notes hastily typed

HEIGHTS OF ABELIAN VARIETIES

TALK GIVEN BY N. DOGRA

ABSTRACT. Definition of Faltings height. Explicit formula for Faltings height of an elliptic curve. Comparison with Weil height in $\mathcal{A}_g$. Faltingss finiteness theorems and sketch of how Faltings height is used. Statement of Masser-Wüstholz isogeny theorem. Silverman specialisation theorem.

CONTENTS

1. WEIL HEIGHT

1.1. Heights on $\mathbb{P}^n$. Let $\frac{a}{b} \in \mathbb{Q}$ such that $(a,b)=1$, then

$$H(\frac{a}{b}) = \max\{|a|,|b|\}$$

For points like $P = [z_0 : \dots, z_n]$ such that $g.c.d(z_0, \dots z_n) = 1$ we set

$$H = \max\{|z_i|\}$$

We also set

$$h(P) := \log H(P).$$

Let $[Z_0, \dots, z_n]$ be any rep of P,

$$h(P) := \sum_v \log \max\{|z_i|_v\}$$

More generally, for points defined over an arbitrary finite extension $K/\mathbb{Q}$, for $P = [z_i] \in \mathbb{P}^n(K)$ we set

$$h(P) := \frac{1}{[K:\mathbb{Q}]} \sum_v \log \max\{|z_i|_v\}.$$

It is independent of the choices: given $x \in K$,

$$\sum_v \log \max\{|z_i|_v\} - \sum_v \log \max\{|xz_i|_v\} = \sum_v \log |x|_v = 0$$

where the last equality holds by the product formula. So we have an height function

$$h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}.$$

1.2. Heights on varieties. Given $P \in V(K)$, for V a smooth projective variety, we want to define $h(P)$. The problem is that $\mathtt{Aut}(\mathbb{P}^n)$ acts transitively on $\mathbb{P}^n$, so any functorial definition of height needs some extra data/assumptions.

Exercise. Given $P \in PGL_n(K)$, $h(gP) = h(P) + o(1)$, so $h(gP)$ differs from $h(P)$ by a bounded amount.

Possible slogan: any two notions of height are asymptotically the same.

Theorem 1.1 (Weil height machine)**.** *We can associate to pairs (V, D), V smooth variety and D a divisor on V, functions*

$$h_{V,D} : V(\overline{K}) \to \mathbb{R}$$

with the following properties

- *normalization:* $h_{\mathbb{P}^n, \mathcal{O}(1)}(P) = h(P) + o(1)$*,*
- *functoriality:* $f : V \to W$*,* $h_{V,f^*D}(P) = h_{W,D} + o(1)$*,*
- *additivity: for all* D, E *divisors in* V*,* $h_{V,D+E} = h_{V,D} + h_{V,E} + o(1)$
- *finiteness: if* D *is ample, then for every finite extension* L/K *and for every* $B > 0$*, the cardinality of the set*

  $$\{P \in V(L) | h_{V,D}(P) < B\}$$

  is finite.

2. Arakelov theory

2.1. Metrized line bundle.

Definition 2.1. A metrized line bundle on $\mathcal{O}_K$ is a pair $\overline{L} = (L, ||-||)$ consisting of a projective $\mathcal{O}_K$-module L of rank 1, together with $\forall v|\infty$ a norm $||-||_v$ on $L \otimes_{\mathcal{O}_K} K_v$.

Definition 2.2. Arakelov degree of $\overline{L}$ to be $\deg_A(\overline{L})$

$$\deg_A(\overline{L}) := -\log|L/x\mathcal{O}_K| + \sum_{v|\infty} \log ||x||_v$$

(the degree $K_v : \mathbb{R}$ is hidden in the def of norm).

It is independent of the choices: enough to show that for all $r \in \mathcal{O}_K - 0$, then we get the same answer when we replace x by rx:

$$\log|L/rx\mathcal{O}_K| + \sum_{v|\infty} \log||rx||_v - \log|L/x\mathcal{O}_K| - \sum \log||x||_v =$$

$$= \log|\mathcal{O}_K/r\mathcal{O}_K| + \sum \log|r|_v = 0$$

again by the product formula.

Remark. If $\mathrm{Pic}(\mathcal{O}_K) = 0$, let x be a generator of L, then $\deg_A(L) = \sum_{v|\infty} ||x||_v$.

2.2. Metrized height function.

Let V/K smooth projective variety. Suppose we have a model $\mathcal{V}/\mathcal{O}_K$ of V, and a line bundle $\mathcal{L}/\mathcal{V}$. Then for every $x \in V(K)$, get a projective $\mathcal{O}_K$-module of rank 1 (namely $x^*\mathcal{L}$). Metric on a line bundle L over $V_{\mathbb{C}}$: it is a continuous function

$$L(\mathbb{C}) \to \mathbb{R}$$

which fibrewise is a norm. A metrized line bundle structure on $\mathcal{V}/\mathcal{O}_K$ is a line bundle $\mathcal{L}$ on $\mathcal{V}$, together with, $\forall v|\infty$ a metric on $\mathcal{L}_{\overline{K_v}}$.

