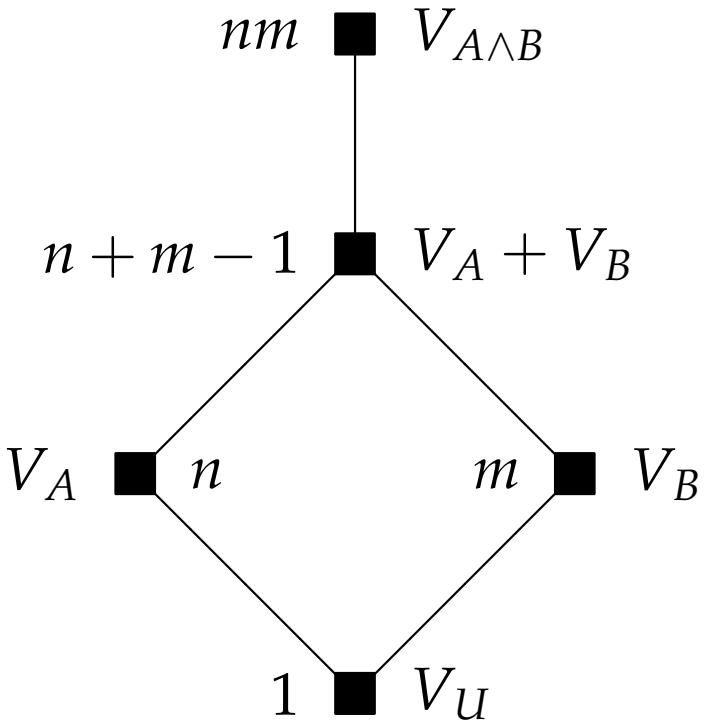

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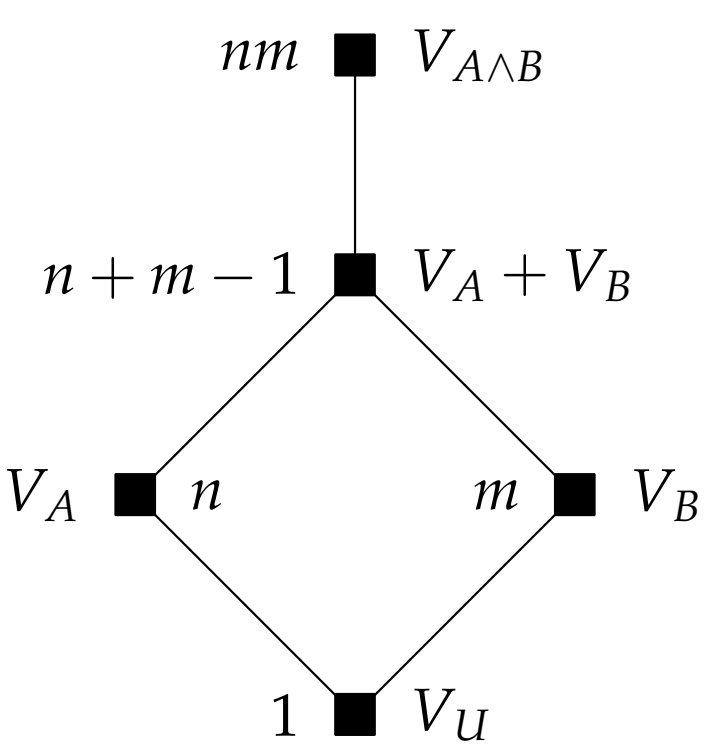

full model

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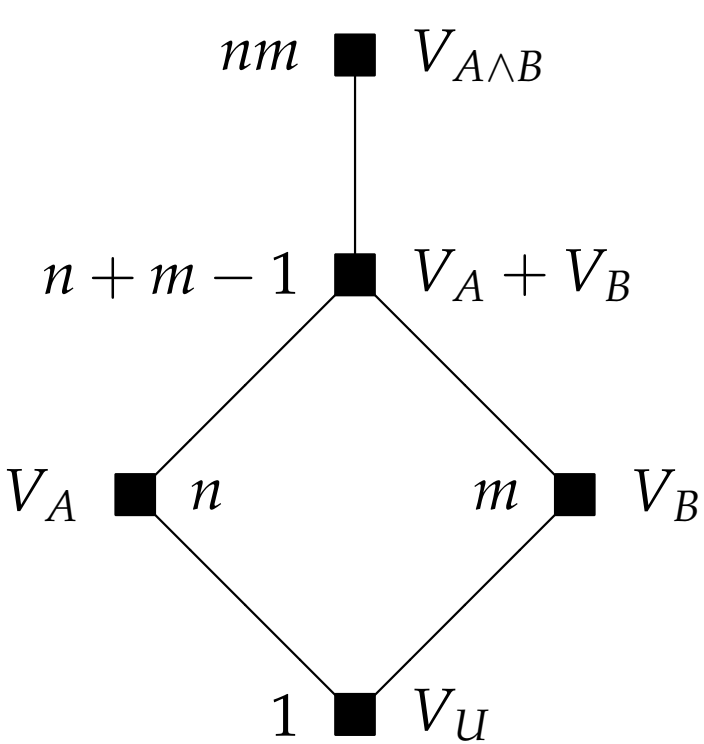

