Precursors for Critical Transitions: Natural vs. Engineered Systems, and a Role for Control Theory

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Eyad H. Abed

United Arab Emirates University
and University of Maryland, College Park

Outline

1. Motivation and Examples
2. Bifurcation Control
3. Instability Warning Systems and Connections with Control
4. Modal Participation Factor Analysis: New Approach
5. Concluding Remarks

Motivation: Differing Issues for Natural vs. Engineered Systems

- In a natural system, we may hope to predict the onset of a transition, and take precautions that may help influence it. Often, however, we have no means to influence the transition.

- However, in engineered systems, we have more options, starting at the stage of system design.

Motivation: Differing Issues for Natural vs. Engineered Systems

- For an engineered system, we can:
  - Design the system to not have a transition in the intended operating range.
  - Design an augmented system that provides a warning to the operator near a transition.
    - Based on a good detailed model,
    - Or based on smart signal processing for an uncertain system.
  - Design a control system that improves behavior at a transition *or* gives a warning signal prior to a transition.

EXAMPLES

(AGRICULTURAL) TRACTORS!

United States Patent [19] [11] **4,284,987**

Gibson et al. [45] **Aug. 18, 1981**

[54] **SLOPE STABILITY WARNING DEVICE FOR ARTICULATED TRACTORS**

[75] Inventors: **Harry G. Gibson; Benjamin C. Thorner,** both of Morgantown, W. Va.; **Jack W. Thomas,** LaGrande, Oreg.

[73] Assignee: **The United States of America as represented by the Secretary of Agriculture,** Washington, D.C.

[21] Appl. No.: **73,474**

[22] Filed: **Sep. 7, 1979**

[51] **Int. Cl.**[3] **G08B 21/00**

[52] **U.S. Cl.** .... **340/689; 33/264;** 33/366; 33/391; 180/282; 116/DIG. 13; 200/61.52; 340/52 H; 340/684; 340/685

[58] **Field of Search** .... 340/52 H, 689, 684, 340/685; 200/61.52; 116/DIG. 13; 33/391, 366, 283, 264; 180/282; 280/466 R, 6 R

OTHER PUBLICATIONS

Gibson, H. G. et al., "Slope Stability of Logging Tractors and Forwarders", *Transactions of the ASAE* vol. 17, No. 2, 1974, pp. 245–250.

Gibson, H. G. et al., "Side Slope Stability of Articulated–Frame Logging Tractors", *Journal of Terramechanics,* vol. 8, No. 2, pp. 65–79, 1971, Great Britain.

Primary Examiner—Glen R. Swann III
Attorney, Agent, or Firm—M. Howard Silverstein; David G. McConnell

[57] **ABSTRACT**

A tip-over warning system for vehicles of the articulated type utilizes a swinging pendulum pivoted in a frame which is mounted on and simulates the stability triangle of the vehicle. The pivotal connection of the pendulum is at a scale distance relative to the frame corresponding to the location of the vehicle center of

JET AIRCRAFT

FA6 11:15

NONLINEAR STABILIZATION OF HIGH ANGLE-OF-ATTACK FLIGHT DYNAMICS USING BIFURCATION CONTROL

Eyad H. Abed and Hsien-Chiarn Lee

Department of Electrical Engineering
and the Systems Research Center
University of Maryland, College Park, MD 20742 USA

Abstract

We consider the problem of designing stabilizing control laws for flight over a broad range of angles-of-attack which also serve to signal the pilot of impending stall. The paper employs bifurcation stabilization coupled with more traditional linear control system design. To focus the discussion, a detailed analysis is given for a model of the longitudinal dynamics of an F-8 Crusader.

I. Introduction

Several authors have studied the nonlinear phenomena that arise commonly in aircraft flight at high angle-of-attack (alpha). The literature on high alpha flight dy-

a bifurcation occurring in a one-parameter family of systems

$$\dot{x} = f_\mu(x, u). \tag{1}$$

These control laws exist generically, even if the critical eigenvalues of the linearized system at the equilibrium of interest are uncontrollable. (The critical eigenvalues are those lying on the imaginary axis.) This approach has been employed in the design of stabilizing control laws for a tethered satellite system in the station-keeping mode.

III. Bifurcation Control of Longitudinal

American Control Conf., Boston, 1990, pp. 2235-2238.

