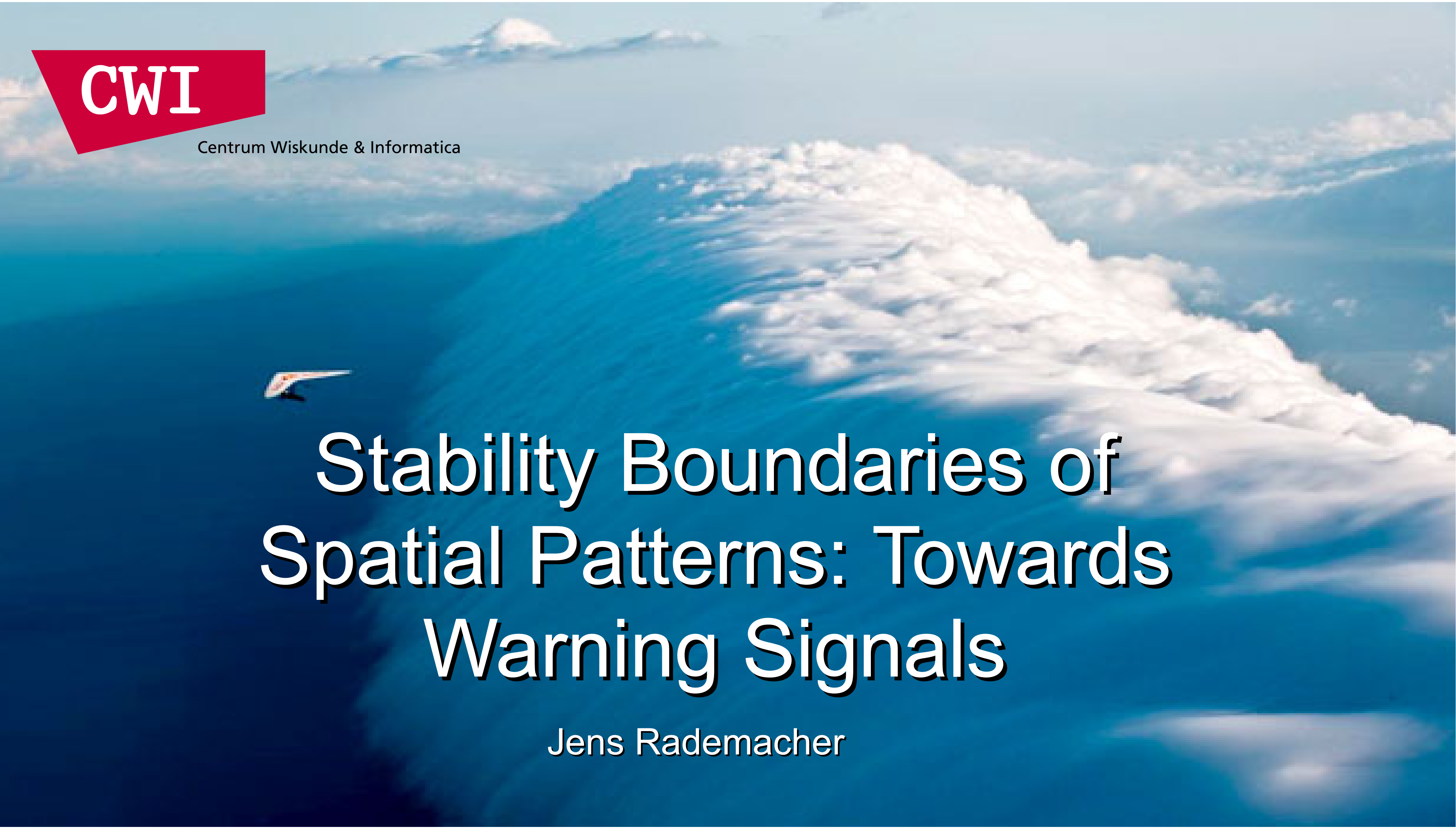


CWI
Centrum Wiskunde & Informatica

Stability Boundaries of Spatial Patterns: Towards Warning Signals

Jens Rademacher

Morning glory “roll clouds” are up to 1000km long and 1-2km high, 100m-200m over ground they travel with speeds up to 60km/h.

Imperial College, London 20.3.2012

I. Self organization and pattern formation: coherence within complexity

Example: reactive media

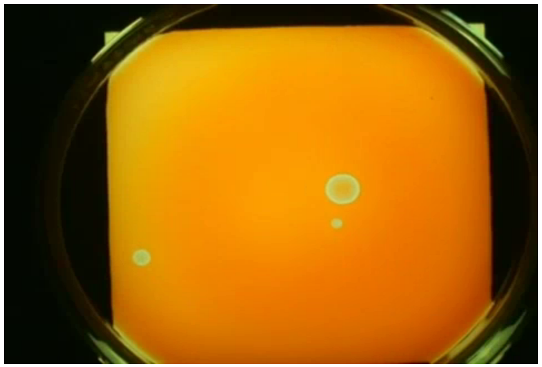
II. Stability regions and wavetrains

1. Turing-instability and Eckhaus-region
2. Hopf-dance and Busse-Balloons

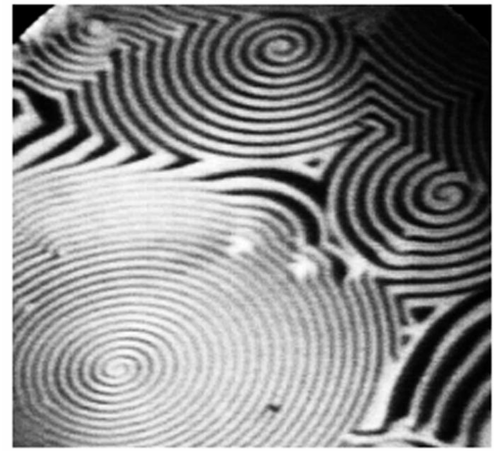
III. Towards warning signals

Critical transitions and Busse balloons
and some other aspects

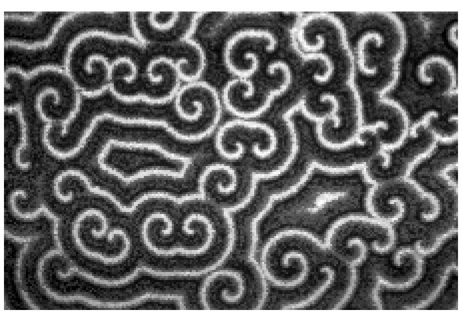
Non-linear waves in reactive media



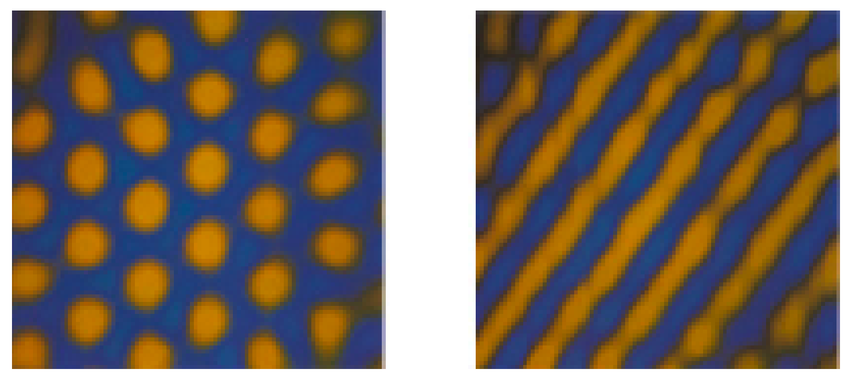
Belousov-Zhabotinsky Reaction



CO Oxidation
on Platin



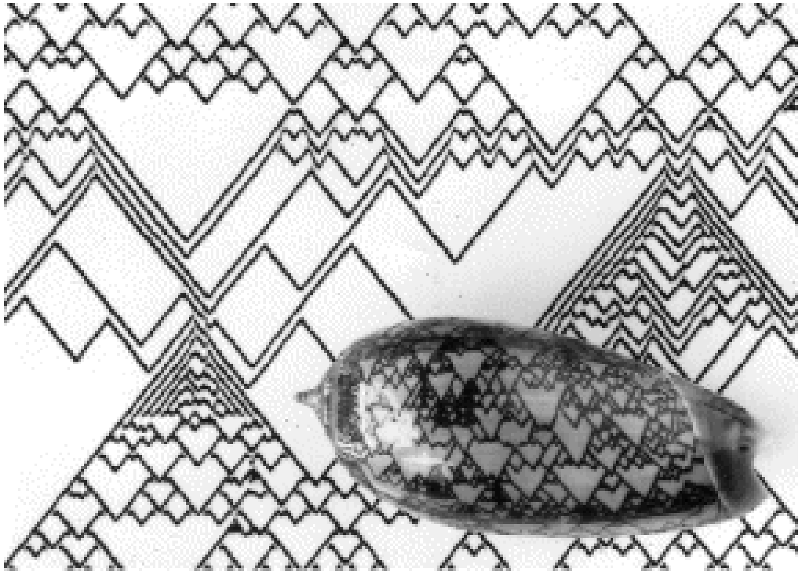
Slime mold colony



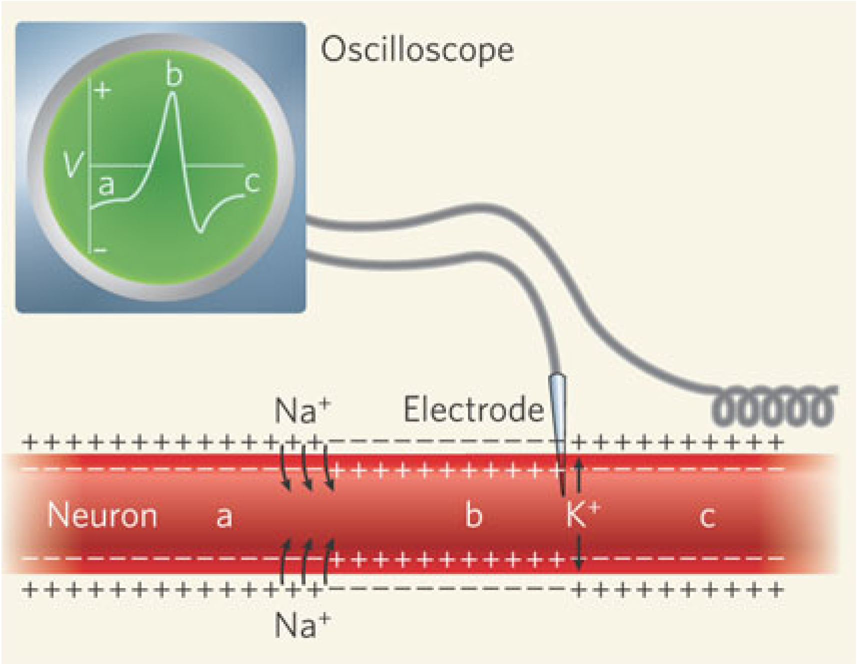
CIMA Reaction (`Turing instability')