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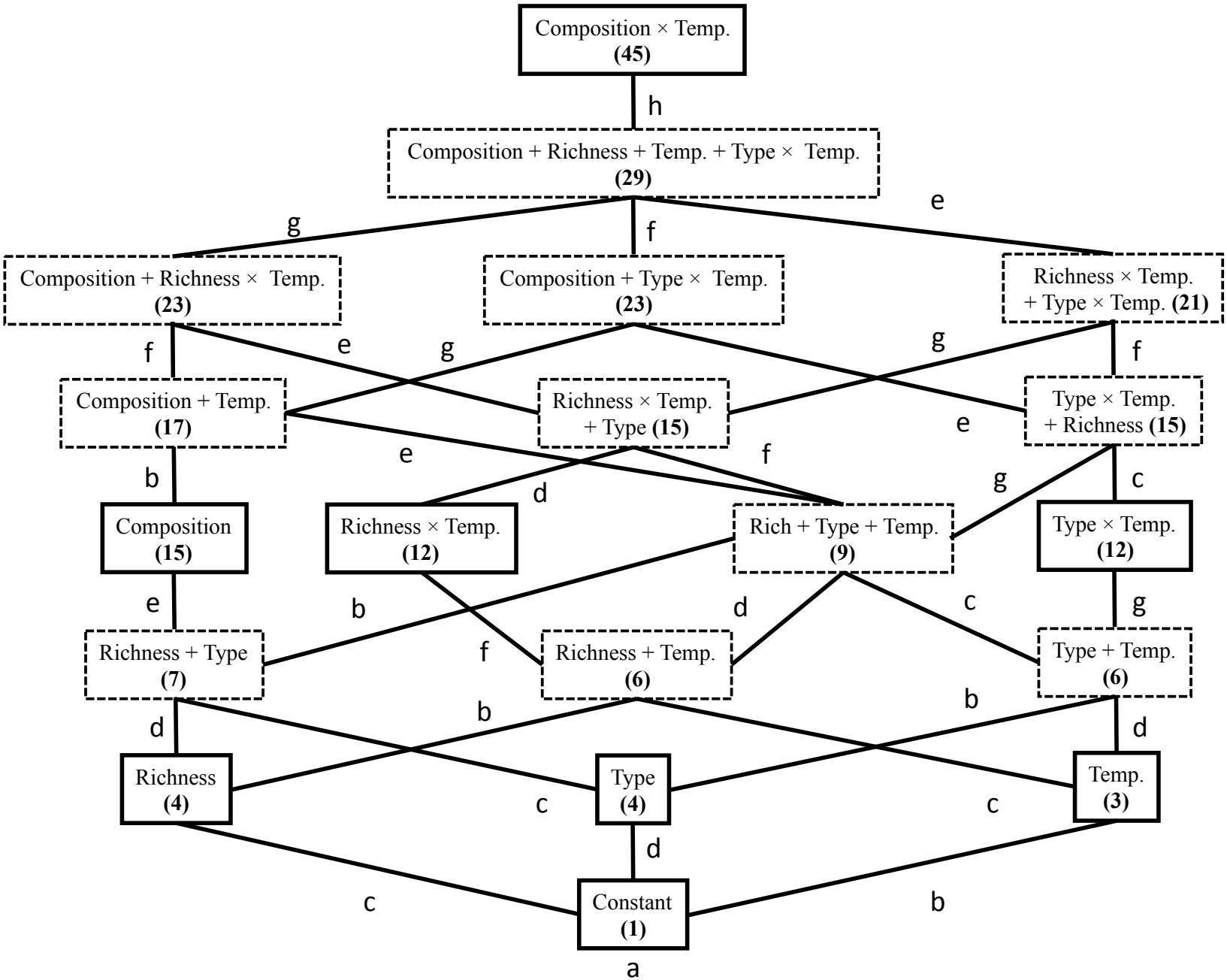
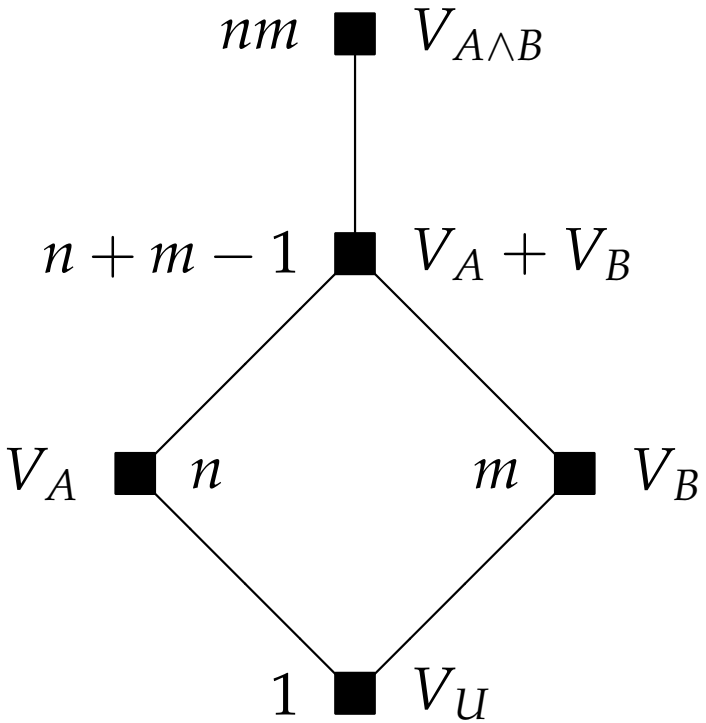
For complicated families of models,
non-mathematicians may find the Hasse diagram
easier to understand than the equations.

