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ANDRÉ-OORT CONJECTURE FOR THE MODULI SPACE OF P.P. ABELIAN VARIETIES

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ABSTRACT. Non-density of positive-dimensional weakly special subvarieties. Putting the ingredients together to get AO for $\mathcal{A}_g$. Statement of average Colmez conjecture. If time permits: Proof of Galois bound from average Colmez conjecture

CONTENTS

INTRODUCTION

Let $x \in \mathcal{A}_g$ be a special point. Today we aim to prove that

$$|\operatorname{disc}(R_x)| \prec |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|.$$

We may assume that A_x, the abelian variety corresponding to x, is simple and with CM given by the maximal order (the general case then follows). The aim of today's talk is to prove the following.

Theorem 0.1 (Large Galois Orbits)**.** *Let $g \geq 1$. There exists $\delta_g > 0$ such that if E is a CM field of degree $2g$, Ψ a primitive CM type for E, A an abelian variety with CM by $(\mathcal{O}_E, \Psi)$, then the field of moduli of A satisfies*

$$[\mathbb{Q}(A) : \mathbb{Q}] \gg_g |\operatorname{disc}(E)|^{\delta_g}.$$

1. OTHER INGREDIENTS WE NEED

1.1. (Stable) Faltings height (for CM abelian varieties).

Theorem 1.1 (Faltings Height can't be too small)**.** *There exists a constant c_g depending only on g s.t.*

$$h_{Fal}(A) \geq b_g$$

(b_g is negative! is something like $-2g$).

From now on we work with simple CM abelian varieties.

Theorem 1.2 (Faltings height depends only on (E, Ψ))**.** *Let (E, Ψ) be a CM type and A an abelian variety with $\operatorname{End}(A) = \mathcal{O}_E$ and CM by (E, Ψ). Then its Faltings height depends only on (E, Ψ).*

We denote by $h_{Fal}(E, \Psi)$ the Faltings height of any such abelian variety.

Corollary 1.3. *Let A be a CM abelian variety with CM by $\mathcal{O}_E$ (and some CM type). We have*

$$h_{Fal}(A) \leq -(2^g - 1)c_g + \sum_{\Psi} h_{Fal}(E, \Psi).$$

[1]Notes for a talk of the Unlikely intersections study group (UCL)

Proof. From Theorem 1.1 we have

$$h_{\text{Fal}}(A) - b_g \geq 0.$$

There are 2^g possible CM types of E, for each of them we have

$$h_{Fal}(E, \Psi) - b_g \geq 0.$$

Since the Faltings height of A is one of $h_{Fal}(E, \Psi)$ we have

$$0 \leq h_{\text{Fal}}(A) - b_g \leq \sum_{\Psi}(h_{Fal}(E, \Psi) - b_g) = \sum_{\Psi} h_{Fal}(E, \Psi) - 2^g b_g.$$

the corollary is proven. □

1.1.1. *Masser-Wüstholz Isogeny Theorem.*

Theorem 1.4. *LetA, B be abelian varieties of dimension g over a number field k, and suppose that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are isogenous. Then if we let N be the minimal degree of an isogeny between them over $\mathbb{C}$, we have*

$$N \ll_g \max(h_{Fal}(A), [k : \mathbb{Q}])^{c_g},$$

where c_g is a positive constant depending only on g.

1.2. **Brauer-Siegel.**

Theorem 1.5.

Bound on Hecke L function:

2. Overview

Each theorem implies the following one.

Theorem 2.1 (Averaged Colmez Conjecture)**.** *We have*

$$\sum_{\Phi} h_{Fal}(E, \Phi) = \sum_{\Phi} \left(\sum_{\rho} c_{\rho,\Phi} \left(\frac{L'(0, \rho)}{L(0, \rho)} + \log f_\rho \right) \right),$$

the outer sum is over all 2^g CM types of E, and ρ ranges over irreducible complex representations of $\mathrm{Gal}(E^{\textit{normal cl}}/\mathbb{Q})$ *for which $L(0, \rho) \neq 0$, $c_{\rho,\Phi}$ are rational numbers depending only on the finite combinatorial data given by Ψ and* $\mathrm{Gal}(E^{\textit{normal cl}}/\mathbb{Q})$*, and f_ρ is the Artin conductor of ρ.*

Theorem 2.2 (Large Galois Orbits)**.** *Let $g \geq 1$. Then, for a special point $x \in \mathcal{A}_{\mathrm{g}}$,*

$$|\operatorname{disc}(R_x)| \prec |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|,$$

where R_x is the centre of the endomorphism ring of the abelian variety corresponding to x, and the implied constant depends only on g.