Animal skin, vegetation, neural patches, heart muscle...

One space dimension

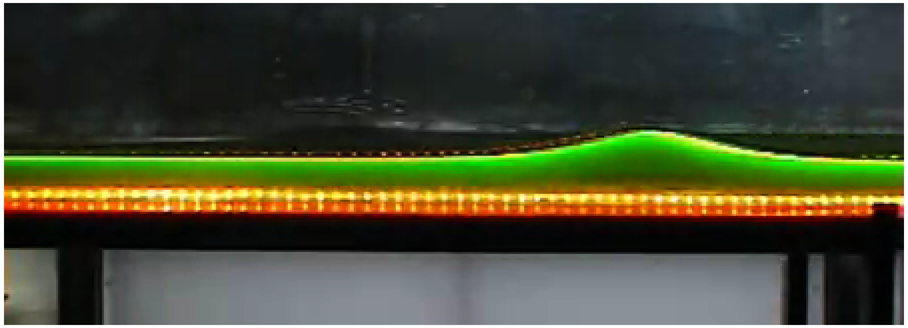


sea shell: space-time-diagram



nerve / axon

Soliton in fluid channel



Paradigms of complex systems

Self-organization and emergent properties:

- Pattern-forming instabilities (Turing, Hopf, Kelvin-Helmholtz, …)
- Stationary and travelling waves (Stripes, spirals, hexagons, ...)
- Localized structures (spots, phase transitions, …)
- Interaction between these.

`Critical transitions' and the transition from laminar to turbulent result from instabilities of different patterns and coherent structures.

→ Study stability boundaries of patterns

Mathematical questions

Existence of patterns and coherent structures

Bifurcation, singular limits, energy landscape, ...

↕

Stability and stability boundaries

spectral theory, types of instabilities, parameter variation, ...

Some model equations

Reactive Media: Reactions-Diffusion-Systems (RDS)

$$\partial_t u = D\Delta u + F(u), u(t,x) \in \mathbb{R}^N$$

Semi-conductor, cold plasma: Drift-Diffusion with poisson equation

$$\partial_t \mathbf{n} = D\Delta \mathbf{n} + \nabla C(\mathbf{n}, E) + F(\mathbf{n}, E),\ \nabla E = G(\mathbf{n})$$

Nanomagnets: Landau-Lifschitz-Gilbert equation

$$\partial_t M = M \times (\alpha \partial_t M - H_{\text{eff}}(M)),\ M(x,t) \in S^2$$

Prototype in the following are RDS

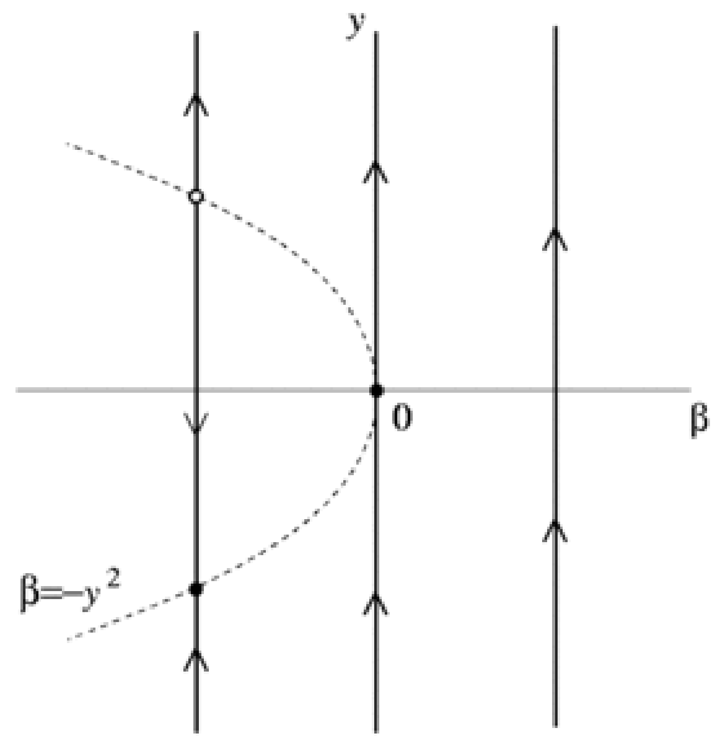
Existence through bifurcation

Evolution equation $\partial_t u = F(\partial_x, u)$

Equilibrium $0 = F(\partial_x, u)$

Linearization $\partial_t v = \mathcal{L} v = D_u F(\partial_x, u) v$

point spectrum $\lambda v = \mathcal{L} v$



Center manifold reduction:

ODE of dimension $\#\{\mathrm{spec}_{\mathrm{pt}}(\mathcal{L}) \cap \mathrm{i}\mathbb{R}\}$

Simplest example: saddle-node

$$\dot{y} = \beta + y^2$$

Note: emergence of a *temporal* heteroclinic orbit.

Existence through bifurcation

Evolution equation $\partial_t u = F(\partial_x, u)$

Equilibrium $0 = F(\partial_x, u) \Rightarrow \partial_x u = G(u)$

Linearization $\partial_t v = \mathcal{L}v = D_u F(\partial_x, u)v$

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ODE of dimension $\#\{\mathrm{spec}_{\mathrm{pt}}(\mathcal{L}) \cap \mathrm{i}\mathbb{R}\}$

Simplest example: saddle-node

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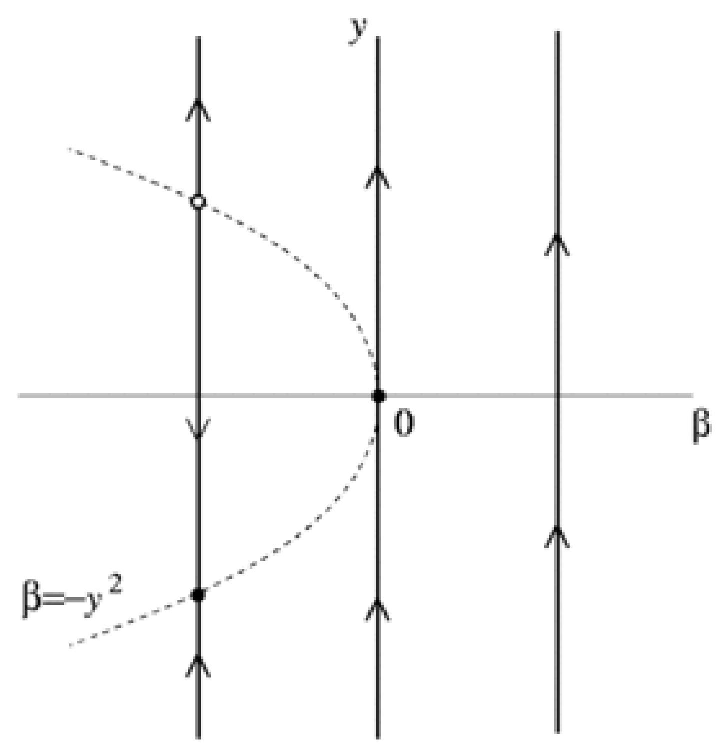


Existence through bifurcation

Evolution equation $\partial_t u = F(\partial_x, u)$

Equilibrium $0 = F(\partial_x, u) \Rightarrow \partial_x u = G(u)$

Linearization $\partial_x v = \mathcal{A}v = D_u G(u)v$

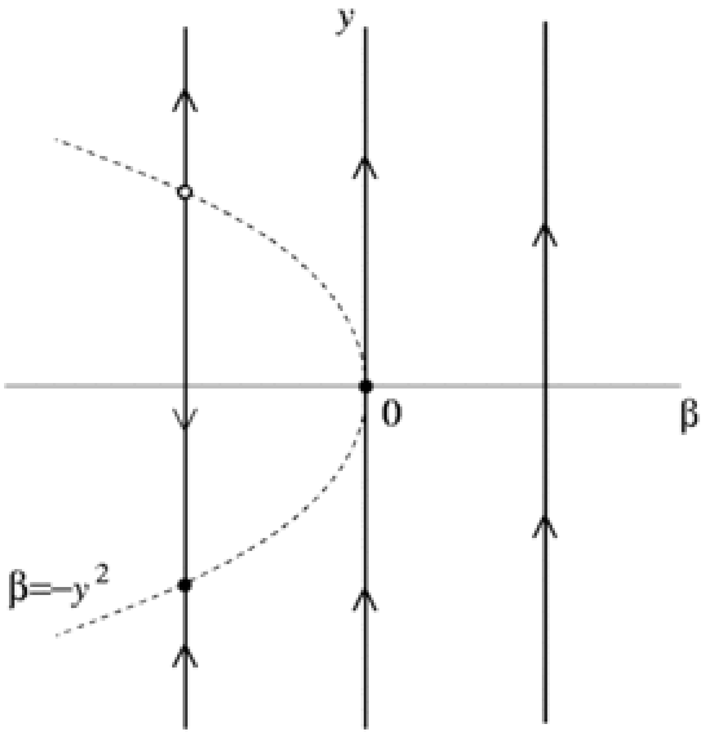
point spectrum $\nu v = \mathcal{A}v$

Center manifold reduction:

