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AX-LINDEMANN-WEIERSTRASS THEOREM FOR $\mathcal{A}_{\mathrm{g}}$

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ABSTRACT. Definition of $\mathbb{H}_g$, fundamental set. Weakly special subvarieties of $\mathcal{A}_{\mathrm{g}}$ (subsection 2.4, statement of Lemma 2.4). Volume of curves in Hermitian symmetric domains (sections 4 and 5). Pila-Wilkie theorem with blocks (Pila, O-minimality and the André-Oort. Conjecture for $\mathbb{C}^n$, Theorem 3.6). Application of Pila-Wilkie, stabiliser of an algebraic set (section 6).

CONTENTS

1. INTRODUCTION: FIRST RESULTS IN TRANSCENDENCE THEORY

Theorem 1.1 (Lindemann-Weierstrass, 1882)**.** *Let $z_1, \dots, z_n$ be algebraic numbers, which are linearly independent over $\mathbb{Q}$, then*

$$\operatorname{tr.deg.}_{\mathbb{Q}}(\exp(z_1), \dots \exp(z_n)) = n.$$

In other therms the smallest $\overline{\mathbb{Q}}$-subvariety of $\mathbb{A}^n$ containing $(\exp(z_1), \dots \exp(z_n))$ is $\mathbb{A}^n$ itself.

It's better to look at the transcendental map

$$\mathbb{C} \to \mathbb{C}^*, z \mapsto \exp(2\pi i z),$$

for example it sends rational points to torsion points.

Theorem 1.2 (Ax, 1970)**.** *Consider*

$$\pi : \mathbb{C}^n \to \mathbb{C}^{*n}, \ \ (z_1, \dots, z_n) \mapsto (\exp(2\pi i z_1), \dots, \exp(2\pi i z_n)).$$

Let $Z \subset \mathbb{C}^n$ be an algebraic subset, which is not contained in a translate of a rational linear subspace. Then $\pi(Z)$ is Zariski dense in $\mathbb{C}^{*n}$.

What kind of varieties are 'bi-algebraic'? I.e there is a transcendental map $\pi : H \to S$, $V \subset S$ is bialgeebraic if it is algebraic and an analytical component of $\pi^{-1}(V)$ is also algebraic.

In the abelian case we have the following (cf. talk on Manin-Mumford).

Theorem 1.3 (Ax, Pila-Zannier, Orr, Peterzil-Starchenko)**.** *Consider π the map uniformizing an abelian variety A*

$$\pi^n_{\mathbb{C}} \to \mathbb{C}^n/\Lambda = A.$$

Let $Z \subset \mathbb{C}^n$ be an algebraic subvariety. The Zariski closure of $\pi(Z)$ is a translate of an abelian subvariety (aka 'weakly special').

Consequence of this: Bi-algebraic=weakly special.

2. The hyperbolic case

We consider quotients like this: $S := \Gamma\backslash X$, where X is a Hermitian symmetric space, and Γ is an arithmetic lattice in a semisimple group $G(\mathbb{R})$ where $G/\mathbb{Q}$ and $X = G(\mathbb{R})/K_\infty$. Γ has to be a congruent subgroup, to be more precise.

Exercise. $G = \mathrm{SL}_2$, $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = \mathbb{H}$ (it's the unit disc), and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We have

$$\pi = j : \mathbb{H} \to \mathbb{C} = S \subset \mathbb{P}^1(\mathbb{C}).$$

A theorem of Baily-Borel says that such S is always an algebraic quasi-projective variety.

Definition 2.1. Let $Z \subset S$ be an algebraic variety (irreducible). Z is called *special* if it is in the above form, i.e. there exists a Shimura variety $S' = \Gamma'\backslash X'$ associated to a subgroup $H \subset G$, and $\Gamma' = H(\mathbb{Q}) \cap \Gamma$ s.t. Z is the image of S' in S.

Definition 2.2. Z is called weakly special if either Z special and there exists a special subvariety $S' = S_1 \times S_2$, s.t. $Z = S_2 \times \{x\}$ for some $x \in S_2$.

Remark. Intersection of ws is ws.

Exercise. Pairs of elliptic curves: $S = \mathbb{C} \times \mathbb{C}$, $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. $\mathbb{C} \times \{special\ point\}$ is a special subvariety, but also $Y_0(n) \subset S$ and product of special points.

