ANDRE-OORT CONJECTURE FOR $Y(1)^n$

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1. FIXING NOTATIONS AND DEFINITIONS

Recall that by $Y(1)$ we mean the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. The j-function

$$j\colon \mathbb{H} \to \mathbb{C}$$

descends to an isomorphism

$$j\colon Y(1) \xrightarrow{\sim} \mathbb{C}.$$

The Theorem we are going to prove is the following

Theorem 1.1 (Andre-Oort for $Y(1)^n$)**.** *Let $V \subset Y(1)^n$ be a subvariety. Then V contains only a finite number of maximal, special subvarieties of $Y(1)^n$.*

This is equivalent to the following

Theorem 1.2 (Andre-Oort for $Y(1)^n$, second version)**.** *Let $\Sigma \subset Y(1)^n$ an arbitrary set of special points and denote by V_Σ its Zariski closure. Then irreducible components of V_Σ are special subvarieties.*

Let's prove the equivalence of this two versions:
Let V as in Theorem 1.1, and consider Σ the set of special points conrained in V. Then by Theorem 1.2 we must have that every irreducible component of V_Σ is a special subvariety, and there are finitely many components as V_Σ is a subvariety. For the converse, consider a random set of speicial point Σ and V_Σ its Zariski closure. By Theorem 1.1 we must have that V_Σ contains finitely many maximal, special subvarieties. But then V_Σ must consists of the union of those, since special points are dense in special subvarieties. Let's now define the objects in the statements above.

Definition 1.3 (Special Points)**.** Let $(c_1, \cdots, c_n) \in Y(1)^n$ be a point. We say that it is special if every c_i is the j-invariant of an elliptic curve with CM.

Definition 1.4 (Special Subvarieties)**.** Let $S_0 \cup S_1 \cup \cdots \cup S_w$ be a partition of n, where only S_0 is allowed to be empty. Consider

- j_i a special point of $Y(1)$ for every $i \in S_0$,
- s_i the smallest element in S_i, $i > 0$,
- for every $j \in S_i - \{s_i\}$, an integer $N_{ij} > 0$.

A special subvariety of $Y(1)^n$ is an irreducible component of a subvariety of the form

$$\{(c_1, \cdots, c_n) \in Y(1)^n \colon c_1 = j_i,\ i \in S_0,\ \Phi_{N_{ij}}(c_{s_i}, c_j) = 0, j \in S_i, j \neq s_i,\ i = 1, \cdots, w\}.$$

Note that the dimension of Y is w.
To explain the definition a bit, consider the following map

$$\begin{aligned} \mathbb{H} &\to \mathbb{C} \times \mathbb{C} \\ \tau &\mapsto (j(\tau), j(N\tau)) \end{aligned}$$

where $N \in \mathbb{Z}_{>0}$. We want the image to be special, and indeed it is with our definition, since it corresponds to

$$\{(c_1, c_2) \in \mathbb{C} \times \mathbb{C} \colon \Phi_N(c_1, c_2) = 0\}.$$

Let's now define the pre-analogues:

Definition 1.5 (Pre-special points)**.** Let $(\tau_1, \cdots, \tau_n) \in \mathbb{H}^n$ be a point. We say that it is a pre-special point iff every τ_i is quadratic, i.e. $[\mathbb{Q}(\tau_i) \colon \mathbb{Q}] = 2$.

Equivalently, τ is pre-special if the elliptic curve $E_\tau := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ has CM.

Definition 1.6 (pre-special subvarieties)**.** Let $S_0 \cup S_1 \cup \cdots \cup S_w$ be a partition of n, where only S_0 is allowed to be empty. Consider

- h_i a pre-special point of $\mathbb{H}$ for every $i \in S_0$,
- s_i the smallest element in S_i, $i > 0$,
- for every $j \in S_i - \{s_i\}$, an element $g_{ij} \in \mathrm{GL}_2(\mathbb{Q})^+$.