ODE of dimension $\#\{\mathrm{spec}_{\mathrm{pt}}(\mathcal{A}) \cap \mathrm{i}\mathbb{R}\}$

Simplest example: saddle-node

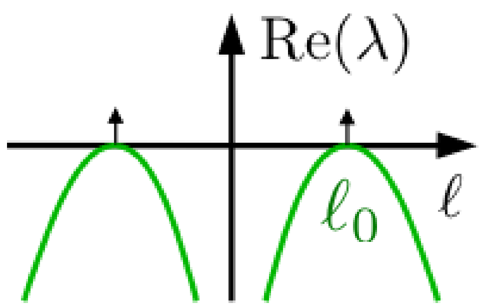
$$\dot{y} = \beta + y^2$$



Note: emergence of a *spatial* heteroclinic orbit.

Turing-instability and wavetrains

Extended domain ($x \in \mathbb{R}$): $\mathcal{L}$ also has essential spectrum
No center manifold reduction if critical... But spatially maybe fine!



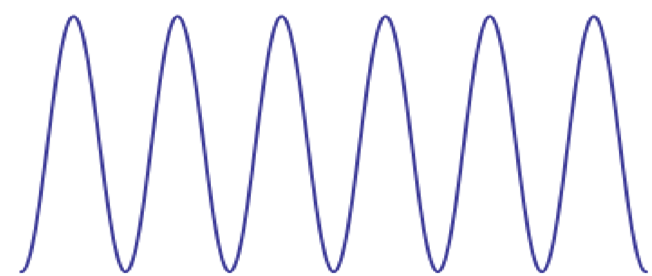
$\mathrm{Im}(\lambda(\ell)) = 0$

eigenmodes $\mathrm{e}^{\mathrm{i}\ell} u_0$

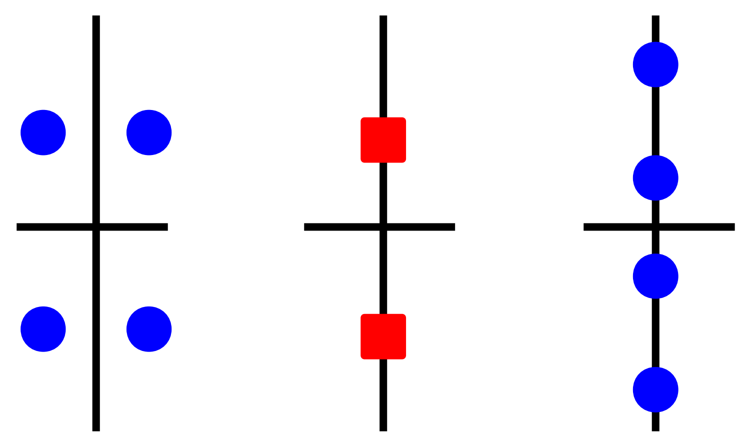
Bifurcation of spatial oscillations

$$u(x,t) = u_*(kx)$$
$$u_*(2\pi) = u_*(0)$$

wavenumber $k \sim \ell_0$



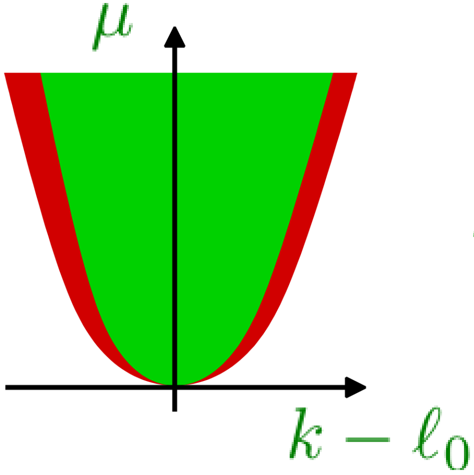
Spatial dynamics: Turing bifurcation
is reversible Hopf bifurcation



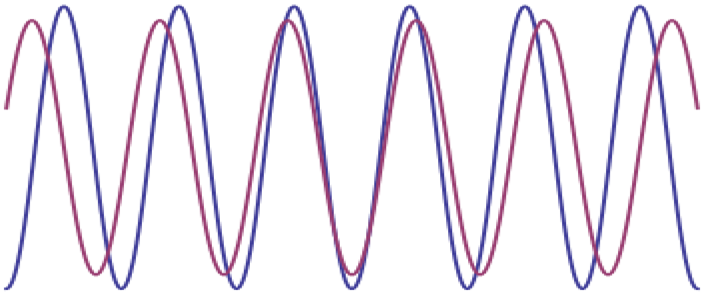
Turing-patterns and Eckhausband

At supercritical Turing-Instabilities with parameter : μ
(well known theory)

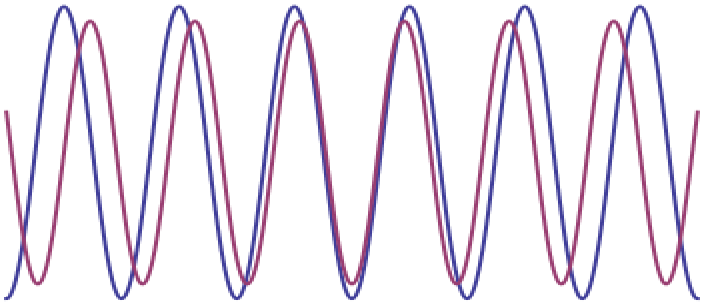
Existence- and
stabilityregion



$$u(x,t) = u_*(kx)$$
$$u_*(2\pi) = u_*(0)$$



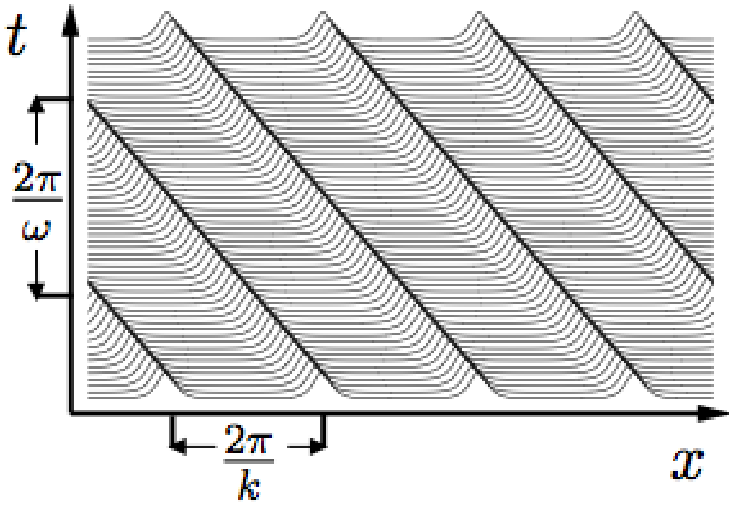
$k < \ell_0$



$k > \ell_0$

General wavetrains

$$u(x,t) = u_*(kx - \omega t),\ u_*(2\pi) = u_*(0)$$



(temporal) frequency ω

phase-velocity $c = \omega / k$

non-linear dispersion-relation
in regular wavetrain: $\omega(k)$

In Reaction-Diffusion-Systems (RDS):

$$\partial_t u = D\partial_x^2 u + F(u,\mu)$$

$$0 = k^2 D\partial_\xi^2 u_* + \omega\partial_\xi u_* + F(u_*,\mu)$$

Spectrum of wavetrains

Linearization in wavetrains for RDS:

$$\mathcal{L}v = k^2 D v_{\xi\xi} + \omega v_\xi + \partial_u F(u_*(\xi), \mu) v$$

Eigendata-problem: $\lambda v = \mathcal{L}v \Leftrightarrow V_\xi = A(\xi, \lambda) V$

Spectrum of wavetrains

Linearization in wavetrains for RDS:

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Eigendata-problem: $\lambda v = \mathcal{L}v \Leftrightarrow V_\xi = A(\xi, \lambda)V$

Theorem [e.g. Gardner 90s]

Let $\Phi(\lambda)$ be the period map of $V_\xi = A(\xi, \lambda)V$ and

$$d(\lambda, \ell) = \det(\Phi(\lambda) - \mathrm{e}^{2\pi \mathrm{i}\ell}) = 0$$

the Dispersion-relation. Then, e.g. in $(L^2(\mathbb{R}))^N$:

$$\mathrm{spec}(\mathcal{L}) = \mathrm{spec}_{\mathrm{ess}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \exists \ell \in \mathbb{R} : d(\lambda, \ell) = 0\}$$

