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MANIN-MUMFORD CONJECTURE FOLLOWING PILA AND ZANNIER

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ABSTRACT. Pila and Zannier's proof of the Manin-Mumford conjecture, filling in some of the details from the outline in the first talk. Statement of the Pila-Wilkie theorem (classical form i.e. Theorem 1.6 in Pila-Wilkie, The rational points of a definable set). Different statements of the Ax-Lindemann-Weierstrass theorem for abelian varieties. Proof of Ax-Lindemann-Weierstrass theorem for abelian varieties following Orr's article. Depending on time: some of the proof of Galois bounds, maybe in a special case (for example, following Habegger's article in the LMS volume).

CONTENTS

1. MANIN-MUMFORD

Theorem 1.1. *Let $A/\overline{\mathbb{Q}}$ be an abelian variety and $V \subset A$ an irreducible subvariety defined over $\overline{\mathbb{Q}}$. If V does not contain any translate of an abelian subvariety of A then V contains only finitely many torsion points.*

It was first proven by Raynaud in 1983, but we will follow the proof given by Pila and Zannier (2008) which uses O-minimality.

Remark. From a specialisation argument, the above theorem implies the version where everything is defined over $\mathbb{C}$.

2. O-MINIMALITY

2.1. O-minimal structures.

Example (Semi-algebraic sets)**.** It is a subset of $\mathbb{R}^n$ defined by polynomial equations and polynomial inequalities. Intersections, products, unions of semialgebraic are semialgebraic. Moreover the projection on the first n-th factors from $\mathbb{R}^{n+1}$ sends semialgebraic to semialgebraic subsets. This last property does not hold for algebraic subsets!

Definition 2.1 (Structure)**.** A *structure* over $\mathbb{R}$ is a sequence S_n, $n \geq 1$ such that

1. S_n is a boolean algebra of subsets of $\mathbb{R}^n$, i.e. $\mathbb{R}^n, \emptyset \in S_n$ and S_n is stable w.r.t. finite unions;
2. If $S \in S_n$ and $S' \in S_m$, then $S \times S' \in S_{n+m}$;
3. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection, then $\pi(S) \in S_n$ holds for all $S \in S_{n+1}$;
4. S_n contains all semialgebraic subsets of $\mathbb{R}^n$.

Remark. Axiom (4) is equivalent to the following:

(4') $\{(x, y, z)|x + y = z\}$, $\{(x, y, z)|xy = z\} \in S_3$, $\{(x, y)!x < y\} \in S_2$, $\{x\} \in S_1$ for all x.

Definition 2.2 (O-minimal structure)**.** A structure is *O-minimal* if in addition satisfies the following:

(5) The sets in S_1 are finite unions of intervals and points, i.e. $S_1 =$ semialgebraic subsets of $\mathbb{R}$.

Notation: $S \in S_n$ for some n is called definable. $f : S \to S'$ is definable, if its graph does.

Example. $\mathbb{Z}$ and $\sin$ are not definable in a O-minimal structure.

Proposition 2.3. *The following hold in an arbitrary structure:*

- *Composition of definable functions is definable;*

- *The image of a definable set, via a definable map is definable;*
- *The interior, closure and boundary of a definable set are definable;*
- *Let S be a definable set, then S has finitely many connected components;*
- *'The dimension of definable sets is definable', i.e. Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $k \in \mathbb{N}$, then the set $\{x \in \mathbb{R}^n | \dim(f^{-1}(x)) = k\}$ is definable.*

To prove the last point you need cell decompositions, in particular you need a well defined notion of 'dimension' for definable sets.

Example. $\mathbb{R}_{\text{an}}$ is the smallest structure containing all restricted analytic functions, i.e. the functions of the form $f_{|[-1,1]^n}$ where $f : U \to \mathbb{R}$ is an analytic functions, and U is a neighbourhood of $[-1, 1]^n$. An hard theorem in logic says that it is indeed an O-minimal structure.

Example. $\mathbb{R}_{\text{an,exp}}$.

2.2. Pila-Wilkie's Theorem.

Definition 2.4 (Multiplicative heights)**.** We define

- $H_{:}\mathbb{Q} \to \mathbb{R}_{>0}$ by $H(a/b) = \max\{|a|, |b|\}$ where a, b are assumed to be coprime;
- We can extend it to $\mathbb{Q}^n$: $H(x_1, \dots, x_n) = \max_i H(x_i)$;
- Let $X \subset \mathbb{R}^n$, we set $N(X, T)$ as the cardinality of $x \in X \cap \mathbb{Q}^n$ such that $H(x) \leq T$;
- $X \subset \mathbb{R}^n$, X^{alg} is the union of the semialgebraic subsets of X which are connected and of positive dimension.

Theorem 2.5 (Pila-Wilkie)**.** *Let $X \subset \mathbb{R}^n$ be a definable set in an arbitrary O-minimal structure. Let $\epsilon > 0$, then there exists a constant $c = c(X, \epsilon) > 0$ such that*

$$N(X - X^{alg}, T) \leq cT^{\epsilon}.$$

3. Proof of Manin-Mumford

Consider the complex uniformisation of a complex abelian variety A of dimension g:

$$\pi : \mathbb{C}^g \to A.$$

We know that π is periodic by a lattice Λ, $\pi^{-1}(A_{\text{tors}}) = \mathbb{Q}\lambda$ and if $P \in A_{\text{tors}}$ has order n, then every point in $\tilde{P} \in \pi^{-1}(P)$ has n as denominator (i.e. n is the smallest positive integer such that $n\tilde{P}$ lies in Λ).

