Bases of primitive linear groups II

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Abstract

We correct and improve a result in [3], giving the structure of finite primitive linear groups of unbounded base size. This confirm a well-known conjecture of Pyber on base sizes of primitive permutation groups in the case of affine groups whose associated linear group is primitive.

1 Introduction

Let $V$ be a finite vector space, and $H$ a subgroup of $GL(V)$. A base for $H$ is a subset of $V$ whose pointwise stabilizer in $H$ is trivial. Denote by $b(H)$ the minimal size of a base for $H$. Theorem 1 of [3] gives an upper bound for $b(H)$ in the case where $H$ acts irreducibly and primitively on $V$, of the form $b(H) \leq \frac{18\log|H|}{\log|V|} + c$, where $c$ is an explicit absolute constant. This confirms part of a well-known conjecture of Pyber [4] on base sizes of primitive permutation groups.

The proof of [3, Theorem 1] relied on Theorem 2 of that paper, a result which gives the structure of primitive linear groups of unbounded base size. Unfortunately this theorem is not correctly stated: the tensor product in part (i) of the conclusion is supposed to be defined over the prime field $\mathbb{F}_p$, but this is not possible in general, as a tensor decomposition of a vector space over an extension of $\mathbb{F}_p$ does not yield a tensor decomposition over $\mathbb{F}_p$. The purpose of this paper is to prove a corrected and improved version of [3, Theorem 2]. This is done in Theorem 1 and Proposition 2 below: these correspond to parts (i) and (ii) of [3, Theorem 2].

Corollary 3, which is a very slightly amended version of Theorem 1 of [3], can readily be deduced from these results, and we do this at the end of the paper.

Before stating the results we need a few definitions. If $V = V_d(q)$ is a vector space of dimension $d$ over the finite field $\mathbb{F}_q$ of characteristic $p$, and $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$, then $\text{Cl}_d(q_0)$ denotes the normalizer in $GL_d(q)$ of one of the insoluble classical groups $SU_d(q_0)$, $Sp_d(q_0)$, $\Omega_d(q_0)$ (where in the last case $q_0$ is odd if $d$ is odd, and both types $\Omega^+_d(q_0)$ are included if $d$ is even). For the symmetric group $\text{Sym}(m)$ of degree $m$, by the natural module over $\mathbb{F}_q$ we mean the nontrivial irreducible constituent of the usual $m$-dimensional permutation module over $\mathbb{F}_q$; it has dimension $m' = m - \delta(p,m)$, where $\delta(p,m)$ is 2 if $p|m$ and 1 otherwise. We denote by $\text{Alt}(m)$ the alternating group of degree $m$.

For $H \leq GL(V)$, define $b^*(H)$ to be the minimal size of a set $B$ of vectors such that any element of $H$ which fixes every 1-space $\langle v \rangle$ with $v \in B$ is necessarily a
scalar multiple of the identity. By [3, 3.1] we have \( b(H) \leq b^*(H) \leq b(H) + 1 \). Also \( H^{(\infty)} \) denotes the last term in the derived series of \( H \).

Let \( V = V_d(q) \) have a tensor decomposition \( V = V_1 \otimes \cdots \otimes V_t \) over \( \mathbb{F}_q \). For subgroups \( H_i \leq GL(V_i) \) (1 \( \leq i \leq t \)), define \( H_1 \otimes \cdots \otimes H_t = \bigotimes_{i=1}^t H_i \) to be the subgroup of \( GL(V) \) consisting of all elements \( h_1 \otimes \cdots \otimes h_t \) \((h_i \in H_i)\), defined by setting
\[
(v_1 \otimes \cdots \otimes v_t)(h_1 \otimes \cdots \otimes h_t) = v_1 h_1 \otimes \cdots \otimes v_t h_t
\]
for \( v_i \in V_i \).