A pre-special subvariety of $\mathbb{H}^n$ is a variety of the form

$$N = \{(\tau_1, \cdots, \tau_n) \in \mathbb{H}^n \colon \tau_i = h_i,\ i \in S_0,\ \tau_j = g_{ij}\tau_{s_i}, j \in S_i, j \neq s_i,\ i = 1, \cdots, w\}.$$

One can prove (see Edixhoven, special points on products of modular curves) that under $j \colon \mathbb{H}^n \to Y(1)^n$, pre-special subvarieties correspond to special subvariety. If Y is a special subvarieties, then $j^{-1}(Y)$ consists of translates under $\Gamma := \mathrm{SL}_2(\mathbb{Z})^n$ of a pre-special subvariety. We will refer to it as a pre-special locus.
If we take $n = 2$, $S_0 = \emptyset$ and $S_1 = \{1, 2\}$ and g_{12} to be the matrix

$$\begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}$$

with $N > 0$, then we obtain obtain a pre-special subvariety of $N \subset \mathbb{H}^2$ of dimension 1 such that $j(N)$ is exactly the special subvariety in the example above.

Definition 1.7 (weakly special subvarieties)**.** A weakly special subvariety of $Y(1)^n$ is, by definition, a variety as set out in Definition 1.4, except that the points j_i for $i \in S_0$ need not to be special.

Definition 1.8 (weakly pre-special subvarieties)**.** A weakly pre-special subvariety of $\mathbb{H}^n$ is, by definition, a variety as set out in Definition 1.6, except that the points h_i for $i \in S_0$ need not to be pre-special.

- Note that under $j\colon \mathbb{H}^n \to Y(1)^n$, weakly pre-special subvarieties correspond to pre-special subvarieties.
- if a weakly (pre-)special subvariety contains a (pre-)special point, then it is (pre-)special.
- the inverse image of a weakly special subvariety of $Y(1)^n$ is a union of translates under Γ of weakly pre-special subvarieties. We will refer to it a weakly pre-special locus.

2. O-MINIMAL STRUCTURES ON $\mathbb{R}$

We will work on the o-minimal structure given by the union of $\mathbb{R}_{an}$ and $\mathbb{R}_{exp}$ and denote it $\mathbb{R}_{an,exp}$. The fact that also this is o-minimal is proved by Van der Dries and Macintyre in the paper 'Geometric categories and o-minimal structures'. We tacitly will use the fact that every map and sets used from now on are definable in this o-minimal structure. For a proof that the map $j\colon \mathbb{F} \to Y(1)$ is definable, see Peterzil-Starchenko's paper 'Uniform definability of the Weierstrass $\mathcal{P}$ funcion'.

3. RATIONAL AND ALGEBRAIC POINTS OF DEFINABLE SETS

Definition 3.1 (Algebraic part)**.** Let $Z \subset \mathbb{R}^v$. The algebraic part of Z, denoted by Z^{alg} is the union of all connected, positive-dimensional semi-algebraic sebsets of Z.

Definition 3.2 (Counting function)**.** For a set $Z \subset \mathbb{R}^v$, an integer $k \geq 1$ and a real number $T \geq 1$, define

$$Z(k,T) := \{z = (z_1, \cdots, z_v) \in Z\colon \max_i [\mathbb{Q}(z_i)\colon \mathbb{Q}] \leq k \ , \max_i H(z_i) \leq T\},$$

where H denotes the absolute multiplicative height of an algebraic number. Then, one sets

$$N(Z,k,T) := \#Z(k,T).$$

Theorem 3.3 (Pila-Wilkie)**.** *Let $Z \subset \mathbb{R}^v$ be definable. Let $k \geq 1$ an integer and $\epsilon > 0$. Then there is a constant $c(Z,k,\epsilon)$ such that for every $T \geq 1$ we have*

