# A Mathematical Framework for Critical Transitions: From Bifurcations to Complex Systems

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# Outline

1. Introduction & Motivation
2. Fast-Slow Systems and Definitions
3. A Theorem on Scaling for Stochastic Systems
4. Examples (predator-prey, epidemics, epileptic seizures)
5. Stability and Instability in Complex Systems
6. More Examples (Games on Networks, Epidemics Data)

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5. Stability and Instability in Complex Systems
6. More Examples (Games on Networks, Epidemics Data)

- Work on epileptic seizures joint with:
  **C. Meisel, MPI-PKS & CGC Hospital, Dresden**
- Work on networks / complex systems joint with:
  **G. Zschaler, MPI-PKS, Dresden** and **T. Gross, U. Bristol**

# Critical Transitions and Tipping Points in Applications

Geoscience (climate change, climate subsystems, earthquakes)

- Lenton et al., *Tipping elements in the Earth's climate system*. PNAS, 2008
- Wieczorek et al., *Excitability in ramped systems*. Proc. R. Soc. A, 2010
- Thompson and Sieber, *Predicting climate tipping as a noisy bifurcation: a review*. IJBC, 2011

Ecology (extinction, desertification, ecosystem control)

- Drake and Griffen. *Early warning signals of extinction in deteriorating environments*. Nature, 2010
- Dakos et al. *Slowing down in spatially patterned systems at the brink of collapse*. Am. Nat., 2011
- Veraart et al., *Recovery rates reflect distances to a tipping point in a living system*. Nature, 2012

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Human Physiology (epileptic seizures, asthma attacks)

- Mormann et al., *Seizure prediction: the long and winding road*. Brain, 2007
- Venegas et al., *Self-organized patchiness in asthma as a prelude to catastrophic shifts*. Nature, 2005

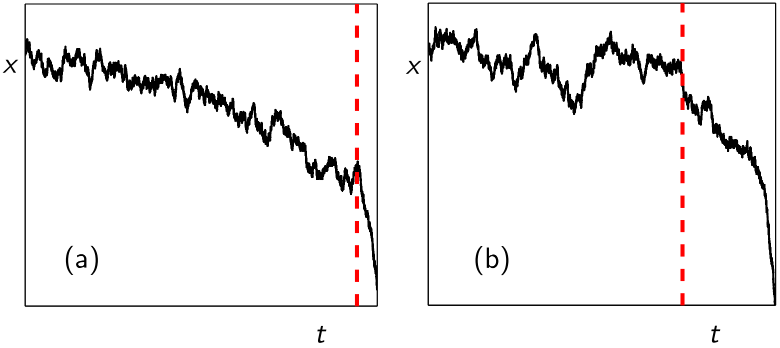
Finance (market crashes, risk analysis, asset price bubbles)

- D. Sornette, *Why Stock Markets Crash*. PUP, 2003
- Jarrow et al., *How to detect an asset bubble*. SIAM J. Finan. Math., 2011

Engineering (voltage collapse, fracture)

- Chertkov et al., *Voltage collapse and ODE approach to power flows*. 2011

# Goals and Perspectives...



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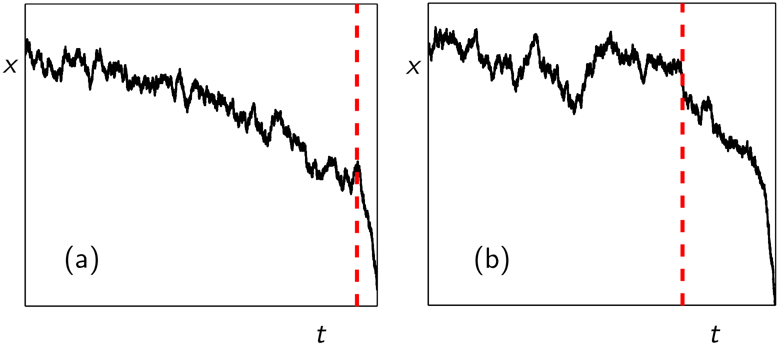


- Prediction of the "critical transition" or "tipping point".
- Dynamical mechanisms and mathematical models.
- Time series analysis and data assimilation.
- Data availability, experiments, theory of extreme events, ...