We define a constant \( C \) just as in [3, p.98], as follows. First, it is shown in [3, 3.6] that if \( H \) is a primitive subgroup of \( GL(V) \) such that the Fitting subgroup \( F(H) \) is irreducible on \( V \), then \( b^*(H) \) is bounded above by an absolute constant; define \( C_1 \) to be the maximum value of \( b^*(H) \) over all such \( H, V \). Next, [3, 2.2] shows that if \( H \leq GL(V) \) with \( E(H) \) quasisimple and absolutely irreducible on \( V \) (where \( E(H) \) is the group generated by all quasisimple subnormal subgroups of \( H \)), and \( E(H) \) is not \( Alt(m) \) or \( Cl_d(q_0) \) with \( V \) the natural module over \( \mathbb{F}_q \), then \( b^*(H) \) is bounded above by an absolute constant; define \( C_2 \) to be the maximum value of \( b^*(H) \) over all such \( H, V \). Finally, set
\[
C = \max\{C_1, C_2, 33\}.
\]

**Theorem 1** Let \( V = V_d(q) \), and let \( H \) be a subgroup of \( \Gamma L(V) \) such that \( H \) acts primitively on \( V \) and \( H^0 := H \cap GL(V) \) is absolutely irreducible on \( V \). Suppose that \( b^*(H^0) > C \). Then
\[
H^0 \leq H_0 \otimes \bigotimes_{i=1}^s \text{Sym}(m_i) \otimes \bigotimes_{i=1}^t Cl_{d_i}(q_i),
\]
where \( s + t \geq 1 \) and the following hold:

(i) \( H_0 \leq GL_{d_0}(q) \) with \( b^*(H_0) \leq C \)

(ii) each factor \( \text{Sym}(m_i) \) is \( GL_{m'_i}(q) \), where \( m'_i = m_i - \delta(p, m_i) \) as above

(iii) each factor \( Cl_{d_i}(q_i) \leq GL_{d_i}(q) \) as above

(iv) \( d = d_0 \cdot \prod_{i}^s m'_i \cdot \prod_{i}^t d_i \)

(v) the integers \( m'_1, \ldots, m'_s, d_1, \ldots, d_t \) are all distinct

(vi) \( F^*(H^0) \) contains \( \prod_{i}^s Alt(m_i) \cdot \prod_{i}^t Cl_{d_i}(q_i)^{(\infty)} \).

Note that any irreducible primitive linear group \( H \leq GL_n(p) \) (\( p \) prime) satisfies the hypotheses of the first sentence of this theorem: for if we choose \( q = q' \) maximal such that \( H \leq \Gamma L_d(q) \leq GL_n(p) \), where \( n = d r \), then \( H^0 := H \cap GL_d(q) \) is absolutely irreducible on \( V_d(q) \) by [2, 12.1].

**Proposition 2** Let \( H, H^0 \) be as in Theorem 1, with \( b^*(H^0) > C \). Take \( m'_s = \max(m'_i : 1 \leq i \leq s) \) and \( d_t = \max(d_i : 1 \leq i \leq t) \) (define these to be 0 if \( s = 0 \) or \( t = 0 \), respectively).

(i) Suppose \( t \geq 1 \) and \( m'_s \leq d_t \), and let \( q = q'_t \). Then \( d < d_t^2 \), and
\[
b^*(H^0) \leq b^*(GL_{d/d_t}(q) \otimes GL_{d_t}(q_t)) \leq \frac{9d_t^2}{dr} + 22.
\]
(ii) Suppose $s \geq 1$ and $m'_s > d_t$, and let $q = p^r$. Then $d < (m'_s)^2$, and

$$b^*(H^0) \leq b^*(GL_{d/m'_s}(q) \otimes \text{Sym}(m_s)) \leq \frac{3m_s \log_p m_s}{dr} + 22.$$ 

**Corollary 3** Suppose $H \leq GL(V)$ is an irreducible, primitive linear group on a finite vector space $V$. Then either

(i) $b(H) \leq C + 1$, or

(ii) $b(H) < 18 \frac{\log |H|}{\log |V|} + 30$.

2 Proofs

**Proof of Theorem 1**

The proof goes by induction on $\dim V$. Assume first that there is a tensor decomposition $V = V_1 \otimes V_2$ over $\mathbb{F}_q$ with $\dim V_i > 1$ such that $H \leq N_{GL(V)}(GL(V_1) \otimes GL(V_2)) := N$. For $i = 1, 2$ let $\phi_i$ be the natural map $N \to PTL(V_i)$, define $H_i$ to be the full preimage in $\Gamma L(V_i)$ of $H \phi_i$, and let $H^{1,i} := H_i \cap GL(V_i)$, so that $H^0 \leq H^{0,1} \otimes H^{0,2}$.