Diagram from a paper in *Global Change Biology*



Main effects and interaction



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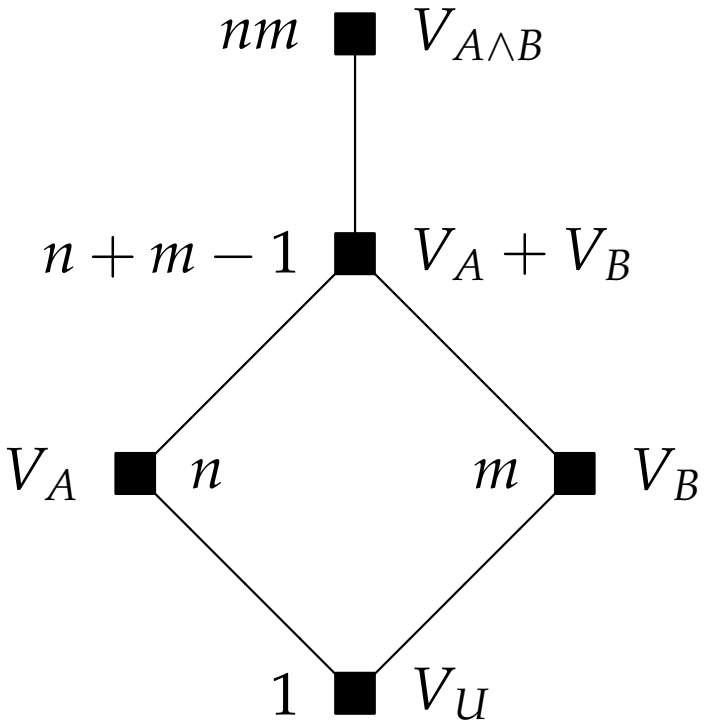

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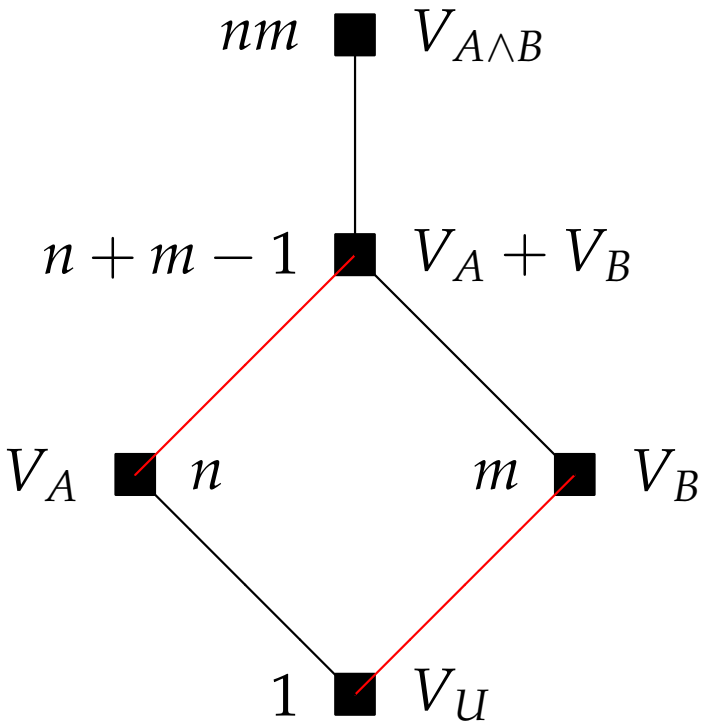

The main effect of factor B is the difference between the vector of fitted values in V_B and the vector of fitted values in V_U.

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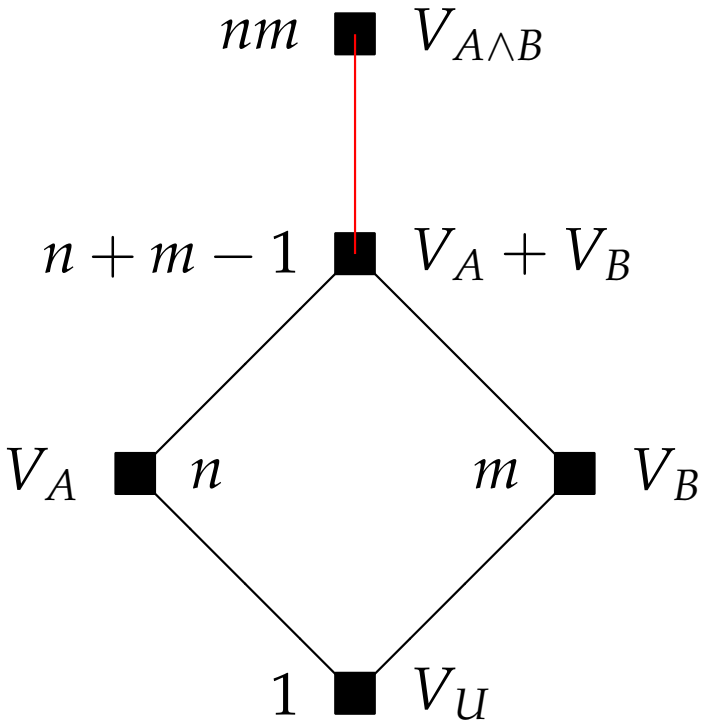

