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MASSER-WÜSTHOLZ ISOGENY THEOREM

TALK GIVEN BY M. ORR

ABSTRACT. Degrees of abelian subvarieties, statement of period theorem (main theorem of Periods). Sketch of proof of isogeny theorem (Isogeny estimates sections 2-5). Notion and importance of polarised isogenies. Bounds for polarised isogenies (M. Orr, 'On compatibility between isogenies and polarisations of abelian varieties', Theorem 1.3).

CONTENTS

INTRODUCTION

For a general overview see also Martin's blog!

The aim of today's talk is to describe the following.

Theorem 0.1 (Isogeny theorem)**.** *For each integer g there are constants $c, \kappa > 0$ such that if A, B are principally polarised abelian varieties of dimension g over a number field k, such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are isogenous, then if we let N be the minimal degree of an isogeny between them over $\mathbb{C}$, we have*

$$N \leq c \max(h_{Fal}(A), [k : \mathbb{Q}])^{\kappa}.$$

Gaudron and Remond have found explicit c, κ (of reasonable size).

1. POLARISATIONS OF ABELIAN VARIETIES

Let A be a complex abelian variety of dimension g. Consider the uniformisation

$$\exp : \mathbb{C}^g \twoheadrightarrow A.$$

The kernel of $\exp$, $\Omega(A) \subset \mathbb{C}$, is a lattice (i.e. a free $\mathbb{Z}$-module such that $\mathbb{C}^g/\Omega(A) \cong A$ has finite volume). The rank of $\Omega(A)$ is $\dim_{\mathbb{R}} \mathbb{C}^g = 2g$.

Given $\Omega \subset \mathbb{C}^g$, when is $\mathbb{C}^g/\Omega$ an abelian variety? $\mathbb{C}^g/\Omega$ is a compact complex Lie group, when is it a projective algebraic variety? Does it have an ample line bundle?

We denote by $\Im(--)$ the imaginary part of a complex number.

Theorem 1.1 (Appell-Humbert)**.** *Let $A = \mathbb{C}^g/\Omega$, we have a s.e.s.*

$$0 \to A^\vee \to \mathrm{Pic}(A) \to \{\textit{hermitian forms } r \textit{ on } \mathbb{C}^g \textit{ s.t. } \Im(r(\Omega \times \Omega)) \subset \mathbb{Z}\}$$

where $A^\vee$ is the dual complex torus.

Example. Elliptic curves: $\Omega = \mathbb{Z} + \tau\mathbb{Z}$, we have

$$r(u, v) = n\frac{u\overline{v}}{\Im(\tau)}$$

for some integer n.

Remark. Recall that a line bundle is ample iff its associated Riemann form is positive definite. So Riemann forms are the first avatar of polarisations.

In particular $\mathbb{C}^g/\Omega$ is an abelian variety iff there exists positive definite Riemann form for Ω.

1.1. **Dual abelian variety.** $A^\vee = V^\vee/\Omega^\vee$, where

$$V^\vee = \{f : \mathbb{C}^g \to \mathbb{C} : f(u+v) = f(u) + f(v), af(u) = f(\bar{a}u) \text{ for } a \in \mathbb{C}\},$$
$$\Omega^\vee = \{f \in V^\vee : \Im(f(\Omega)) \subset \mathbb{Z}\}.$$

Notice that:

- Hermitian forms r on $\mathbb{C}^g$ correspond to linear maps $f_r \mathbb{C}^g \to V^\vee$. This dual space is the only other $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-form of the classic dual.
- Riemann forms correspond to linear maps f_r mapping Ω into $\Omega^\vee$.

So f_r induces

$$\psi_r : A \to \mathbb{A}^\vee.$$

If r comes from a line bundle $\mathcal{L}$, $\psi_r(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$.

- If r is positive definite then f_r is an isomorphism. Hence ψ_r is an isogeny.

Over an arbitrary field: if $\mathcal{L}$ is an ample line bundle, then $\psi_{\mathcal{L}}(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is an isogeny

$$A \to A^\vee = \mathrm{Pic}^0(A).$$

1.2. **Polarisations.** A polarisation of A/k is an isogeny $A \to A^\vee$ of the form $\psi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$ on $A_{\bar{k}}$.

Definition 1.2. The degree of a polarisation is the degree of $\psi_{\mathcal{L}}$ as an isogeny, which is equal to the following numbers:

$$\mathrm{disc}(Re(r_{|\Omega})) = \mathrm{disc}(\Im(r_{|\Omega})) = [(\mathcal{L}^g)/g!].$$

where the first $r_{|\Omega}$ is seen as a symmetric form, and the second as sympletic form.