We claim that $H^1$ is primitive on $V_i$, and that $H^{0,i}$ is absolutely irreducible on $V_i$, for $i = 1, 2$. The first assertion is straightforward, since if, say, $H^1$ preserves a direct sum decomposition $V_i = \bigoplus_j X_j \otimes V_2$, then $H$ preserves the decomposition $V = \bigoplus_j X_j \otimes V_2$, and so $r = 1$ as $H$ is primitive. For the second assertion, observe that if $K = C_{GL(V)}(H^{0,i})$, then $K \otimes 1$ centralizes $H^{0,1} \otimes H^{0,2}$, hence centralizes $H^0$; since $H^0$ is absolutely irreducible this implies that $K = \mathbb{F}_q^*$, hence $H^{0,i}$ is absolutely irreducible.

By the claim, we can apply induction to the groups $H^{0,i} \leq GL(V_i)$ for $i = 1, 2$. This gives

$$H^{0,1} \leq H_0^{(1)} \otimes \bigotimes \text{Sym}(m_i) \otimes \bigotimes \text{Cl}_{d_i}(q_i),$$

$$H^{0,2} \leq H_0^{(2)} \otimes \bigotimes \text{Sym}(m'_i) \otimes \bigotimes \text{Cl}_{d'_i}(q'_i),$$

where $b^*(H_0^{(i)}) \leq C$ for $i = 1, 2$. As argued on p.110 of [3], we can assume that all the numbers $m_i, m'_i, d_i, d'_i$ are distinct; also $b^*(H_0^{(1)} \otimes H_0^{(2)}) \leq C$ by [3, 3.3(ii)]. Since $H^0 \leq H^{0,1} \otimes H^{0,2}$, the conclusion of Theorem 1 therefore holds.

Hence we may assume from now on that there is no nontrivial tensor decomposition of $V$ over $\mathbb{F}_q$ preserved by $H$. By [2, 12.2], it follows that if $N$ is a normal subgroup of $H$ such that $N \leq H^0$ and $N \not\subseteq Z(H^0)$, then $V \downarrow N$ is absolutely irreducible.

Now $H$ is insoluble, since otherwise $b(H) \leq 4$ by [5]. Let $Z = Z(H^0)$ and let $S$ be the socle of $H/Z$. Write $S = M_1 \times \cdots \times M_k$ where each $M_i$ is a minimal normal subgroup of $H/Z$. Let $R$ be the full preimage of $S$ in $H$, and $P_i$ the preimage of $M_i$, so that $R = P_1 \cdots P_k$, a commuting product. Clearly $R \cap H^0 \neq 1$, so we may take $P_1 \leq H^0$. By the previous paragraph, $V \downarrow P_1$ is absolutely irreducible.

If $P_1/Z$ is abelian then $b^*(H^0)$ is bounded, by [3, 3.6] — indeed, $b^*(H^0) \leq C$ by definition of $C$, which is a contradiction.

Hence $P_1/Z \cong T^t$, where $T$ is a non-abelian simple group. If $t > 1$, then [1, 3.16, 3.17] implies that $P_1$ preserves a tensor decomposition $V = V_1 \otimes \cdots \otimes V_t$ with $\dim V_i$ independent of $i$, and $H^0 \leq N_{GL(V)}(\bigotimes GL(V_i))$; but then $b(H^0) \leq 4$ by [3, 3.5].
Hence $t = 1$. Now [3, 2.2], together with the definition of $C$, implies that $E(H^0)$ is either $\text{Alt}(m)$ (with $d = m - \delta(p, m)$) or $\text{Cl}_d(q_0)$, as in the conclusion of Theorem 1. This completes the proof.

**Proof of Proposition 2**

The proof runs along similar lines to that of [3, Theorem 2(ii)], but there are a few differences, so we give it in full here. Let $H, H^0$ be as in Theorem 1, with $b^*(H^0) > C$. The proof goes by induction on $s + t$.