Bifurcation diagrams in the aircraft example: Uncontrolled and controlled

FIGURE 1a. α AT OPEN-LOOP EQUILIBRIA

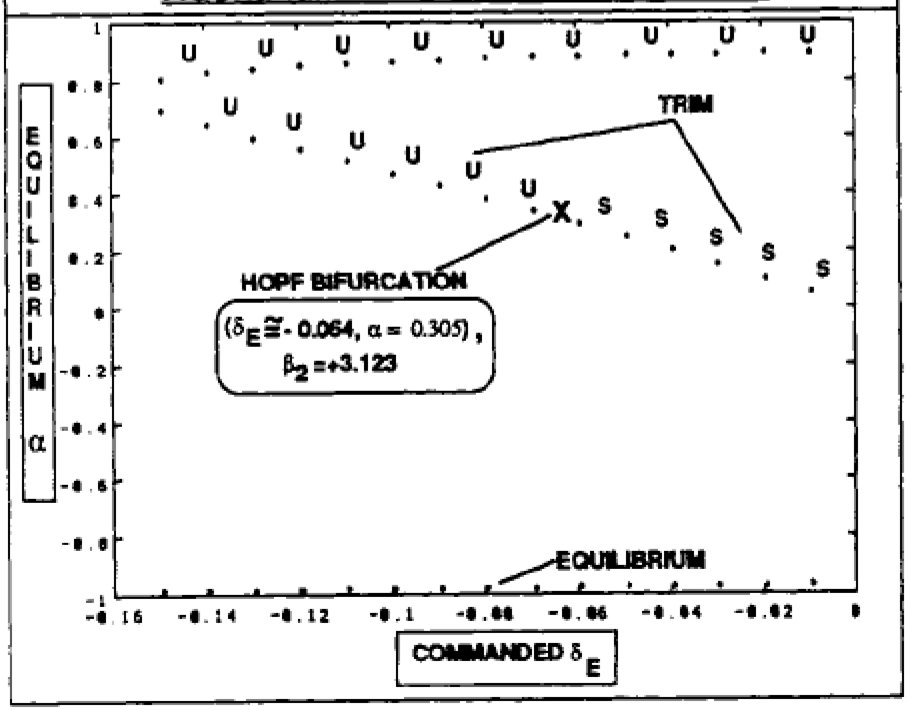


FIGURE 2a. α AT EQUILIBRIA UPON LINEAR FEEDBACK

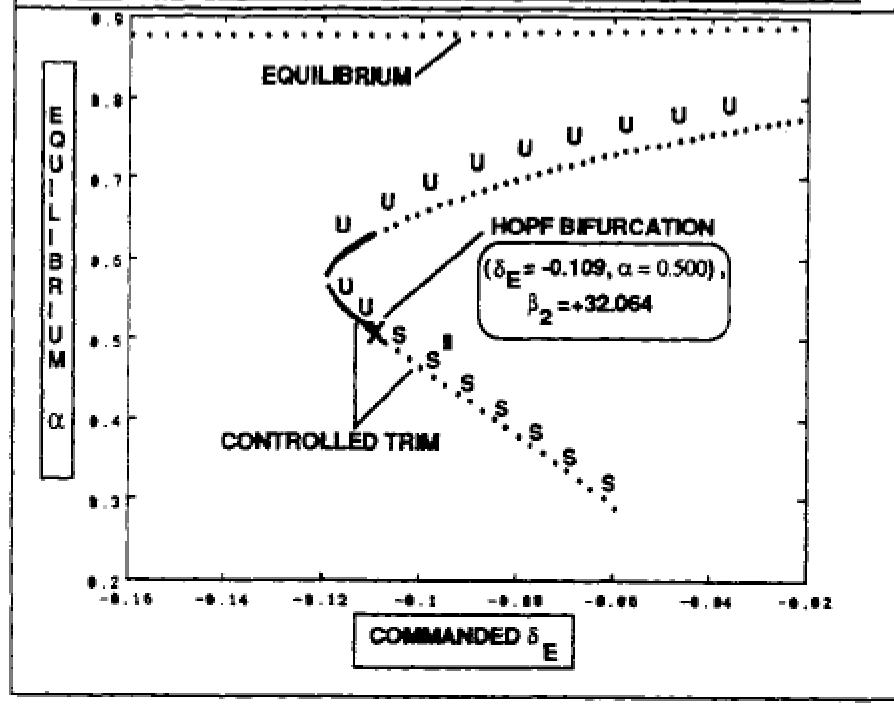


FIG. 4. SHOWING DIVERGENCE UNDER LINEAR FEEDBACK

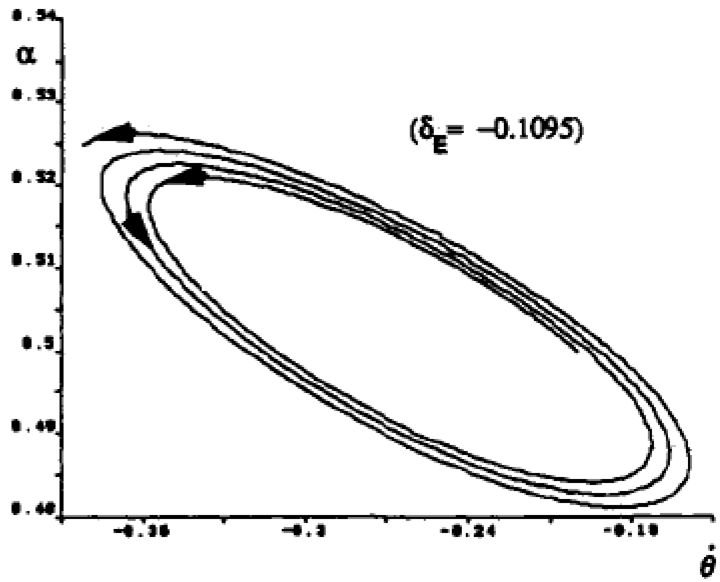
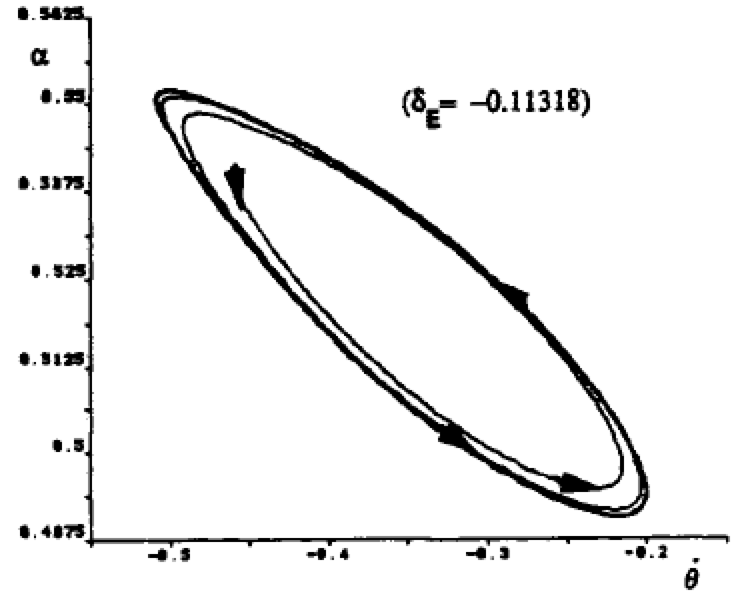


FIG. 5. SHOWING CONVERGENCE TO LIMIT CYCLE UNDER LINEAR-PLUS-NONLINEAR FEEDBACK



JET ENGINE STALL

Annu. Rev. Fluid Mech. 2001. 33:491–517

COMPRESSION SYSTEM STABILITY AND ACTIVE CONTROL

JD Paduano, EM Greitzer, and AH Epstein
Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139; e-mail: paduano@mit.edu, greitzer@mit.edu, epstein@mit.edu

2.1 Waves as Precursors to Stall

The long-length-scale description of rotating stall inception starts with a two-dimensional representation of the idealized compression system, which is shown in Figure 2. A linearized stability analysis captures many of the characteristics of rotating stall inception. This stability analysis has also been extended to address

ELECTRIC POWER NETWORKS

- As noted by Hauer (APEx 2000): "[recurring problem of system oscillations and voltage collapse] is due in part to … system behavior not well captured by the models used in planning and operation studies"
- In the face of component failures, system models quickly become mismatched to the physical network, and are only accurate if they're updated using a powerful and accurate failure detection system.

- In several papers, Hauer has discussed large system experiments using probe signal injection and ambient noise effects for stability and oscillation studies. This includes HVDC modulation at mid-level (125MW) for probing of inividual oscillation modes, and low-level (20MW) for broadband probing.

- Hauer used *prony analysis* of streaming measurements to estimate system modes, and the calculated dominant mode was used as an indicator of impending instability.

HUNTING OF RAILWAY VEHICLES

ELSEVIER

Journal of Sound and Vibration 315 (2008) 766–780

www.elsevier.com/locate/jsvi

Stabilization and utilization of nonlinear phenomena based on bifurcation control for slow dynamics

Hiroshi Yabuno

Department of Mechanical Engineering, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Yokohama 223-8522, Japan

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Abstract

Mechanical systems may experience undesirable and unexpected behavior and instability due to the effects of nonlinearity of the systems. Many kinds of control methods to decrease or eliminate the effects have been studied. In particular, bifurcation control to stabilize or utilize nonlinear phenomena is currently an active topic in the field of nonlinear dynamics. This article presents some types of bifurcation control methods with the aim of realizing vibration control and motion control for mechanical systems. It is also indicated through every control method that slowly varying components in the dynamics play important roles for the control and the utilizations of nonlinear phenomena. In the first part, we deal with stabilization control methods for nonlinear resonance which is the 1/3-order subharmonic resonance in a nonlinear spring–mass–damper system and the self-excited oscillation (hunting motion) in a railway vehicle wheelset. The

EARTHQUAKES
(OK, this isn't an engineered system)

Pure appl. geophys. 157 (2000) 2365–2383
0033–4553/00/122365–19 $ 1.50 + 0.20/0

Pure and Applied Geophysics

Development of a New Approach to Earthquake Prediction: Load/Unload Response Ratio (LURR) Theory

Xiang-Chu Yin,[1,2] Yu-Cang Wang,[1] Ke-Yin Peng,[1,2] Yi-Long Bai,[1]
Hai-Tao Wang[1] and Xun-Fei Yin[1,3]

Abstract—The seismogenic process is nonlinear and irreversible so that the response to loading is different from unloading. This difference reflects the damage of a loaded material. Based on this insight, a new parameter-load/unload response ratio (LURR) was proposed to measure quantitatively the proximity to rock failure and earthquake more than ten years ago. In the present paper, we review the