Translation-symmetry in ξ : $d(0,0) = 0$

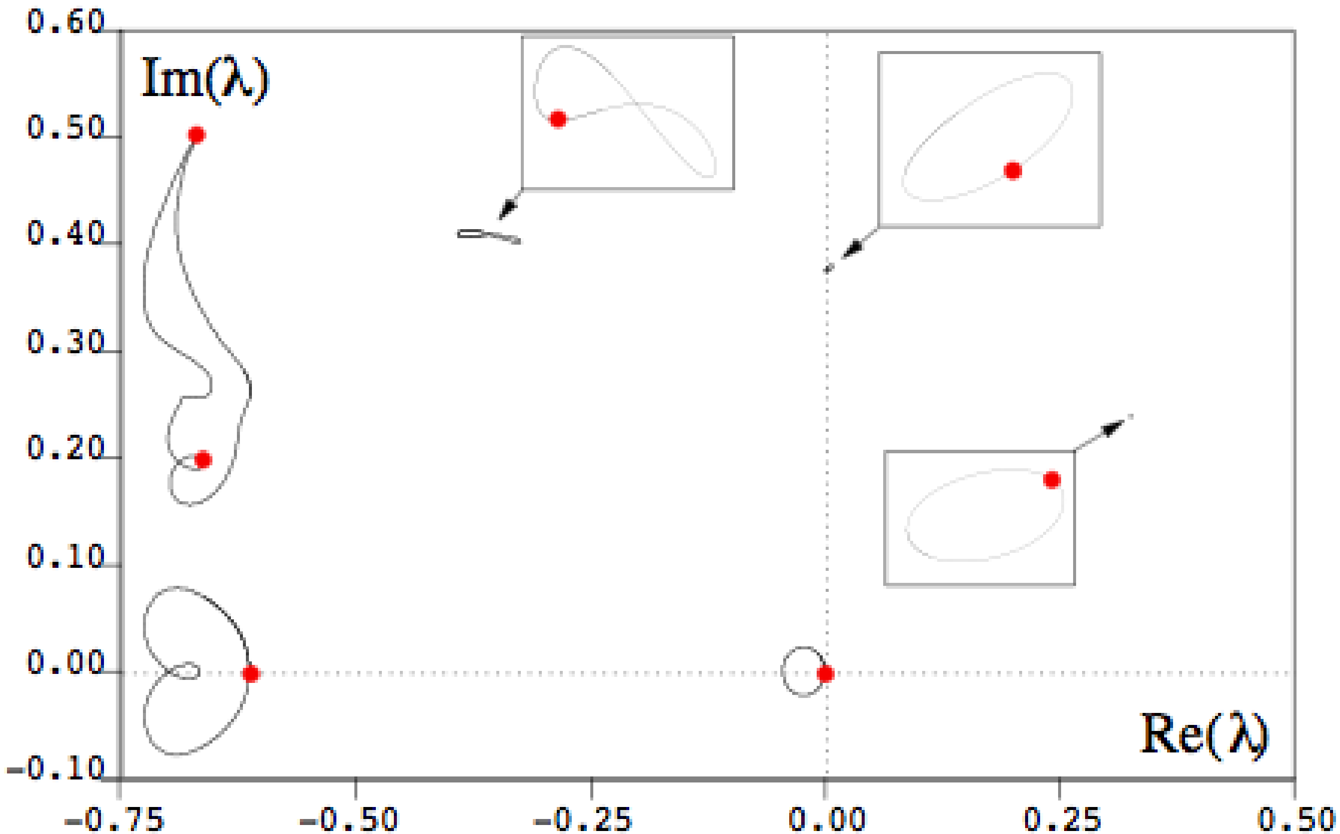
Numerical computation

Solve $d(\lambda, \ell) = 0$ numerical by continuation in ℓ of

$$V_\xi = A(\xi, \lambda)V \quad \text{mit} \quad V(2\pi) = e^{2\pi i \ell} V(0)$$

Example:

wavetrain in Schnakenberg model, $u(x,t) \in \mathbb{R}^2$



R., Sandstede, Scheel. Physica D 229 (2007) 166-183
R. SIAM J. Appl. Dyn. Sys. 5 (2006) 634-649

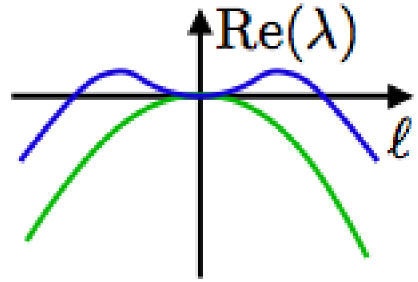
Instabilities of wavetrains

Sideband-instability:

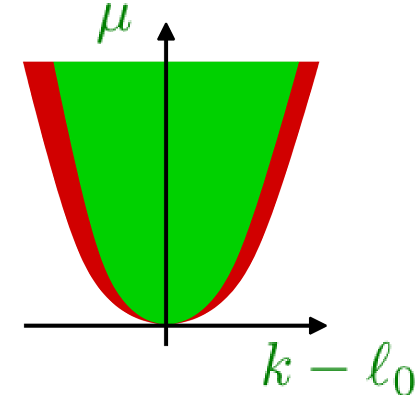
Since $0 \in \mathrm{spec}(\mathcal{L})$

sign change of $\partial_\ell^2 \lambda_0(0)$

is codimension-1

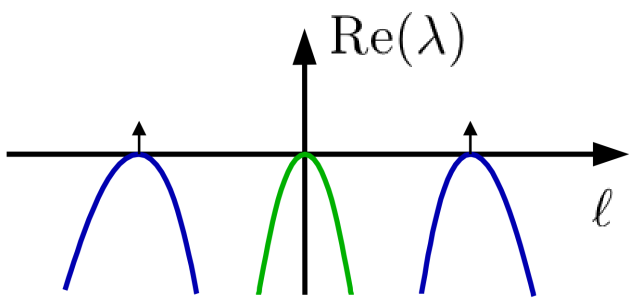


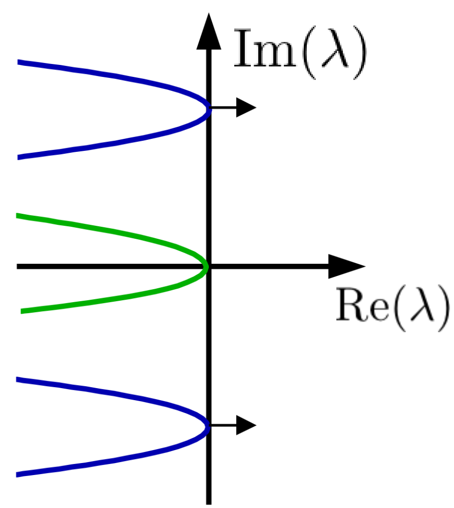
Occurs at bdry of Eckhaus-band :



Hopf-Instability:

Spatially and/or temporally oscillating critical modes





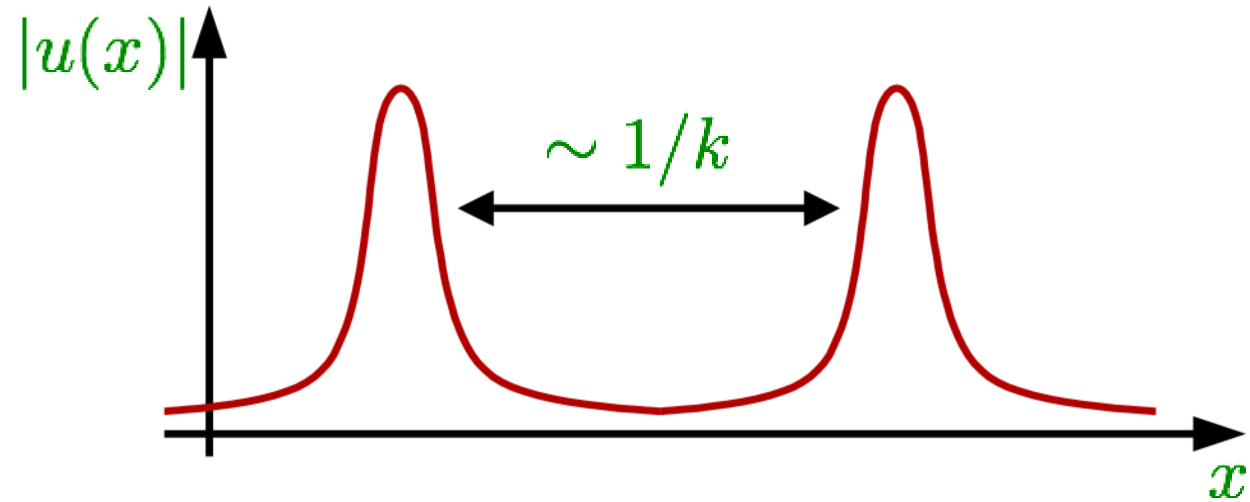
R., Scheel. Int. J. Bif. Ch. 17 (2007) 2679-2691

Busse-Balloon

How do stability-regions ("Busse-Ballons") in (k, μ)-space look like globally?