The interaction between factors A and B is the difference between the vector of fitted values in $V_{A\wedge B}$ and the vector of fitted values in $V_A + V_B$.

The main effect of factor B is the difference between the vector of fitted values in V_B and the vector of fitted values in V_U.

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Example with two treatment factors: feeding chickens

Four diets for feeding newly-hatched chickens were compared. The diets consisted of all levels of Protein (groundnuts or soya bean) with two levels of Fishmeal (added or not). Each diet was fed to two chickens, and they were weighed at the end of six weeks.

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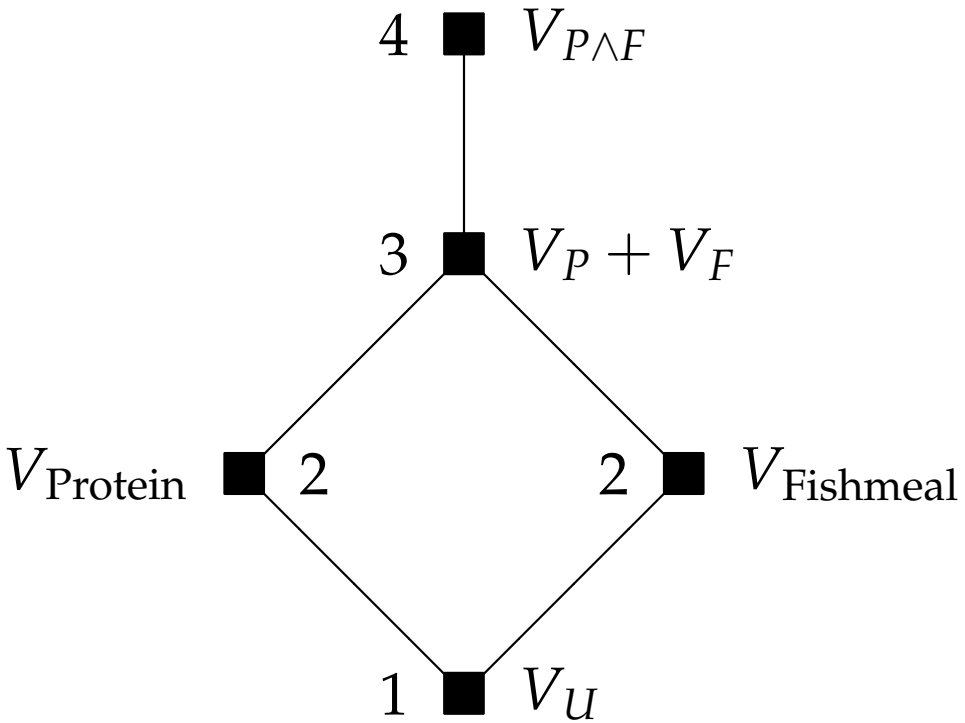

Chicken example: anova

Source	SS	df	MS	VR
Protein	4704.5	1	4704.50	35.57
Fishmeal	3120.5	1	3120.50	23.60
Protein $\wedge$ Fishmeal	128.0	1	128.00	0.97
residual	529.0	4	132.25	

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You know how to interpret the anova table:
do the scientists who did the experiment know how to?

