Elimination of Loops and Multiple Arrows in Coupled Cell Systems

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Motivation

I. Stewart, M. Golubitsky, "Synchrony-breaking bifurcation at a simple real eigenvalue for regular networks I: 1-dimensional cells", Preprint (22-02-2011)

- "The examples of degenerate bifurcation that we construct arise in networks with few cells but having arrows of high multiplicity"
- "Multiple arrows can be removed by appealing to the Lifting Theorem" ⇒ (larger) single arrow network - "lift"
- "It seems plausible that in most, if not all, cases, the lift can be chosen to keep the critical eigenvalue simple (...) However, this has not been proved"

General questions: Given a network with loops or multiple arrows,

⇒ **Question 1:** is it possible to construct a lift with no loops and no multiple arrows keeping the multiplicity of each eigenvalue?

⇒ **Question 2:** does any bifurcation associated to this network can be studied as a bifurcation problem associated to a network with no loops and no multiple arrows?

Coupled Cell Networks

Formal theory developed by I. Stewart, M. Golubitsky, M. Pivato, A. Török (2003, 2005)

Coupled cell system. finite collection of interacting cells (systems of differential equations)

System associated to cell j.

$$\dot{x}_j = f_j(x_j; x_{i_1}, x_{i_2}, ..., x_{i_m})$$

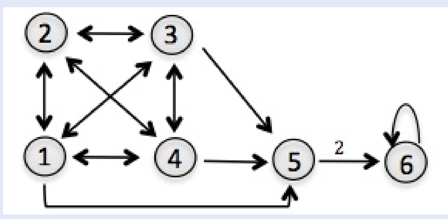
The general theory allows:

- loops. $i_q = j$, for some q
- multiple arrows. $i_{q_1} = i_{q_2}$, for some q_1, q_2

Network. directed graph whose nodes represent cells and whose arrows represent couplings

Example.

$$\begin{cases} \dot{x}_1 = f(x_1; \overline{x_2, x_3, x_4}) \\ \dot{x}_2 = f(x_2; \overline{x_1, x_3, x_4}) \\ \dot{x}_3 = f(x_3; \overline{x_1, x_2, x_4}) \\ \dot{x}_4 = f(x_4; \overline{x_1, x_2, x_3}) \\ \dot{x}_5 = f(x_5; \overline{x_1, x_3, x_4}) \\ \dot{x}_6 = f(x_6; \overline{x_5, x_5, x_6}) \end{cases}$$



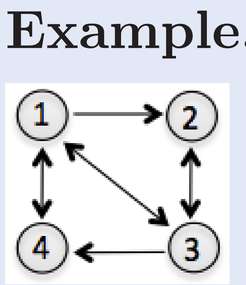
Regular Networks

Each cell has the same differential equation (up to reordering coordinates) and one kind of coupling

Quotient Networks

Synchrony subspace. subspace defined by the equality of some cell coordinates which is flow invariant, for every admissible vector field

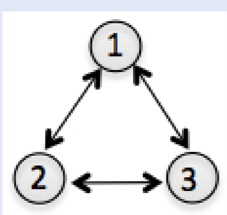
Example.



$$\begin{cases} \dot{x}_1 = f(x_1; \overline{x_3, x_4}) \\ \dot{x}_2 = f(x_2; \overline{x_1, x_3}) \\ \dot{x}_3 = f(x_3; \overline{x_1, x_2}) \\ \dot{x}_4 = f(x_4; \overline{x_1, x_3}) \end{cases}$$

$\Delta = \{x_2 = x_4\}$ is a synchrony subspace because identifying cells 2 and 4 we obtain:

$$\begin{cases} \dot{x}_1 = f(x_1; \overline{x_3, x_2}) \\ \dot{x}_2 = f(x_2; \overline{x_1, x_3}) \\ \dot{x}_3 = f(x_3; \overline{x_1, x_2}) \end{cases}$$



Quotient network. network obtained with the cell identification in a synchrony subspace