From Turing-instability (small amplitude, $k \sim \ell_0$)

to homoclinic bifurcation ($k \searrow 0$)

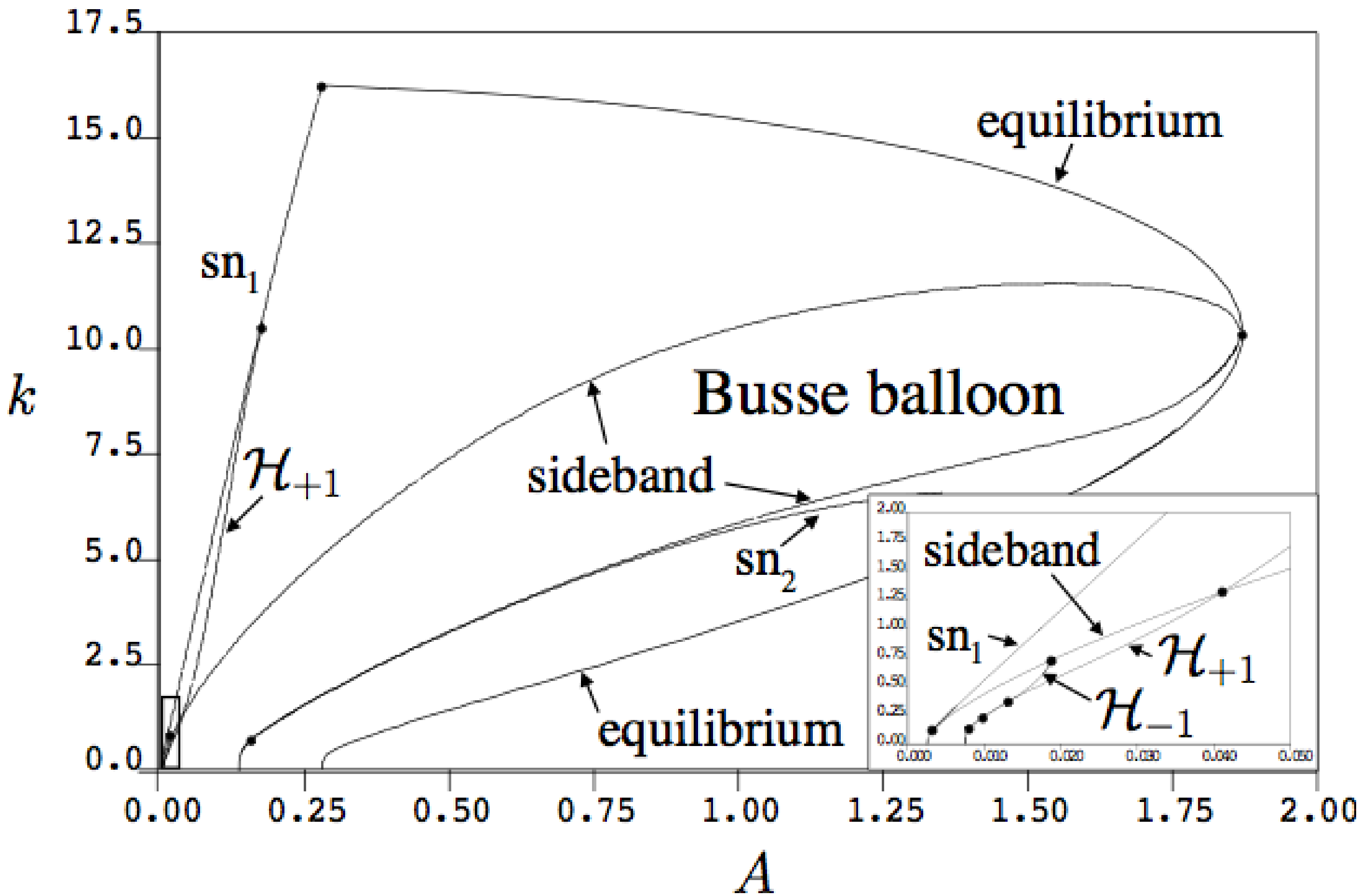


Busse balloon numerically

Example: Gray-Scott-model

$$U_t = U_{xx} + A(1-U) - UV^2$$

$$\varepsilon^2 = 0.001,\ B = 0.26 \qquad V_t = \varepsilon^2 V_{xx} - BV + UV^2$$



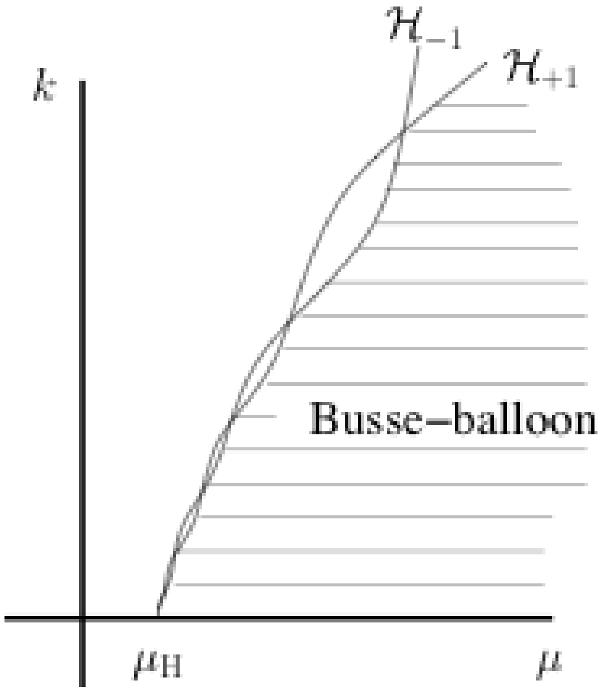
Doelman, R., van der Stelt. Discr. Cont. Dyn. Sys. 5 (2012) 61-92

Hopf-dance and Busse-Balloon

Theorem [D.,R.,vd S.]

In a class of singularly perturbed RDS with two components it holds that:

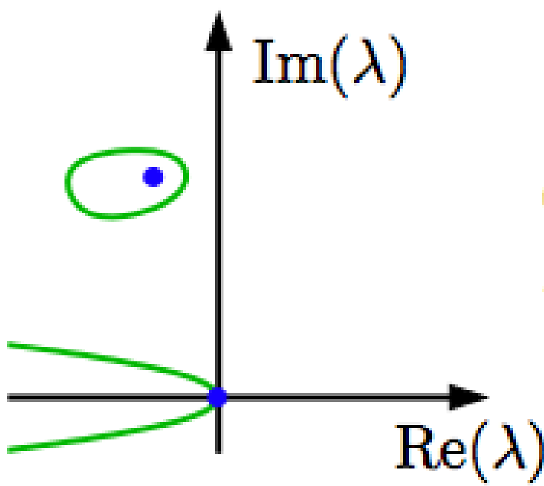
A Hopf-bifurcation of a pulse generates two curves of Hopf-instabilities of the bifurcating wavetrains. These oscillate about each other and define the stability boundary for small wavenumber.



Idea of proof: Taylor-expansion of the dispersion-relation in the doubly singular limit $0 < \varepsilon, k \ll 1$

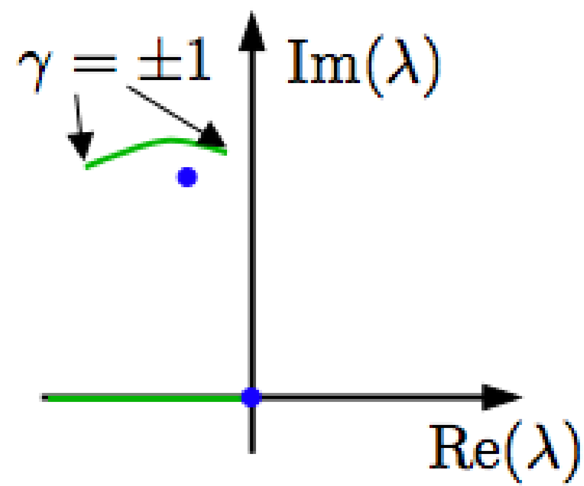
Doelman, R., van der Stelt. Discr. Cont. Dyn. Sys. 5 (2012) 61-92

Hopf dance ingredients I



$\omega = 0$ + symmetry:

$u_*(\xi) = u_*(-\xi)$

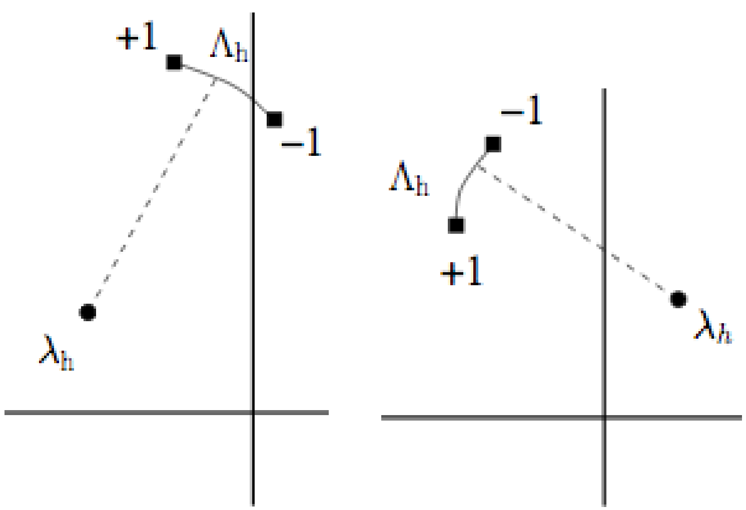


Hopf eigenvalue of pulse generates loops of essential spectrum for nearby wavetrains [Gardner; Sandstede, Scheel; Doelman, vd Ploeg]. Parametrize loop by $\gamma = \exp(\mathrm{i}\ell) \in \mathrm{S}^1$