Scaling the Hasse diagram of expectation subspaces

Suppose that V_1 and V_2 are expectation subspaces,
with $V_1 < V_2$,
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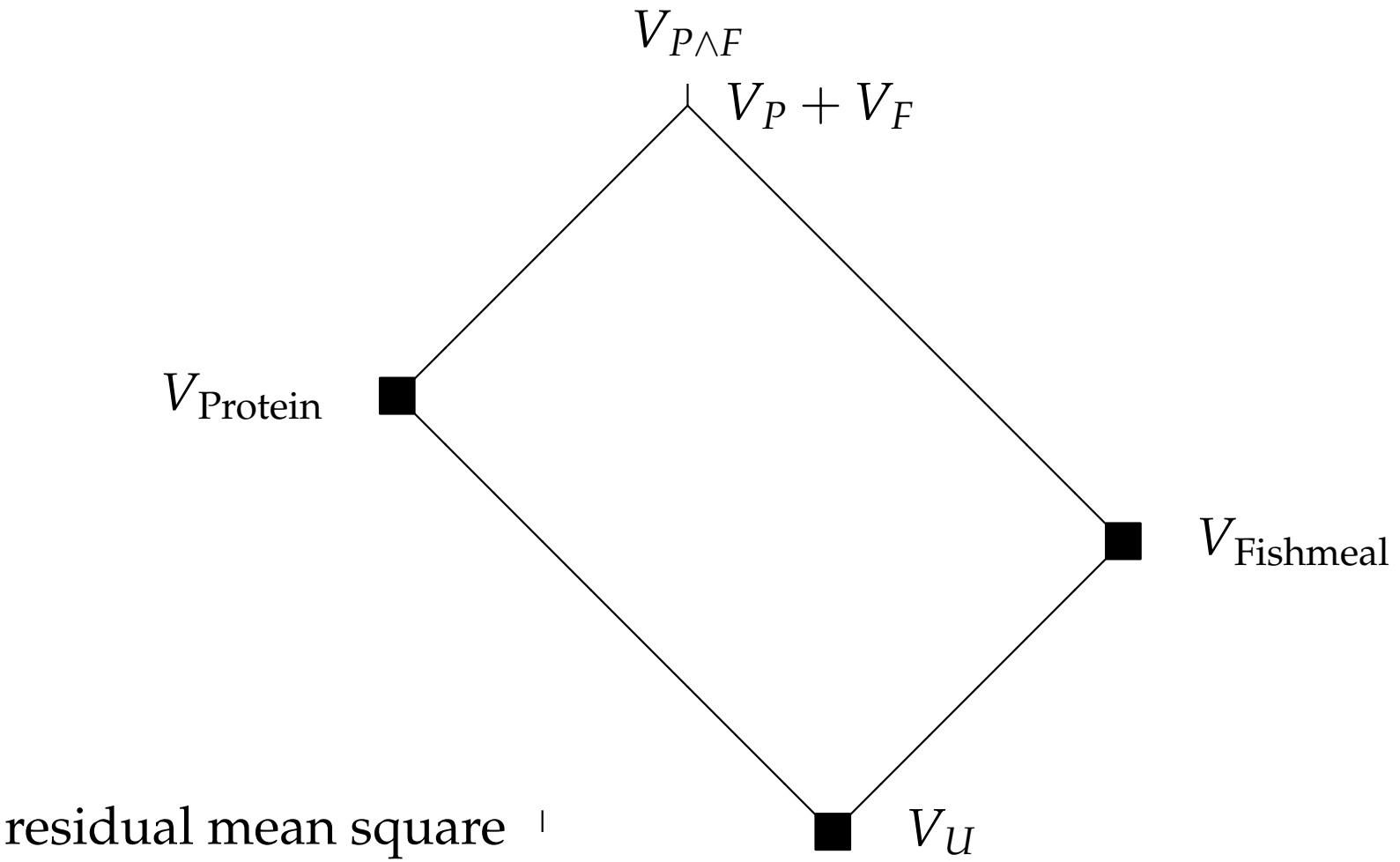
Scale the Hasse diagram so that each edge has length
proportional to the relevant mean square,
and show the residual mean square to give a scale.

Chickens: scaled Hasse diagram of expectation subspaces



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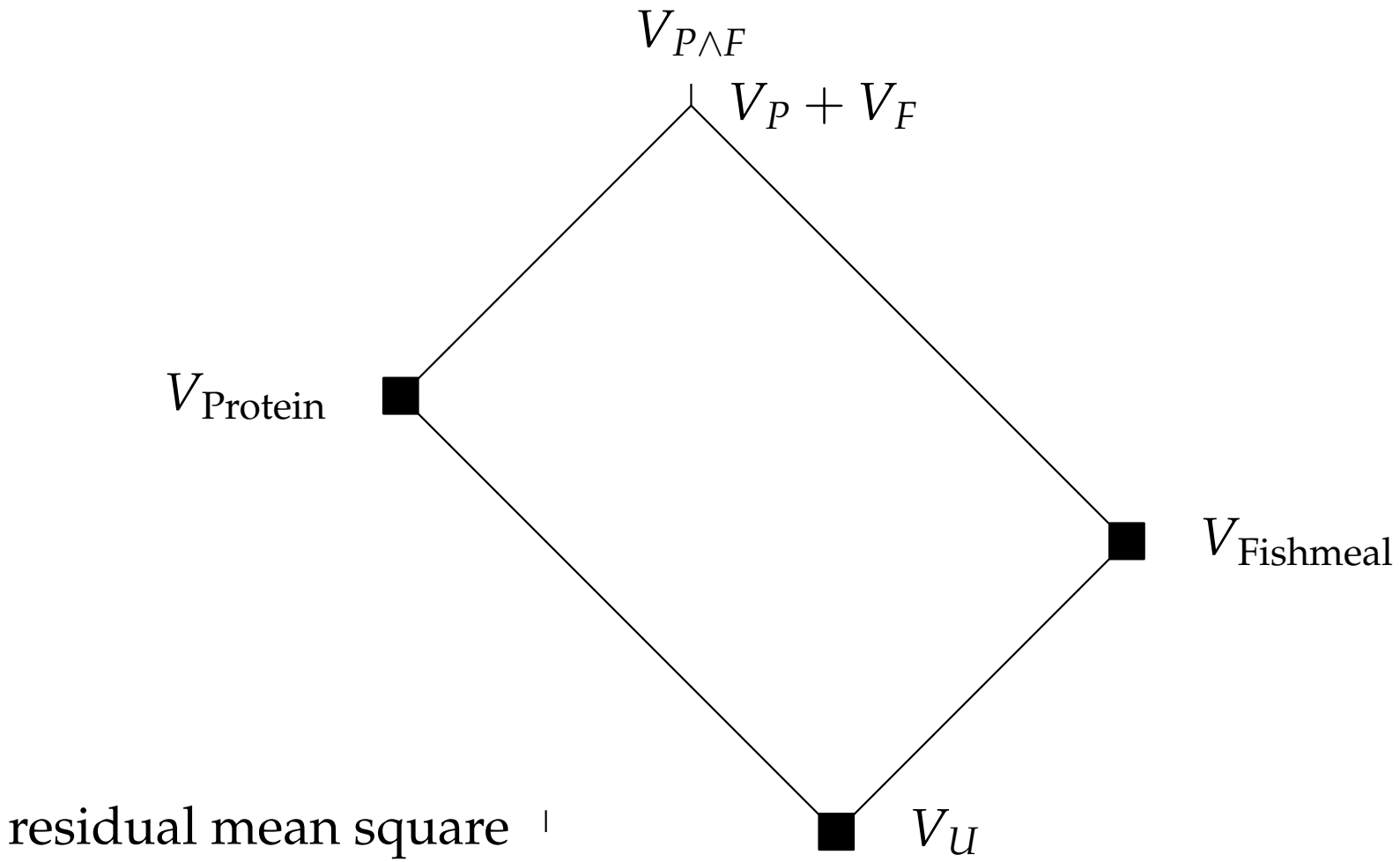