Theorem 2.3. *The André-Oort conjecture holds for $\mathcal{A}_{\mathrm{g}}$, for any $g \geq 1$, i.e. an irreducible closed algebraic subvariety of $\mathcal{A}_{\mathrm{g}}$ is special if and only if it contains a dense set of special points.*

We can also put together what we discuss in the first two talks, in the following theorem.

Theorem 2.4 (AO-MM)**.** *Let*

$$X = S \times A \times \mathbb{G}_m^n$$

where S is a Shimura variety of abelian type and A an abelian variety. Define special subvarieties of S as the subvarieties of the form $M \times N \times P$, where M is a special subvariety in S, N a special subvariety in A and P in $\mathbb{G}_m^n$. Then a subvariety of X is special if and only if it contains a dense set of special points.

Remark. Theorem 2.2 also implies AO for any mixed Shimura variety whose pure part is a Shimura subvariety of $\mathcal{A}_{\mathrm{g}}$.

3. Averaged Colmez implies the Galois bound

The main goal of this note is to prove the following, from the Colmez conjecture, and then deduce Theorem 2.2 from this easier version.

Theorem 3.1 (Large Galois Orbits). *Let $g \geq 1$. There exists $\delta_g > 0$ such that if E is a CM field of degree $2g$, Ψ a primitive CM type for E, A an abelian variety with CM by $(\mathcal{O}_E, \Psi)$, then the field of moduli of A satisfies*

$$[\mathbb{Q}(A) : \mathbb{Q}] \gg_g |\operatorname{disc}(E)|^{\delta_g}.$$

3.1. **Classical CM theory.** Let E be a CM field with totally real subfield E_0 and set $g = [E_0 : \mathbb{Q}]$. Let Φ be a CM type of E, i.e. $\Psi \subset \operatorname{Hom}(E, \mathbb{C})$ such that $\Psi \cup \overline{\Psi} = \operatorname{Hom}(E, \mathbb{C})$ and $\Psi \cap \overline{\Psi} = \emptyset$.

Let

$$S(E, \Psi)$$

be the set of complex abelian varieties A, up to iso, such that $\mathcal{O}_E \subset \operatorname{End}_{\mathbb{C}}(A)$ and the induced action of E on $T_0 A(\mathbb{C}) = \mathbb{C}^g$ is given by Ψ.

Theorem 3.2 (Main thm of CM). *Fields of moduli of abelian varieties in $S(E, \Psi)$ are the same.*

Moreover we also often use the following facts:

- Any abelian variety with complex multiplication is defined over a number field.
- Any abelian variety A with complex multiplication over a number field F, there exists a finite field extension L/F such that A_L has everywhere good reduction. We always compute the Faltings height of A over such a number field L!

Theorem 3.3. *The set $S(E, \Psi)$ is finite and its cardinality is equal to the cardinality of the class group of E.*

Moreover we may apply Brauer-Siegel:

$$|S(E, \Psi)| \gg_g |\operatorname{disc}(E)|^{1/4 - o_g(1)}.$$

Indeed the cardinality of $S(E, \Psi)$ is bounded below by $|\mathrm{Cl}(E)|/|\mathrm{Cl}(E_0)|$, and $|\operatorname{disc}(E)| \geq |\operatorname{disc}(E_0)|^2$.

From now on assume $\boxed{\Psi \text{ is a primitive CM type}}$, so that the abelian varieties have endomorphism rings equal to $\mathcal{O}_E$ and isogenies between them correspond to ideals in $\mathcal{O}_E$. Since there are $n^{o(1)}$ ideals of norm n in $\mathcal{O}_E$, we have (by taking $n = |\operatorname{disc}(E)|^{1/4 - o_g(1)}$)

Proposition 3.4 (Isogenies of large degree). *There are two elements in $S(E, \Psi)$ such that the minimal degree of an isogeny between them over $\mathbb{C}$ is at least $|\operatorname{disc}(E)|^{1/4 - o_g(1)}$.*

If we want to use MW to conclude we are left to prove something like this:

$$h_{\mathrm{Fal}}(A) \leq |\operatorname{disc} E|^{o_g(1)}.$$

Indeed the strategy is the following: Let $A \in S(E, \Psi)$, we want to see get a bound for $[\mathbb{Q}(A) : \mathbb{Q}]$ using MW. All the abelian varieties in $S(E, \Psi)$ are defined over the same field $\mathbb{Q}(A)$, so we can assume we have two abelian varieties A and B as in the above proposition, i.e. the degree of the minimal isogeny between A and B is N, and

$$N \geq |\operatorname{disc} E|^{1/4 - o_g(1)},$$

together with MW we have

$$\max(h_{Fal}(A), [\mathbb{Q}(A) : \mathbb{Q}])^{c_g} \geq |\operatorname{disc} E|^{1/4 - o_g(1)}.$$

If $h_{\mathrm{Fal}}(A) \leq |\operatorname{disc} E|^{o_g(1)}$, we get that

$$[\mathbb{Q}(A) : \mathbb{Q}]^{c_g} \geq |\operatorname{disc} E|^{1/4 - o_g(1)}.$$

which gives the result, whenever $\delta_g < 1/4c_g$.