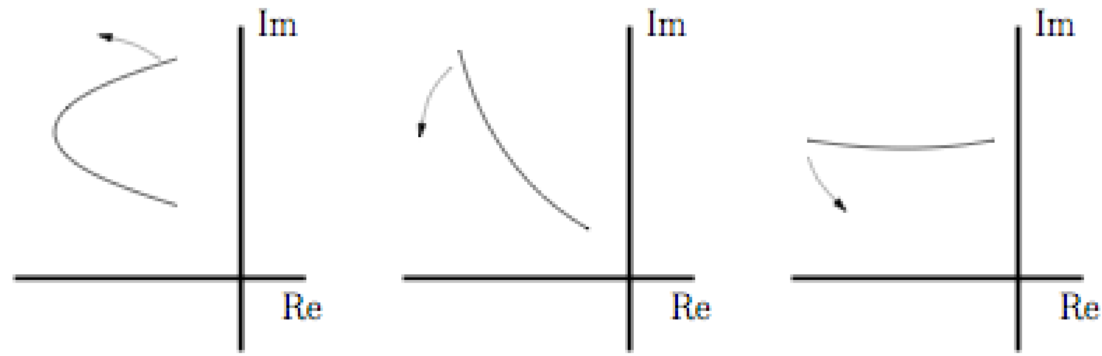
Hopf dance ingredients II

$$\lambda(\gamma, L; \mu) = \lambda_{\mathrm{h}}(\mu) + 2\frac{E_0(L;\mu) - \gamma_{\mathrm{r}} E_{\mathrm{h}}(L;\mu)}{\mathcal{S}'(\lambda_{\mathrm{h}};\mu)} \quad \text{+ h.o.t.}$$

$$E_{\mathrm{h}}(L;\mu) = \mathrm{e}^{-2L\sqrt{\mu+\lambda_{\mathrm{h}}}}, \quad E_0(L;\mu) = \mathrm{e}^{-2L\sqrt{\mu}}$$



$\mathrm{Re}(\mathcal{S}') > 0$ $\quad$ $\mathrm{Re}(\mathcal{S}') < 0$

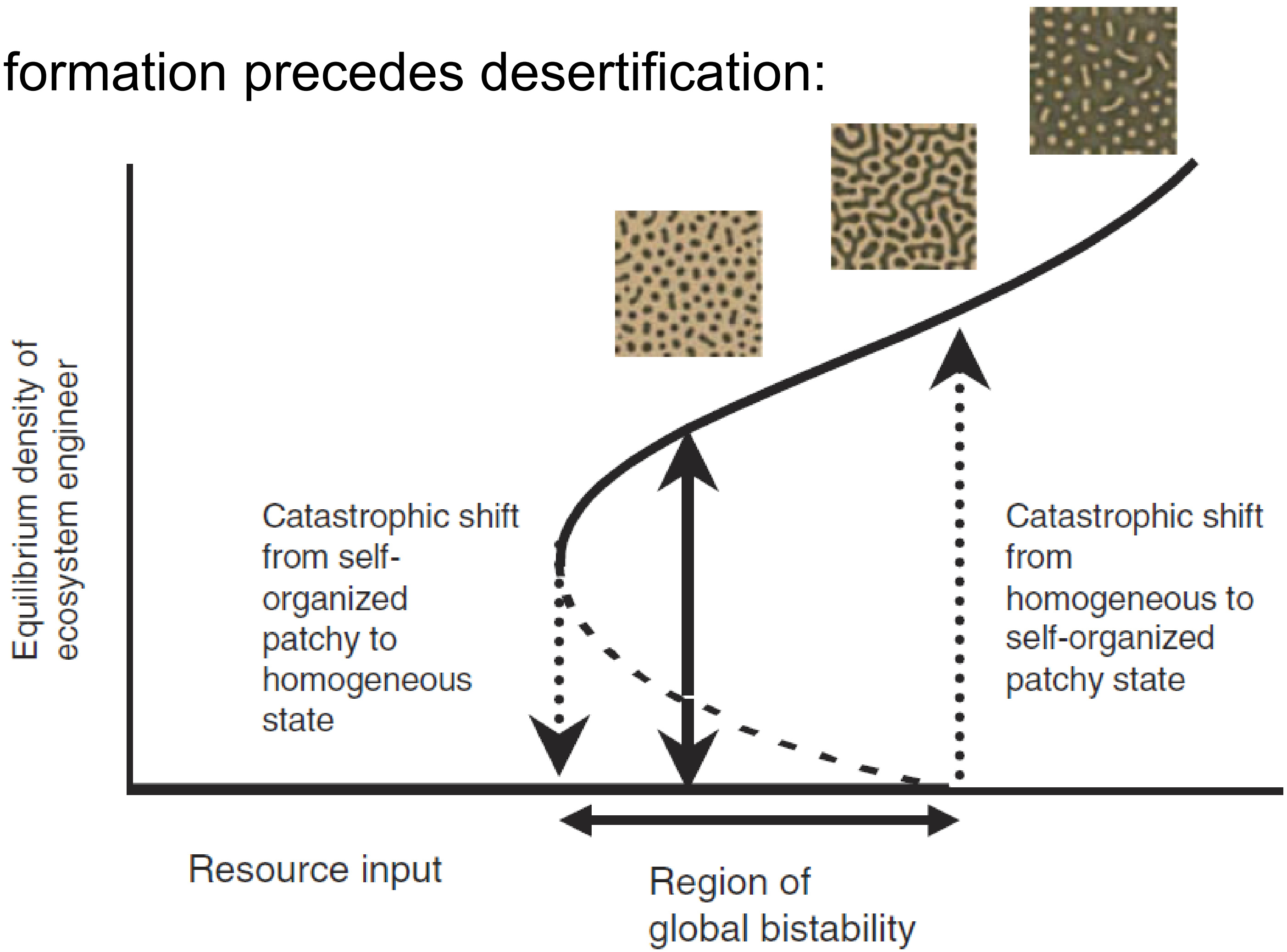


Next order: `belly dance'

Relation to pulse's Hopf eigenvalue

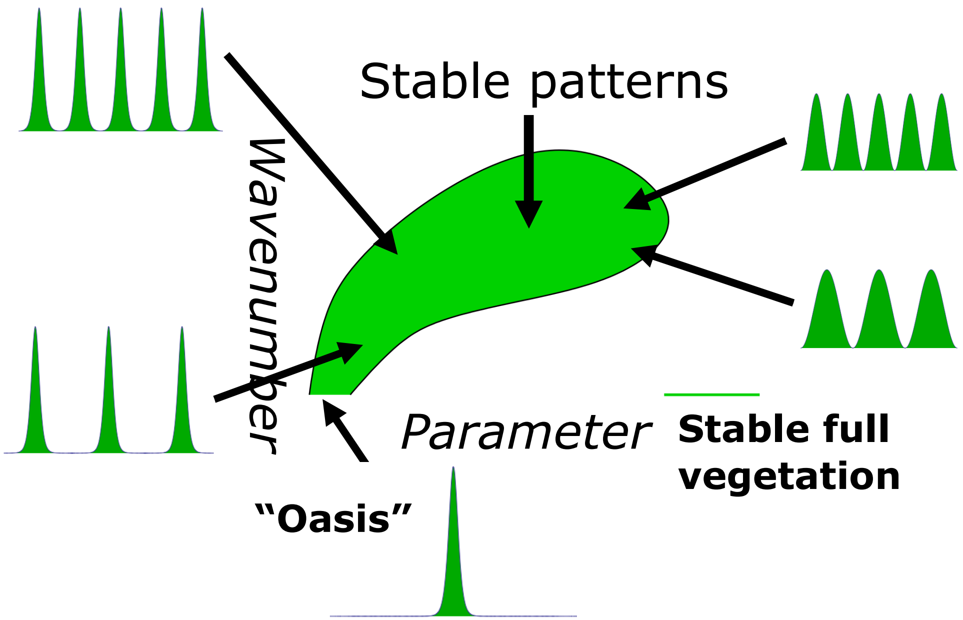
Application: vegetation patterns

Pattern formation precedes desertification:



NWO grant: ``Critical transitions and early warning signals in spatial ecology”
with Arjen Doelman (Leiden), Max Rietkerk & Maarten Eppinga (Utrecht).
PhD's: Eric Siero & Koen Siteur → Poster at this workshop

Relation to Busse balloon



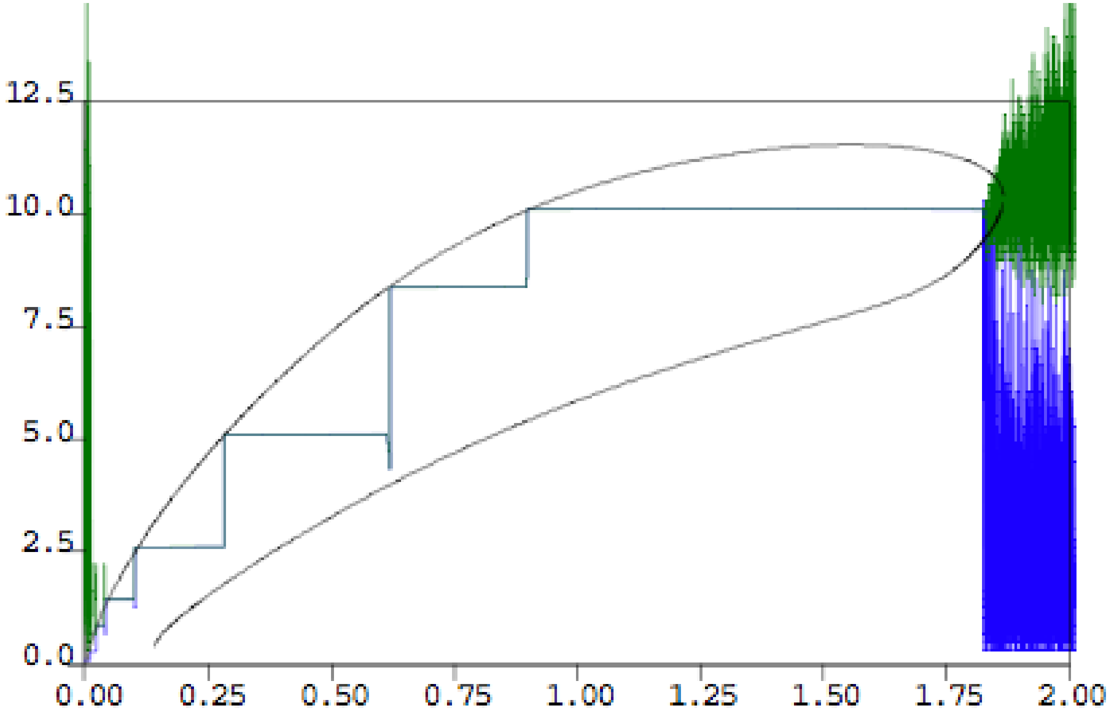
Conceptual model: Generalized Gray-Scott-Klausmeier

