Tipping between phases of stochastic complex systems

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Outline

1. Space-time phases
2. Uniqueness or not
3. Effects of nudges and small parameter changes
4. Theory in the weakly dependent regime
5. Comments on R-tipping

1. Space-time phases

- The space-time phases for a spatially extended dynamic system are the probability distributions for state as a function of position and time, for realisations started in the infinite past.
- e.g. climate for the atmosphere-ocean-biosphere system
- cf. time-phases derived from
  - stationary probabilities for a finite state homogeneous Markov process, or
  - Sinai-Ruelle-Bowen measures for a finite-dimensional deterministic dynamical system

2. Uniqueness or not

- Finite-state Markov process with a unique communicating component has a unique stationary probability and unique time-phase
- For discrete-time need to add aperiodicity of the communicating component
- Basin of a uniformly hyperbolic topologically mixing attractor of a smooth deterministic finite-dimensional dynamical system has unique SRB measure and unique time-phase.
- Trivial examples with non-unique phase: systems with more than one communicating component, or more than one attractor

Spatially extended systems

- For “weakly dependent” systems one can prove there is a unique space-time phase (and they are exponentially mixing).
- But some infinite systems, even with unique communicating component or indecomposable dynamics, have more than one space-time phase.
- Some proved examples [demos]:
  - Contact processes above threshold have in addition to the all-healthy phase an endemic disease phase
  - Majority voter models with “eroder” neighbourhoods and sufficiently small error rate have at least two phases
  - Coupled map lattice analogues of the above
- Moreover, the phenomenon is robust to small changes in these models.

3. Effects of nudges and small parameter changes

- In the exponentially mixing regime, the effects of nudges decay away exponentially in time (open question: range in space of a local nudge?)
- Also, for slow parameter changes, the space-time phase remains unique and tracks the instantaneous one.
- But in the strongly coupled regime, nudges and parameter changes can tip the system from one phase to another [demo]
- This goes back to Bennett & Grinstein, Phys Rev Lett 55 (1985) 657
- I think this is the right framework for Zeeman's ideas about control of riots etc. (rather than catastrophe theory), though qualitatively the same conclusions apply.

Theory

- Difficult to do theory for the strongly coupled regime
- But can deal with the weakly dependent one in some detail, and I consider it important to establish this firmly first.

4. Theory in the weakly dependent regime

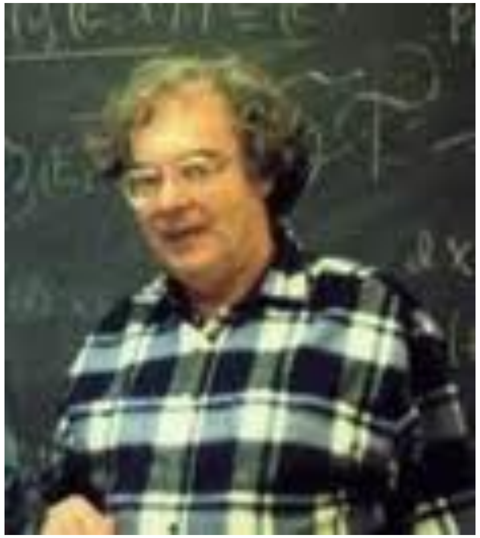
- Suppose inhomogeneous finite-state Markov chain with time-dependent transition operator P_t, near an exponentially mixing one, then there is a unique time-phase with marginals π_t and it is differentiable with respect to smooth parameter change:

  $\pi'_t = \pi_{t-1}P'_{t-1} + \pi_{t-2}P'_{t-2}P_{t-1} + \pi_{t-3}P'_{t-3}P_{t-2}P_{t-1} + \ldots$
- To make sense of this for spatially extended systems, e.g. probabilistic cellular automata (PCA), need a suitable metric on space of probabilities on a large product space.