# Typical Characteristics of Critical Transitions

Following Scheffer et al, Nature, 2009:

(1) A qualitative change from regular dynamics occurs.

(2) Rapid change in comparison to the regular dynamics.

(3) The system crosses a special threshold near a transition.

(4) The new state is far away from its previous state.

(5) There is often small noise i.e. observed data has a major deterministic component with "random fluctuations".

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Towards a mathematical theory:

- Define critical transitions / tipping points using (1)-(5).
- Verify the known early-warning signs are indeed generic.
- Go beyond basic elements and extend the theory.

# Identify Generic Models: Fast-Slow Systems

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \epsilon \ll 1$.

$$\left\{ \begin{array}{lll} \frac{dx}{dt} = x' & = & f(x,y) \\ \frac{dy}{dt} = y' & = & \epsilon g(x,y) \end{array} \right. \quad \overset{\epsilon t = s}{\longleftrightarrow} \quad \left\{ \begin{array}{lllll} \epsilon \frac{dx}{ds} & = & \epsilon \dot{x} & = & f(x,y) \\ \frac{dy}{ds} & = & \dot{y} & = & g(x,y) \end{array} \right.$$

$$\downarrow \; \epsilon = 0 \qquad\qquad \downarrow \; \epsilon = 0$$

$$\left\{ \begin{array}{lll} x' & = & f(x,y) \\ y' & = & 0 \end{array} \right. \qquad\qquad \left\{ \begin{array}{lll} 0 & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{array} \right.$$

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fast subsystem — slow subsystem

- $C := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- $C$ is normally hyperbolic if $D_x f$ has no zero-real-part eigenvalues.
- **Fenichel's Theorem:** Normal hyperbolicity $\Rightarrow$ "nice" perturbation.
- Loss of Normal hyperbolicity $\Rightarrow$ complicated dynamics.

Example - A fold bifurcation of the fast subsystem with slow drift:

$$\begin{aligned} \epsilon \dot{x} &= -y - x^2, \\ \dot{y} &= 1. \end{aligned}$$

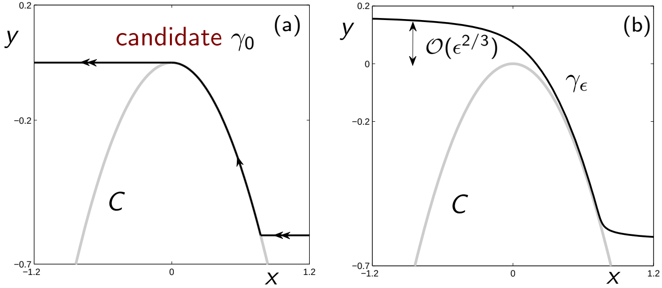


Figure: (a) Singular limit $\epsilon = 0$. (b) $\epsilon = 0.02$.

# A Natural Definition...

Let $p = (x_p, y_p) \in C_0$ where $C_0$ is not normally hyperbolic then $p$ is a critical transition if $\exists$ a candidate $\gamma_0$ and times $t_{j-1}, t_j$ s.t.

(C1) $\gamma_0(t_{j-1}, t_j) \subset S_0 \subset C_0$, $S_0$ normally hyperbolic attracting,

(C2) $p = \gamma_0(t_j)$ is a transition point between subsystems,

(C3) and $\gamma_0(t_{j-1}, t_j)$ is oriented from $\gamma_0(t_{j-1})$ to $\gamma_0(t_j)$.