BIFURCATION CONTROL

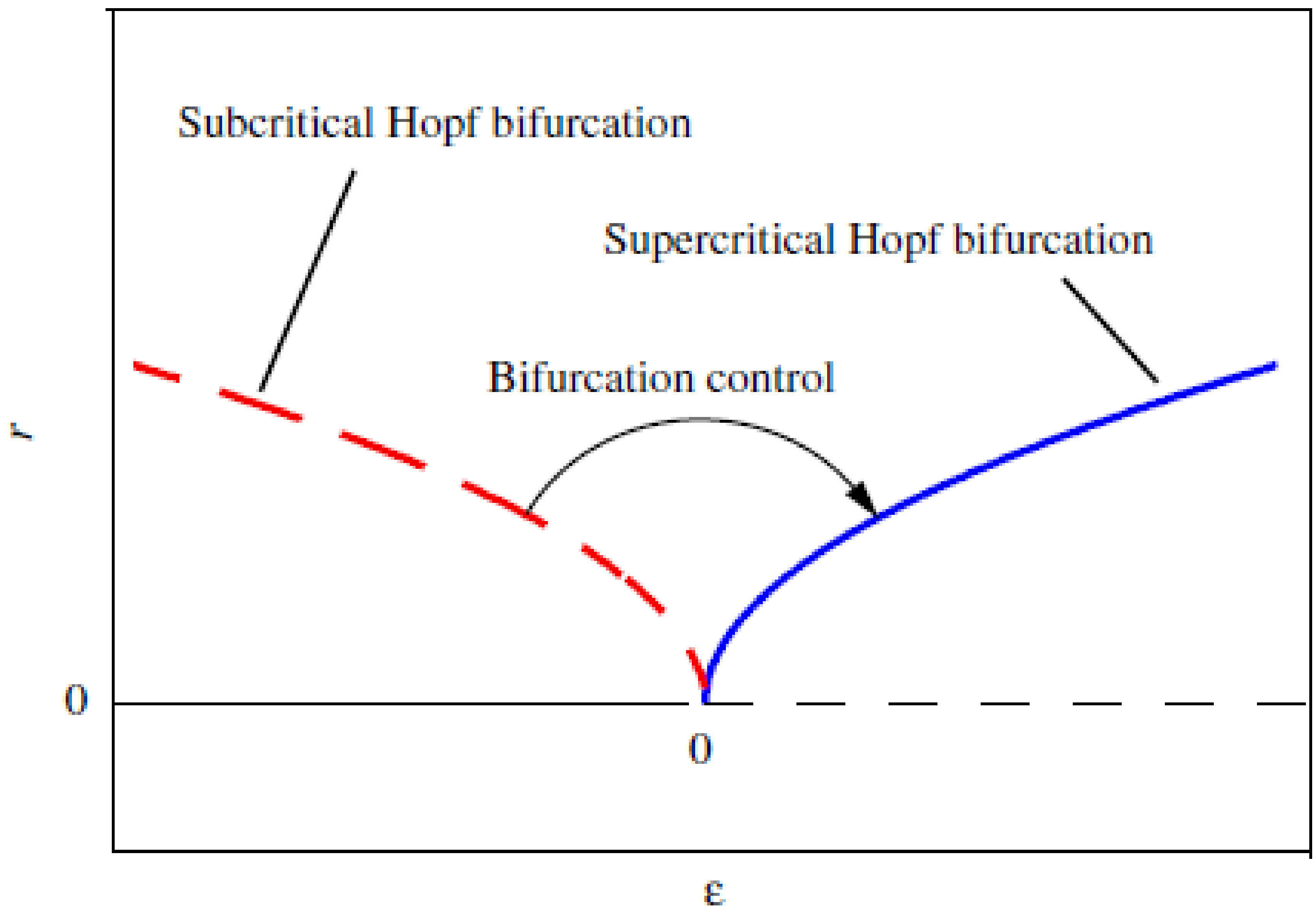
- Bifurcation control is an analytical methodology for designing feedback to achieve desired bifurcation behavior.

- Often one wishes to render a subcritical bifurcation supercritical by feedback.

- The theory has been applied to models in many contexts, from engineering to epileptic seizures.

- Dynamic feedback structures using “washout filters” (special high pass filters) allow control without affecting equilibrium path locations, even without having an accurate system model!

(Ref: Abed and Fu, Systems and Control Letters, 1986 and 1987, Lee and Abed 1991, and subsequent work.)



Radius of limit cycle and nonlinear control (—— stable, – – – unstable).

INSTABILITY WARNING SIGNALS AND CONNECTIONS WITH CONTROL

Bifurcation Control for Introducing Bifurcation Warning Signals

- Methods should be developed for redesigning systems and system controllers so that new distinguishing features occur as bifurcation boundaries are approached.

- These features can serve as warning signals for impending bifurcation.

- This can involve designing more tame bifurcations into the dynamics, such as supercritical Hopf bifurcations.

- The jet aircraft example above is pertinent. “Wing rock” would be a good warning signal to “departure.”

Stability Monitoring Should be Detection of *Impending* Bifurcation and Instability

vs.

(Early) Detection of *Incipient* Bifurcation and Instability

Most stability detection in engineering focuses on detecting incipient instability:

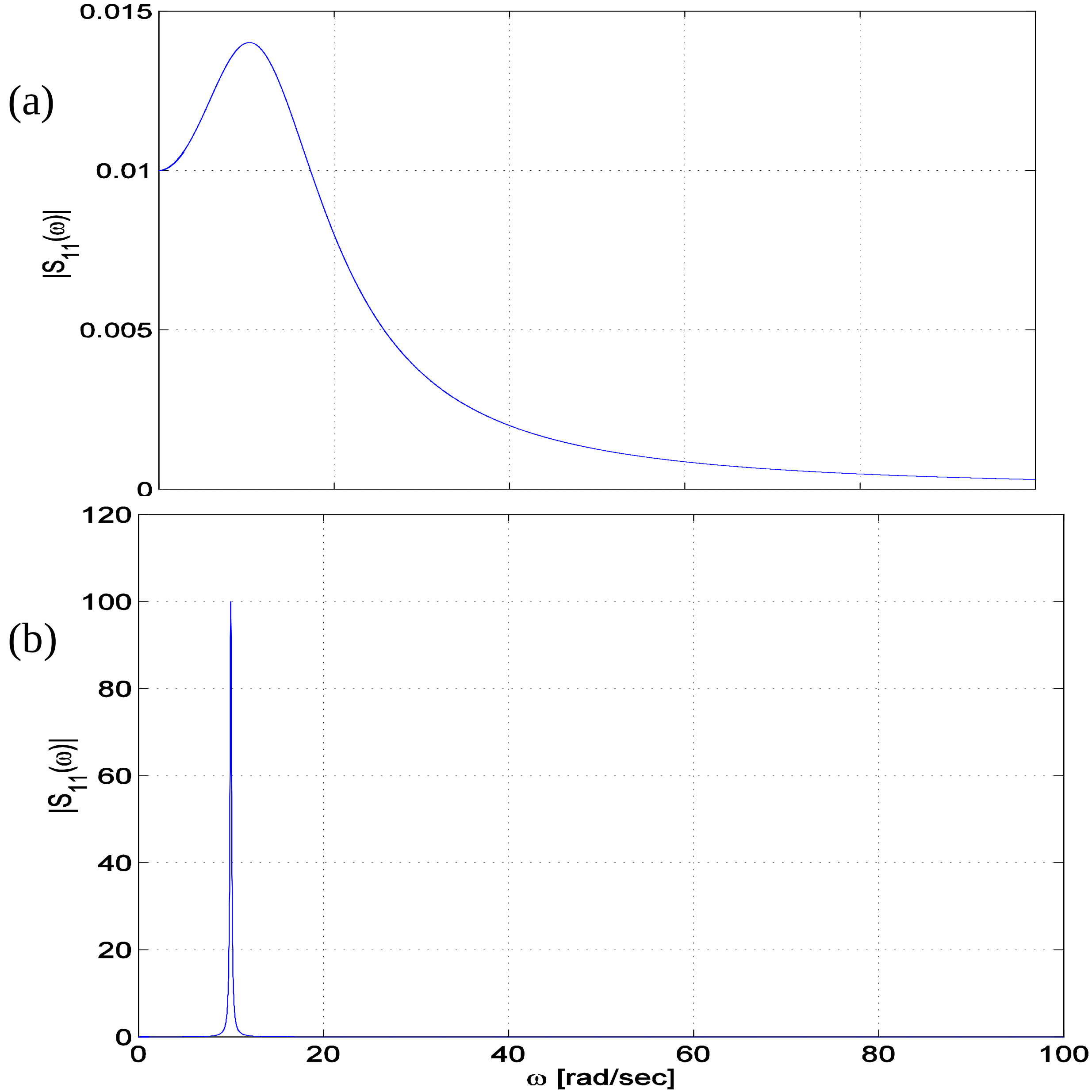
- **Incipient instability** is an instability that has already begun
- It may be too late at that stage to take adequate control action to save the system from collapse