1.3. **Abelian subvarieties.** Let $B \subset A$ be an abelian subvariety. $\exp_A^{-1}(B)$ is a $\mathbb{C}$-vector subspace, $W \subset \mathbb{C}^g$, such that

$$B = W/W \cap \Omega(A),$$

so $W \cap \Omega(A)$ has full rank in W. Conversely if $W \subset \mathbb{C}^g$ is a subspace s.t. $W \cap \Omega(A)$ has full rank in W, then $W/W \cap \Omega(A)$ is an abelian subvariety of A.

Fix now a polarisation for A. It restricts to a polarisation for B. The normalised degree of B is defined as $\sqrt{deg(\psi_{|B})} = (\mathcal{L}_{|B}^{\dim B})/\dim(B)!$. We denote it by $\Delta(B)$.

Example. Let $A = E \times E'$, $f : E \to E'$ an isogeny, Γ_f = graph of f. Then

$$\Delta(\Gamma_f) = 1 + \deg(f).$$

2. Proof of MW for elliptic curves

general proof: 1993. elliptic curves: 1990.

Let E, E' two isogenous elliptic curves. Look at non-split abelian subvariety of $E^m \times E'^n$, i.e. not in the form $U \times V$ where $U \subset E^m$ and $V \subset E'^n$.

Lemma 2.1. *If there exists a non split subvariety $B \subset E^m \times E'^n$, then there exists an isogeny $E \to E'$ of degree $\ll \Delta(B)^2$.*

2.1. **Masser-Wüstholz Period Theorem.**

Theorem 2.2 (Period theorem)**.** *Let A be an abelian variety defined over a number field k with a principal polarisation λ. For any non-zero period ω of A, the smallest abelian subvariety A_ω of A whose tangent space contains ω satisfies*

$$\deg_\lambda A_\omega \leq C \max([k : \mathbb{Q}], h_F(A), r_\lambda(\omega,\omega))^\kappa$$

where C and κ are constants depending only on $\dim A$.

Can we choose a period ω such that $r_\lambda(\omega,\omega)$ is not too big and A_ω is non-split?

Let's prove that the period thm implies the isogeny theorem.

2.2. **Geometry of numbers.** Ω_E and $\Omega_{E'}$ have small bases in terms of $\Im(\tau)$ (where $\Omega_E = \mathbb{Z}+\tau\mathbb{Z}$), and it is possible to relate $\Im(\tau)$ to $h_F(E)$.

Let $\{\omega_1, \omega_2\}$ be a basis for Ω_E, and $\{\omega_1', \omega_2'\}$ be a basis for $\Omega_{E'}$. Let $\psi : E' \to E$ be an isogeny of degree N. We have an induced map $\psi_* \Omega_{E'} \to \Omega_E$ and write $\psi_*(\omega_1') = a\omega_1 + b\omega_2$.

Now look at

$$\omega := (\omega_1, \omega_2, \omega_1') \in \Omega(E^2 \times E')$$

and the set of periods lying in

$$W := \{(z_1, z_2, z') \in \mathbb{C}^3 | az_1 + bz_2 = \psi_*(z')\}$$

Consider

$$B := W/W \cap \Omega(A) = \{(x_1, x_2, y) \in E^2 \times E' : ax_1 + b_2 = \psi(y)\}$$

$\omega \in \Omega(B)$ so $A_\omega \subset B$ forces A_ω to be non-split.

2.3. **Weak form of the period theorem.**

Proposition 2.3. *There exists $C \subset B$ non-zero abelian subvariety s.t.*

$$\Delta(C) \ll (h \cdot \max\{1, r(\omega, \omega)\} \cdot [k : \mathbb{Q}])^\kappa$$

where $h = \max\{1, h_{Fal}(A), \log|a|, \log|b|\}$ (a and b are the coeff coming from B).

The coeff $|a|, |b|$ are $\ll N^{1/2} h_{Fal(A)}$. Controlling these coefficients is the deepest part for a general g (in the tangent space lemma the Faltings height appears…).

This would give an isogeny $E \to E'$ of degree bounded by $\max\{[k : \mathbb{Q}], h_{Fal}(E), \log N\}$

To prove the proposition we need some method from transcendence theory: construct an auxiliary polynomial $P \in k[X_0, \dots, X_n]$ and $A \subset \mathbb{P}^n$ such that $P_{|A} \neq 0$ and $\deg(P) = D \sim (h \cdot \max\{1, r(\omega, \omega)\} \cdot [k : \mathbb{Q}])^{g-1}$ and that the height of the coefficients of P is $\ll T \sim (h \cdot \max\{1, r(\omega, \omega)\} \cdot [k : \mathbb{Q}])^g$. So P vanishes to big order at 0 along B. Then P vanishes to (not quite so) big order at $\exp(l\omega/L)$ for all $l \in Z$ and suitable $L \subset \mathbb{Z}$.

Finally the zero lemma of Philippon: if there exists such a P, then there exists an algebraic subgroup of controlled degree.

3. Interactions between isogenies and polarisations