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- Covers (1)-(4) of the phenomenological features:

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| regular dynamics | $\leftrightarrow$ | slow subsystem flow |
| rapid change | $\leftrightarrow$ | slow-to-fast transition |
| special threshold | $\leftrightarrow$ | fast subsystem bifurcation ($\Rightarrow$ genericity) |
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- Extension to stochastic systems immediate $\gamma_\epsilon \rightarrow \gamma_{\epsilon,\sigma}$.
- $\gamma_{0,0}$ yields B-tipping, $\gamma_{0,\sigma}$ N-tipping ($\rightarrow$ Ashwin); $\gamma_{\epsilon,\delta}$ perturbations.
- Natural: time series correspond to sample paths $\gamma_{\epsilon,\sigma}$.
- Extensions to PDEs, DDEs, etc. are possible via paths.

# What about Noise? - Stochastic Fast-Slow Systems...

Consider the fast-slow stochastic differential equation

$$\begin{aligned} dx_s &= \frac{1}{\epsilon} f(x_s, y_s) ds + \frac{\sigma}{\sqrt{\epsilon}} F(x_s, y_s) dW_s, \\ dy_s &= g(x_s, y_s) ds. \end{aligned} \tag{1}$$

where $W_s$ is standard Brownian motion.

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Suppose (1) has a deterministic attracting slow manifold

$$C_\epsilon = \{(x, y) \in \mathbb{R}^{m+n} : x = h_\epsilon(y) = h_0(y) + \mathcal{O}(\epsilon)\}$$

**Theorem (Berglund and Gentz)**

*Sample paths of* (1) *stay near* $C_\epsilon$ *with high probability.*

# Towards Early-Warning Signs

(W1) The system recovers slowly from perturbations: slowing down.

(W2) The autocorrelation increases before a transition.

(W3) The variance increases near a critical transition.

(W4) . . .

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*Example:* Deterministic fold bifurcation $x' = -y - x^2$, $y' = \epsilon$.

- Slow subsystem $\epsilon = 0$, attracting critical manifold $x = \sqrt{-y}$.
- Fast subsystem $\epsilon = 0$, parameterized family $x' = -y - x^2$.
- Variational equation:

$$X' = -2\sqrt{-y}X \qquad \Rightarrow \quad X(t) = X(0)e^{-2\sqrt{-y}t}$$

Slowing down as $y \to 0^-$.

# Useful Stochasticity - Variance near a Fold

$$\begin{aligned} dx_t &= \frac{1}{\epsilon}(-y_t - x_t^2)\ dt + \frac{\sigma}{\sqrt{\epsilon}} dW_t, \\ dy_t &= 1\ dt. \end{aligned}$$

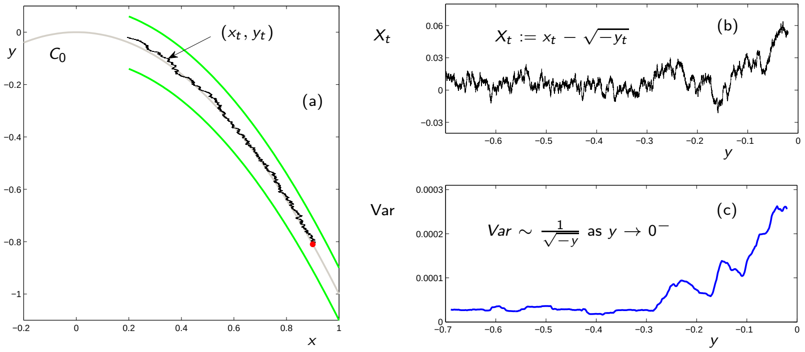


Figure: $(x_0, y_0) = (0.9, -0.9^2)$ [red dot], $\sigma = 0.01$, $\epsilon = 0.01$.

# Sketch of Proof: Early-Warning for Folds ($\rightarrow$ Berglund)

- Focus on attracting slow manifold $C_\epsilon = \{x = h_0(y)\}$.
- Variational equation for linearized process:

$$d\xi_s^l = \frac{1}{\epsilon}(-2h_0(y_s)\xi_s^l)ds + \frac{\sigma}{\sqrt{\epsilon}}dW_s$$

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- Define $X_s := \sigma^{-2}\text{Var}(\xi_s^l)$ and "observe"

$$\begin{aligned} \epsilon\dot{X} &= -4h_0(y)X + 1, \\ \dot{y} &= 1. \end{aligned}$$

- Conclusion (up to leading order)

$$\text{Var}(x_s) = \frac{\sigma^2}{4\sqrt{-y}} = \mathcal{O}\left(\frac{1}{\sqrt{-y}}\right)$$

as $y \rightarrow 0^-$ and $\sigma$ fixed.