Detection of impending bifurcation and instability:

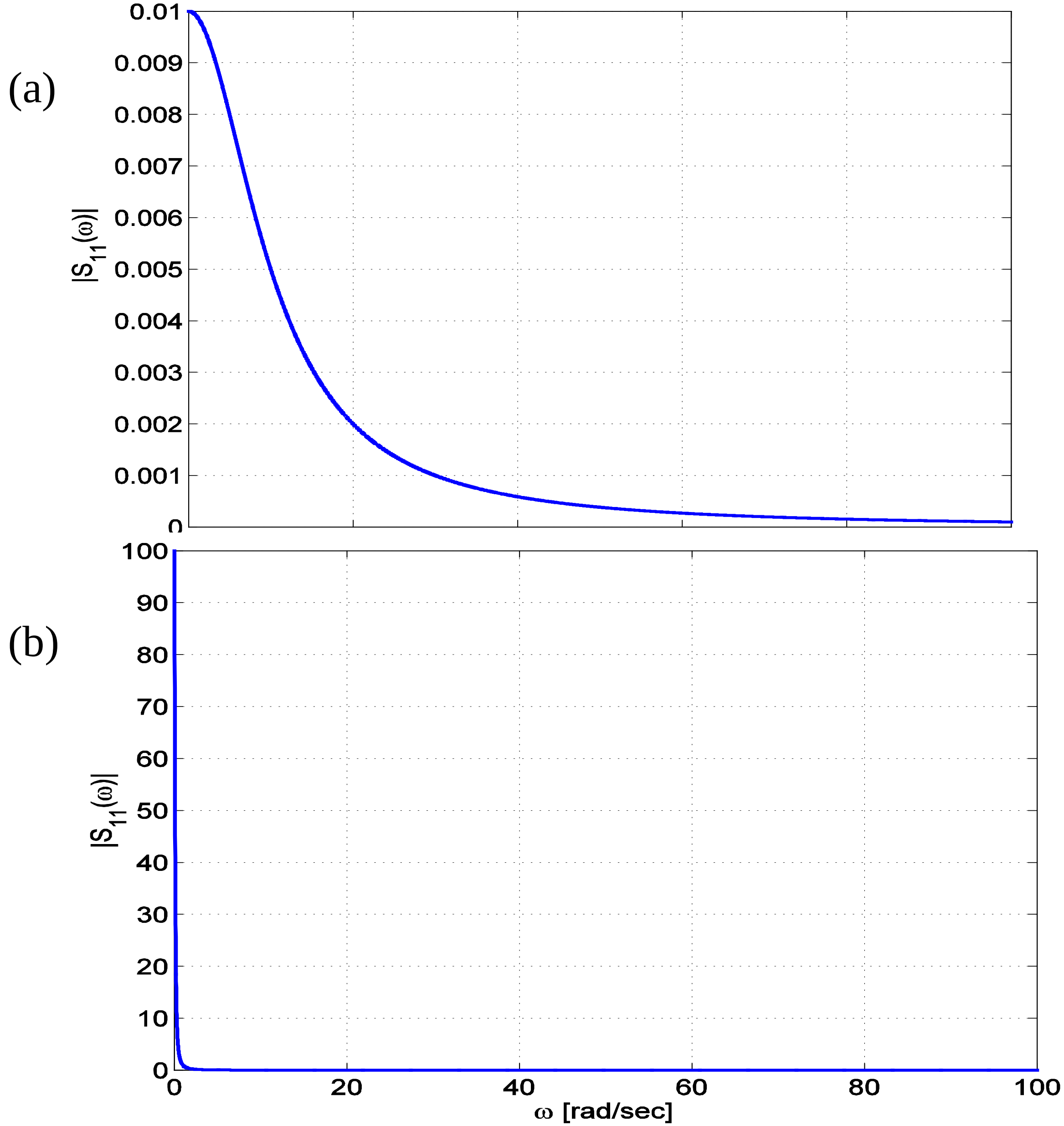
- An **impending bifurcation** is one that is nearby in parameter space and about ready to occur
- **"Noisy precursors"** give a *robust, nonparametric indicator of impending instability.*

Stability Monitoring and Noisy Precursors

- We consider instability monitoring using probe signals (e.g., additive white Gaussian noise).
- “Noisy precursors” were studied by Kurt Wiesenfeld (1985, J. Stat. Phys.) in the context of noise amplification near criticality (stability boundary). Wiesenfeld found different noisy precursors for different bifurcations, assuming a small white noise disturbance.
- It is important to note that noisy precursors also give a nonparametric indicator of impending instability.
- Noisy precursors are observed as rising peaks in the power spectral density of a measured output signal of a system with a persistent noise disturbance --- the rising peak is seen as one or more eigenvalues approach the imaginary axis.



Power spectrum magnitude for Hopf bifurcation when ω_0=10 for two values of ε: (a) ε=10, (b) ε=0.1 [Kim & Abed 2000].



Power spectrum magnitude for stationary bifurcation for two values of ε: (a) ε=10, (b) ε=0.1 [Kim & Abed 2000].

Monitoring Systems using Noisy Precursors

- Kim and Abed (IEEE Trans. Circuits and Systems, 2000) developed a feedback system in which the physical system and a model are run side by side.

- The model system is designed to enhance detectability of impending instability.

- Actions are taken as needed to protect the physical system.

Feedback Tuning of Bifurcations

- Moreau and Sontag (Systems and Control Letters, 2003) studied algorithms that can drive a system to a bifurcation point.

- A feedback tuning of bifurcations algorithm could be attached to a physical system.

- When the algorithm is seen to be quickly approaching bifurcation, this could be taken as a warning signal.

Extensions of Identification Theory

- In identification theory, parameter identification for a linear system is possible under a “persistent excitation” condition for a probe signal.

