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PROOF OF AVERAGED COLMEZ CONJECTURE

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ABSTRACT.

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1. INTRODUCTION

Colmez conjectured a formula for Faltings height of a CM abelian variety in terms of special values of L-functions. The averaged version was proven in two independent papers:

- Andreatta, Goren, Howard and Madapusi-Pera;
- Yuan and Zhang.

We will discuss the former.

1.1. Chowla-Selberg formula.

Chowla-Selberg formula. Colmez translated the Chowla-Selberg formula in terms of the Faltings height: Let A be an elliptic curve, E an imaginary quadratic field such that $\mathrm{End}(A) = \mathcal{O}_E$. Then

$$h_{Fal}(E) = -\frac{1}{2}\frac{L'(\chi,0)}{L(\chi,0)} - \frac{1}{4}\log|D_E| - \frac{1}{2}\frac{\zeta'(0)}{\zeta(0)}.$$

where χ is a Dirichlet character associated with quadratic extension $E/\mathbb{Q}$.

1.2. Colmez conjecture.

Colmez conjecture. From now on we work with simple CM abelian varieties with CM given by the maximal order of a CM field E of degree $2g$. We write $\mathrm{Gal}(E/\mathbb{Q})$ for the galois group . E_0 is its totally real extension.

Theorem 1.1 (Faltings height depends only on (E, Ψ))**.** *Let (E, Ψ) be a CM type and A an abelian variety with $\mathrm{End}(A) = \mathcal{O}_E$ and CM by (E, Ψ). Then its Faltings height depends only on (E, Ψ).*

We denote by $h_{Fal}(E, \Psi)$ the Faltings height of any such abelian variety.

Conjecture 1.1. *We have*

$$h_{Fal}(A) = \sum_{\eta} a(\eta)\left(\frac{L'(\eta,0)}{L(\eta,0)} + \frac{1}{2}\log f_\eta\right)$$

where the sum if over the Artin character η of $\mathrm{Gal}(E/\mathbb{Q})$ and $a(\eta)$ are explicit constants.

Theorem 1.2 (Averaged Colmez Conjecture)**.** *We have*

$$\sum_{\Phi} h_{Fal}(E,\Phi) = \sum_{\Phi}\left(\sum_{\rho} c_{\rho,\Phi}\left(\frac{L'(0,\rho)}{L(0,\rho)} + \log f_\rho\right)\right) = -\frac{1}{2}\frac{L'(\chi,0)}{L(\chi,0)} - \frac{1}{4}\log\frac{|D_E|}{|D_{E_0}|} - \frac{g}{2}\frac{\zeta'(0)}{\zeta(0)}$$

the outer sum is over all 2^g CM types of E, and ρ ranges over irreducible complex representations of $\mathrm{Gal}(E^{\text{normal cl}}/\mathbb{Q})$ for which $L(0,\rho) \neq 0$, $c_{\rho,\Phi}$ are rational numbers depending only on the finite combinatorial data given by Ψ and $\mathrm{Gal}(E^{\text{normal cl}}/\mathbb{Q})$, and f_ρ is the Artin conductor of ρ.

2. Strategy

Let $E^{\#}$ be a CM-algebra (i.e. product of CM-fields), called the total reflex algebra. We have

$$h_{Fal}(E^{\#}, \Psi^{\#}) = \frac{1}{2g} \sum_{\Psi} h_{Fal}(E, \Psi).$$

Construct a 0-dimensional Shimura variety Y_0, with abelian scheme

$$A^{\#} \to Y_0$$

such that every fibre of $A^{\#}$ has CM by $E^{\#}$. We have

$$h_{Fal}(E^{\#}, \Psi^{\#}) = \frac{1}{\deg_{\mathbb{C}}(Y_0)} \hat{\deg}(\hat{\omega}_0)$$

for an arithmetic line bundle $\hat{\omega}_0$ on Y_0. If $L \subset E$ is a lattice, we get an associated orthogonal Shimura variety M_L and a finite cover

$$Y_L \to Y_0$$

such that $Y_L \hookrightarrow M_L$. We also have an automorphic line bundle $\hat{\omega}$ on M_L such that

$$\hat{\omega}_{|Y_L}$$

is related to $\hat{\omega}_0$.