There is no evidence of any interaction, so we can simplify to the additive model.

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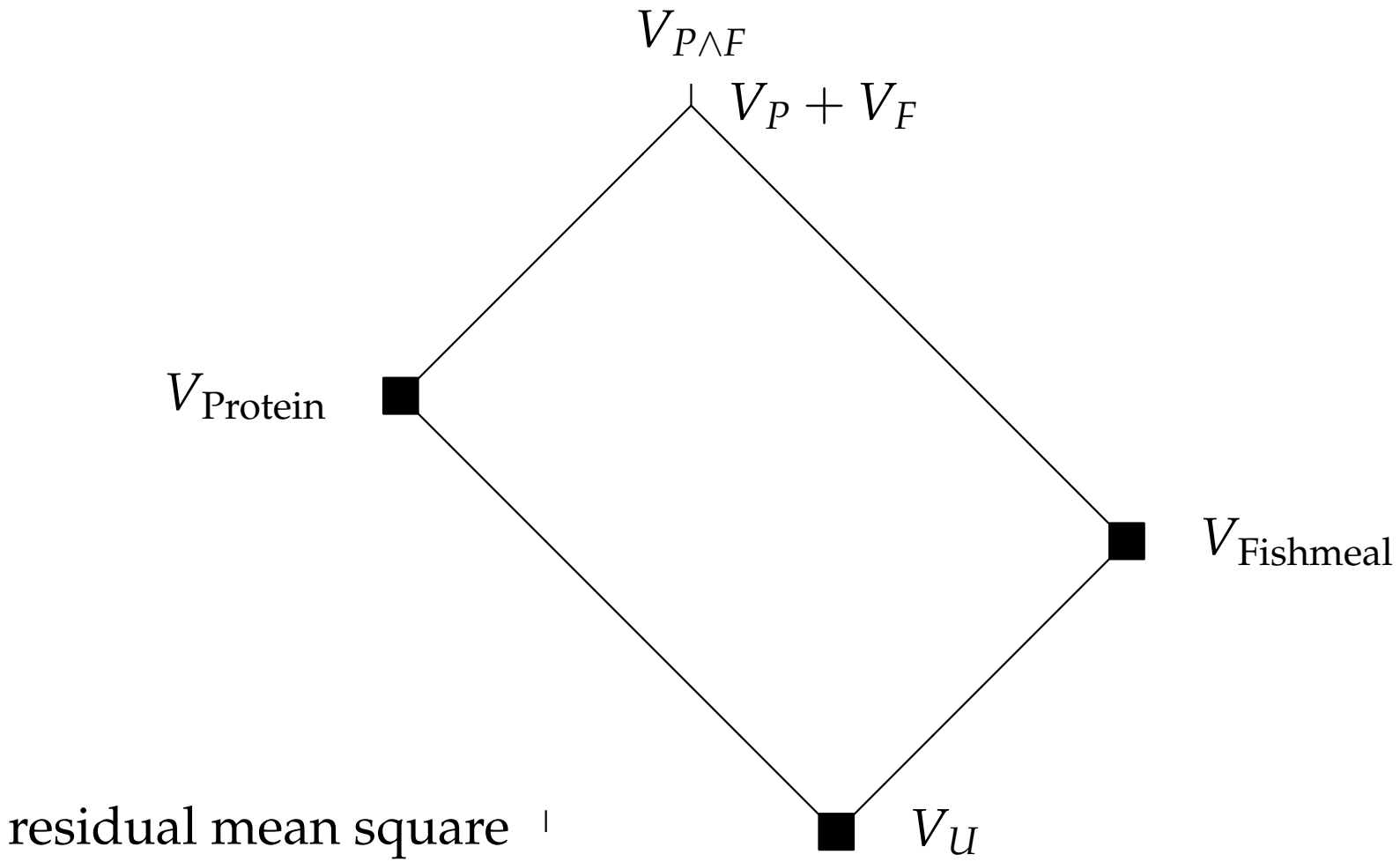

There is no evidence of any interaction, so we can simplify to the additive model.
Neither main effect is zero, so we cannot simplify further.