$$N(Z - Z^{alg}, k, T) \leq c(Z,k,\epsilon)T^{\epsilon}.$$

We need a notion of complexity for pre-special points. First note that if $\tau \in \mathbb{H}$ is pre-special, then it is the solution of a unique polynomial $ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$ and $(a,b,c) = 1$. We define the discriminat of τ to be $D(\tau) := b^2 - 4ac$. One can easily see that the elliptic curve defined by the lattice $\mathbb{Z} + \tau\mathbb{Z}$ has CM by the field $\mathbb{Q}(\sqrt{D(\tau)})$. Moreover, if we write $D(\tau) = f^2\Delta$, where Δ is the discriminant of the CM field, then the ring of endomorphisms of this elliptic curve is given by the order

$$\mathcal{O} = \mathbb{Z} + f\mathcal{O}_{\mathbb{Q}(\sqrt{D(\tau)})}$$

.

Definition 3.4. Let $\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{H}^n$ be a pre-special point, and denote by D_i the discriminat $D(\tau_i)$. We define the complexity of τ to be

$$\Delta(\tau) := \max_i |D_i|.$$

Theorem 3.5 (height and complexity)**.** *Let $\tau \in \mathbb{F}^n$ be a pre-special point. Then there exists a constant c_{height} such that*

$$H(\tau) \leq c_{height}\Delta(\tau).$$

Theorem 3.6 (Siegel's corollary)**.** *Let $\tau \in \mathbb{H}^n$ be a pre-special point. Let $\epsilon > 0$. Then there exists a constant $c_{Siegel}(\epsilon)$ such that*

$$[\mathbb{Q}(j(\tau)) \colon \mathbb{Q}] \geq c_{Siegel}(\epsilon)|\Delta(\tau)|^{1/2-\epsilon}.$$

Proof. Indeed, by the Theory of complex multiplication, we have that the field $\mathbb{Q}(j(\tau_i))$ is nothing but the ring class field associated to the unique order with discrmininant D_i. So we have

$$[\mathbb{Q}(j(\tau)) \colon \mathbb{Q}] = h(D_i) \geq c_{\text{Siegel}}(\epsilon)|D_i|^{1/2-\epsilon}.$$

The last inequality is the Brauer-Siegel Theorem when the order happens to be the maximal one. For the proof of the general case, see the paper of Lenstra, 'Factorising integers with elliptic curves'.
Remember that a baby-form of Brauer-Siegel theorem asserts that if K_i, $i \in \mathbb{N}$, are different immaginary quadratic fields of discriminant D_i, then

$$\lim_{i\to\infty} \frac{\log h(K_i)}{\log \sqrt{|D_i|}} = 1.$$

□

4. The algebraic part & ALW

In this section, we characterize maximal algebraic subsets of

$$\mathcal{Z} = j^{-1}(V) \subset \mathbb{H}^n.$$

Definition 4.1. Let $U \subset \mathbb{C}^n$ be an open domain, which is semi-algebraic when seen as a subset of $\mathbb{R}^{2n}$ and let $\mathcal{Z} \subset U$ be a complex analytic subset. We define a *complex algebraic component* of $\mathcal{Z}$ to be a componend Y of positive dimension of $W \cap \mathcal{Z}$, where W is an irreducible complex algebraic set of $\mathbb{C}^n$. The complex algebraic part of $\mathcal{Z}$ is then defined to be $\mathcal{Z}^{\text{ca}}$, the union of those.

Proposition 4.2. *Let $\mathcal{Z} := j^{-1}(V) \subset \mathbb{H}^n$. Then $\mathcal{Z}^{ca} = \mathcal{Z}^{alg}$.*

This is not difficult, but yet not obvious. For a proof, see Lemma 2.1 of Pila's paper 'Rational points of definable sets and results of Andre-Oort-Manin-Mumford type'.
We will call a component of $\mathcal{Z}^{\text{ca}}$ maximal if it is not contained in any other components.