We can calculate $\hat{\deg}(\hat{\omega}_{|Y_L})$:

- $\hat{\omega}$ can be written as a combination of Heegner divisor $Z(m, \mu)$ (Borcherds, Brunier).
- calculate arithmetic intersections $Z(m, \mu).Y_L$ (Brunier-Kudla-Yang conjecture)

3. Main ingredients

3.1. **Metrized line bundles.**

Let $X/\mathbb{C}$ be a smooth proper curve, $\mathcal{L}$ a line bundle on X, we define

$$\deg(\mathcal{L}) := \sum_{p \in X(\mathbb{C})} v_p(s)$$

for a section s of $\mathcal{L}$ (if a section exists). Since, for $f \in \mathbb{C}(X)$, $\sum_p v_p(f) = 0$ (product formula for absolute values on $\mathbb{C}(X)$), the degree does not depend on the choice of s.

Arithmetic version of this: Let K be a number field and consider the one dimensional scheme (not proper)

$$\mathrm{Spec}(\mathcal{O}_K).$$

A line bundle on $\mathrm{Spec}(\mathcal{O}_K)$ is a projective rank 1 $\mathcal{O}_K$-module. One could try

$$\deg(\mathcal{L}) = \sum_{\mathcal{P}} v_{\mathcal{P}}(s) \log |\mathcal{O}_K/\mathcal{P}|$$

for $s \in \mathcal{L} - 0$. But it does not work: it depends on s. We need to compactify $\mathrm{Spec}(\mathcal{O}_K)$. Indeed the product formula for absolute values on K involves the archimedean absolute values as well as finite places. So we need to add some archimedean information.

For each $\sigma : K \to \mathbb{C}$, choose a Hermitian form on $\mathcal{L} \otimes_{K,\sigma} \mathbb{C}$. Then

$$\hat{\deg}(\hat{\mathcal{L}}) = \sum_{\mathcal{P}} v_{\mathcal{P}}(s) \log |\mathcal{O}_K/\mathcal{P}| - \sum_{\sigma} ||s||_{\sigma} \in \mathbb{R}.$$

This generalises to higher dimensions, but it is enough for us in this setting. We will only look at

$$\hat{\deg}(\hat{\omega}_{|Y_L})$$

where $Y_L/\mathcal{O}_E$ has relative dimension zero. So for each point of $Y_L(\mathbb{C})$, $\hat{\omega}$ restricts to a line bundle over $\mathcal{O}_E$.

We can use this to define heights: If $f : X \hookrightarrow \mathbb{P}^n$ (everything over $\mathcal{O}_K$), let $\mathcal{L} := f^*\mathcal{O}(1)$. It is a very ample line bundle on X, and comes naturally with a metric (from the metric of $\mathcal{O}(1)$).

If $s : \mathrm{Spec}(\mathcal{O}_K) \to X$, we set $h(f(s)) = \hat{\deg}(s^*\mathcal{L})$.

3.2. **Faltings height.** Faltings height of an abelian variety can be defined in a similar way: there is a line bundle ω on $\mathcal{A}_{\rm g}$ (considering its compactification over $\mathbb{Z}$ and taking care of the stacky issue), with a metric (with log singularities on boundary). If A is an abelian variety corresponding to $s \in \mathcal{A}_{\rm g}(\mathcal{O}_K)$, then

$$h_{\rm Fal}(A) = \frac{1}{[K:\mathbb{Q}]} \hat{\deg}(s^* \hat{\omega}).$$

3.3. **Orthogonal Shimura varieties.** Let $V = \mathbb{Q}$ as $\mathbb{Q}$-vector space of dimension $2g$. Choose $\lambda \in F$ such that $\sigma_0(\lambda) < 0$ and $\sigma_i(\lambda) > 0$ for $i = 1, \dots, g-1$, where $\sigma_i : F \to \mathbb{R}$. We can define a quadratic form $V \times V \to \mathbb{Q}$

$$B(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \overline{y})$$

of signature $(2g-2, 2)$.

