Study group on unlikely intersections in Shimura varieties

Draft schedule – December 2017

(1) **Introduction**

Statement of conjectures such as Manin–Mumford, André–Oort, Zilber–Pink.

History and motivation.

Overview of known results, and methods used.

High-level outline of the steps in the Pila–Zannier strategy.

If time permits: Definition of o-minimal structure, idea of "tame topology". Examples: $\mathbb{R}_{\mathrm{alg}}$, $\mathbb{R}_{\mathrm{exp}}$, $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{an,exp}}$.

Programme of the study group.

(2) **Manin–Mumford conjecture following Pila and Zannier**

J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, Rend. Lincei (9) Mat. Appl. **19** (2008), no. 2, 149–162.

M. Orr, *Introduction to abelian varieties and the Ax–Lindemann–Weierstrass theorem*, in: A. J. Wilkie and G. O. Jones (eds.), *O-minimality and Diophantine geometry*, LMS Lecture Note Series 421 (2015), 100–128

Pila and Zannier's proof of the Manin–Mumford conjecture, filling in some of the details from the outline in the first talk.

Statement of the Pila–Wilkie theorem (classical form i.e. Theorem 1.6 in Pila–Wilkie, *The rational points of a definable set*).

Different statements of the Ax–Lindemann–Weierstrass theorem for abelian varieties.

Proof of Ax–Lindemann–Weierstrass theorem for abelian varieties following Orr's article.

Depending on time: some of the proof of Galois bounds, maybe in a special case (for example, following Habegger's article in the LMS volume).

(3) **André–Oort conjecture for $Y(1)^n$**

J. Pila, *O-minimality and the André–Oort conjecture for* $\mathbb{C}^n$, Ann. of Math. (2) **173** (2011), no. 3, 1779–1840.

(Consider $Y(1)^n$ only, not abelian varieties and $\mathbb{G}_m$.)

Special and weakly special points and subvarieties in $Y(1)^n$. (Definitions 1.2.1, 1.3.1, 6.3.1, 6.5.1. "weakly special" = "quasi-special")

Application of Pila–Wilkie theorem (sections 10 and 11).

Statement of Ax–Lindemann–Weierstrass for j (Theorem 6.8 – omit proof because the next talk proves a more general statement).

Large Galois orbits in $Y(1)^n$ (Propositions 5.7 and 5.8).

(4) **Ax–Lindemann–Weierstrass theorem for $\mathcal{A}_g$**

J. Pila and J. Tsimerman, *Ax–Lindemann for $\mathcal{A}_g$*, Ann. of Math. (2) **179** (2014), no. 2, 659–681.

Definition of $\mathcal{H}_g$, fundamental set.
Weakly special subvarieties of $\mathcal{A}_g$ (subsection 2.4, statement of Lemma 2.4).
Volume of curves in Hermitian symmetric domains (sections 4 and 5).
Pila–Wilkie theorem with blocks (Pila, *O-minimality and the André–Oort conjecture for* $\mathbb{C}^n$, Theorem 3.6).
Application of Pila–Wilkie, stabiliser of an algebraic set (section 6).

(5) **Heights of abelian varieties**

Definition of Faltings height.
Explicit formula for Faltings height of an elliptic curve.
Comparison with Weil height in $\mathcal{A}_g$.
Faltings's finiteness theorems and sketch of how Faltings height is used.
Statement of Masser–Wüstholz isogeny theorem.
Silverman specialisation theorem.

(6) **André–Oort conjecture for $\mathcal{A}_g$**

J. Pila and J. Tsimerman, *Ax–Lindemann for $\mathcal{A}_g$*, Ann. of Math. (2) **179** (2014), no. 2, 659–681.
E. Ullmo, *Applications du théorème d'Ax–Lindemann hyperbolique*, Compositio Math. **150** (2014), no. 2, 175–190.
J. Tsimerman, *The André–Oort conjecture for $\mathcal{A}_g$*, to appear in Ann. of Math., `arxiv.org/abs/1506.01466`

Non-density of positive-dimensional weakly special subvarieties (can use Pila–Tsimerman section 7 or Ullmo, but include Ullmo Theorem 4.1).
Putting the ingredients together to get André–Oort for $\mathcal{A}_g$.
Statement of average Colmez conjecture.
If time permits: Proof of Galois bound from average Colmez conjecture (Tsimerman). Some remarks on proof of average Colmez conjecture.

(7) **Masser–Wüstholz isogeny theorem**

D. Masser and G. Wüstholz, *Periods and minimal abelian subvarieties*, Ann. of Math. (2) **137** (1993), no. 2, 407–458.
D. Masser and G. Wüstholz, *Isogeny estimates for abelian varieties, and finiteness theorems*, Ann. of Math. (2) **137** (1993), no. 3, 459–472.

Degrees of abelian subvarieties, statement of period theorem (main theorem of *Periods*).
Sketch of proof of isogeny theorem (*Isogeny estimates* sections 2–5).
Notion and importance of polarised isogenies.
Bounds for polarised isogenies (M. Orr, *On compatibility between isogenies and polarisations of abelian varieties*, Theorem 1.3).

(8) **Unlikely intersections in families of abelian varieties**

F. Barroero and L. Capuano, *Linear relations on families of powers of elliptic curves*, Alg. & Number Th. **10** (2016), no. 1, 195–214.

F. Barroero and L. Capuano, *Unlikely Intersections in Products of Families of Elliptic Curves and the Multiplicative Group*, preprint, `arxiv.org/abs/1606.02063`.

Statements of theorems of Masser–Zannier and Barroero–Capuano.

Statement of Pila–Wilkie theorem with a path (Habegger–Pila, *O-minimality and certain atypical intersections*, Corollary 7.2).

Proof of one of the theorems (the most convenient might be to prove the theorem from *Linear relations* using Habegger–Pila's Corollary 7.2 instead of Proposition 4.3).

(9) **André–Pink conjecture**

M. Orr, *Families of abelian varieties with many isogenous fibres*, J. Reine Angew. Math. **705** (2015), 211–231.

Proof that Zilber–Pink implies André–Pink (Lemma 2.2).

Partial action of $\mathrm{GL}_{2g}(\mathbb{R})$ on $\mathcal{H}_g$, isogenies and polarised isogenies (section 3.2).

Application of Pila–Wilkie (section 3).

Height bounds for rational representations of endomorphisms (section 4 – deduce Proposition 4.1 from Proposition 4.2 and Lemma 4.3. Further details are optional).

(10) **Zilber–Pink conjecture for $Y(1)^3$**

P. Habegger and J. Pila, *Some unlikely intersections beyond André–Oort*, Compositio Mathematica **148** (2012), 1–27.

Could maybe look at:

M. Orr, *Unlikely intersections with Hecke translates of a special subvariety*, preprint, `arxiv.org/abs/1710.04092`.

Special subvarieties of $Y(1)^3$, modular curves. Maybe generalise to Hecke correspondences in $\mathcal{A}_g \times \mathcal{A}_g$.

Statement of Habegger–Pila, Theorem 1 and Orr, Theorem 1.3.

Application of Pila–Wilkie to Hecke translates (Orr, section 3).

Faltings and Weil heights, large Galois orbits for an asymmetric curve (Habegger–Pila, lemma 4.2 or Orr, section 5).

(11) **Conditional proof of the Zilber–Pink conjecture**

C. Daw and J. Ren, *Applications of the hyperbolic Ax–Schanuel conjecture*, preprint, `arxiv.org/abs/1703.08967`.

Anomalous and optimal subvarieties.

Statement of Ax–Schanuel for Shimura varieties.

Daw and Ren's arithmetic conjectures.