Theorem 4.3 (Ax-Lindemann-Weierstrass for $Y(1)^n$)**.** *Let $Y \subset \mathcal{Z} := j^{-1}(V)$ be a maximal complex algebraic component. Then Y is weakly pre-special.*

This theorem, which is the core-result of Pila's paper, will be proved in the next talk by Andrei in a much more general form.

5. Basic pre-special components

This is one nice application of o-minimality. We need another family of subvarieties of $\mathbb{H}^n$. In the same way that a weakly pre-special subvariety is basically a pre-special subvarieties with no conditions on the points indexed by S_0, we are now allowing also le g_{ij} to vary in $\mathrm{SL}_2(\mathbb{R})$.

Definition 5.1 (Linear Subvarieties)**.** Let $S_0 \cup S_1 \cup \cdots \cup S_w$ be a partition of n, where only S_0 is allowed to be empty. Consider

- h_i a point of $\mathbb{H}$ for every $i \in S_0$,
- s_i the smallest element in S_i, $i > 0$,
- for every $j \in S_i - \{s_i\}$, an element $g_{ij} \in \mathrm{SL}_2(\mathbb{R})$.

A linear subvariety of $\mathbb{H}^n$ is a subvariety of the form

$$Y = \{(\tau_1, \cdots, \tau_n) \in \mathbb{H}^n \colon \tau_1 = h_i,\ i \in S_0,\ \tau_j = g_{ij}\tau_{s_i} = 0, j \in S_i, j \neq s_i,\ i = 1, \cdots, w\}.$$

Of course any weakly pre-special subvariety is a linear subvariety.
The union of gY for $g \in \Gamma$ will be called a linear locus. If S_0 is empty, we call the corresponding linear subvariety (or locus) *basic*. The data $S_1, \cdots, S_w$, g_{ij} determines a basic linear subvariety (locus) B in

$$\prod_{i \notin S_0} \mathbb{H}$$

and we say that the linear subvariety (locus) Y is a translate of B by the elements $h_i \in S_0$.

Proposition 5.2. *Let $V \subset Y(1)^n$ be a subvariety. Then there are only finitely many basic pre-special subvarieties having a translate that is a maximal weakly pre-special subvarieties contained in $\mathcal{Z} = j^{-1}Y$*

Proof. First of all, we prove that the two following sets are the same:

(1) the basic pre-special subvarieties having a translate maximal among weakly pre-special subvarieties contained in $\mathcal{Z}$.
(2) the basic linear subvarieties having a translate maximal among linear subvarieties contained in $\mathcal{Z}$.

The fact the 1) $\subset$ 2) is obvious. Now, let B be as in 2) and let $h + B$ be one of its translates that occour as a maximal linear subvariety in $\mathcal{Z}$. Since $h + B$ is algebraic, it must be contained in a maximal algebraic component of $\mathcal{Z}$. By ALW, this is weakly pre-special. Hence B is pre-special as well.
Now we note that the set of basic linear subvarieties is definable, it is basically a product of copies of $\mathrm{SL}_2(\mathbb{R})$ indexed by the possible partitions of n. The subset M of these that have a maximal translate is definable too (this is what Pila says). But we notice that M must be contained (in the product of the $\mathrm{SL}_2(\mathbb{R})$'s) in the image of $\mathrm{GL}_2(\mathbb{Q})^+$, by the first part of this proof. So M is countable and definable, which implies finite. □

6. Proof of Andre-Oort for $Y(1)^n$

Definition 6.1. Let $V \subset Y(1)^n$ a subvariety. The special set of V, denoted V^{sp}, is the union of special subvarieties of positive dimension contained in V.