For a lattice $L \subset V$, we can define a Shimura variety M_L associated with $GSpin(V)$ and arithmetic subgroup $GSpin(L)$, of dimension $2g-2$.

$GSpin(V)$ has two natural representations:

- on V: get a family of HS on M_L of K3 type, i.e. with $\dim V^{2,0} = 1$ and $\dim V^{1,1} = 2g-2$
- on $C^+(V)$: get a family of HS with $\dim C^+ V^{0,1} = 2^{2g-2}$.

Outcome: $V^{2,0}$ gives a line bundle on M_L, $C^+(V)$ gives a family of AVs over M_L and $E^\# \hookrightarrow C^+(V)$ induces $Y_L \to M_L$.

The authors construct an integral model for M_L over $\mathrm{Spec}(\mathbb{Z})$, relying on Kisin-Vasiu, who constructed a model over $\mathbb{Z}[1/2\Delta_L]$.

There is a natural metric on ω associated with $V^{2,0}$, and, using de Rham realisations, ω has a model over $\mathcal{O}_E$. Under suitable conditions on $L \subset E$, also $Y_L \to M_L$ has a model over $\mathcal{O}_E$ and suitable compatibilities with automorphic line bundles. So: $\hat{\omega}$ on M_L pulls back to an arithmetical line bundle on Y_L which we can understand.

3.4. **Hegneer divisors.** If $\lambda \in V$, $B(\lambda, \lambda) > 0$, then its orthogonal $V_\lambda := \lambda^\perp \subset V$ is a quadratic vector space of signature $(2g-3, 2)$. Call $L_\lambda := \lambda^\perp \subset L$. Get an orthogonal Shimura variety M_λ from $Gspin(V_\lambda), L_\lambda$. We also get

$$GSpin(V_\lambda) \to GSpin(V)$$

but $GSpin(L_\lambda)$ does not map into $GSpin(L)$, so we get 'almost' a map of Shimura varieties $M_\lambda \to M_L$. Formally we can 'sum over' M_λ for λ in an $O(L)$-orbit:

$$\sum_\lambda M_\lambda \to M_L.$$

We define Hegneer divisor $Z(m, \mu)$ as the image of this, where the sum is over $\lambda \in \wedge^+ \mu$ s.t. $B(\lambda) = m$, for $m \in \mathbb{Q}$ $\mu \in V$. They generalise Hegneer points on modular curves. They admit integral models $\mathcal{Z}(m, \mu)$.

4. Arithmetic intersection calculations

Let f be a suitable weakly holomorphic modular form

$$f = \sum_{\mu \in L^\vee / L} \sum_{m \in \Delta_L^{-1}\mathbb{Z}} c_f(m, \mu) q^m \varphi_\mu$$

and let

$$Z(f) = \sum_{\mu, m > 0} c(-m, \mu) Z(m, \mu)$$

and call ω_f the line bundle associated with $Z(f)$ on M_L. The theory of Borcherds lift gives a metric on ω_f (using the construction of Burnier-Yang).

Theorem 4.1 (Hormann)**.** *For the right choice of f, we have*

$$\hat{\omega}_f = \hat{\omega}^{\otimes c(0,0)} + \hat{\mathcal{E}}$$

where $\hat{\mathcal{E}}$ is an error term supported at primes dividing the discriminant δ_L.

By varying the lattices, we can get rid of the error term.

Theorem 4.2.
$$\frac{\hat{\deg}(\hat{\omega}_{f|Y_L})}{\deg_{\mathbb{C}} Y_L} = \frac{-2\Lambda'(\chi, 0)}{\Lambda(\chi, 0)} c(0, 0)$$
where Λ is the completed L-function (up to some small errors).

The archimedean part was computed by Brunier-Kudla-Yang. AGHMP computed the finite part.

Conclusion: from these calculations can obtain $h_{Fal}(E^{\#}, \Psi^{\#})$ up to small errors at bad primes. We can do this for different choices of lattices L to get the exact formula.