It's time to use Colmez!

3.2. **Proof of the bound for the Faltings height.**

Theorem 3.5. *The averaged Colmez conjecture implies that*

$$h_{Fal}(A) \leq |\operatorname{disc}(E)|^{o_g(1)}$$

for every abelian variety A with CM given by $(\mathcal{O}_E, \Psi)$.

Recall that from Cor 1.3 we are left to prove

$$\sum_{\Psi} h_{Fal}(E, \Psi) \leq |\operatorname{disc}(E)|^{o_g(1)}.$$

Using the Colmez we can prove the theorem by known bounds. In particular:

(1) Conductor-discriminant formula: For any irreducible Artin representation ρ, we have $f_\rho \leq |\operatorname{disc} E|$, so

$$\log f_\rho \leq |\operatorname{disc}(E)|^{o_g(1)};$$

(2) Brauer theorem on induced character + Brauer-Siegel:

$$L(1, \overline{\rho}) = |\operatorname{disc}(E)|^{o_g(1)};$$

(3) Cauchy's theorem + standard convexity estimates:

$$L'(1, \overline{\rho}) \leq |\operatorname{disc}(E)|^{o_g(1)};$$

(4) This is enough to conclude since, from the functional equation of the Artin L functions, we know

$$\frac{L'(1,\rho)}{L(1,\rho)} + \frac{L'(0,\overline{\rho})}{L(0,\overline{\rho})} = O_g(\log f_\rho).$$

3.3. **Final reduction.** From the previous theorem we actually need to prove this.

Theorem 3.6 (Large Galois Orbits)**.** *Let $g \geq 1$. Then, for a special point $x \in \mathcal{A}_g$,*

$$|\operatorname{disc}(R_x)| \prec |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|,$$

where R_x is the centre of the endomorphism ring of the abelian variety A_x corresponding to x, and the implied constant depends only on g.

The point is that A_x is isogenous to a product of abelian varieties with complex multiplication. And we need to reduce the statement to a (simple) abelian variety with complex multiplication by a maximal order.

This implies also the following corollary.

Corollary 3.7. *For all positive integers g and n there are only finitely many isomorphism classes of complex abelian varieties A of dimension g of CM type defined over number fields of degree n.*

Using Hermite-Minkowski, the above theorem immediately implies the corollary for absolutely simple principally polarised abelian varieties; the general case requires some more work (Zarhin's trick + if you have an order R in $\prod_i F_i$, then $\operatorname{disc} R \geq \prod \operatorname{disc} F_i$).

Remark. One can wonder if the following stronger statement is also true: Let $C > 0$ and $g > 0$ be fixed constants. The set of isomorphism classes of complex CM abelian varieties of dimension g and Faltings height bounded by C is finite. But to prove this one needs the Colemz conjecture and the Artin conjecture (the averaged version is not enough).

Proof of the reduction. Shall I prove this? □

4. IF TIME PERMITS: HOW TO APPLY ALW

First some recaps.

4.1. **O-minimality.** $\mathbb{R}_{\text{an,exp}}$ is an O-minimal structure. A countable definable set is finite.

Definition 4.1 (Algebraic part)**.** Let $Z \subset \mathbb{R}^n$. The algebraic part of Z, denoted by Z^{alg} is the union of all connected, positive-dimensional semi-algebraic subsets of Z.

Definition 4.2 (Counting function)**.** For a set $Z \subset \mathbb{R}^n$, an integer $k \geq 1$ and a real number $T \geq 1$, define

$$Z(k,T) := \{z = (z_1, \cdots, z_v) \in Z \colon \max_i [\mathbb{Q}(z_i) \colon \mathbb{Q}] \leq k \ , \max_i H(z_i) \leq T\},$$

where H denotes the absolute multiplicative height of an algebraic number. Then, one sets

$$N(Z,k,T) := \#Z(k,T).$$

Theorem 4.3 (Pila-Wilkie)**.** *Let $Z \subset \mathbb{R}^n$ be definable. Let $k \geq 1$ an integer and $\epsilon > 0$. Then there is a constant $c(Z,k,\epsilon)$ such that for every $T \geq 1$ we have*

$$N(Z - Z^{alg}, k, T) \leq c(Z,k,\epsilon) T^{\epsilon}.$$

Sub polynomial growth.