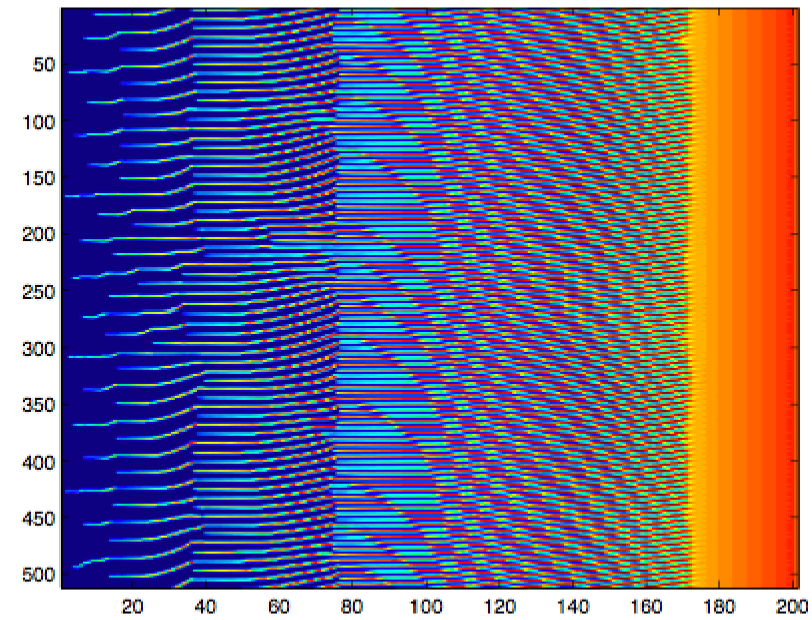
$$U_t = (U^\gamma)_{xx} + CU_x + A(1-U) - UV^2$$

$$V_t = \varepsilon^2 V_{xx} - BV + UV^2$$

van der Stelt, Doelman, Hek, R. Preprint 2012.

Critical transitions and Busse balloon





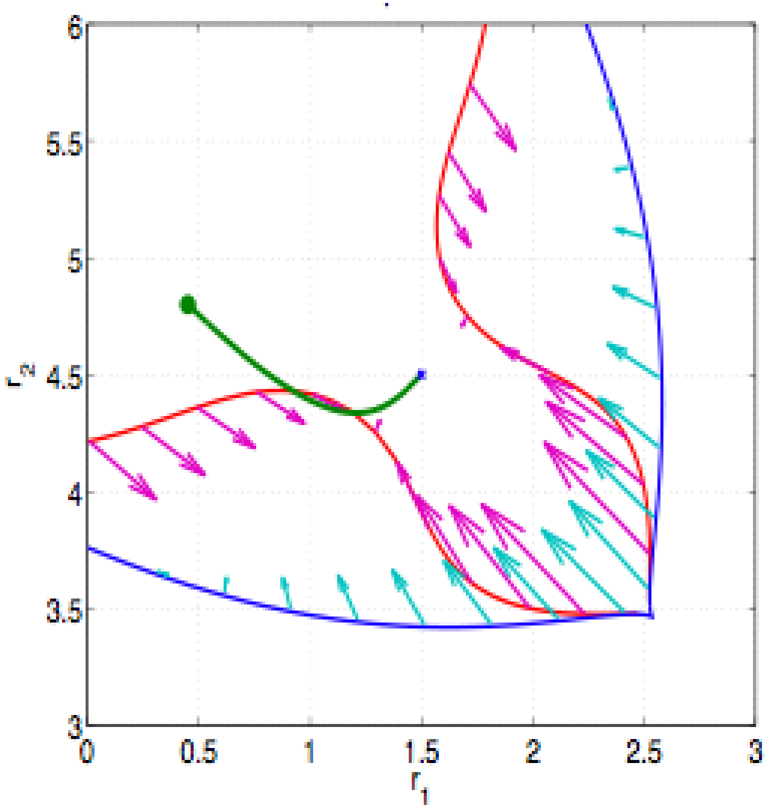
Simulation by Eric Siero of slowly decreasing `rainfall' parameter, large flat terrain for linear diffusion.

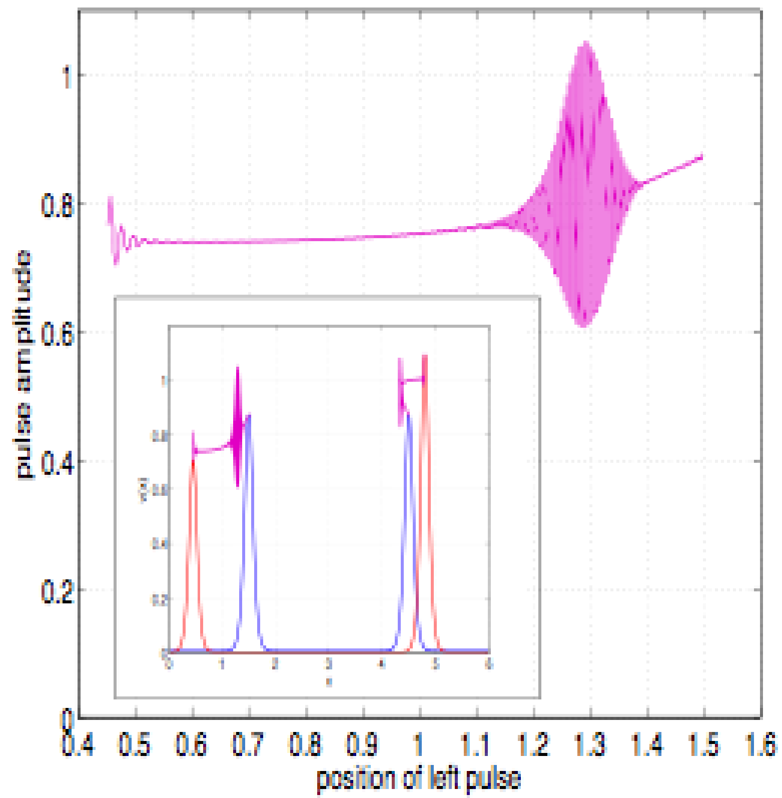
Noise added each few steps to shorten delay in Turing bifurcation.

Preliminary results:

At moderate variation speeds: transitions nearly spatial period doubling.
At slow speeds smaller steps, unclear how small possible.
At fast speeds can have `desertification' at first instability.

Metastable pulse patterns





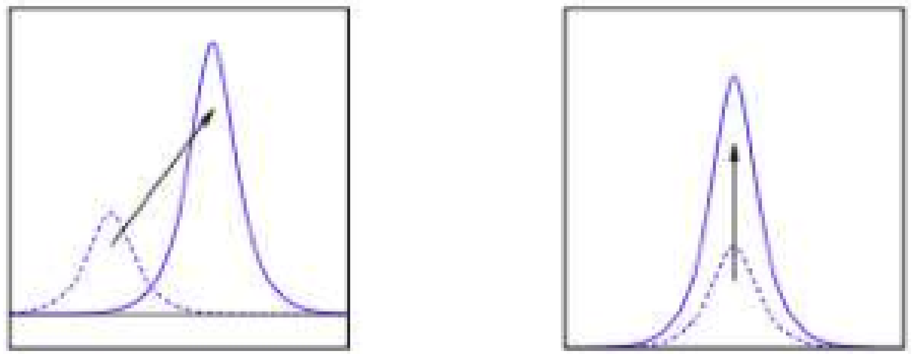
Schnakenberg model (essentially same as Gray-Scott):

near singular limit pulses move on slow manifold (arrows).

Red: Hopf stability boundary, blue: fold (again noise added)

Joint work with M. Wolfrum (WIAS Berlin) and J. Ehrt (HU Berlin).

