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AX-LINDEMANN-WEIERSTRASS THEOREM FOR $\mathcal{A}_g$

TALK GIVEN BY D. VALLONI

ABSTRACT. Proof that Zilber-Pink implies André-Pink (Lemma 2.2). Partial action of $Gl_{2g}(\mathbb{R})$ on $\mathbb{H}_g$, isogenies and polarised isogenies (section 3.2). Application of Pila-Wilkie (section 3). Height bounds for rational representations of endomorphisms (section 4 - deduce Proposition 4.1 from Proposition 4.2 and Lemma 4.3. Further details are optional).

CONTENTS

1. INTRODUCTION

Let $S := \mathcal{A}_g$ be the Siegel moduli space of ppav of dimension g. We call the defect of a subvariety $Z \subset S$ the codimension of Z in its special closure.

Conjecture 1.1 (Zilber-Pink)**.** *Let $Z \subset S$, suppose that there is a Zariski dense set of points of defect d in Z, then the defect of Z is $\leq d$.*

Proposition 1.1. *ZP implies the result of today.*

2. PRELIMINARIES

2.1. **Definable blocks.** We work in the O-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Definition 2.1. A subset $W \subset \mathbb{R}^n$ is a *definable block of dimension w* if it is a connected definable subset of $\mathbb{R}^n$ of dimension w, regular at every point, and there exists a subset $A \subset \mathbb{R}^n$ of dimension w, regular at every point, such that $W \subset A$.

Definition 2.2. By a definable block families we mean a definable subset $W \subset \mathbb{R}^n \times \mathbb{R}^m$, such that for all $\eta \in \mathbb{R}^m$, the fibre W_η is a definable block.

Theorem 2.3 (Pila)**.** *Let $Z \subset \mathbb{R}^n$ a definable set, and fix $\epsilon > 0$. There are only a finite number $J := J(Z, \epsilon)$ and block families $W^{(i)} \subset \mathbb{R}^n \times \mathbb{R}^m$, $j = 1, \ldots, J$ and a constant $c := c(Z, \epsilon)$ such that*

- *for all $\eta \in R^m$, $W^{(j)}_\eta \subset Z$ for all j,*
- *for all $T \geq 1$ we have that the set*

$$\{x \in Z \cap \mathbb{Q}^n | H(x) \leq T\}$$

is contained in a union of at most cT^ϵ blocks of the form $W^{(j)}_\eta$ for some η, j.

2.2. **Bi-algebricity.** Then we need the characterisation of weakly special subvarieties, after Ullmo-Yafaev.

Theorem 2.4. *Weakly special= bi-algebraic.*

2.3. **MW.**

Theorem 2.5. *Let K be a finitely generated field of char zero, and A/K an abelian variety. There exist constants $c := c(A, K)$ and $K := K(\dim A)$ such that:*

If B/L is an abelian variety and $[L : K] < \infty$ geometrically isogenous to A, then there exists a $\overline{K}$-isogeny between A and B of degree $< c[L : K]^K$.

3. Proof

Let π be the complex uniformisation of $\mathcal{A}_g$, and F_g a fundamental domain. Let $Z \subset \mathcal{A}_g$ be a curve, and $\widetilde{Z} := \pi^{-1}(Z) \cap F_g$. Fix $\tilde{s} \in \mathbb{H}_g$ and $s := \pi(\tilde{s})$.

We define the complexity inside the isogeny class, as the minimal degree of the isogeny between the point and A_s.

Proposition 3.1. *Fix $\epsilon > 0$. There exists a constant $c(Z, \tilde{s}, \epsilon)$ such that for all $n \geq 1$, there is a collection of cn^ϵ definable blocks $W_i \subset \widetilde{Z}$ such that their union contains all the points of complexity $\leq n$.*

Consider the map

$$\gamma : \mathrm{GL}_{2g}(\mathbb{R}) \Leftarrow \mathbb{H}_g$$

sending (A, B, C, D) to $\frac{A\tilde{s}+B}{C\tilde{s}+D}$. It is defined over a Zariski open subset of $\mathrm{GL}_{2g}(\mathbb{R})$, which we call $\mathcal{U}$ and the preimage along γ of the points in the lift of the isogeny class have rational coordinates. Moreover γ is semialgebraic.

Set $Y := \gamma^{-1}(\widetilde{Z}) = \gamma^{-1}(\pi^{-1}(Z) \cap F_g)$. So Y is definable.

Lemma 3.2. *There exist constants c, k depending only on $\dim A$ and $\tilde{s}$ such that for all $\tilde{t} \in \widetilde{Z} \cap$ (the lift of the isogeny class) of complexity n, there exists $\gamma \in Y$ with rational entries, mapping to $\tilde{t}$ and height (as a matrix) bounded by cn^k.*

With the lemma, we may apply Pila's theorem to Y: for every $\epsilon > 0$ there are J block families $W^{(j)} \subset Y \times \mathbb{R}^m$ and a constant $c = c(Y, \epsilon)$ such that

> all the points in Y of height at most $T > 1$ are contained in at most $c_1 T^\epsilon$ blocks of the form $W^{(j)}_\eta$, $\eta \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$, $j = 1, \ldots, J$.

Now, since $\gamma : Y \to \widetilde{Z}$ is semialgebraic, we have that the image of definable blocks is again a definable block. Since the blocks containing the rational points of Y of height at most $\leq T$ come from J families, it follows that the image $\gamma(W^{(j)})$ of a block is a union of at most $N > 0$ blocks, where N only depends on j.

Recall that $\gamma(\bigcup_j W^j)$ is the union of at most cT^ϵ blocks, then, if $T = cn^k$, we have that the image

$$\gamma(\bigcup_j W^j)$$

contains all the points of complexity $\leq n$.

We just proved that: for $\epsilon > 0$ there exists a constant c such that for all $n \geq 1$, there is a collection of cn^ϵ blocks in $\widetilde{Z}$ containing all the points of complexity $\leq n$.

Strategy: We have more points than blocks: so there is a block of dimension ≥ 1! Now use bi-algebricity.

Proposition 3.3. *Let Δ the isogeny class we are dealing with. Let $\Delta_1 \subset \Delta$ the subset given by*

$$\Delta_1 = \{t \in \delta | \exists W \subset \widetilde{Z} \text{ block of positive dim s.t. } t \in \pi(W)\}.$$

If Δ is Zariski dense in Z, then Δ_1 is Zariski dense in Z as well.

Now let $t \in \Delta \cap Z$ be a point of complexity n, denote by $K(t)$ the field of moduli of the abelian variety A_t. Working up to a field extension of bounded degree depending only on g, field of def=field of moduli. So A_t is defined over $K(t)$ and we may apply MW:

$$c_1 n^{1/k} \leq [K(t) : K].$$

Applying the previous proposition, we have that for $n >> 0$

$$cn^{1/2k} \leq c_1 n^{1/k} \leq [K(t) : K].$$

Notice: $cn^{1/2k}$ is the number of blocks that contain points of complexity $\leq n$, $[K(t) : K]$ is the number of Galois conjugates of a point t. So we must have a block $W \subset \widetilde{Z}$ such that $\pi(W)$ contains a Galois conjugates of t. But blocks are connected, therefore $\dim W > 0$.

The theorem is proven!