Theorem 2.3 ((Hyperbolic) Ax-Lindemann-Weierstrass)**.** *Consider the projection map*

$$\pi : X \to \Gamma\backslash X = S.$$

Let $Y \subset X$ be an algebraic *variety, then the Zariski closure of $\pi(Y)$ is weakly special.*

By algebraic in X we mean the following: X can be realised as a bounded subset of $\mathbb{C}^n$ where $n = \dim X$. For example $\mathbb{H}$ can be realised as open unit disc in $\mathbb{C}$, and now use the notion of being algebraic in $\mathbb{C}^n$. So fix a realization $X \subset \mathbb{C}^n$, Y is *algebraic* in X if there exists an algebraic subset $\tilde{Y} \subset \mathbb{C}^n$ such that $Y = X \cap \tilde{X}$. We call Y irreducible if Y is an *irreducible* analytic component of $X \cap \tilde{Y}$.

3. ALW in Pila-Zannier strategy for AO

Conjecture 3.1 (André-Oort)**.** *Let $Z \subset S$ a subvariety which contains a Zariski-dense set of special points.*

We have

$$\pi : X \to \Gamma\backslash X = S$$

and a fundamental domain $F \subset X$. Let $\Sigma \subset Z$ be the Zariski dense set of special points, and $x \in \Sigma$. Consider $\tilde{Z} := \pi^{-1}(Z) \cap F$, $\tilde{x} := x\pi^{-1}(x) \cap F$. Points in Σ are defined over $\overline{\mathbb{Q}}$!

Theorem 3.1. *Everything is defined in a O-minimal structure.*

To a point $x \in \Sigma$ we associate a number d_x, the size of the image of the reciprocity map associated to x. For abelian shimura variety, when x correspond to a CM abelian variety A_x, it is the discriminant of the endomorphism ring associated to A_x. Important: $d_x \to \infty$, whenever we have infinitely many special x.

We need two bounds: Lower bound

$$|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \gg d_x^\alpha \text{ for some } \alpha.$$

Upper bound to apply Pila-Wilkie:

$$H(\tilde{x}) \ll d_x^A.$$

Since d_x is the same for all points in $\operatorname{Gal}(\overline{\mathbb{Q}}/K) \cdot x$, $\tilde{Z}$ contains $\gg d_x^\alpha$ points in $\overline{\mathbb{Q}}^n$ of degree smaller than d_x^A, and $d_x \to \infty$. So Pila-Wilkie says that $\tilde{Z}$ contains an algebraic subset of positive dimension. ALW implies that Z contains a Zariski dense set of weakly special subvarieties. A theorem of Ullmo, which again uses ALW, says that there exists $S' = S_1 \times S_2 \subset S$ ($\dim S_1 > 0$) such that $Z = S_1 \times Z'$ for some $Z' \subset S_2$. Now work by induction: you can apply the same to S_2 and Z'.

4. Idea of proof of ALW

Consider
$$\pi : X \to \Gamma\backslash X = S,$$
and F the fundamental domain. First $\pi_{|F}$ is definable in $\mathbb{R}_{\text{an,exp}}$. We have $V \subset S$, $Y \subset \pi^{-1}(V)$ maximal algebraic. We may assume that V and Y are not contained in any proper weakly special subvarieties, and that V is the smallest algebraic subvariety containing $\pi(Y)$.

We are reduced to showing that $\pi(Y)$ is Zariski dense. Let C be an algebraic curve in Y. Let
$$\Sigma(C) := \{g \in G(\mathbb{R}) | \dim(g.C \cap \pi^{-1}(V) \cap F) = 1\}.$$
It is a definable set in $\mathbb{R}_{\text{an,exp}}$(dimension does!). Moreover
$$\Sigma(C) \cap \Gamma = \{\gamma \in \Gamma : \gamma^{-1}F \cap C \neq \emptyset\},$$
since $\pi^{-1}(V)$ is Γ-invariant! Recall that $G \subset \mathrm{GL}_n$, and $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$, so we can define $H(\gamma)$, for $\gamma \in \Gamma$, as the maximum of the absolute values of the entries of γ (seen as a matrix).

We want to show that the stabilizer is big. We have the following counting function:
$$N_Y(T) = |\{\gamma \in \Sigma(C) \cap \Gamma : H(\sigma) \leq T\}|.$$
Essential part of the proof is showing that
$$N_Y(T) \gg T^{\alpha}.$$
Assuming this, we can use PW: there exists a semialgebraic subset $\mathcal{S}$ in $\Sigma(Y)$, and $\mathcal{S}$ contains infinitely many rational points. Therefore there exists a $H \subset G$ defined over $\mathbb{Q}$ such that
$$H(\mathbb{Q}).Y = Y,$$
by using maximality of Y. By general theory of Shimura varieties we have: Let $\tilde{V}$ be a component of $\pi^{-1}(V)$ and $\Gamma' = \mathrm{Stab}_{\Gamma}(\tilde{V})$. Γ' is Zariski-dense in G. So one can use the assumption that V is the smallest algebraic subvariety containing $\pi(V)$ to show that Γ' normalises H. It follows that $H = G$, therefore $Y = V = S$.