Absolute vs. convective instability

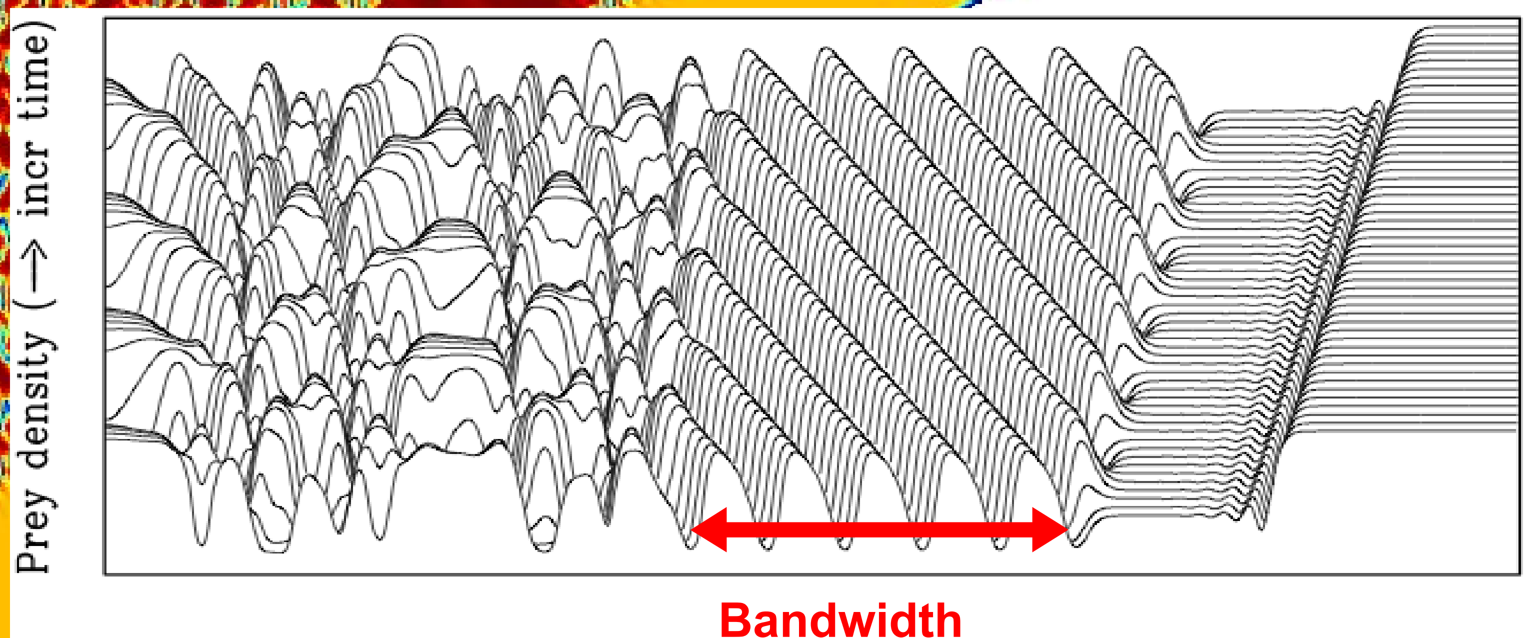


Can be distinguished by certain solutions of dispersion relation:

$$d(\lambda, \nu_{j_\star}) = 0$$

$$\partial_\nu d(\lambda, \nu_{j_\star}) = 0$$

Bandwidth of unstable oscillatory invasion



Bandwidth
measures degree of coherence despite instability.

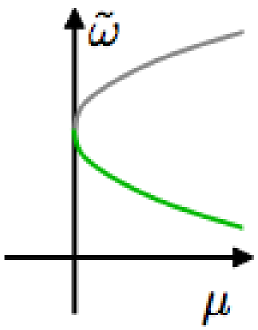
J.A. Sherratt, M.J. Smith, R.
Locating the transition from periodic oscillations to spatiotemporal chaos in the wake of invasion.
Proc. Nat. Acad. Sc. 106: 10890-10895 (2009)

Thanks for your attention!

Oscillatory patterns

Critical case:

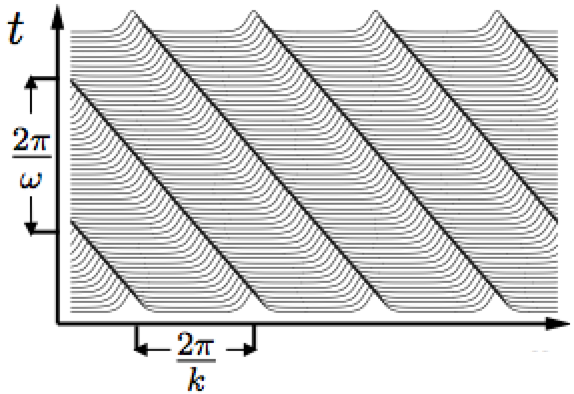
Saddle-node bifurcation of homogeneous oscillation of which one is stable (e.g. near Hopf-Bautin point)



Emergence of spatio-temporal patterns?
Which, how and are there stable ones?

Rademacher, Scheel. J. Dyn. Diff. Eqns. 19 (2007) 479-496
Rademacher, Scheel. Manuskript.

Oscillatory patterns

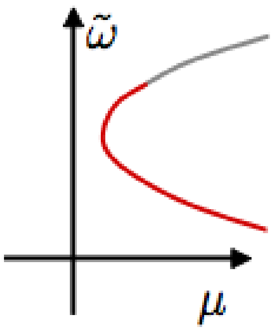


Theorem [R.,S.]

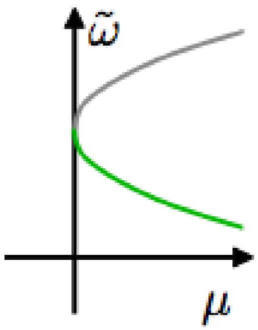
In RDS there are precisely two typical cases:

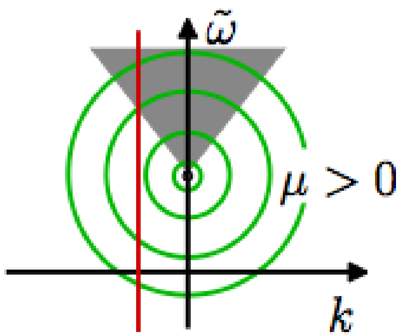
1. elliptic ~ supercritical
2. hyperbolic ~ subcritical

There are always also stable wavetrains



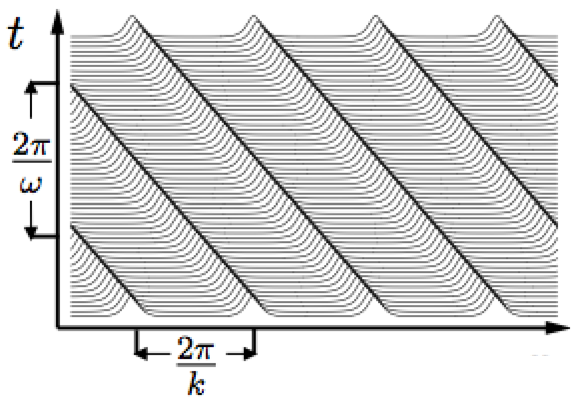
gray = stable





elliptic

Oscillatory patterns



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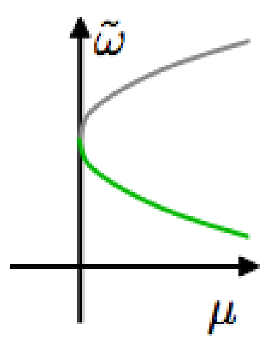
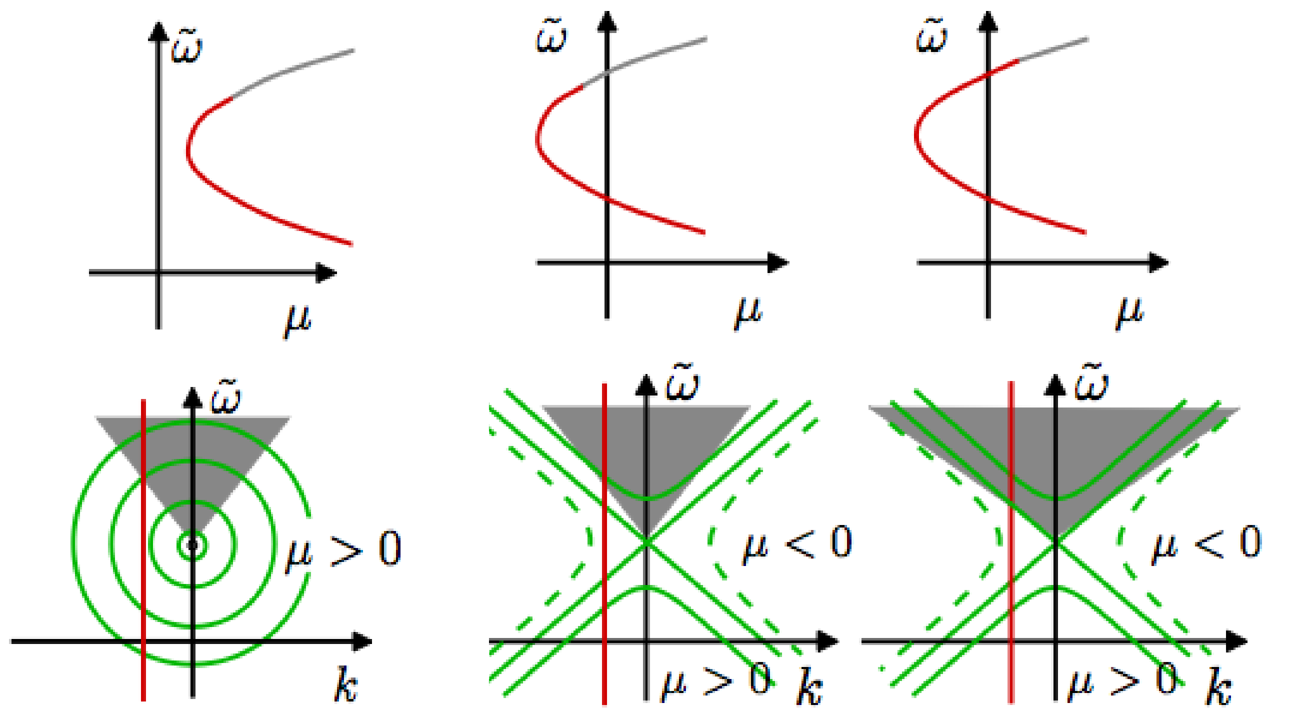
gray = stable





elliptic hyperbolic