# Main Result - Overview

## Theorem (K. 2011)

*Classification of generic critical transitions for all fast subsystem bifurcations up to codimension two:*

- *Fold, Hopf, (transcritical), (pitchfork)*
- *Cusp, Bautin, Bogdanov-Takens*
- *Gavrilov-Guckenheimer, Hopf-Hopf*

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*The main results are:*

1. *(Existence:) Conditions on slow flow to get a critical transition.*
2. *(Scaling:) Leading-order covariance scaling $H_\epsilon(y)$ for*

$$Cov(x_s) = \sigma^2[H_\epsilon(y)] + \mathcal{O}(\sigma^4).$$

3. *($(\epsilon, \sigma)$-expansion:) Higher-order calculations for the fold.*
4. *(Technique:) Covariance estimates without martingales.*

# Example 1: The Bazykin Predator-Prey System

$$
\begin{aligned}
dx_1 &= \left[x_1 - \frac{x_1 x_2}{1+y_1 x_1} - 0.01 x_1^2\right] dt + \frac{\sigma_1}{\sqrt{\epsilon}} dW^{(1)}, \\
dx_2 &= \left[-x_2 + \frac{x_1 x_2}{1+y_1 x_1} - y_2 x_2^2\right] dt + \frac{\sigma_2}{\sqrt{\epsilon}} dW^{(2)}, \\
dy_1 &= \epsilon g_1(x, y) dt, \\
dy_2 &= \epsilon g_2(x, y) dt,
\end{aligned}
$$

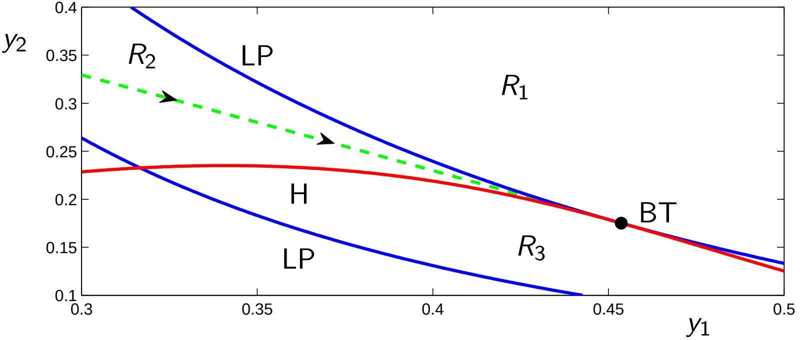



Figure: Partial bifurcation diagram; $\epsilon = 0 = \sigma_1 = \sigma_2$.

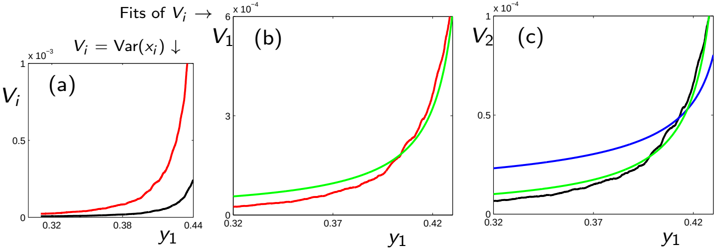
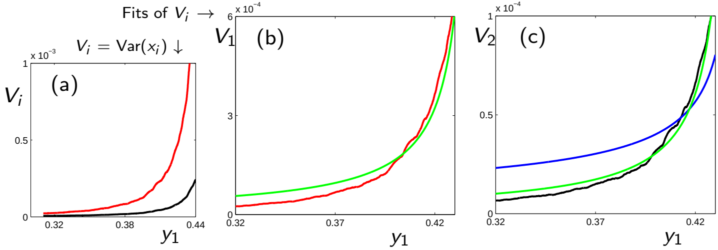



Figure: Averaged over 50 sample paths $(\epsilon, \sigma) = (3 \times 10^{-5}, 1 \times 10^{-3})$; $V_i = \mathrm{Var}(x_i(y))$ for $i \in 1, 2$; $V_1$ (red) $V_2$ (black).