Consider the base case $s + t = 1$ we have $H^0 \leq H_0 \otimes M$ where $M = \text{Cl}_{d_1}(q_1)$ or $\text{Sym}(m_1)$. Write $m = d_1$ or $m_1'$, respectively, so that $d = d_0m$. By [3, 3.7], we have $b(M) \leq \frac{3m}{r} + 5$ (where $q = q_1^t$) or $\frac{\log m}{r} + 5$ (where $q = p^r$), respectively.

Assume $d_0 > m$. If $b^*(H_0) > m$ then by [3, 3.3(ii)],

$$b^*(H^0) \leq \max\{b^*(H_0), b^*(M)\} \leq \max\{b^*(H_0), m + 1\} = b^*(H_0) \leq C,$$

which is a contradiction. And if $b^*(H_0) \leq m$ then [3, 3.3(iv)] implies that $b(H^0) \leq 3$, also a contradiction.

Therefore $d_0 \leq m$. Also $b^*(M) > d_0$, again by [3, 3.3(iv)]. Hence [3, 3.3(iii)] gives $b(H^0) \leq 3(1 + \frac{b^*(M)}{d_0})$. If $M = \text{Cl}_{d_1}(q_1)$, then $b^*(M) \leq b(M) + 1 \leq \frac{3m}{r} + 6$, so this gives

$$b(H^0) \leq 3(1 + \frac{3m + 6r}{rd_0}) \leq \frac{9m^2}{rd} + 21,$$

which yields part (i) of the proposition. Similarly part (ii) holds when $M = \text{Sym}(m_1)$.

Now assume $s + t \geq 2$. Let $m$ be the maximum of $d_1$ and $m_1'$, and write $M$ for the corresponding group $\text{Cl}_{d_1}(q_1)$ or $\text{Sym}(m_1)$. Let $N$ be the tensor product of $H_0$ and the other factors $\text{Cl}_{d_1}(q_1), \text{Sym}(m_1)$, so that $H^0 \leq N \otimes M$. If $b^*(N) \leq C$ the conclusion follows as in the $s + t = 1$ case, so assume $b^*(N) > C$.

Let $m'$ be the largest among the dimensions $d_1, m_1'$ omitting $m$, and write $N_1$ for the corresponding group $\text{Cl}_{d_1}(q_1)$ or $\text{Sym}(m_1)$.

Consider the case where $N_1 = \text{Cl}_{d_1}(q_1)$. Let $q = q_1^t$. By induction we have

$$b^*(N) \leq 9\frac{d_1^2m}{d_1u} + 22 \leq 9\frac{d_1}{u} + 22.$$

Suppose $d \geq m^2$. Then $b^*(N) \geq m$ by [3, 3.3(iv)], so [3, 3.3(iii)] implies that

$$b^*(H^0) \leq 3(1 + \frac{9d_1 + 22u}{um}).$$

Since $m \geq d_1$ and $m > 22$ (otherwise [3, 3.3] can easily be used to deduce that $b^*(H^0) < C$), this yields $b^*(H^0) < 33 \leq C$, a contradiction. Hence $d < m^2$ in this case. Now the conclusion of the proposition follows by the argument given for the $s + t = 1$ case.

Finally, consider the case where $N_1 = \text{Sym}(m_1)$. Let $q = p^r$. By induction,

$$b^*(N) \leq \frac{(3m_1 \log p m_1) \cdot m}{dr} + 22 \leq \frac{3 \log p m_1}{r} + 22.$$

Now the argument of the previous paragraph gives the conclusion.
This completes the proof of Proposition 2.

Proof of Corollary 3
Let $V = V_n(q_0)$, and suppose $H \leq GL(V)$ acts primitively and irreducibly on $V$. Choose $q = q_0^{\ast}$ maximal such that $H \leq \Gamma L_d(q) \leq GL_n(q_0)$, where $n = dr$. Write $H^0 = H \cap GL_d(q)$ and $V = V_d(q)$. By [2, 12.1], $H^0$ is absolutely irreducible on $V$.

If $b^*(H^0) \leq C$ then $b(H) \leq C + 1$, as in part (i) of the corollary. So assume that $b^*(H^0) > C$. Then $H^0$ is given by Theorem 1 of this paper. Now the proof that $H$ satisfies (ii) proceeds just as in [3, p.112].

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References