- For system stability monitoring, we want to focus on identifying the dominant mode(s) only. What is the theoretical condition for this to be possible?

(B. Hamzi of Imperial College is currently working on this problem.)

Application to a Power System Model

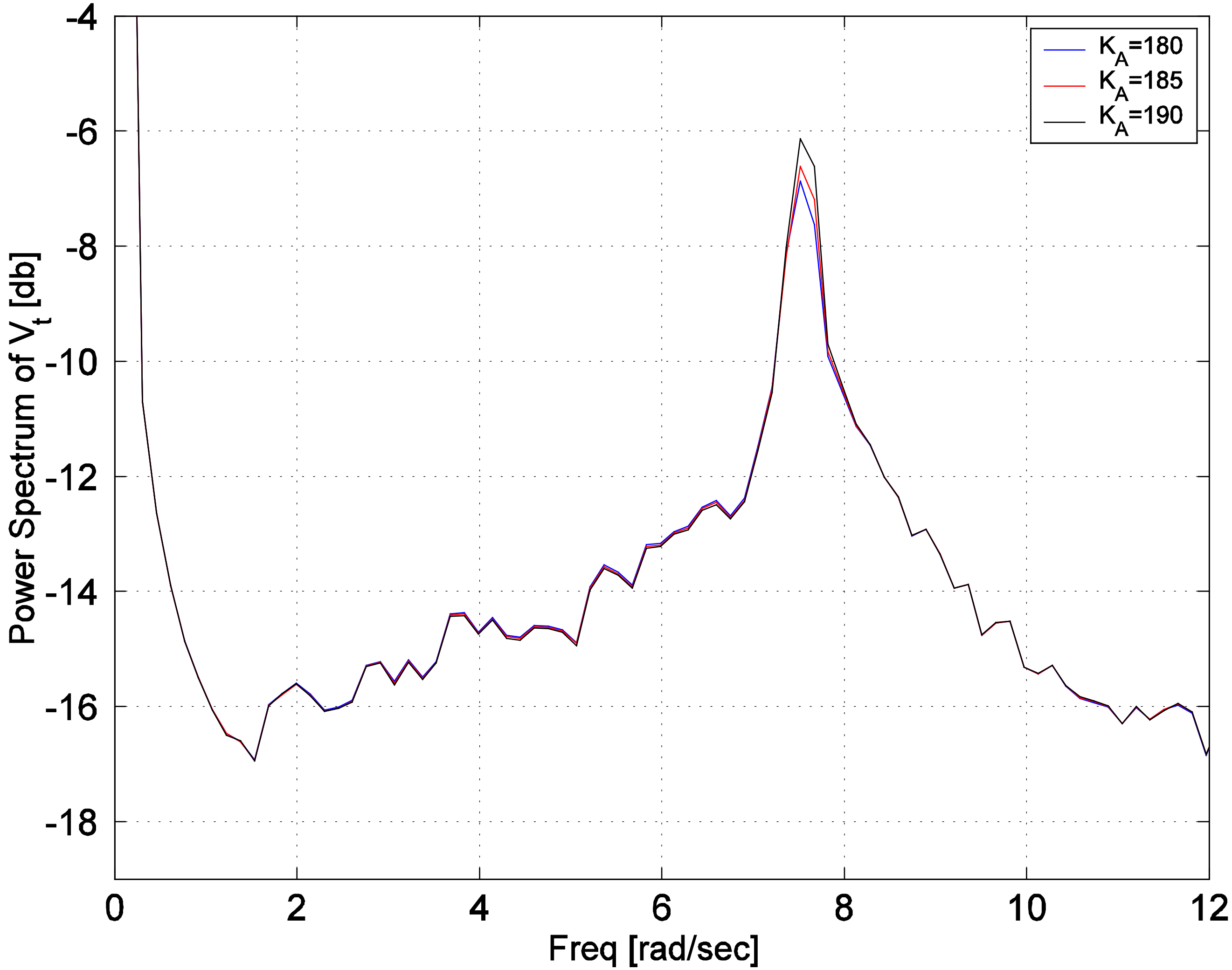
Consider a synchronous machine connected to an infinite bus together with excitation control [Abed & Varaiya, 1984]. It was shown that this system undergoes a subcritical Hopf bifurcation as the control gain in the excitation system is increased beyond a critical value.

The dynamics of the generator is given by:

$$\begin{aligned}
&\dot{\delta} = \omega \\
&2H\dot{\omega} = -D\omega + \omega_0(P_m - P_e) \\
&\tau'_{d0}\dot{E}'_q = E_{FD} - E'_q - (X_d - X'_d)i_d
\end{aligned}$$

The dynamics of the generator is given by:

$$\begin{aligned}
&\tau_E\dot{E}_{FD} = -K_E E_{FD} + V_R - E_{FD}S_E(E_{FD}) \\
&\tau_F\dot{V}_3 = -V_3 + \frac{K_F}{\tau_E}(-K_E E_{FD} + V_R - E_{FD}S_E(E_{FD})) \\
&\tau_A\dot{V}_R = -V_R + K_A(V_{REF} - V_t - V_3)
\end{aligned}$$



Output spectrum with noise probe signal, as instability is approached. (Critical K_A = 193.7.)

Stability Monitoring and Noisy Precursors, cntd.

- Noisy precursors, like critical slowing down and increasing variance, are valid precursors of instability.

- However, for engineered systems we should often be able to have a better warning signal designed into the system or into its controller.

- Biologically motivated precursors can also be considered, since nature has other wonderful precursors.

MODAL PARTICIPATION FACTOR ANALYSIS

(Ref: Hashlamoun, Hassouneh and Abed, IEEE Trans. Automatic Control, 2009.)

- Modal participation factors were introduced at MIT in the early 1980's by Verghese, Perez-Arriaga and Schweppe.

- They've been used widely especially in power systems, including for selection of sites for measurement and sites for control.

- We've recently revisited the concept, and come up with two distinct calculations of participation factors, and the implications for measurement and control are being investigated.

Modal Analysis of Linear Systems

In linear system theory, we start out by considering the linear time-invariant continuous-time system

$$\dot{x} = Ax(t)$$

where $x \in R^n$ and A is a real $n \times n$ matrix.

Assume that A has a set of n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

The system state *x(t)* is known to be a linear combination of exponentials functions $x(t) = exp(\lambda_i t)\, c^i$ where the vectors c^i are determined by the system initial condition. These functions are the *system modes*.

Participation Factors: Original Definition

Consider again the linear time-invariant system

$$\dot{x} = Ax(t)$$

where $x \in R^n$ and A is a real $n \times n$ matrix.

Assume that A has a set of n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Let $(r^1, r^2, \ldots, r^n)$ be right eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively.

Let $(l^1, l^2, \ldots, l^n)$ denote left (row) eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively.

Participation Factors: Original Definition, cont'd

The right and left eigenvectors are taken to satisfy the normalization

$$l^i r^j \quad = \quad \delta_{ij}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The solution to $\dot{x} = Ax$ starting from an initial condition $x(0) = x^0$ is

$$x(t) \quad = \quad e^{At} x^0 = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i$$

The *k*-th state is given by

$$x_k(t) \quad = \quad \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r_k^i$$

Participation Factors: Original Definition, cont'd

<u>Relative participation of the *i*-th mode in the *k*-th state:</u>

The special choice for the initial condition $x^0 = e^k$, the unit vector along the *k*-th coordinate (Perez-Arriaga, Verghese & Schweppe), yields

$$x_k(t) = \sum_{i=1}^{n} (l_k^i r_k^i) e^{\lambda_i t} =: \sum_{i=1}^{n} p_{ki} e^{\lambda_i t}.$$

The quantities

$$p_{ki} := l_k^i r_k^i$$

were defined as the mode-in-state participation factors.