- Theory for Bogdanov-Takens point predicts:

$$\mathrm{Cov}(x_s(y)) = \sigma^2 \begin{pmatrix} \mathcal{O}(1/y_1) & K \\ K & \mathcal{O}(1/\sqrt{-y_1}) \end{pmatrix}.$$

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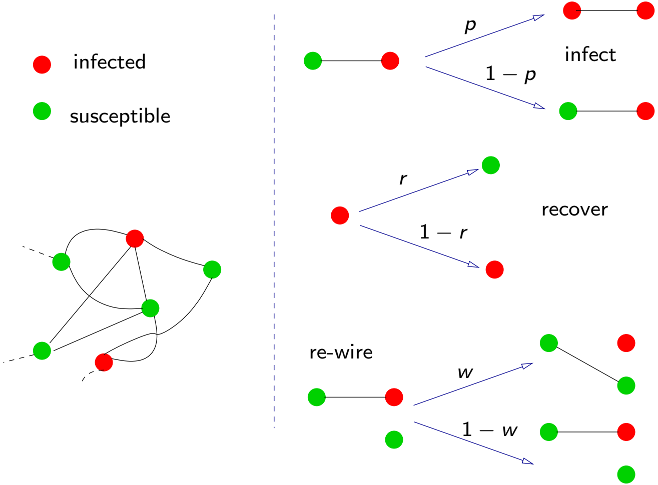
- NOT normal form: hidden scaling law ⇒ “unpredictable” $(\mathcal{O}(1/y_1) + \mathcal{O}(1/\sqrt{-y_1}) = \mathcal{O}(1/y_1)$ as $y_1 \to 0^-)$

# Example 2: Epidemics on Adaptive Networks

SIS dynamics on adaptive networks [Gross et al., 2006]:



A moment closure pair-approximation ($l_{abc} = \frac{l_{ab} l_{bc}}{b}$) yields:

$$
\begin{aligned}
i' &= p(\tfrac{\mu}{2} - l_{II} - l_{SS}) - ri, \\
(l_{II})' &= p(\tfrac{\mu}{2} - l_{II} - l_{SS}) \left( \frac{\frac{\mu}{2} - l_{II} - l_{SS}}{1-i} + 1 \right) - 2rl_{II}, \\
(l_{SS})' &= (r+w)(\tfrac{\mu}{2} - l_{II} - l_{SS}) - \frac{2p(\frac{\mu}{2} - l_{II} - l_{SS}) l_{SS}}{1-i}.
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where we assume $\mu = 2\frac{\#\text{links}}{\#\text{nodes}} = 20$, $r = 0.002$ and $w = 0.4$.

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Often moment-closure works, and sometimes it doesn't but certainly there is finite-size noise:

$$
\begin{aligned}
dx_1 &= \frac{1}{\epsilon}\left[ y(\tfrac{\mu}{2} - x_2 - x_3) - rx_1 \right] ds + \frac{\sigma_1}{\sqrt{\epsilon}} dW^{(1)}, \\
dx_2 &= \frac{1}{\epsilon}\left[ y(\tfrac{\mu}{2} - x_2 - x_3) \left( \frac{\frac{\mu}{2} - x_2 - x_3}{1-x_1} + 1 \right) - 2rx_2 \right] ds + \frac{\sigma_2}{\sqrt{\epsilon}} dW^{(2)}, \\
dx_3 &= \frac{1}{\epsilon}\left[ (r+w)(\tfrac{\mu}{2} - x_2 - x_3) - \frac{2y(\frac{\mu}{2} - x_2 - x_3)x_3}{1-x_1} \right] ds + \frac{\sigma_3}{\sqrt{\epsilon}} dW^{(3)}, \\
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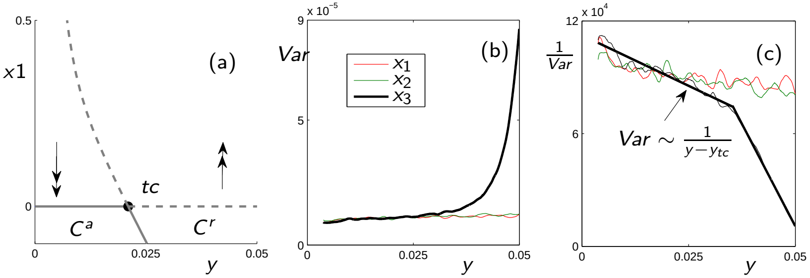
# Results on Adaptive SIS Epidemics



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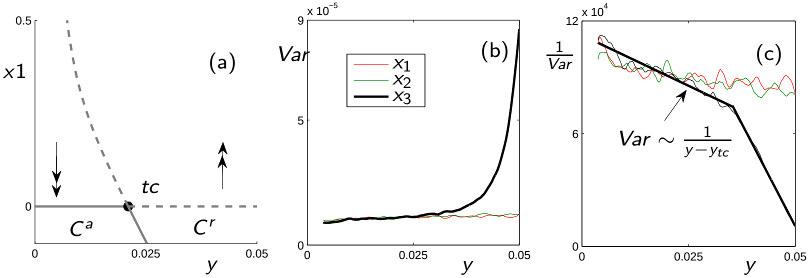
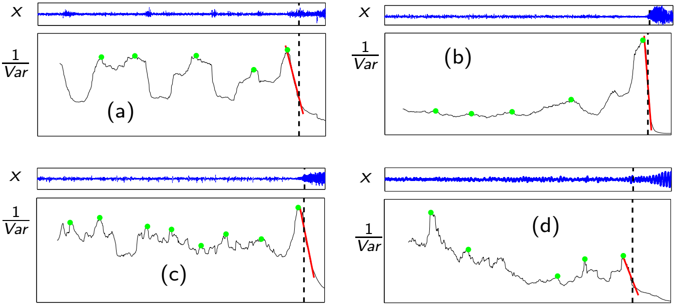



- Observe theory for scaling law for the transcritical bifurcation $dx_s = (x_s y_s - x_s^2)ds + \sigma dW$

$$\mathrm{Var}(x_s) = \sigma^2 \mathcal{O}\left((y_s - y_{tc})^{-1}\right) \qquad \text{as } y_s \to y_{tc}.$$

- **Important:** Early-warning sign in the link density only!
- Also observe a delay ($\rightarrow$ way-in way-out function).

# Example 3: ECoG Data and Epileptic Seizures

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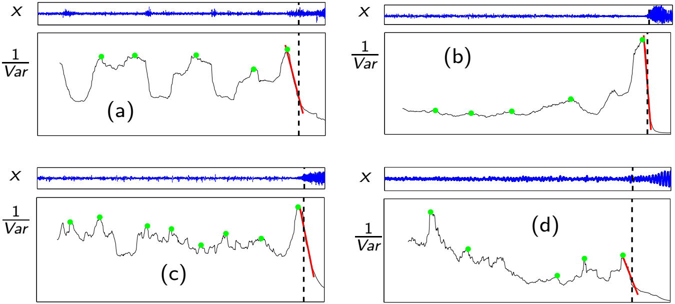


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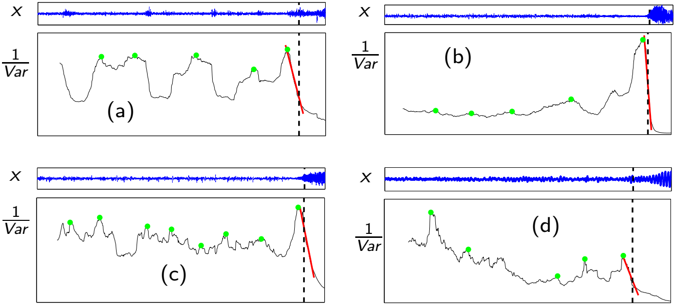


Figure: Avg. ECoG data (blue), seizure (black dashed), scaling law (red).