4.2. **Hyperbolic Ax-Lindemann-Weierstrass.** Recall that, from the number theoretical inputs obtained in the previous section, to prove Andre-Oort we needed also the following.

Let $\pi : \mathbb{H}_g \to \mathcal{A}_{\text{g}}$ be the complex uniformization and $F_g \subset \mathbb{H}_g$ be the fundamental domain for the action of Sp_{2g}.

Theorem 4.4. *The pullback of a special point $x \in \mathcal{A}_{\text{g}}$ under $\pi_{|F_g}$ is an algebraic point y of degree bounded by $2g$. Moreover the naive height $H(y)$ is polynomially bounded in $|\operatorname{disc} R_x|$.*

Theorem 4.5 (ALW)**.** *Bialgebraic=weakly special.*

4.3. **The theorem.** Let (G,X) be a Shimura datum, X^+ a connected component of X, and $G(\mathbb{Q})_+$ the stabilizer of X^+ in $G(\mathbb{Q})$. Let $K \subset G(\mathbb{A}_f)$ be a compact open. We work with the connected component of $\mathrm{Sh}_K(G,X)$ given by $S := \Gamma \backslash X^+$, where $\Gamma := G(\mathbb{Q})_+ \cap K$.

$V \subsetneq S$ be a Hodge generic subvariety, which is not of the form $V = S_1 \times V'$, whenever $S = S_1 \times S_2$, for some subvariety V' in S. Using the ALW theorem and the fact that $\pi : X^+ \to S$, restricted to a fundamental domain F, is definable in $\mathbb{R}_{\text{an,exp}}$ is definable, we can prove the following.

Theorem 4.6. *Let $V \subsetneq S$ be a Hodge generic subvariety. The set $\mathcal{E}_{>0}(V)$ of weakly special subvarieties of V of positive dimension is not Zariski dense in V.*

This is thm 4.1 in Ullmo. See also section 7 of Pila-Tsimermann.

Sketch of the proof. We first recall a general statement about weakly special subvarieties.

Fact 4.1. *Let $Z \subset S$ be a weakly-special subvariety, there exist a semisimple group $D_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and a $z_0 \in F$ such that $Z = \pi(D(\mathbb{R})^+.z_0)$.*

We may suppose V contains a weakly special subvariety of dimension d, so that we may fix D and z_0 such that $\pi(D(\mathbb{R})^+.z_0) \in \mathcal{E}_d(V)$.

(1) $\pi(tD(\mathbb{R})t^{-1}.z)$ is a weakly special subvariety of V, for every $(t,z) \in G(\mathbb{R}) \times F$ such that $\pi(tD(\mathbb{R})^+t^{-1}.z)$ is contained in V (use ALW).

(2) The set

$$C(D,V) := \{tD(\mathbb{R})^+t^{-1}\}_{t \in G(\mathbb{R}) | \exists z \in F \ s.t.\ \pi(tD(\mathbb{R})^+t^{-1}.z) \subset V}$$

is definable and countable, hence finite. Hint: the set of $\mathbb{Q}$-algebraic subgroups of G is countable and use the above fact.

(3) Let m be the dimension of the maximal weakly special subvariety contained in V. The set $\mathcal{E}_m(V)$ is not Zariski dense in V. Let's prove this only in a simpler case: Let $D'_{\mathbb{Q}}$ be a subgroup of $G_{\mathbb{Q}}$ such that $G \cong D' \times D''$, then the set $\{\pi(D'(\mathbb{R})^+.z)\}_{z \in F}$ is not Zariski dense in V. Indeed it consists of weakly special subvariety of the form $S_1 \times x_z$, where $S = S_1 \times S_2$ is the decomposition induced by $G \cong D' \times D''$. So the Zariski closure of the set is $S_1 \times V'$ for $V' = \overline{\{x_z\}_z}$, but we assumed that V has not this shape. To reduce to this case use the previous step.

(4) This is enough to conclude. Idea: the following set is finite

$$\{tD(\mathbb{R})t^{-1}, t \in G(\mathbb{R}) | \exists z \in F \text{ s.t. } \pi(tD(\mathbb{R})^{+}t^{-1}.z) \subset V \text{ and } \pi(tD(\mathbb{R})t^{-1}.z) \not\subseteq \mathcal{E}_d(V)\}.$$

□