Example: an experiment about protecting metal

An experiment was conducted to compare two protective dyes for metal, both with each other and with no dye. Ten braided metal cords were broken into three pieces. The three pieces of each cord were randomly allocated to the three treatments. After the dyes had been applied, the cords were left to weather for a fixed time, then their strengths were measured, and recorded as a percentage of the nominal strength specification.

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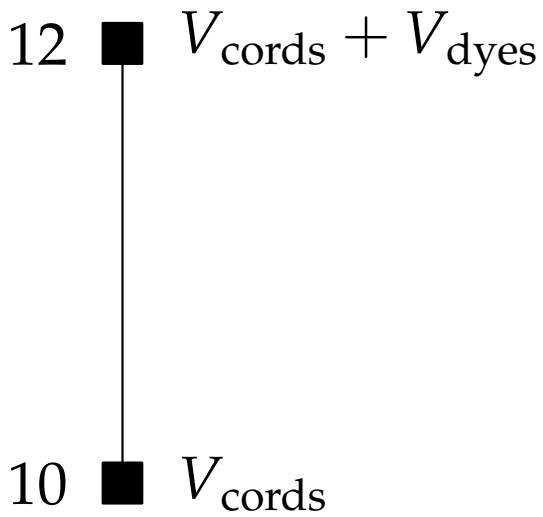
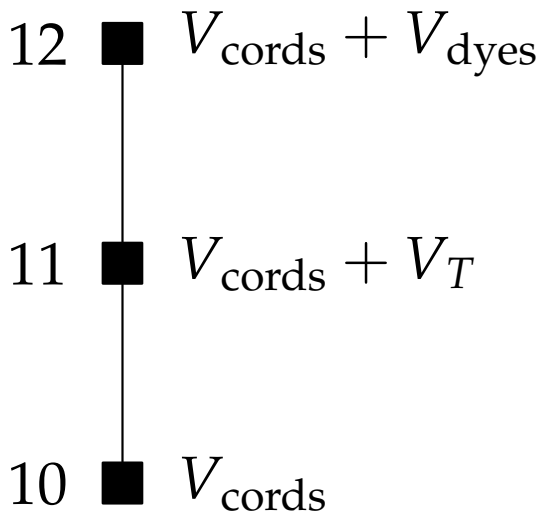
Factors: Dye, with three levels (no dye, dye A, Dye B);
Cords, with ten levels;
U, with one level; E, with 30 levels.

Cords: Hasse diagram of expectation subspaces



We assume that there are differences between cords, so all the models that we consider include V_{cords}.

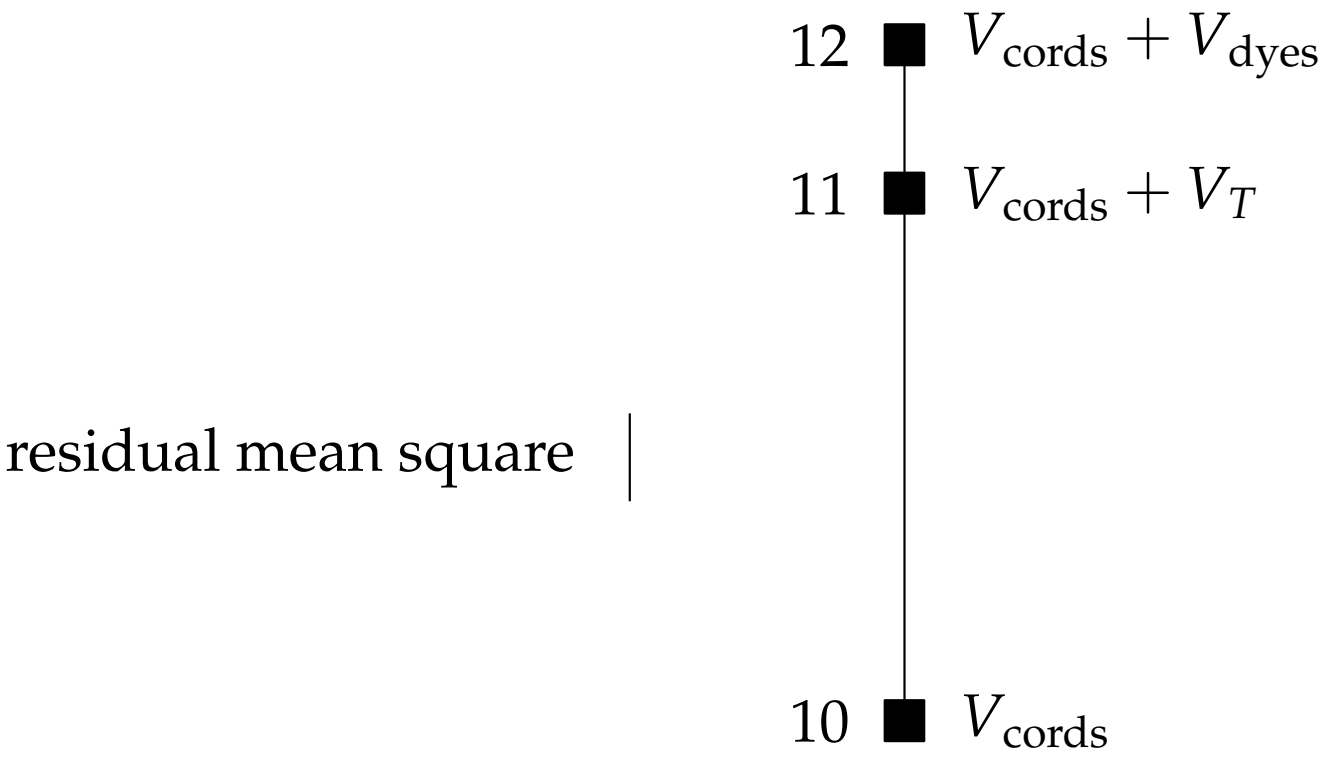
Cords: Hasse diagram of expectation subspaces