We are ready to prove

Theorem 6.2 (Andre-Oort, baby version)**.** *Suppose that V is defined over a number field K that contains the field of definition of X and assume that V^{sp} is a variety. Then $V - V^{sp}$ contains only finitely many special points.*

Proof. Let as usual $\mathcal{Z} = j^{-1}V$. Then $\mathcal{Z}^{\text{alg}}$ consists of $\mathcal{Z}^{\text{ps}} = j^{-1}(V^{\text{sp}})$ together with other weakly pre-special subvarieties that contain no pre-special point. This is because of AXL for $Y(1)^n$. Put $Z^{\text{ps}} = \mathcal{Z}^{\text{ps}} \cap \mathbb{F}$. Denote by

$$N_2^{\text{prespecial}}(W,T)$$

the number of pre-special point in a set W up to height T. For $\epsilon > 0$ and $T \geq 1$ we have

$$N_2^{\text{prespecial}}(Z - Z^{\text{ps}}, T) = N_2^{\text{prespecial}}(Z - Z^{\text{alg}}, T) \leq N_2(Z - Z^{\text{alg}}, T) \leq c(Z, 2, \epsilon)T^{\epsilon}.$$

Suppose now that $Z - Z^{\text{sp}}$ contains a pre-special point u of complexity Δ. Then $x = j(u) \in V - V^{\text{sp}}$ is special. Using Siegel's corollary (with exponent, say, $1/3$) we have that the point $j(u)$ has at least

$$[K:\mathbb{Q}]^{-1}c_{\text{Siegel}}\Delta^{1/3}$$

conjugates x' which also lies in $V - V^{\text{sp}}$. These conjugates have distinc pre-images $u' \in Z - Z^{\text{sp}}$ which are pre-special points of same complexity, hence by Therem 3.5 we have

$$H(u') \leq c_{\text{height}}\Delta.$$

Now we must have,

$$[K:\mathbb{Q}]^{-1}c_{\text{Siegel}}\Delta^{1/3} \leq N_2(Z - Z^{\text{alg}}, c_{\text{height}}\Delta) \leq c(Z,2,\epsilon)(c_{\text{height}}\Delta)^{\epsilon}$$

by the computation above. If we choose $\epsilon < 1/3$, we see that this disequality produces a contraddiction once $\Delta \to \infty$. So we must have that for every pre-special point $u \in Z - Z^{\text{sp}}$, its complexity $\Delta(u)$ is bounded, hence they are finitely many. □

We are now ready to prove Theorem 1.1. We are going to this by induction on n.

Proof. First, we can find a subvariety $\overline{V} \subset V$ containing all the algebraic points of V and defined over $\overline{\mathbb{Q}}$. So that we can assume V to be defined over $\overline{\mathbb{Q}}$ as well. We prove the theorem by induction on n. If $n = 1$ there is nothing to prove. If $n = 2$, there there are two cases: if V is consists of points, then we are done again. If V is proper and with some positive-dimensional components, then V^{sp} is a subvariety and we can apply our baby-version again. So assume that $n \geq 3$ and V is proper. We know, thanks again to the baby version, that AO is true whenever V^{sp} is a variety. So that what we are going to prove is that, using the induction argument that AO is true for smaller dimension, V^{sp} is a variety. Now, there are only finitely many basic special subvarieties whose translates occour as maximal special subvarieties. We only have to prove that finitely many of these translates occour as maximal special subvarieties. So consider B a basic special subvariety in some variables $\{1, 2, \cdots, n\} - S_0$. We must have $0 \leq \#S_0 < n$. The translations of B are hence parametrised by the variety $\mathbb{C}^{\#S_0}$. Now the set

$$\{x \in \mathbb{C}^{\#S_0} : x + B \subset V\}$$

is an algebraic subvariety V' of $\mathbb{C}^{\#S_0}$, defined over $\overline{\mathbb{Q}}$. The points $x \in V'$ such that (x, B) is maximal and special are the special points of $V' - (V')^{sp}$; this is because, take a component of

C of $(V')^{sp}$, then the image of the map

$$C \times B \to Y(1)^n, \ (c, b) \mapsto c + b$$

is a special subvariety, contraddicting maximality. By induction there are finitely many of these points, so we are done. □