Let $V \subset A$ be an irreducible subvariety defined over $\overline{\mathbb{Q}}$ and assume that V does not contain any translate of an abelian subvariety of A. To apply Pila-Wilkie we need $\mathcal{F}$, a fundamental domain for Λ, to work on $X := \pi^{-1}(V) \cap \overline{\mathcal{F}}$. This last set is indeed definable in $\mathbb{R}_{\text{an}}$. What is its algebraic interior? With a nice description of X^{alg} we would like to use the fact $N(X - X^{\text{alg}}, T) \leq cT^{\epsilon}$, to get Manin-Mumford.

Strategy of the proof:

- $\pi(X^{\text{alg}})$ is contained in the union of the weakly special subvarieties of X of positive dimensions.
- Since V does not contain any translate of an abelian subvariety of A, we get that $X^{\text{alg}} = \emptyset$
- $N(X, T) \leq cT^{\epsilon}$. Masser proved that if $P \in A_{texttors}$ is a torsion point of order T and K a number field such that V and A are defined over K, then

  $$d(P) = [K(P) : \mathbb{Q}] \geq c'(A)T^{\rho}$$

  where $c'(A)$ depends on A and ρ only on $\dim(A)$.

The first two points will be proven using the so called Ax-Lindemann theorem. We first need to define what a weakly special subvariety is.

3.1. Special and Weakly special subvarieties.

Definition 3.1. Let A be an abelian variety. We define

- A subvariety $Y \subset A$ is *special* if it is a component of an algebraic subgroup, i.e. $Y = x + B$ with $x \in A_{\text{tors}}$ and B an abelian subvariety of A;
- A subvariety $Y \subset A$ is *weakly special* if $Y = x + B$ with B an abelian subvariety of A.

Remark. Special points are nothing but torsion points.

3.2. Ax-Lindemann.

Theorem 3.2. *Let $A/\mathbb{C}$ be an abelian variety of dimension g, $V \subset A$ an irreducible subvariety. Consider the usual map $\pi : \mathbb{C}^g \to A$. Let $Y \subset \mathbb{C}^g$ be a complex algebraic subvariety maximal among those contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is weakly special in A.*

Originally proved by Ax. We can prove it in a different way using O-minimality (and Pila-Wilkie theorem)!

Lemma 3.3. *Let $Z \subset \mathbb{C}^g$ be a complex analytic subvariety of $\mathbb{C}^g$ and $W \subset Z$ a semialgebraic set (irreducible w.r.t the topology given by real semialgebraic subsets). Then there is a a complex algebraic subvariety W' of $\mathbb{C}^g$ such that $W \subset W' \subset Z$.*

Proof of Theorem 3.2. Consider the set

$$\Sigma(Y) := \{z \in \mathbb{C}^g | (Y + z) \cap \mathcal{F} \neq \emptyset \text{ and } Y + z \subset \pi^{-1}(V)\}.$$

We have that V is the Zariski closure of $\pi(Y)$.

Lemma 3.4. *The set $\Sigma(Y)$ is definable in $\mathbb{R}_{an}$.*

Proof. Notice that

$$\Sigma(Y) = \{z \in \mathbb{C}^g | \dim(Y + x \cap \mathcal{F} \cap \pi^{-1}(V)) > 0\}$$

and apply Proposition 2.3. □

Lemma 3.5. *The cardinality of*

$$\{z \in \Sigma(Y) \cap \Lambda | H(z) < T\}$$

is bigger or equal than $T/2$ for T big enough.

Proof. Think about the picture and recall that Y is unbounded... □

In virtue of Lemma 3.5 we can apply Pila-Wilkie to get that $\Sigma(Y)$ contains a positive dimensional semialgebraic set containing some $\lambda \in \Lambda$. We need another lemma.

Lemma 3.6. *Let $W \subset \Sigma(Y)$ be a semialgebraic connected subset such that $W \cap \Lambda \neq \emptyset$. Then there exists $\lambda \in W \cap \Lambda$ such that $W + Y - \lambda \subset Y$.*

Proof. Consider $Y \subset W + Y - \lambda \subset \pi^{-1}(V)$. By Lemma 3.3 we have that

$$Y \subset W + Y - \lambda \subset Y' \subset \pi^{-1}(V),$$

where Y' is a complex algebraic subvariety, WMA Y to be maximal, so we get that $Y = Y'$. □

Lemma 3.7. *Let Θ be the stabilizer of $Y \subset \mathbb{C}^g$ and H the connected component of the identity of the stabilizer of V in A. Then $\pi(\Theta) = H$.*

Finally consider $Y \subset \mathbb{C}^g \to Y' \subset \mathbb{C}^g/\Theta$, which projects onto $V \subset A \to V' \subset A/H$. Going through the same proof for $Y' \subset \mathbb{C}^g/\Theta \to A/H$ implies that Y' is a point. This ends the proof. □