- Scaling law + genericity $\Rightarrow$ Hopf ($\rightarrow$ Terry, U. Sheffield).
- Early-warning signs for excitable systems (FitzHugh-Nagumo).
- Comparison: neuron (micro), cluster (meso), network (macro).
- Network measures based upon wavelet decomposition.

# Critical Transitions in (unstructured) Complex Systems

Consider $x \in \mathbb{R}^N$, $N \gg 1$

$$\frac{dx}{dt} = f(x).$$

At a steady state $x^*$, we have

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- Large system ($\rightarrow$ May) take $A$ a random real symmetric matrix, use semi-circle law

$$\mathbb{P}(x^* \text{ (un-)stable}) = \left(\frac{1}{2}\right)^N, \qquad \mathbb{P}(x^* \text{ saddle}) = 1 - \left(\frac{1}{2}\right)^{N-1}.$$

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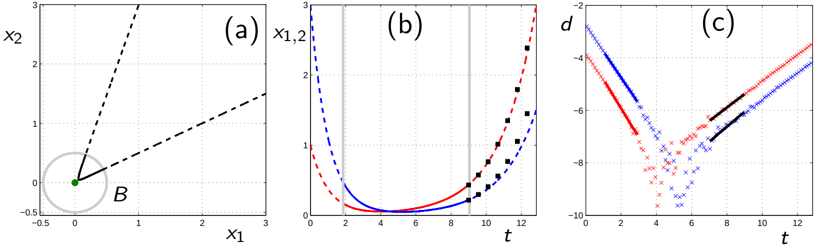
- Needs strong assumptions on structure of the matrix $A$!
- Proof of full circular law ($\to$ Tao and Van Vu 2008).

Expected abundance of saddle points in complex systems.

# Metastability and Critical Transitions near Saddle Points

Trivial case: planar saddle in $\mathbb{R}^2$ at $x = (0,0)$ locally

$$x' = Ax \quad \Rightarrow x(t) = y_1(0)e^{\lambda_s t}v_1 + y_2(0)e^{\lambda_u t}v_2.$$



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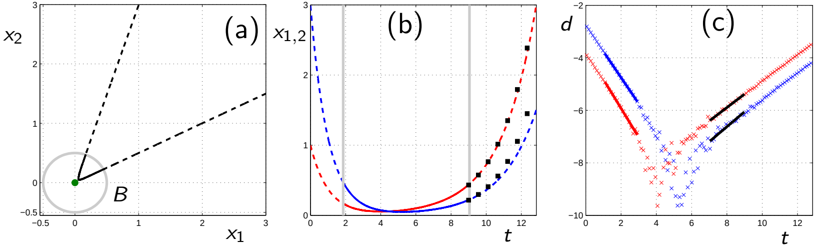


If $x(0)$ close to $W^s(0)$ then logarithmic distance reduction

$$\ln \|x(t_2) - x(t_1)\| \approx \lambda_s t_1 + k_1.$$

# Example 4: Evolutionary Game on a Network

Agents/nodes play snowdrift game each time step

$$M = \begin{pmatrix} b - c/2 & b - c \\ b & 0 \end{pmatrix} \qquad \text{c=cost, b=benefit.}$$

Re-wire ($p$), adopt ($1 - p$) based upon performance.