Lifts

Lift. G is a *lift* of Q when Q is a quotient of G

Extra eigenvalues. Elements of $S_G - S_Q$

- Given $\lambda \in S_Q$, G is **λ-preserving** when
  $\forall \mu \in S_G - S_Q,\ Re(\mu) \neq Re(\lambda)$
- G is **spectrum-preserving** when
  $\forall \lambda \in Q,\ G$ is λ-preserving

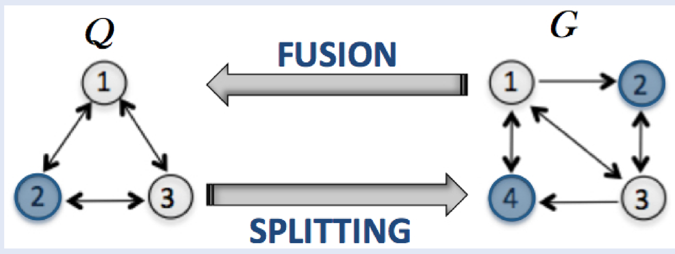
Cellular Splitting

Any lift is interpreted as resulting from a cellular splitting of the initial network

Example.

Q is a quotient of G / G is a lift of Q

$\Delta = \{x_2 = x_4\} \Rightarrow Q = G/\Delta$

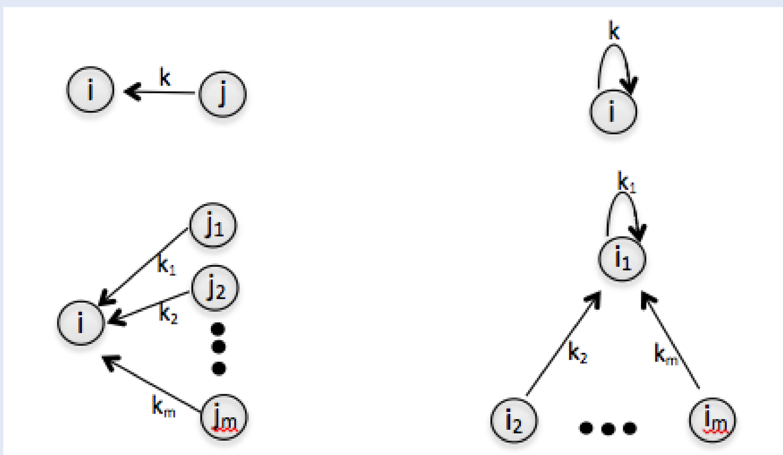


Splitting cells. cells that split (in the quotient)

Splitted cells. cells resulting from a splitting (in the lift)

Fundamental Property

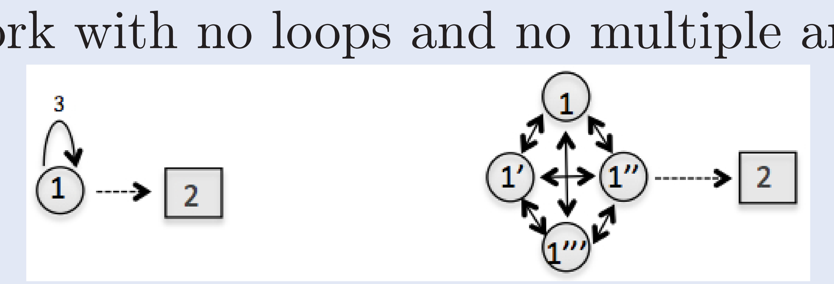
If i is a cell that receives k arrows from cell j then, after the splitting, cell i or each of its splitted cells if i is a splitting cell, receives k arrows either from cell j or, if j splits, from the set of splitted cells associated to cell j

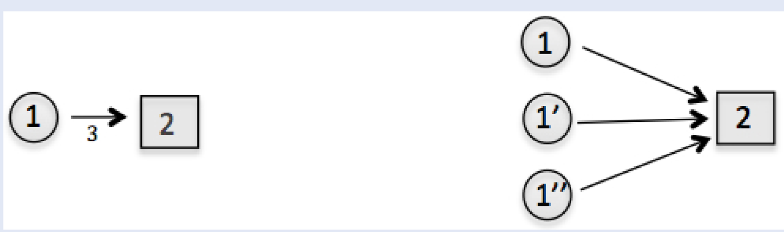


- j splits into m cells: j_1, j_2,..., j_m.
- $k_i \geq 0$ and $k = \sum_i k_i$.