The scalars p_{ki} are dimensionless

Participation Factors: Original Definition, cont'd

<u>Relative participation of the *k*-th state in the *i*-th mode:</u>

Applying the similarity transformation

$$z := V^{-1}x \qquad \text{with} \qquad V^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ \vdots \\ l^n \end{bmatrix}$$

to $\dot{x} = Ax$ yields

$$\dot{z}(t) = V^{-1}AVz(t) = \Lambda z(t),$$

where $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

The evolution of the new state vector components z_i, $i = 1, \ldots, n$ is given by

$$z_i(t) = z_i^0 e^{\lambda_i t} = l^i x^0 e^{\lambda_i t}.$$

Clearly, $z_i(t)$ represents the evolution of the i-th mode.

Participation Factors: Original Definition, cont'd

Relative participation of the *k*-th state in the *i*-th mode:

The special choice for the initial condition $x^0 = r^k$, the right eigenvector corresponding to λ_i, (Perez-Arriaga, Verghese & Schweppe), yields

$$z_i(t) \quad = \quad l^i r^i e^{\lambda_i t} = \sum_{k=1}^{n} l_k^i r_k^i e^{\lambda_i t} =: \sum_{k=1}^{n} p_{ki} e^{\lambda_i t}.$$

As before, the quantities

$$p_{ki} := l_k^i r_k^i$$

were defined as the *state-in-mode* participation factors.

This formula is identical to the formula for mode-in-state participation factors.

Motivating Examples Showing Inadequacy of Participation Factors Formula as a Measure of State in Mode Participation

Example 1 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where a, b and d are constants.

The eigenvalues of A are given by $\lambda_1 = a$ and $\lambda_2 = d$.

The right and left eigenvectors corresponding to λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, r^2 = \begin{bmatrix} 1 \\ \frac{d-a}{b} \end{bmatrix}$$

and

$$l^1 = \begin{bmatrix} 1 & \frac{b}{a-d} \end{bmatrix}, l^2 = \begin{bmatrix} 0 & \frac{-b}{a-d} \end{bmatrix}.$$

Example 1, cont'd

The evolution of the mode corresponding the λ_1 can be written explicitly:

$$\begin{aligned} z_1(t) &= l^1 x^0 e^{\lambda_1 t} = \begin{bmatrix} 1 & \dfrac{b}{a-d} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} e^{\lambda_1 t} \\ &= \left(x_1^0 + \frac{b}{a-d} x_2^0 \right) e^{\lambda_1 t}. \end{aligned}$$

Note that the evolution of mode 1 is influenced by both $x^0{}_1$ and $x^0{}_2$, with the relative degree of influence depending on the values of the system parameters a, b and d.

Example 1, cont'd

Participation Factors Based on Original Definition:

Participation of state 1 in mode 1: $p_{11} = l_1^1 r_1^1 = 1$

Participation of state 2 in mode 1: $p_{21} = l_2^1 r_2^1 = 0$

Thus, the original definition of participation factors for state in mode participation indicates that state x_2 has much smaller (even zero) influence on mode 1 compared to the influence coming from state x_1, regardless of the values of system parameters a, b and d.

This is in stark contradiction to what we observed using the explicit formula for mode 1's evolution!

$$z_1(t) = \left(x_1^0 + \frac{b}{a-d} x_2^0 \right) e^{\lambda_1 t}.$$

Motivating Examples, cont'd

Example 2 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -d & -d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where d is a constant.

The eigenvalues of *A* are given by $\lambda_1 = 0$ and $\lambda_2 = 1 - d$.

The right and left eigenvectors corresponding to λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \, r^2 = \begin{bmatrix} 1 \\ -d \end{bmatrix}$$

and

$$l^1 = \begin{bmatrix} \frac{-d}{1-d} & \frac{-1}{1-d} \end{bmatrix}, \, l^2 = \begin{bmatrix} \frac{1}{1-d} & \frac{1}{1-d} \end{bmatrix}.$$

Example 2, cont'd

The evolution of the system modes can be written explicitly:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{-d}{1-d}x_1^0 - \frac{1}{1-d}x_2^0\right)e^{\lambda_1 t} \\ \frac{1}{1-d}\left(x_1^0 + x_2^0\right)e^{\lambda_2 t} \end{bmatrix}.$$

Observe that state x_1 and state x_2 *participate equally* in mode 2 since z_2(t) depends on the initial condition x^0 through the sum $x_1{}^0 + x_2{}^0$.

Participation Factors Based on Original Definition:

Participation of state 1 in mode 2: $p_{12} = r_1^2 l_1^2 = \frac{1}{1-d}$

Participation of state 2 in mode 2: $p_{22} = r_2^2 l_2^2 = \frac{-d}{1-d}$

Clearly $p_{12} \neq p_{22}$

which IS NOT in agreement with the explicit expression for mode 2.

Participation Factors: New Approach and New Definitions

The linear system

$$\dot{x} = Ax(t)$$

usually represents the small perturbation dynamics of a nonlinear system near an equilibrium.

- The initial condition for such a perturbation is usually viewed as being an uncertain vector of small norm.
- We have introduced a new definition of state-in-mode and of mode-in-state participation factors using deterministic and probabilistic uncertainty models.
- By averaging the effect of system initial conditions to re-define each notion of modal participation, we find that the formula

$$p_{ki} = l_k^i r_k^i$$

is quite reasonable for measuring participation of modes in states, but that it is better replaced by a new (more complex) formula for participation of states in modes.

Participation Factors: New Approach and New Definitions, cont'd

Mode in State Participation Factors:

In the set-theoretic formulation, the participation factor measuring relative influence of the mode associated with λ_i on state x_k is defined as

$$p_{ki} \quad := \quad \underset{x^0 \in \mathcal{S}}{\operatorname{avg}} \; \frac{(l^i x^0) r_k^i}{x_k^0}$$

whenever this quantity exists.

Here S is the initial condition uncertainty set.

Mode In State Participation Factors, cont'd

In the probabilistic formulation, the participation factor measuring relative influence of the mode associated with λ_i on state x_k is defined as

$$p_{ki} \quad := \quad E\left\{\frac{(l^i x^0) r_k^i}{x_k^0}\right\}$$

whenever this quantity exists.

The expectation is evaluated using some assumed joint probability density function $f(x^0)$ for the initial condition uncertainty.

Mode In State Participation Factors, cont'd

Under any of the following conditions (1-3)

1. The initial condition x^0 is taken to lie in an uncertainty set $\mathcal{S}$ which is symmetric with respect to each of the hyperplanes $x_k^0 = 0$, $k = 1, 2, \ldots, n$,

2. The initial condition components are independent random variables with marginal density functions which are symmetric with respect to $x_k^0 = 0$, $k = 1, 2, \ldots, n$,

3. The initial condition components, x_j^0, $j = 1, 2, \ldots, n$, are jointly uniformly distributed over a sphere centered at the origin,

the new definitions of mode in state participation factors yield the following formula

$$p_{ki} = l_k^i r_k^i$$

same as formula originally introduced by Perez-Arriaga, Verghese and Schweppe.

New Definition of State in Mode Participation Factors

We've introduced a new definition for *state in mode* participation factors using a probabilistic approach. Let's focus on the case of distinct real eigenvalues first.