How to measure distance between multivariate probability distributions

- S countable set
- For s in S, (X_s, d_s) Polish (complete separable metric) space of diameter $\leq \Omega$
- $X = \Pi X_s$ with product topology
- Ⓟ = Borel probabilities on X; want a metric on Ⓟ
- All standard metrics are useless when |S| is large, e.g. "Total variation convergence essentially never occurs for particle systems" (Liggett, 1985). Same for Jeffreys-Jensen-Shannon, Hellinger, Fisher information, projective, transportation (Vasserstein, Kantorovich, Rubinstein) metrics.

Dobrushin metric



- BC = bounded continuous functions $f: X \rightarrow R$
- $\Delta_s(f) = \sup (f(x)-f(y))/d_s(x_s,y_s)$
  over x,y in X with $x_r = y_r$ for all $r \neq s$, $x_s \neq y_s$.
- $|f| = \Sigma\, \Delta_s(f)$, *Dobrushin semi-norm*
- F = {f in BC: $|f| < \infty$}, *Dobrushin's functions*
- Z = Borel zero-charge measures μ on X, i.e. $\mu(X)=0$
- $|\mu| = \sup \mu(f)/|f|$ over non-constant f in F
- $(Z,|.|)$ is a Banach space
- For ρ,σ in Ⓟ:
  $D(\rho,\sigma) = |\rho-\sigma|$, *Dobrushin metric,*
  makes Ⓟ a complete metric space (of diameter = $\sup \mathrm{diam}_s(X_s)$)
- Not purely information theoretic; reflects metrics on the X_s.

Applications to PCA

- Probability p_s^x on X_s for new state x_s' of site s in S given current state x in X
- Transition probability $p^x = \Pi\, p_s^x$
- Transition operator P on f in BC:

  $(Pf)(x) = p^x(f)$
- Induces P on ρ in Ⓟ by $(\rho P)(f) = \rho(Pf)$
- Want to bound |P| on Z

Dobrushin's dependency matrix

- For ρ,σ probabilities on X_r, let
  $D_r(\rho,\sigma) = \sup (\rho(g)-\sigma(g))/|g|$
  over non-constant Lipschitz functions $g: X_r \rightarrow R$, $|g|$ = best Lipschitz constant
- For r,s in S, let $K_{rs} = \sup D_r(p_r^x,p_r^y)/d_s(x_s,y_s)$
  over x,y in X with $x_q = y_q$ for all $q \neq s$, $x_s \neq y_s$.
- Then $|P| \leq |K|_\infty$.
- In particular, $|K|_\infty < 1$ implies P has a unique stationary probability π and it attracts exponentially
- e.g. Stavskaya for $\lambda > ½$, NEC voter for λ in $(⅓,⅔)$
- Same if $|K^t|_\infty \leq Cr^t$ for some $r<1$, $D(\sigma P^t,\pi) \leq Cr^t D(\sigma,\pi)$

More on the exponentially mixing regime

- Exponentially attracting stationary probability is stable to perturbation:

  $D(\sigma P^t,\pi) \leq C(r+C|P-P_0|)^t D(\sigma,\pi)$
- Can use Dobrushin metric to define C^1 dependence of P on parameters λ and deduce C^1 dependence of π on λ with $\pi' = \pi P' (I-P)^{-1}$.
- And time-dependent response formula converges
- Question: range of control?

Beyond the exponentially mixing regime

- For S infinite can get non-unique stationary probability, or non-mixing ones, even if finite truncations would not.
- e.g. Stavskaya with λ small, NEC voter with λ small or near 1.
- Questions of control to influence selection, range of control... [demonstrate effect of boundary control on NEC voter]

5. Comments on R-tipping

- The work of Ashwin et al. on R-tipping is valid, but bear in mind that
  - it is well known that rate of parameter variation can lead to big changes, e.g. parametric excitation
  - the safety criterion of Bishnani & MacKay, Dyn Sys 18 (2003) 107-129, gives sufficient conditions for the response of an equilibrium to arbitrary time-dependent forcing to remain in a safe region
  - the safety criterion is being generalised to other types of attractor, using normal hyperbolicity