Uniform network

Network with no loops and no multiple arrows





Results

Given a regular network Q with loops or multiple arrows

A: it is possible to construct a uniform lift with -1 and 0 as unique possible extra eigenvalues

B: if $\lambda \in S_Q$ and $Re(\lambda) \neq 0$, it is possible to construct a λ-preserving uniform lift

C: if there is a cell that has no loops, that sends multiple arrows and that forms a trivial strongly connected component then all uniform lifts have 0 as extra eigenvalue

Steady-state bifurcations

- Q: n-cell regular network
- F: family of admissible vector fields for Q

$$\dot{x} = F(x, \lambda),$$

with $x = (x_1, \ldots, x_n) \in (R^k)^n$, $\lambda \in R$ (k is the dimension of the internal dynamics)

- Given the *bifurcation problem*:

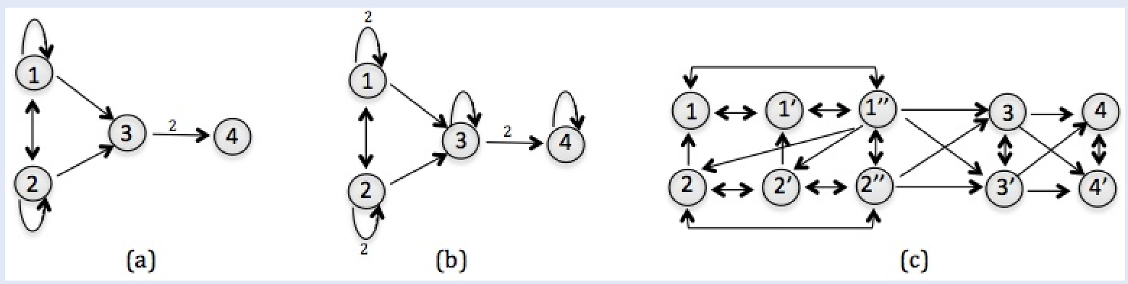
$$F(x, \lambda) = 0,$$

F has a steady-state bifurcation at the origin for $\lambda = 0$, i.e., $F(0,0) = 0$ and $(dF)_{(0,0)}|_{E^c}$ has a zero eigenvalue ($E^c \equiv$ center subspace)

Main result

Given a regular network with loops or multiple arrows, there is always an ODE-equivalent[a] network that admits a spectrum-preserving uniform lift

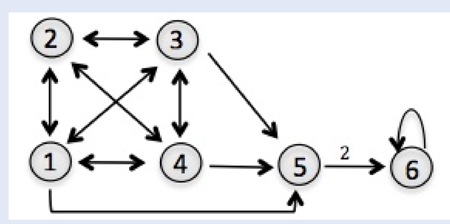
Example. (a) and (b) are ODE-equivalent networks. No uniform lift of (a) keeps the eigenvalue 0 multiplicity. (c) is a spectrum-preserving uniform lift of (b).



[a]Two networks are ODE-equivalent if they give rise to the same space of admissible vector fields (for a suitable choice of cell phase spaces)

Conclusions

Answer 1: Not all cases (Result C). For example, the following network with respect to the eigenvalue 0



Answer 2: Yes (main result), although the lift may not be a lift of the initial network

References

- M. Golubitsky, I. Stewart, A. Török, "Patterns of synchrony in coupled cell networks with multiple arrows", *SIAM J. Appl. Dynam. Sys.* **4** (2005) 78-100.
- I. Stewart, M. Golubitsky, M. Pivato, "Symmetry grupoids and patterns of synchrony in coupled cell networks", *SIAM J. Appl. Dynam. Sys.* **2** (2003) 609-646.

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