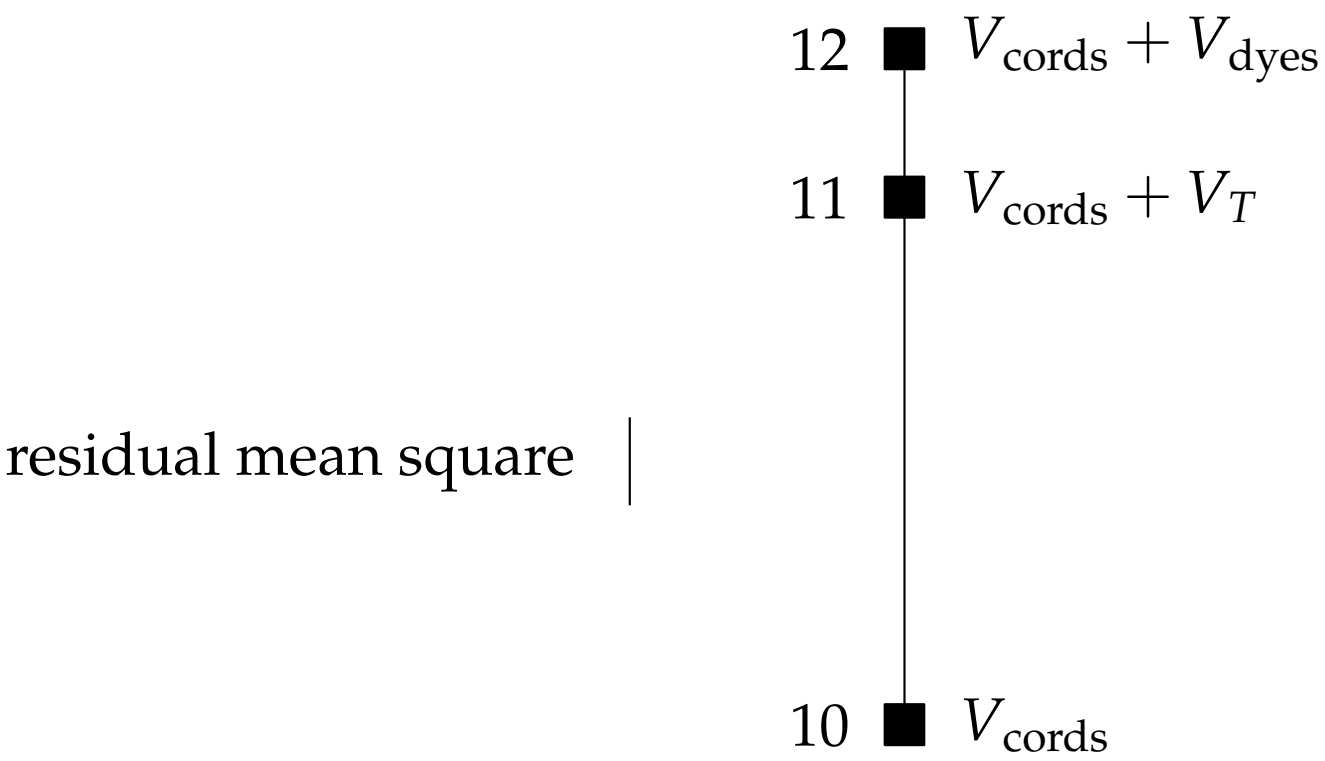
We assume that there are differences between cords,
so all the models that we consider include V_{cords}.

There is another factor T (To-dye-or-not-to-dye).
It has one level on 'no dye' and another level on both real dyes.

Cords: Scaled Hasse diagram of expectation subspaces



Cords: Scaled Hasse diagram of expectation subspaces



There is no evidence of a difference between dye A and dye B; but there is definitely a difference between no dye and real dyes.

Using scaled Hasse diagrams

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These diagrams can be extended to deal
with non-orthogonal models,
and with situations with more than one residual mean square
(use different colours for the corresponding edges).

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