The participation factor of state x_k in mode i is defined as

$$\pi_{ki} \quad := \quad E\left\{\frac{l_k^i x_k^0}{\sum_{j=1}^n (l_j^i x_j^0)}\right\} = E\left\{\frac{l_k^i x_k^0}{z_i^0}\right\}$$

whenever this expectation exists.

Here the notation $z_i^0 = z_i(0)$

where $z_i(t)$ is the i^{th} system mode:

$$z_i(t) \quad = \quad e^{\lambda_i t} l^i x^0 = e^{\lambda_i t} \sum_{j=1}^n (l_j^i x_j^0).$$

State in Mode Participation Factors, cont'd

$$
\begin{aligned}
\pi_{ki} &= E\left\{\frac{l_k^i x_k^0}{z_i^0}\right\} \\
&= E\left\{\frac{l_k^i \sum_{j=1}^n r_k^j z_j^0}{z_i^0}\right\} \\
&= E\left\{\frac{l_k^i r_k^i z_i^0}{z_i^0}\right\} + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\left\{\frac{z_j^0}{z_i^0}\right\} \\
&= l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\left\{\frac{z_j^0}{z_i^0}\right\}.
\end{aligned}
$$

- Note that the first term in the expression for π_{ki} coincides with p_{ki}, the original participation factors formula.

- However, the second term does not vanish in general. This is true even when the components $x_1^0, x_2^0, \ldots, x_n^0$ representing the initial conditions of the state are assumed to be independent.

State in Mode Participation Factors, cont'd

To obtain a simple closed-form expression for the state in mode participation factors π_{ki} using the general definition, we need to find an assumption on the probability density function $f(x^0)$ governing the uncertainty in the initial condition x^0 that allows us to explicitly evaluate the integrals inherent in the definition.

We assume that the probability density function $f(x^0)$ is such that the components $x^0_1, x^0_2, \ldots, x^0_n$ are jointly uniformly distributed over the unit sphere in R^n centered at the origin:

$$f(x^0) = \begin{cases} k & ||x^0|| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

K is chosen to ensure the normalization condition:

$$\int_{||x^0|| \leq 1} f(x^0) dx^0 = 1.$$

State in Mode Participation Factors, cont'd

The following lemma will be used in obtaining a closed form expression for π_{ki}.

Lemma

For vectors $a, b \in R^n$ with $b \neq 0$ we have

$$\int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x \quad = \quad \frac{a^T b}{b^T b} V_n$$

where $d_n x$ denotes the differential volume element $dx_1 dx_2 \cdots dx_n$, and V_n is the volume of a unit sphere in R^n which is given by

$$V_n = \begin{cases} 2, & n = 1 \\ \pi, & n = 2 \\ \frac{2\pi}{n} V_{n-2}, & n \geq 3 \end{cases}$$

Using this Lemma, we obtain a closed-form expression for state in mode participation factors denoted by π_{ki}.

State in Mode Participation Factors, cont'd

A Closed-Form Expression for State in Mode Participation Factors (Case of real eigenvalues):

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j \frac{l^j (l^i)^T}{l^i (l^i)^T}. \qquad (1)$$

Note that in general

$$\pi_{ki} \neq p_{ki}.$$

$\pi_{ki} = p_{ki}$ only if the left eigenvectors of the system matrix A are mutually orthogonal which is a very restrictive case.

Another expression equivalent to (1) is

$$\pi_{ki} = \frac{(l_k^i)^2}{l^i (l^i)^T} = \frac{(l_k^i)^2}{\sum_{j=1}^{n} (l_j^i)^2}$$

State in Mode Participation Factors, cont'd

A Closed-Form Expression for State in Mode Participation Factors (general case):

Definition 1: For a linear time-invariant continuous-time system (25), the participation factor for the kth state in the ith mode is

$$\pi_{ki} := \begin{cases} E\left\{ \frac{l_k^i x_k^0}{z_i^0} \right\}, & \text{if } \lambda_i \text{ is real} \\ E\left\{ \frac{(l_k^i + l_k^{i*}) x_k^0}{z_i^0 + z_i^{0*}} \right\}, & \text{if } \lambda_i \text{ is complex} \end{cases} \quad (27)$$

whenever this expectation exists.

Finally, we have:

$$\pi_{ki} = \mathrm{Re}\{l_k^i\} \frac{(\sum_{j=1}^{n} r_k^j l^j)(\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T} \quad (54)$$

This formula can be rewritten as

$$\pi_{ki} = \mathrm{Re}\{l_k^i\} \frac{(r_k^i l^i)(\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T} + \sum_{j=1, j \neq i}^{n} \frac{\mathrm{Re}\{l_k^i\} r_k^j l^j (\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T} \quad (55)$$

Examples 1 and 2 Revisited

Example 1 revisited: Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The evolution of the mode corresponding the λ_1 can be written explicitly:

$$\begin{aligned} z_1(t) &= l^1 x^0 e^{\lambda_1 t} = \begin{bmatrix} 1 & \dfrac{b}{a-d} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} e^{\lambda_1 t} \\ &= \left(x_1^0 + \frac{b}{a-d} x_2^0 \right) e^{\lambda_1 t}. \end{aligned}$$

Example 1, cont'd

	Based on Original Formula	Based on New Formula
Participation of state 1 in mode 1	$p_{11} = 1$	$\pi_{11} = \dfrac{(a-d)^2}{(a-d)^2 + b^2}$
Participation of state 2 in mode 1	$p_{21} = 0$	$\pi_{21} = \dfrac{b^2}{(a-d)^2 + b^2}$

The coupling between state x_2 and state x_1 in the system dynamics is not reflected in the original formula for participation factors (the p_{ki}), whereas this coupling between state variables is reflected in the result of applying the new formula (for the π_{ki}).

Examples 1 and 2 Revisited, cont'd

Example 2 revisited: Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -d & -d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The evolution of the system modes can be written explicitly:

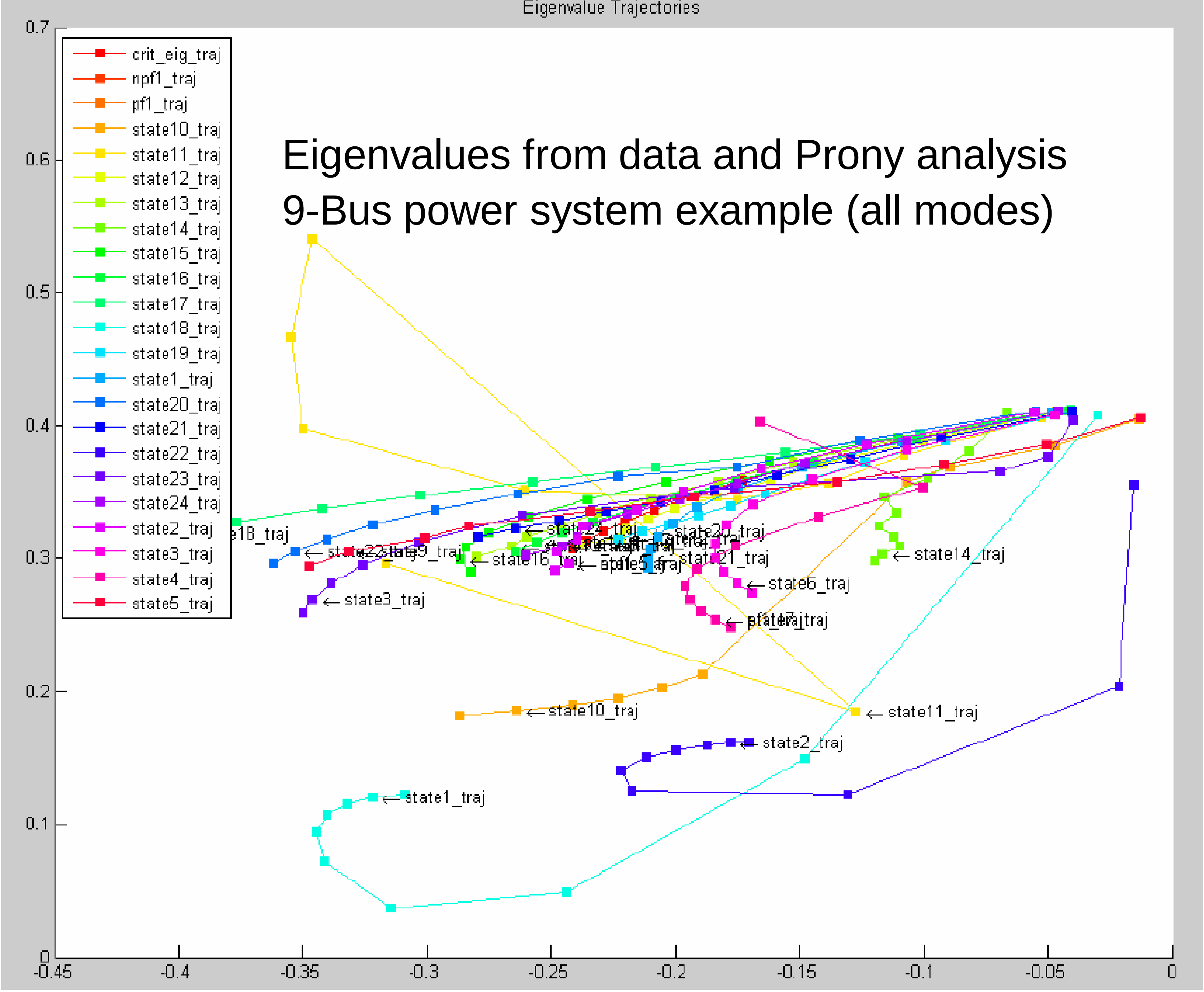
$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{-d}{1-d} x_1^0 - \frac{1}{1-d} x_2^0\right) e^{\lambda_1 t} \\ \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t} \end{bmatrix}.$$

Example 2, cont'd

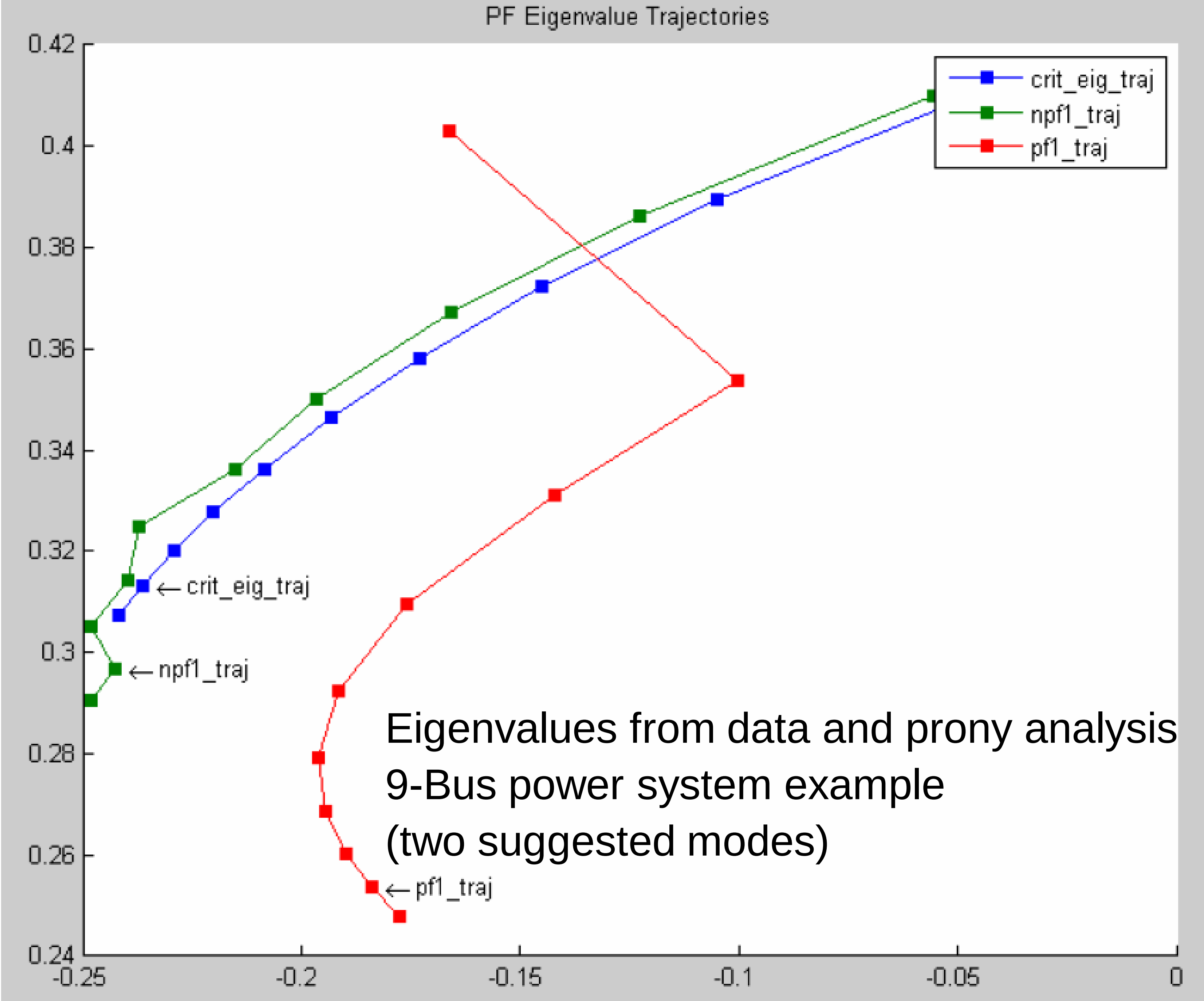
	Based on Original Formula	Based on New Formula
Participation of state 1 in mode 2	$p_{12} = \dfrac{1}{1-d}$	$\pi_{12} = \dfrac{1}{2}$
Participation of state 2 in mode 2	$p_{22} = \dfrac{-d}{1-d}$	$\pi_{22} = \dfrac{1}{2}$

The results using the new formula more faithfully reflect the relative contributions of the initial conditions of the two state variables to the evolution of mode 2, which is given explicitly by the formula

$$z_2(t) = \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t}$$



The results for all of the system state variables.



The results for just the participation factor suggested states. Note the strange outlier.

Concluding Remarks

- Critical transition prediction issues differ in natural and engineered systems.
- Engineers (especially control engineers) have dealt with system design for delaying bifurcation or detecting bifurcation for some time. Usually the methods and results depend very much on the system.
- Precursors of bifurcation can be designed into a system or into its controller.
- For large complex networked systems, modal participation analysis may assist in determining where to measure and where to apply control.
- Signal processing methods are needed for connecting model-based and nonparametric approaches to system stability monitoring.
- Applications to power networks have just begun, but other applications such as computer networks and social networks are yet to be considered.