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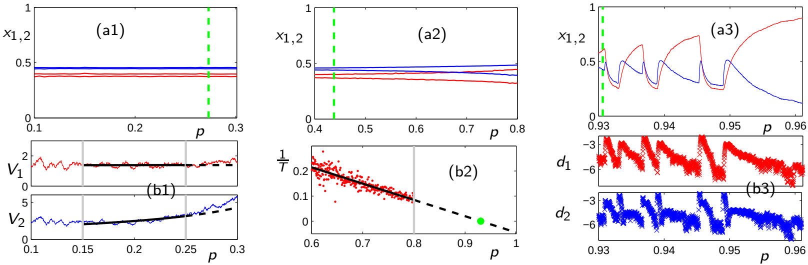

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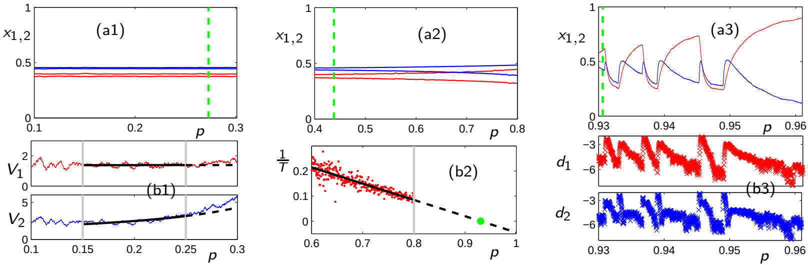


- For near-full cooperation with high re-wiring: saddle point.
- Early-warning signs: period blow-up and log-distance.

# Example 5: Back to Epidemics

Measles epidemics in cities in the UK between 1944 and 1966.

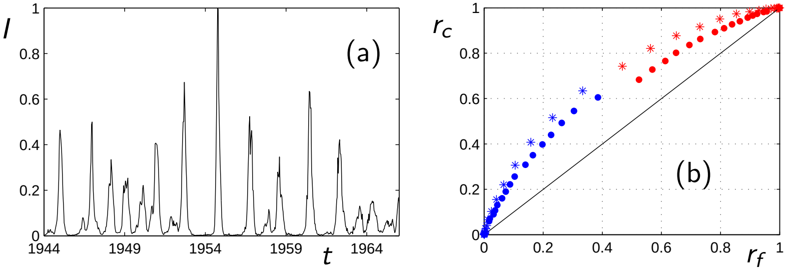


Figure: (a) Example time series. (b) ROC($d$)-curve (dots, crosses = different forecast lengths); blue=true instability, red=weak stability.

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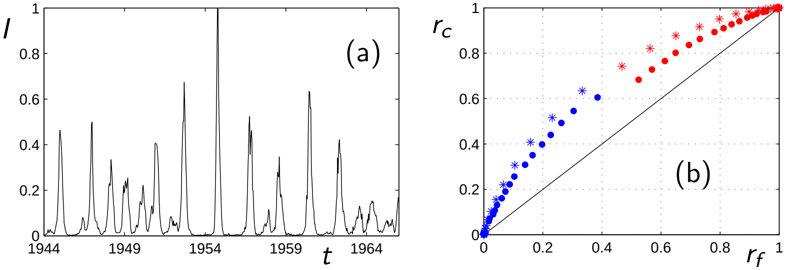


Figure: (a) Example time series. (b) ROC($d$)-curve (dots, crosses = different forecast lengths); blue=true instability, red=weak stability.

Test logarithmic distance indicator to estimate $\lambda_u$, threshold $d$.
Receiver-operator-characteristic curve

$$r_c = \frac{\#\text{correct predictions}}{\#\text{events/outbreaks}} \quad \text{and} \quad r_f = \frac{\#\text{false positives}}{\#\text{non-events}}.$$

# Overview & Conclusions

Mathematical Theory:

- Characterization and definition of critical transitions.
- Useful noise: proof of scaling laws up to codimension two.
- Hidden laws, coarse-grained networks.
- Saddle points as critical transitions.
- Logarithmic distance and period blow-up as warning signs.
- Data analysis, excitable systems, multiplicative noise, ...

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Applications:

- Social connections could be crucial for epidemic prediction.
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**Thank you for your attention.**