Exercise (Fubini-Study metric). $V = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(1)$, $a = \sum a_i X_i$, $P \in \mathbb{P}^n$

$$||P^*a|| = |\frac{|a|^2}{\sum |x_i|^2}(P)|$$

where $P = (x_0, \ldots, x_n)$, we have

$$h_{V,\overline{\mathcal{L}}}(p) = \sum_{v<\infty} \log\max\{|x_i|\} + \sum_{v|\infty} \log|\sum |x_i|^2|$$

3. Faltings height

Let A/K be an abelian variety of dimension g. Let $s : Spec(K) \to A$ be the identity section, then $s^*\Lambda^g\Omega^1_{A/K}$ is a rank one K-vector space. Let $\mathcal{A}/\mathcal{O}_K$ be the Neron model of A. Define

$$\omega_A := s^*\Lambda^g\Omega^1_{\mathcal{A}/K}$$

is a projective rank one $\mathcal{O}_K$-module.

Metrics: For $v|\infty$, $\alpha \in \omega_A$ we define

$$||\alpha||_v := \frac{1}{2}\int_{A(\overline{K_v})} |\alpha \wedge \overline{\alpha}|.$$

Define the Faltings height of A to be

$$\frac{1}{[K:\mathbb{Q}]}\deg_A(\overline{\omega})$$

where $\overline{\omega} := (\omega_A, ||-||)$.

3.1. Stable Faltings height.

If K'/K is a finite extension, A/K abelian variety, how does $h_F(A)$ relate to $h_F(A_{K'})$? At $v|\infty$ it is clear. At $v < \infty$ it is quite hard: Neron-models are not stable under base extensions— But can show that

$$h_F(A_{K'}) \leq h_F(A_K)$$

This is not a problem if A ha semistable reduction over K_v, since Neron-Models become stable under base extension.

Theorem 3.1. *For every A/K there exists a finite extension K'/K finite such that $A_{K'}$ has semistable reduction everywhere.*

Definition 3.2. Stable Faltings height of A/K: $h_S(A) := h_F(A_{K'})$ for a K'/K such that $A_{K'}$ has semistable reduction everywhere.

3.2. Faltings height of an elliptic curve.

Proposition 3.3. *Let E/K be an elliptic curve , with minimal discriminant $\Delta_{E/K} \subset \mathcal{O}_K$ with $\mathbb{C}$ uniformization given by*

$$\forall v|\infty E_{\overline{K_v}} \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_v,$$

then $h_F(E) =$

$$\frac{1}{12[K:\mathbb{Q}]}(\log(N_{K/\mathbb{Q}}\delta_{E/K}) - \sum_{v|\infty}\log|\delta(\tau)Im(\tau)^6|).$$

The function $|\delta(\tau)|Im(\tau)^6$ had appeared before in a paper of chowla and selberg:

Theorem 3.4 (Chowla-Selberg)**.** *$d > 0$ square free*

$$\prod_{[a]\in Cl(\mathbb{Q}(\sqrt{-d}))} |\Delta(\tau_a)|Im(\tau_a)^6 = \frac{1}{2\pi\sqrt{d}}\prod_{0<c<d}\Gamma(\frac{c}{d})^{6\frac{-d}{c}}.$$

where τ_a is defined by $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_a \cong \mathbb{C}/[a]$.

4. Faltings theorems, relation to Weil height

Theorem 4.1 (Faltings)**.** *For all $B > 0$ there are finitely many abelian varieties defined over a number field of bounded Faltings height.*

ETP that Faltings height is a height. Dream: Let $\mathcal{A}_g$ be the moduli space of the ppav of dimension g, L the pullback along the identity section of the canonical boundle of universal AV, show that $h_{\mathcal{A}_g,L}([A]) = h_F(A)$ up to an $o(1)$.

Problems:

- shouldn't be h_F but h_S,
- what if A isn't pp? cheat: $h_S(A) = h_S(A^\vee)$
- what is $\mathcal{A}_g$? smooth stack or singular scheme

- $\mathcal{A}_{\mathrm{g}}$ is not projective: you need to compactify: $\overline{\mathcal{A}_{\mathrm{g}}}$ w.r.t Baily-Borel. Does the metric extend? No, but the Faltings height is related to a metrized height function with log singularities along $\overline{\mathcal{A}_{\mathrm{g}}} - \mathcal{A}_{\mathrm{g}}$.

Proposition 4.2. *U smooth projective variety, $Z \subset V$ proper subset, $U = V - Z$. Let h be a metrized height with log singularities along Z, then there for all $B > 0$*

$$|\{x \in U(K) : h(x) < B\}| < \infty.$$

Theorem 4.3 (Faltings)**.** *Given A/K, $G = \lim G_n \subset A[p^\infty]$ a p-divisible subgroup of $A[p^\infty]$. Define*

$$A_n = A/G_n$$

then $(h(A_n))_n$ takes only finitely many varieties.

It implies the Tate cj!

Original proof. define $h_F(G_n)$, show that

$$h_F(G_n) = h_F(A) - h_F(A_n),$$

characterize $h_F(G_n)$ in terms of the p-divisible groups, suing the theory of p-divisible group. □

Much later: Masser-Wüstholz Isogeny Theorem.

Theorem 4.4. *LetA, B be abelian varieties of dimension g over a number field k, and suppose that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are isogenous. Then if we let N be the minimal degree of an isogeny between them over $\mathbb{C}$, we have*

$$N \ll_g \max(h_{Fal}(A), [k : \mathbb{Q}])^{c_g},$$

where c_g is a positive constant depending only on g.